Exploring the spectrum of the hidden charm strange pentaquark in the SU(4) version of the flavor-spin model

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We study the spectrum of the isoscalar pentaquark $uds c\bar{c}$, of either positive or negative parity, in a constituent quark model with linear confinement and a flavor-spin hyperfine interaction previously extended to SU(4) and used to describe the spectrum of the $uudc\bar{c}$ pentaquarks observed at LHCb in 2019. For positive parity we make a distinction between the case where one unit of angular momentum is located in the subsystem of four quarks and the case where the angular momentum is located in the relative motion between a ground-state four-quark subsystem and the antiquark. The novelty is that we introduce the coupling between different flavor states, due to the breaking of exact SU(4)-flavor symmetry of the Hamiltonian model, both for positive- and negative-parity states. An important consequence is that the lowest state, located at 4404 MeV, has quantum numbers $J^P = 1/2^-$ while without coupling the lowest state has $J^P = 1/2^+$ or $3/2^+$.

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I. INTRODUCTION

The 2019 LHCb observation of the narrow structures $P_c^+(4312)$, $P_c^+(4440)$ and $P_c^+(4457)$ in the $\Lambda_b^0 \to J/\psi K^- p$ decay [1] has given a new impetus to the study of hidden charm pentaquarks. The $J/\psi p$ component suggested that the pentaquark wave functions should have the flavor content $uudc\bar{c}$.

Although observed in the $J/\psi p$ channel, the proximity of the mass of the $P_c^+(4312)$ to the $\Sigma_c^+\bar{D}^0$ threshold (4318 MeV) and of the masses of $P_c^+(4440)$ and $P_c^+(4457)$ to the $\Sigma_c^+\bar{D}^{*0}$ threshold (4460 MeV), favored a molecular S-wave interpretation of the $\Sigma_c^+ + \bar{D}^0$ and $\Sigma_c^+ + \bar{D}^{*0}$ systems respectively [2–15]. In such an interpretation, the binding arises via meson exchanges between point particles and in the elastic channel all resonances acquire a negative parity. However, if one introduces the coupling of the $\Sigma_c^+\bar{D}^{*0}$ and the $\Lambda_c(2595)\bar{D}$ channels, due to the very close proximity of their thresholds, one obtains $J^P(4440) = 3/2^-$ and $J^P(4457) = 1/2^+$ respectively [16].

A more general point of view has been adopted in Ref. [17] where the $P_c(4312)$ signal was analyzed by using some general principles of the S-matrix theory. In this way it was concluded that $P_c(4312)$ is more likely a virtual (unbound) molecular state.

The 2019 LHCb pentaquarks have also been analyzed in compact pentaquark models based on the chromomagnetic interaction of the one-gluon-exchange model, with quark/antiquark correlations [18] or without correlations [19,20]. In both cases the lowest state has negative parity.

Presently, the spin and parity of the narrow structures $P_c^+(4312)$, $P_c^+(4440)$ and $P_c^+(4457)$ remain to be established experimentally.

Anticipating new experiments, the 2019 LHCb successful observation stimulated interest in the theoretical study of analogue pentaquarks in particular of the hidden charm pentaquarks with strangeness: the $udsc\bar{c}$ system. For example, in Ref. [21] it has been analyzed in the framework of a molecular scenario with heavy quark symmetry constraints and in Ref. [22] within the chiral effective theory where the short-range contact interaction, the long-range one-pion-exchange interaction and the intermediate-range two-pion-exchange interaction were included. In Ref. [23] the hidden charm pentaquarks with strangeness have been considered in the hadrocharmonium model.

Predictions for the isoscalar $udsc\bar{c}$ pentaquark have already been made. In Ref. [24] the spectrum of the $udsc\bar{c}$ pentaquark was studied in the compact pentaquark picture in a quark model with either the chromomagnetic, the flavor-spin or the instanton-induced interaction. In all cases it was found that the lowest state has the spin parity $J^P = 1/2^-$. In Ref. [25] the stability of several pentaquark systems has been analyzed in a constituent quark model with a simple chromomagnetic interaction, and the $udsc\bar{c}$ pentaquark has been found among the most stable ones.

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In an SU(4) classification of pentaquarks and its decomposition in SU(3) submultiplets, by selecting those with the charm quantum number $C = 0$, one finds the $udsc\bar{c}$ pentaquark as a member of either an octet with isospin $I = 0$, 1 or as a member of a decuplet with isospin $I = 1$. These SU(3) submultiplets belong to the [421] irreducible representation of SU(4) of dimension 140. The members of the irreducible representation denoted by 140 can have a spin value of either 1/2 or 3/2 [26,27].

The hidden charm pentaquarks that have a strange quark are presently unknown. In principle they can be produced and observed, for example, in the study of the $\Xi_b^- \to J/\psi\Lambda K^-$ reaction [21] or in the decay of Λ_b into $J/\psi\Lambda K^0$ [28]. Their discovery would require much more data relative to the nonstrange hidden charm pentaquarks observed at LHCb [29]. If discovered they may possibly distinguish between the various theoretical pictures.

Here we explore the spectrum of the pentaquark $udsc\bar{c}$ within a quark model [30], which has a flavor-dependent hyperfine interaction. The hyperfine splitting in hadrons is due to the short-range part of the Goldstone-boson-exchange interaction between quarks. The merit of the flavor-spin (FS) model is that it reproduces the correct ordering of positive- and negative-parity states of both nonstrange and strange baryons [30–32] in contrast to the one-gluon-exchange model. However, it cannot explain the hyperfine splitting in mesons, because it does not explicitly contain a quark-antiquark interaction.

It is therefore useful to compare the spectrum of hidden charm nonstrange and hidden charm strange pentaquarks within the same model.

In a previous work [33] the model of Ref. [30] has been generalized from SU(3) to SU(4) in order to incorporate the charm quark. The extension has been made in the spirit of the phenomenological approach of Ref. [34] where, in addition to Goldstone bosons of the hidden approximate chiral symmetry of QCD, the flavor-exchange interaction was augmented by an additional exchange of D mesons between u, d and c quarks and of D_s mesons between s and c quarks. The model provided a satisfactory description of the heavy flavor baryons.

The extended SU(4) flavor-spin model has been applied to the study of $uudc\bar{c}$ pentaquarks. Presently we study the pentaquarks of structure $udsc\bar{c}$ in the same framework considering both positive and negative parities.

The parity of the pentaquark is given by $P = (-)^{\ell+1}$, where ℓ is the orbital angular momentum. As shown in Ref. [33], there are two ways to introduce orbital excitations. For the lowest positive-parity states one way is to introduce an angular momentum $\ell = 1$ in the internal motion of the four-quark subsystem and the other is to introduce a unit of angular momentum in the relative motion between a ground-state four-quark subsystem and the antiquark. According to the Pauli principle, in the first case the four-quark subsystem must be in a state of orbital symmetry $[31]_O$. In the second case the four-quark subsystem is in the ground state $[4]_O$.

In Ref. [33], in the context of a schematic flavor-spin interaction, i.e., exact SU(4) symmetry, it was shown that the lowest pentaquark state has a positive parity with the orbital excitation in the internal motion of the four-quark subsystem. Although the kinetic energy of such a state is higher than that of the totally symmetric $[4]_O$ state of negative parity, the flavor-spin interaction overcomes this excess and generates a lower eigenvalue for the $[31]_O$ state with an s^3p configuration than for $[4]_O$ with an s^4 configuration.

In the exact SU(4) limit the strength of the interaction is the same for all pairs, independent of the quark masses, and it is a constant as a function of the relative distance between the interacting quarks. The model Hamiltonian introduced in the next section breaks the SU(4)-flavor symmetry through the quark masses and the radial dependence of the interaction potential. We calculate the masses of the lowest positive- and negative-parity states of the pentaquarks of structure $udsc\bar{c}$ considering states with flavor symmetry $[22]_F$, $[31]_F$ and $[211]_F$. The SU(4)-flavor symmetry breaking implies the mixing of wave functions containing $[31]_F$ and $[211]_F$ parts. It is shown that this mixing affects the ordering of positive- and negative-parity states and that the lowest-state $udsc\bar{c}$ pentaquark has quantum numbers $J^P = 1/2^-$.

The paper is organized as follows. In Sec. II we introduce the model Hamiltonian and the two-body matrix elements of the FS interaction corresponding to SU(4). Section III describes the orbital part of the four-quark subsystem constructed to be translationally invariant both for positive- and negative-parity states. Sections IV,V and VI summarize analytic formulas. Section VII contains the numerical results for the spectrum and a comparison with relevant previous studies of hidden charm strange pentaquarks. The last section is devoted to conclusions. Appendix A is a reminder of useful group theory formulas for SU(n). Appendix B exhibits a variational solution for the baryon masses relevant for the present study. In Appendix C we present explicit forms of the flavor states of content $udsc$ in the Young-Yamanouchi-Rutherford basis, for specific irreducible representations $[f]_F$.

II. THE HAMILTONIAN

Here we closely follow the description of the model as presented in Ref. [33]. The parameters required by the incorporation of the strange quark were added.

The FS model Hamiltonian has the general form [30]

$$H = \sum_i m_i + \sum_i \frac{\vec{p}_i^2}{2m_i} - \frac{(\sum_i \vec{p}_i)^2}{2\sum_i m_i} + \sum_{i<j} V_{\text{conf}}(r_{ij}) + \sum_{i<j} V_\chi(r_{ij}), \tag{1}$$

with m_i and $\vec{p}_i$ denoting the quark masses and momenta respectively and r_{ij} is the distance between the interacting quarks i and j. The Hamiltonian contains the internal kinetic energy and the linear confining interaction

$$V_{\mathrm{conf}}(r_{ij}) = -\frac{3}{8}\lambda_i^c \cdot \lambda_j^c C r_{ij}. \tag{2}$$

The hyperfine part $V_\chi(r_{ij})$ has a flavor-spin structure that was extended to SU(4) in Ref. [33]. One has

$$\begin{aligned} V_\chi(r_{ij}) = \Bigg\{ &\sum_{F=1}^{3} V_\pi(r_{ij})\lambda_i^F\lambda_j^F + \sum_{F=4}^{7} V_K(r_{ij})\lambda_i^F\lambda_j^F \\ &+ V_\eta(r_{ij})\lambda_i^8\lambda_j^8 + V_{\eta'}(r_{ij})\lambda_i^0\lambda_j^0 \\ &+ \sum_{F=9}^{12} V_D(r_{ij})\lambda_i^F\lambda_j^F + \sum_{F=13}^{14} V_{D_s}(r_{ij})\lambda_i^F\lambda_j^F \\ &+ V_{\eta_c}(r_{ij})\lambda_i^{15}\lambda_j^{15} \Bigg\} \vec{\sigma}_i \cdot \vec{\sigma}_j, \end{aligned} \tag{3}$$

with the SU(4) generators λ_i^F $(F = 1, 2, \ldots, 15)$ and $\lambda_i^0 = \sqrt{2/3}\,\mathbf{1}$, where $\mathbf{1}$ is the 4×4 unit matrix.

In the SU(4) version the interaction (3) contains $\gamma = \pi, K, \eta, D, D_s, \eta_c$ and η' meson-exchange terms. Every $V_\gamma(r_{ij})$ is a sum of two distinct contributions: a Yukawa-type potential containing the mass of the exchanged meson and a short-range contribution of opposite sign, the role of which is crucial in baryon spectroscopy. For a given meson γ the meson-exchange potential is

$$\begin{aligned} V_\gamma(r) = \frac{g_\gamma^2}{4\pi}\frac{1}{12 m_i m_j}\Bigg\{ &\theta(r - r_0)\mu_\gamma^2 \frac{e^{-\mu_\gamma r}}{r} \\ &- \frac{4}{\sqrt{\pi}}\alpha^3 \exp(-\alpha^2 (r - r_0)^2) \Bigg\}. \end{aligned} \tag{4}$$

In the present calculations we use the parameters of Ref. [31] to which we add the μ_D and μ_{D_s} masses and the coupling constants $\frac{g_{Dq}^2}{4\pi}$ and $\frac{g_{D_s q}^2}{4\pi}$. These are

$$\begin{aligned} &\frac{g_{\pi q}^2}{4\pi} = \frac{g_{\eta q}^2}{4\pi} = \frac{g_{Dq}^2}{4\pi} = \frac{g_{D_s q}^2}{4\pi} = 0.67, \qquad \frac{g_{\eta' q}^2}{4\pi} = 1.206, \\ &r_0 = 0.43\ \mathrm{fm}, \qquad \alpha = 2.91\ \mathrm{fm}^{-1}, \qquad C = 0.474\ \mathrm{fm}^{-2}, \\ &\mu_\pi = 139\ \mathrm{MeV}, \qquad \mu_\eta = 547\ \mathrm{MeV}, \\ &\mu_{\eta'} = 958\ \mathrm{MeV}, \qquad \mu_K = 495\ \mathrm{MeV}, \\ &\mu_D = 1867\ \mathrm{MeV}, \qquad \mu_{D_s} = 1968\ \mathrm{MeV}. \end{aligned}$$

The meson masses correspond to the experimental values from the Particle Data Group [35]. As discussed in the following, we ignore the η_c exchange.

The model of Ref. [31] has previously been used to study the stability of open-flavor tetraquarks [36] and open-flavor pentaquarks [37,38]. Accordingly, for the quark masses we take the values determined variationally in Refs. [36,37]

$$\begin{aligned} &m_{u,d} = 340\ \mathrm{MeV}, \qquad m_s = 440\ \mathrm{MeV}, \\ &m_c = 1350\ \mathrm{MeV}. \end{aligned} \tag{5}$$

They were adjusted to satisfactorily reproduce the average mass $\bar{M} = (M + 3M^*)/4 = 2008$ MeV of the D mesons and the mass 2.087 MeV of D_s.

After integration in the flavor space, the two-body matrix elements containing contributions due to light, strange and charm quarks are [33]

$$V_{ij} = \vec{\sigma}_i \cdot \vec{\sigma}_j \begin{cases} V_\pi + \frac{1}{3}V_\eta^{uu} + \frac{1}{6}V_{\eta_c}^{uu}, & [2]_F,\ \ I = 1 \\ 2V_K - \frac{2}{3}V_\eta^{us},\ \ 2V_D^{uc} - \frac{1}{2}V_{\eta_c}^{uc} & [2]_F,\ \ I = \frac{1}{2} \\ 2V_{D_s}^{sc} - \frac{1}{2}V_{\eta_c}^{sc} & [2]_F,\ \ I = 0 \\ \frac{4}{3}V_\eta^{ss} + \frac{3}{2}V_{\eta_c}^{cc} & [2]_F,\ \ I = 0 \\ -2V_{D_s}^{sc} - \frac{1}{2}V_{\eta_c}^{sc} & [11]_F,\ \ I = 0 \\ -2V_K - \frac{2}{3}V_\eta^{us},\ \ -2V_D^{uc} - \frac{1}{2}V_{\eta_c}^{uc} & [11]_F,\ \ I = \frac{1}{2} \\ -3V_\pi + \frac{1}{3}V_\eta^{uu} + \frac{1}{6}V_{\eta_c}^{uu}, & [11]_F,\ \ I = 0 \end{cases} \tag{6}$$

In Eq. (6) the pair of quarks ij is either in a symmetric $[2]_F$ or an antisymmetric $[11]_F$ flavor state and the isospin I is defined by the quark content. The upper index of V exhibits the flavor of the two quarks interchanging a meson specified by the lower index. In order to keep close to the notations of Ref. [30] the upper index of π and K is not indicated. Obviously, in every sum/difference of Eq. (6) the upper index is the same for all terms.

To calculate the matrix elements of the hyperfine interaction (3) between quarks the first step is to decouple the flavor and spin parts of the wave function of partition $[f]_{FS}$ by using Clebsch-Gordan coefficients of the permutation group S_4 [39]. With the usual spin wave functions and the flavor wave functions given in Appendix C, one can reduce the calculation of four-body matrix elements to that of two-body matrix elements. Implementing the expressions (6) one obtains the matrix elements of the flavor-spin interaction (3) for four-quark states in the flavor-spin space. The diagonal matrix elements are presented in Table I.

In the case of $udsc\bar{c}$ pentaquarks there are also nonvanishing off-diagonal matrix elements. These are

TABLE I. The hyperfine interaction V_χ, Eq. (3), integrated in the flavor-spin space, for the quark subsystem $udsc$ with $I=0$. $V_\gamma^{q_a q_b}$ are defined in Eq. (6) where the upper index $q_a q_b$ indicates the flavor of the interacting quark pair.

State	V_χ
$\vert 1\rangle = \vert[31]_O[22]_F[22]_S[4]_{FS}\rangle$	$9\ V_\pi - V_\eta^{uu} - 2V_{\eta'}^{uu} - \frac{1}{2}V_{\eta_c}^{uu} + 6V_K + 6V_D^{uc} + 6V_{D_s}^{sc} + \frac{3}{2}V_{\eta_c}^{sc} - 2V_{\eta'}^{sc}$
$\vert 2\rangle = \vert[31]_O[31]_F[31]_S[4]_{FS}\rangle$	$9\ V_\pi - V_\eta^{uu} - 2V_{\eta'}^{uu} - \frac{1}{2}V_{\eta_c}^{uu} + 6V_K + 6V_D^{uc} + 2V_{D_s}^{sc} - \frac{1}{2}V_{\eta_c}^{sc} + \frac{2}{3}V_{\eta'}^{sc}$
$\vert 3\rangle = \vert[4]_O[211]_F[22]_S[31]_{FS}\rangle$	$\frac{14}{3}V_\pi - \frac{14}{27}V_\eta^{uu} - \frac{28}{27}V_{\eta'}^{uu} - \frac{7}{27}V_{\eta_c}^{uu} + \frac{14}{9}V_K + \frac{14}{27}V_\eta^{us} - \frac{14}{27}V_{\eta'}^{us} + \frac{46}{9}V_D^{uc} + \frac{23}{18}V_{\eta_c}^{uc} - \frac{46}{27}V_{\eta'}^{uc} + \frac{20}{9}V_{D_s}^{sc} + \frac{5}{9}V_{\eta_c}^{sc} - \frac{20}{27}V_{\eta'}^{sc}$
$\vert 3'\rangle = \vert[4]_O[211]'_F[22]_S[31]_{FS}\rangle$	$\frac{13}{3}V_\pi - \frac{13}{27}V_\eta^{uu} - \frac{13}{54}V_{\eta_c}^{uu} - \frac{26}{27}V_{\eta'}^{uu} + \frac{20}{9}V_D^{uc} + \frac{5}{9}V_{\eta_c}^{uc} - \frac{20}{27}V_{\eta'}^{uc} + \frac{52}{9}V_K + \frac{52}{27}V_\eta^{us} - \frac{52}{27}V_{\eta'}^{us} + \frac{10}{9}V_{D_s}^{sc} + \frac{5}{18}V_{\eta_c}^{sc} - \frac{20}{54}V_{\eta'}^{sc}$
$\vert 4\rangle = \vert[4]_O[31]_F[22]_S[31]_{FS}\rangle$	$6V_\pi - \frac{2}{3}V_\eta^{uu} - \frac{4}{3}V_{\eta'}^{uu} - \frac{1}{3}V_{\eta_c}^{uu} + \frac{2}{3}V_D^{uc} + \frac{5}{6}V_{\eta_c}^{uc} - \frac{10}{9}V_{\eta'}^{uc} + \frac{2}{3}V_K + \frac{10}{9}V_\eta^{us} - \frac{10}{9}V_{\eta'}^{us} - \frac{4}{3}V_{D_s}^{sc} + \frac{1}{3}V_{\eta_c}^{sc} - \frac{4}{9}V_{\eta'}^{sc}$

$$\langle 3|V_\chi|3'\rangle = \frac{\sqrt{2}}{9}\bigg(-3V_\pi + \frac{1}{3}V_\eta^{uu} + \frac{1}{6}V_{\eta_c}^{uu} + \frac{2}{3}V_{\eta'}^{uu} + 2V_K + \frac{2}{3}V_\eta^{us} - \frac{2}{3}V_{\eta'}^{us} + 10V_D^{uc} + \frac{5}{2}V_{\eta_c}^{uc} - \frac{2}{3}V_{\eta'}^{uc} - 10V_{D_s}^{sc} - \frac{5}{2}V_{\eta_c}^{sc} + \frac{2}{3}V_{\eta'}^{sc}\bigg), \tag{7}$$

$$\langle 3|V_\chi|4\rangle = \frac{1}{2}\bigg(-6V_\pi + \frac{2}{3}V_\eta^{uu} + \frac{1}{3}V_{\eta_c}^{uu} + \frac{4}{3}V_{\eta'}^{uu} - 8V_K + V_\eta^{us} + \frac{4}{3}V_{\eta'}^{us} - 4V_D^{uc} - 2V_{\eta_c}^{uc} - \frac{4}{3}V_{\eta'}^{uc} + 4V_{D_s}^{sc} - V_{\eta_c}^{sc} + \frac{4}{3}V_{\eta'}^{sc}\bigg), \tag{8}$$

and

$$\langle 3'|V_\chi|4\rangle = \sqrt{2}\bigg(-3V_\pi + \frac{1}{3}V_\eta^{uu} + \frac{1}{6}V_{\eta_c}^{uu} + \frac{2}{3}V_{\eta'}^{uu} + 2V_K + \frac{2}{3}V_\eta^{us} - \frac{2}{3}V_{\eta'}^{us} - 2V_D^{uc} + \frac{1}{2}V_{\eta_c}^{uc} - \frac{2}{3}V_{\eta'}^{uc} + 2V_{D_s}^{sc} - \frac{1}{2}V_{\eta_c}^{sc} + \frac{2}{3}V_{\eta'}^{sc}\bigg). \tag{9}$$

Note that the integration in the orbital space is not yet performed in the diagonal and off-diagonal matrix elements presented above.

To reproduce the exact SU(4) limit one has to take $V_\pi = V_\eta^{uu} = V_{\eta_c}^{uu} = V_D^{uc} = V_{\eta_c}^{uc} = V_K = V_{D_s}^{sc} = V_{\eta_c}^{sc} = -C_\chi$, $V_\eta^{us} = -3/4C_\chi$ and $V_{\eta'}^{uu} = V_{\eta'}^{uc} = V_{\eta'}^{us} = V_{\eta'}^{sc} = 0$. Then, in the exact SU(4) limit, the flavor-spin interaction takes the following form [33]:

$$V_\chi = -C_\chi \sum_{i<j} \lambda_i^F \cdot \lambda_j^F \vec{\sigma}_i \cdot \vec{\sigma}_j, \tag{10}$$

where C_χ is an equal-strength constant for all pairs. Using Appendix A, one can check that the diagonal matrix elements of Table I are $-27C_\chi$, $-21C_\chi$, $-15C_\chi$, $-15C_\chi$ and $-7C_\chi$ respectively. In the exact SU(4) limit the off-diagonal matrix elements of V_χ vanish identically. Thus the lowest state of Table I is $|1\rangle$ because it acquires the largest attraction due to the FS interaction in the exact SU(4) limit. This implies that the lowest state has positive parity, a conclusion which sometimes still holds for broken symmetry, as for example for the $uudd\bar{c}$ pentaquarks [38].

III. ORBITAL SPACE

The orbital wave functions are defined in terms of four internal Jacobi coordinates for pentaquarks chosen as

$$\begin{aligned}
\vec{x} &= \vec{r}_1 - \vec{r}_2, \\
\vec{y} &= (\vec{r}_1 + \vec{r}_2 - 2\vec{r}_3)/\sqrt{3}, \\
\vec{z} &= (\vec{r}_1 + \vec{r}_2 + \vec{r}_3 - 3\vec{r}_4)/\sqrt{6}, \\
\vec{t} &= (\vec{r}_1 + \vec{r}_2 + \vec{r}_3 + \vec{r}_4 - 4\vec{r}_5)/\sqrt{10},
\end{aligned} \tag{11}$$

where 1, 2, 3 and 4 are the quarks and 5 is the antiquark so that t gives the distance between the antiquark and the center-of-mass coordinate of the four-quark subsystem.

For the lowest positive-parity states having $\ell = 1$, there are two ways to introduce orbital excitations [33]. One is to excite the four-quark subsystem, and the other is to include the angular momentum in the relative motion between the four-quark subsystem and the antiquark. Both imply translationally invariant states (no center-of-mass motion).

A. Excited four-quark subsystem, $P = +1$

In this case one has to express the orbital wave functions of the four-quark subsystem of structure s^3p in terms of the internal coordinates $\vec{x}$, $\vec{y}$, $\vec{z}$ for the specific permutation symmetry $[31]_O$. The method of constructing translationally invariant states of definite permutation symmetry containing a unit of angular momentum was first given in Ref. [38] and recently revised in Ref. [33]. The three independent states denoted below by ψ_i, which define the basis vectors of the irreducible representation $[31]_O$ in terms of shell model states $\langle\vec{r}|n\ell m\rangle$ where $n=0$, $\ell=1$, are

$$\psi_1 = \langle\vec{x}|000\rangle\langle\vec{y}|000\rangle\langle\vec{z}|010\rangle, \tag{12}$$

$$\psi_2 = \langle\vec{x}|000\rangle\langle\vec{y}|010\rangle\langle\vec{z}|000\rangle, \tag{13}$$

$$\psi_3 = \langle\vec{x}|010\rangle\langle\vec{y}|000\rangle\langle\vec{z}|000\rangle. \tag{14}$$

In this picture there is no excitation in the relative motion between the cluster of four quarks and the antiquark defined by the coordinate $\vec{t}$. Then the pentaquark orbital wave functions ψ_i^5 are obtained by multiplying each ψ_i from above by the wave function $\langle\vec{t}|000\rangle$ which describes the relative motion between the four-quark subsystem and the antiquark $\bar{c}$. Assuming an exponential behavior we introduce two variational parameters: a for the internal motion of the four-quark subsystem and b for the relative motion between the subsystem $qqqc$ and $\bar{c}$. We explicitly have

$$\psi_1^5 = N \exp\left[-\frac{a}{2}(x^2+y^2+z^2) - \frac{b}{2}t^2\right] z Y_{10}(\hat{z}), \quad (15)$$

$$\psi_2^5 = N \exp\left[-\frac{a}{2}(x^2+y^2+z^2) - \frac{b}{2}t^2\right] y Y_{10}(\hat{y}), \quad (16)$$

$$\psi_3^5 = N \exp\left[-\frac{a}{2}(x^2+y^2+z^2) - \frac{b}{2}t^2\right] x Y_{10}(\hat{x}), \quad (17)$$

where

$$N = \frac{2^{3/2} a^{11/4} b^{3/4}}{3^{1/2}\pi^{5/2}}. \quad (18)$$

B. Excitation between the four-quark subsystem and the antiquark, $P = +1$

The authors of Ref. [24] have studied the $qqqc\bar{c}$ and the $qqsc\bar{c}$ pentaquarks, in three different models, including the FS model. The orbital wave function of the four-quark subsystem has the symmetry $[4]_O$ for both parities. Although the radial wave function was not specified, one can infer that the positive-parity states of Ref. [24] were obtained by including a unit of orbital angular momentum in the relative motion between the four-quark subsystem and the antiquark. The states remain translationally invariant. In this case the orbital wave function takes the form

$$\psi_4^5 = N_4 \exp\left[-\frac{a}{2}(x^2+y^2+z^2) - \frac{b}{2}t^2\right] t Y_{10}(\hat{t}), \quad (19)$$

where

$$N_4 = \frac{8^{1/2} a^{9/4} b^{5/4}}{3^{1/2}\pi^{5/2}}. \quad (20)$$

C. Negative-parity states, $P = -1$

We also need the orbital wave function of the lowest negative-parity state described by the s^4 configuration of the symmetry $[4]_O$ which is

$$\phi_0 = N_0 \exp\left[-\frac{a}{2}(x^2+y^2+z^2) - \frac{b}{2}t^2\right], \quad (21)$$

with

$$N_0 = \left(\frac{a}{\pi}\right)^{9/4}\left(\frac{b}{\pi}\right)^{3/4}. \quad (22)$$

IV. KINETIC ENERGY

The kinetic energy T of the Hamiltonian (1) can be calculated analytically. Below we present the expression of its expectation value for the three cases introduced above.

Case A. In this case the expectation value of the kinetic energy is defined by the average over the three wave functions defined by Eqs. (15)–(17). One obtains

$$\langle T\rangle = \frac{1}{3}[\langle\psi_1^5|T|\psi_1^5\rangle + \langle\psi_2^5|T|\psi_2^5\rangle + \langle\psi_3^5|T|\psi_3^5\rangle] = \hbar^2\left(\frac{11}{2\mu_1}a + \frac{3}{2\mu_2}b\right), \quad (23)$$

with

$$\frac{4}{\mu_1} = \frac{2}{m_q} + \frac{1}{m_s} + \frac{1}{m_Q}, \quad (24)$$

which is the generalization of Eq. (22) of Ref. [33] to include strange quarks and

$$\frac{5}{\mu_2} = \frac{1}{\mu_1} + \frac{4}{m_Q}, \quad (25)$$

where $q = u, d$ and $Q = c$. Here, we have $m_q = 340$ MeV, $m_s = 440$ MeV and $m_c = 1350$ MeV, as defined by Eq. (5). Taking $m_u = m_d = m_s = m_Q = m$ and setting $a = b$, one can recover the identical-particle limit $\langle T\rangle = \frac{7}{2}\hbar\omega$ with $\hbar\omega = 2a\hbar^2/m$.

Case B. In this case there is only one orbital wave function because we deal with the symmetric state $[4]_O$. The orbital excitation is located in the relative motion of the four-quark system and the antiquark. One obtains

$$\langle T\rangle = \hbar^2\left(\frac{9}{2\mu_1}a + \frac{5}{2\mu_2}b\right), \quad (26)$$

where μ_1 and μ_2 are the same as above. Again one can recover the identical-particle limit when $a = b$ but the contributions of the two terms are different because the coefficients $11/2$ and $3/2$ now become $9/2$ and $5/2$ respectively, which is natural because the unit of orbital excitation is no longer located in the four-quark subsystem but in the relative motion between the four-quark subsystem and $\bar{c}$.

Case C. One deals with the symmetric state $[4]_O$ and no orbital excitation. The only orbital state has negative parity and Eq. (21) gives

$$\langle T\rangle = \hbar^2\left(\frac{9}{2\mu_1}a + \frac{3}{2\mu_2}b\right), \tag{27}$$

with μ_1 and μ_2 as above.

V. CONFINEMENT

By integrating in the color space, the expectation value of the confinement interaction (2) has the same form as that of the $uudc\bar{c}$ system [33]

$$\langle V_{\rm conf}\rangle = \frac{C}{2}(6\langle r_{12}\rangle + 4\langle r_{45}\rangle) \tag{28}$$

where $\langle r_{ij}\rangle$ is the interquark distance and the coefficients 6 and 4 account for the number of quark-quark and quark-antiquark pairs, respectively, for all cases A, B and C, but with different expressions for $\langle r_{ij}\rangle$ in each case.

Case A. Here one has

$$\langle r_{ij}\rangle = \frac{1}{3}[\langle\psi_1^5|r_{ij}|\psi_1^5\rangle + \langle\psi_2^5|r_{ij}|\psi_2^5\rangle + \langle\psi_3^5|r_{ij}|\psi_3^5\rangle], \tag{29}$$

where $i,\ j = 1,2,3,4,5$ $(i\neq j)$. An analytic evaluation gives

$$\langle r_{12}\rangle = \frac{20}{9}\sqrt{\frac{1}{\pi a}}, \tag{30}$$

and

$$\langle r_{45}\rangle = \frac{1}{3\sqrt{2\pi}}\left[2\sqrt{\frac{3}{a}+\frac{5}{b}} + \sqrt{5b}\left(\frac{1}{2a}+\frac{1}{b}\right)\right]. \tag{31}$$

Case B. The expectation value of the confinement interaction is given by Eq. (28) with

$$\langle r_{12}\rangle = \sqrt{\frac{4}{\pi a}}, \tag{32}$$

and

$$\langle r_{45}\rangle = \frac{2}{3}\sqrt{\frac{2b}{5\pi}}\left(\frac{3}{4a}+\frac{5}{b}\right). \tag{33}$$

Case C. In this case the four quarks are in the s^4 configuration described by the states $|3\rangle$, $|3'\rangle$ or $|4\rangle$ and there is no orbital excitation at all. The expectation value of the confinement interaction is given by Eq. (28) as well, with

$$\langle r_{12}\rangle = \sqrt{\frac{4}{\pi a}}, \tag{34}$$

and

$$\langle r_{45}\rangle = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{3}{a}+\frac{5}{b}}. \tag{35}$$

VI. FLAVOR-SPIN INTERACTION

In order to integrate the expressions of Table I and Eqs. (7)–(9) in the orbital space one has to decouple the orbital part of the wave function $[f]_O$ from the part containing the other degrees of freedom by using Clebsch-Gordan coefficients of the permutation group S_4 [39]. The next step is to reduce the matrix elements of the hyperfine interaction V^χ of Eq. (3) of the four-quark system to matrix elements of two quarks. Table I gives the diagonal matrix elements and Eqs. (7)–(9) gives the off-diagonal ones. As there are six pairs, the contribution of one pair is one sixth of the above expressions.

For states of type A with one unit of orbital excitation the result is a linear combination of orbital two-body matrix elements of type $\langle ss|V_\gamma^{q_aq_b}|ss\rangle$, $\langle sp|V_\gamma^{q_aq_b}|sp\rangle$ and $\langle sp|V_\gamma^{q_aq_b}|ps\rangle$. For states of type B or C there are two-body matrix elements between single-particle s states, namely $\langle ss|V_\gamma^{q_aq_b}|ss\rangle$. In every term q_aq_b is a pair of quarks from Eq. (6).

VII. RESULTS AND DISCUSSION

We have looked for variational solutions of the Hamiltonian of Sec. II using the orbital part of the wave functions as described in Sec. III, which contain the parameters a and b. The wave functions are the product of the four-quark subsystem states of flavor-spin structure defined in Table I and the charm antiquark wave function denoted by $|\bar{c}\rangle$. The total angular momentum is $\vec{J} = \vec{L} + \vec{S} + \vec{s}_Q$, where $\vec{L}$ and $\vec{S}$ are the angular momentum and spin of the four-quark cluster and $\vec{s}_Q$ is the spin of the heavy antiquark.

We have neglected the contributions of $V_{\eta_c}^{uu}$, $V_{\eta_c}^{uc}$ and $V_{\eta_c}^{sc}$ because little $u\bar{u}$, $d\bar{d}$ and $s\bar{s}$ are expected in η_c. We have also neglected $V_{\eta'}^{uc}$ and $V_{\eta'}^{sc}$ by assuming a little $c\bar{c}$ component in η'. Thus, in the expressions of Table I we took

$$V_{\eta_c}^{uu} = V_{\eta_c}^{uc} = V_{\eta'}^{uc} = V_{\eta_c}^{sc} = V_{\eta'}^{sc} = 0. \tag{36}$$

For Case A the numerical results are presented in Table II. The eigenvalues of $|1\rangle|\bar{c}\rangle$ and $|2\rangle|\bar{c}\rangle$ states are degenerate for the allowed values of J in each case. For $|2\rangle|\bar{c}\rangle$ the states with $J^P = 1/2^+$ and $3/2^+$ have multiplicity 2. The optimal values found for the parameters a and b are the same for both states. We found that the ratio of the matrix elements of the K- and π-meson exchange is about 0.74, close to the quark mass ratio $m_{u,d}/m_s$ and the matrix

TABLE II. Lowest positive -parity $udsc\bar{c}$ pentaquarks of quantum numbers S and J^P and symmetry structure $|1\rangle$ and $|2\rangle$ defined in Table I. Column 1 gives the state, column 2 gives the spin, column 3 gives the parity and total angular momentum, column 4 gives the optimal variational parameters associated to the wave functions defined in Sec. III, and column 5 gives the calculated mass.

			Variational parameters		Mass
State	S	J^P	a (fm^{-2})	b (fm^{-2})	(GeV)
$\vert 1\rangle\vert\bar{c}\rangle$	$\frac{1}{2}$	$\frac{1}{2}^+, \frac{3}{2}^+$	1.798	1.053	4442
$\vert 2\rangle\vert\bar{c}\rangle$	$\frac{1}{2}$	$\frac{1}{2}^+, \frac{3}{2}^+, \frac{5}{2}^+$	1.798	1.053	4495

elements of the K- and D-meson exchange is about 0.34 close to the ratio m_s/m_c.

For Case B the masses and the mixing coefficients of the $1/2^+$ and $3/2^+$ states, obtained from the combination of the basis vectors $|3\rangle|\bar{c}\rangle$, $|3'\rangle|\bar{c}\rangle$ and $|4\rangle|\bar{c}\rangle$ are presented in Table III. The optimal variational parameters are the same as in Table II. The mixing coefficients all turn out to be large for the lowest state of 4493 MeV. The next state at 4614 MeV is dominantly a $|3'\rangle|\bar{c}\rangle$ state and the last eigenstate at 5075 MeV is mostly a combination of $|3\rangle|\bar{c}\rangle$ and $|4\rangle|\bar{c}\rangle$ due to the large off-diagonal matrix element (9) where the dominant π- and K-meson exchanges contribute with the same sign.

Case C corresponding to the negative-parity $1/2^-$ state is shown in Table IV. The mixing coefficients are the same as those of Table III, because they result from the diagonalization of a hyperfine interaction identical to that of Case B. The difference between these cases appears only in the kinetic and the confinement matrices, which are diagonal. Hence, in Case C the masses can be obtained from those of Table III by lowering each of them by 89 MeV which is precisely the difference in the kinetic energy plus the confinement energy between Case B and Case C. The largest mixing is between the states $|3\rangle|\bar{c}\rangle$ and $|4\rangle|\bar{c}\rangle$. The diagonal matrix element of the Hamiltonian $\langle 3\bar{c}|H|3\bar{c}\rangle$ is lowered from 4612 to 4404 MeV and the value of $\langle 4\bar{c}|H|4\bar{c}\rangle$ is increased from 4786 to 4986 MeV.

TABLE III. The mass and the mixing coefficients of states of positive parity $|3\rangle|\bar{c}\rangle$, $|3'\rangle|\bar{c}\rangle$ and $|4\rangle|\bar{c}\rangle$ defined in Table I with $L=1$, $S=0$, $J^P = 1/2^+, 3/2^+$ obtained from the orbital wave function of Case B with $a = 1.798$ fm^{-2} and $b = 1.053$ fm^{-2}.

Mass (MeV)	$\vert 3\rangle\vert\bar{c}\rangle$	$\vert 3'\rangle\vert\bar{c}\rangle$	$\vert 4\rangle\vert\bar{c}\rangle$
4493	0.748	0.324	−0.579
4614	0.326	−0.939	−0.104
5075	−0.578	−0.111	−0.808

TABLE IV. The mass and the mixing coefficients of states of negative parity, Case C, diagonalized in the basis $|3\rangle$, $|3'\rangle$ and $|4\rangle$ defined in Table I with $L=0$, $S=0$, $J^P = 1/2^-$. The variational parameters of the orbital wave function are $a = 1.798$ fm^{-2} and $b = 1.053$ fm^{-2}.

Mass (MeV)	$\vert 3\rangle$	$\vert 3'\rangle$	$\vert 4\rangle$
4404	0.748	0.324	−0.579
4525	0.326	−0.939	−0.104
4986	−0.578	−0.111	−0.808

Looking at Tables II, III and IV one can see that the lowest mass is 4404 MeV. Thus the lowest pentaquark $udsc\bar{c}$ has quantum numbers $J^P = 1/2^-$, in contrast to the lowest pentaquark $uudc\bar{c}$ for which it was found that $J^P = 1/2^+$ in Ref. [33].

The mixing of states $|3\rangle|\bar{c}\rangle$, $|3'\rangle|\bar{c}\rangle$ and $|4\rangle|\bar{c}\rangle$ was first discussed in Ref. [24] with the corresponding notation $|3\rangle \to |1\rangle$, $|3'\rangle \to |1'\rangle$ and $|4\rangle \to |2\rangle$ where the quark model of Ref. [34] with a harmonic oscillator confinement and a simplified hyperfine interaction were used. The mixing was introduced for $J^P = 1/2^-$ only, Case C. There the $J^P = 1/2^-$ state appears at 4084 MeV and the $J^P = 1/2^+$ state at 4291 MeV, i.e., about 200 MeV above the lowest negative-parity state. Thus the lowest $J^P = 1/2^-$ state of Ref. [24] is about 300 MeV lower than in the present case.

The $J^P = 1/2^-$ states found in this study are located within the energy range of the $J^P = 1/2^-$ resonances predicted in Ref. [21]. There only s-wave meson-baryon interactions were considered so that only negative-parity states were discussed. Their coupling to the $J/\psi\Lambda$ channel was found to be small, but large enough to provide convenient production rates. The masses of hidden charm strange pentaquarks with $J^P = 1/2^-$ found in Ref. [22] within a chiral effective field theory are located as well in the energy range predicted in the present work. A similar mass range was found in Ref. [23] in a hadrocharmonium picture, with the difference that the lowest state has positive parity.

VIII. CONCLUSIONS

We have calculated a few of the lowest masses of the hidden charm strange pentaquarks $udsc\bar{c}$, in the SU(4) version of the flavor-spin model introduced in Ref. [33] where it was applied to $uudc\bar{c}$ pentaquarks. The model provides an isospin dependence and an internal structure of pentaquarks. For positive parity the angular momentum can be located in the internal motion of the four-quark subsystem, Case A, or in the relative motion between the four-quark subsystem and the antiquark, Case B.

According to the discussion presented in Ref. [33] for exact SU(4) symmetry the lowest positive pentaquark state has positive parity when the orbital excitation is located in the internal motion of the four-quark subsystem. For broken SU(4) such a result remained valid for the $uudc\bar{c}$ pentaquark. In the present analysis it was found that the lowest state of the $udsc\bar{c}$ pentaquark has negative parity.

This is due to the breaking of SU(4)-flavor symmetry which implies the coupling of states of different flavor symmetries $[f]_F$. This coupling considerably lowers the negative-parity state and not so much the positive-parity ones. As a consequence, the negative-parity state $J^P = 1/2^-$, without any orbital excitation, Case C, was found to have the lowest mass of 4404 MeV, followed by the lowest positive-parity states $J^P = 1/2^+$ or $3/2^+$ with a mass of 4442 MeV.

There is an important difference between $udsc\bar{c}$ and $uudc\bar{c}$ pentaquarks due to the presence of the quark s. The $udsc\bar{c}$ pentaquark has two Weyl tableaux associated to the irreducible representation [211] of the four-quark subsystem at $I = 0$, as shown in Appendix C. Due to the Pauli principle the $uudc\bar{c}$ pentaquark has only one Weyl tableau associated to the irreducible representation [211]. Accordingly, in the $udsc\bar{c}$ pentaquark there are three states which can couple due to the SU(4) breaking, the $|3\rangle$, $|3'\rangle$ and $|4\rangle$, as shown in the present study. As mentioned above, this coupling brings the lowest $J^P = 1/2^-$ state below the lowest positive-parity states $J^P = 1/2^+$ or $3/2^+$.

In the $uudc\bar{c}$ pentaquark, there are only two flavor states which, in principle, can couple due to the breaking of SU(4). They are of type $|3\rangle$ and $|4\rangle$ with appropriate Weyl tableaux. We found that the coupling between the states of symmetry $|3\rangle = |[4]_O[211]_F[22]_S[31]_{FS}\rangle$ and $|4\rangle = |[4]_O[31]_F[22]_S[31]_{FS}\rangle$ vanish identically for the $uudc\bar{c}$ pentaquark. Therefore the lowest state in the $uudc\bar{c}$ pentaquark has positive parity, as shown in Ref. [33]. This conclusion is at variance with the result of Ref. [24] where $|3\rangle$ and $|4\rangle$ mix together. A possible reason for the discrepancy is that the three flavor states of symmetry [31], as defined by Eqs. (A.9)–(A.11) of Ref. [24] do not form a proper Young-Yamanouchi basis for the irreducible representation [31] of the permutation group S_4.

We recall that the parity sequence of the $uudc\bar{c}$ pentaquark studied in the hadrocharmonium model [40] was similar to ours [33], namely that the lowest pentaquark state has $J^P = 1/2^+$ quantum numbers. In the hadrocharmonium model description of Ref. [23] the lowest state of the $udsc\bar{c}$ pentaquark has positive parity, contrary to the present result.

Therefore, in the flavor-spin model the presence of the strange quark brings more richness to the flavor structure and changes the parity order of the lowest two states in the $udsc\bar{c}$ pentaquark relative to the $uudc\bar{c}$ pentaquark.

The J^P quantum numbers of the 2019 LHCb resonances are not yet known. Likewise, for possible future observations the spin and parity will be essential to discriminate between the existing interpretations of pentaquarks, or inspire new developments.

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APPENDIX A: EXACT SU(4) LIMIT

The exact SU(4) limit is useful in checking the integration in the flavor space, made in Table I. In this limit every expectation value of Table I reduces to the expectation value of Eq. (10) and one can use the following formula [27]:

$$\langle \sum_{i<j} \lambda_i^F \cdot \lambda_j^F \vec{\sigma}_i \cdot \vec{\sigma}_j \rangle = 4C_2^{\mathrm{SU}(2n)} - 2C_2^{\mathrm{SU}(n)} - \frac{4}{k} C_2^{\mathrm{SU}(2)} - k\frac{3(n^2-1)}{n} \tag{A1}$$

where n is the number of flavors and k is the number of quarks; here $n = 4$ and $k = 4$. $C_2^{\mathrm{SU}(n)}$ are the Casimir operator eigenvalues of SU(n) which can be derived from the expression [41]

$$C_2^{\mathrm{SU}(n)} = \frac{1}{2}[f'_1(f'_1 + n - 1) + f'_2(f'_2 + n - 3) + f'_3(f'_3 + n - 5) + f'_4(f'_4 + n - 7) + \cdots + f'_{n-1}(f'_{n-1} - n + 3)] - \frac{1}{2n}\left(\sum_{i=1}^{n-1} f'_i\right)^2 \tag{A2}$$

where $f'_i = f_i - f_n$, for an irreducible representation given by the partition $[f_1, f_2, \ldots, f_n]$. Equation (A1) has been previously used for $n = 3$ and $k = 6$ in Ref. [41].

APPENDIX B: THE BARYONS

The masses of ground-state baryons relevant to the study of $udsc\bar{c}$ pentaquarks with isospin $I = 0$ were estimated variationally by using a radial wave function of the form $\phi \propto \exp[-\frac{a}{2}(x^2 + y^2)]$ containing the variational parameter a and the coordinates x and y defined by Eq. (11). The results are indicated in Table V together with the experimental masses. We took $V^{uc}_{\eta_c} = V^{uc}_{\eta'} = V^{sc}_{\eta_c} = V^{sc}_{\eta'} = 0$. The resulting charmed baryon masses are about 100 MeV lower than the experimental values. By increasing the charmed quark mass from $m_c = 1.35$ to $m_c = 1.45$ GeV the agreement with the experiment would be much better. However,

TABLE V. Masses of ground-state baryons with the flavor-spin interaction of Sec. II. Column 1 gives the baryon, column 2 gives the isospin, column 3 gives the spin and parity, column 4 gives the calculated mass, column 5 gives the variational parameter and the last column gives the experimental mass.

Baryon	I	J^P	Calculated Mass (GeV)	a (fm^{-2})	Experimental mass (GeV)
Λ	0	$\frac{1}{2}^+$	1.165	2.484	1.116
Λ_c	0	$\frac{1}{2}^+$	2.180	2.055	2.283
Ξ_c	0	$\frac{1}{2}^+$	2.304	1.797	2.469

we prefer to use the same parameters as in Ref. [33] in order to make a comparison with the $uudc\bar{c}$ pentaquarks.

APPENDIX C: THE FLAVOR WAVE FUNCTIONS

The four-quark flavor states with quark content $udsc$ defining the basis vectors of the irreducible representations $[31]_F$, $[22]_F$, $[211]_F$ and $[1111]_F$ have been given in Ref. [24] for $I = 0$. We have checked them with the method of Ref. [42]. In Ref. [24] the flavor states were defined in the Young-Yamanouchi basis. The order of particles is always 1234 in every term.

In Table VI, except for $[1111]_F$, which is not needed here, we give the correspondence between the Young-Yamanouchi basis and the notation of Ref. [24] for each Yamanouchi symbol which is a compact notation for a Young tableau. For a tableau with n particles it is defined by $Y = (r_n, r_{n-1}, \ldots, r_1)$ where r_i represents the row of the particle i. The Weyl tableaux are indicated for each irreducible representation.

Here we write the flavor states in terms of products of symmetric $\phi_{[2]}(q_a q_b) = (q_a q_b + q_b q_a)/\sqrt{2}$ or antisymmetric $\phi_{[11]}(q_a q_b) = (q_a q_b - q_b q_a)/\sqrt{2}$ quark pair states for the pairs 12 and 34. This allows a straightforward calculation of the flavor-integrated matrix elements (6) and in addition one can easily read off the isospin of the corresponding wave function.

For the irrep [22] there are two basis vectors and their expressions are straightforward because the pairs 12 and 34 are always either in a symmetric or antisymmetric pair. We have

$$|[22]_F 2211\rangle = \frac{1}{2}[\phi_{[2]}(us)\phi_{[2]}(cd) + \phi_{[2]}(cd)\phi_{[2]}(us) - \phi_{[2]}(sd)\phi_{[2]}(uc) - \phi_{[2]}(uc)\phi_{[2]}(sd)] \tag{C1}$$

TABLE VI. The $I = 0$ $udsc$ flavor states in two different notations and the corresponding Weyl tableaux.

Young-Yamanouchi	Ref. [24]	Weyl tableau
$[22]_F 2211$	$[22]_{F_1}$	u s / d c
$[22]_F 2121$	$[22]_{F_2}$	
$[31]_F 2111$	$[31]_{F_1}$	u s c / d
$[31]_F 1211$	$[31]_{F_2}$	
$[31]_F 1121$	$[31]_{F_3}$	
$[211]_F 3211$	$[211]_{F_1}$	u s / d / c
$[211]_F 3121$	$[211]_{F_2}$	
$[211]_F 1321$	$[211]_{F_3}$	
$[211]'_F 3211$	$[211]'_{F_1}$	u c / d / s
$[211]'_F 3121$	$[211]'_{F_2}$	
$[211]\prime_F 1321$	$[211]'_{F_3}$	

and

$$|[22]_F 2121\rangle = \sqrt{\frac{1}{12}}[2\phi_{[11]}(ud)\phi_{[11]}(sc) + 2\phi_{[11]}(sc)\phi_{[11]}(ud) + \phi_{[11]}(uc)\phi_{[11]}(sd) + \phi_{[11]}(sd)\phi_{[11]}(uc) - \phi_{[11]}(us)\phi_{[11]}(cd) - \phi_{[11]}(cd)\phi_{[11]}(us)] \tag{C2}$$

where Eq. (C2) obviously has isospin $I = 0$ which means that the pairs 12 and 34 in Eq. (C1) have to couple to the same isospin as well.

For irrep $[31]_F$ the vectors $[31]_{F_1}$ and $[31]_{F_2}$ have to be combined in the so-called Young-Yamanouchi-Rutherford basis first proposed in the context of nuclear physics [43,44]. It is defined such that the last two particles are either in a symmetric or antisymmetric state. The pair 12 is also in a symmetric or antisymmetric state, which is very advantageous. For more than four particles the problem is more complicated. Here we have [42]

$$|[31]_F \overline{12}11\rangle = \sqrt{\frac{2}{3}}|[31]_F 1211\rangle + \sqrt{\frac{1}{3}}|[31]_F 2111\rangle \tag{C3}$$

where on the left-hand side both pairs 12 and 34 are in a symmetric state and

$$|[31]_F \widetilde{12}11\rangle = \sqrt{\frac{1}{3}}|[31]_F 1211\rangle - \sqrt{\frac{2}{3}}|[31]_F 2111\rangle \tag{C4}$$

where the pair 12 is in a symmetric state and 34 is in an antisymmetric state. Using Eqs. (A.16) and (A.15) of Ref. [24], defining $[31]_{F_2}$ and $[31]_{F_1}$ respectively, one obtains

$$|[31]_F \overline{12}11\rangle = \frac{1}{2}[\phi_{[2]}(us)\phi_{[2]}(cd) - \phi_{[2]}(cd)\phi_{[2]}(us) + \phi_{[2]}(uc)\phi_{[2]}(ds) - \phi_{[2]}(ds)\phi_{[2]}(uc)], \tag{C5}$$

and

$$|[31]_F \widetilde{12}11\rangle = \sqrt{\frac{1}{8}}[\phi_{[2]}(uc)\phi_{[11]}(ds) - \phi_{[2]}(us)\phi_{[11]}(cd) - \phi_{[2]}(cd)\phi_{[11]}(us) - \phi_{[2]}(ds)\phi_{[11]}(uc) - 2\phi_{[2]}(sc)\phi_{[11]}(ud)]. \tag{C6}$$

The state (C6) obviously has $I = 0$ and thus Eq. (C5) should also have $I = 0$.

The third basis vector $[31]_{F_3}$ of Ref. [24] can simply be rewritten as

$$|[31]_F 1121\rangle = \sqrt{\frac{1}{8}}[2\phi_{[11]}(ud)\phi_{[2]}(sc) - \phi_{[11]}(ds)\phi_{[2]}(uc) + \phi_{[11]}(cd)\phi_{[2]}(us) + \phi_{[11]}(us)\phi_{[2]}(cd) + \phi_{[11]}(uc)\phi_{[2]}(ds)], \quad \text{(C7)}$$

where the pair 12 is in an antisymmetric state and 34 is in a symmetric state. The state obviously has $I = 0$.

For the irrep $[211]_F$ the Young-Yamanouchi-Rutherford basis vectors are

$$|[211]_F\overline{1321}\rangle = \sqrt{\frac{2}{3}}|[211]_F 1321\rangle + \sqrt{\frac{1}{3}}|[211]_F 3121\rangle \quad \text{(C8)}$$

where the pair 12 is in an antisymmetric state and 34 is in a symmetric state and

$$|[211]_F\widetilde{1321}\rangle = \sqrt{\frac{1}{3}}|[211]_F 1321\rangle - \sqrt{\frac{2}{3}}|[31]_F 3121\rangle \quad \text{(C9)}$$

where both pairs 12 and 34 are in an antisymmetric state. Using Eqs. (A.20) and (A.19) of Ref. [24] one obtains

$$|[211]_F\overline{1321}\rangle = \sqrt{\frac{1}{24}}[2\phi_{[11]}(ud)\phi_{[2]}(sc) - 3\phi_{[11]}(uc)\phi_{[2]}(ds) - 3\phi_{[11]}(cd)\phi_{[2]}(us) + \phi_{[11]}(us)\phi_{[2]}(cd) - \phi_{[11]}(ds)\phi_{[2]}(uc)] \quad \text{(C10)}$$

and

$$|[211]_F\widetilde{1321}\rangle = \sqrt{\frac{1}{12}}[-2\phi_{[11]}(ud)\phi_{[11]}(sc) + 2\phi_{[11]}(sc)\phi_{[11]}(ud) - \phi_{[11]}(uc)\phi_{[11]}(ds) - \phi_{[11]}(cd)\phi_{[11]}(us) + \phi_{[11]}(us)\phi_{[11]}(cd) + \phi_{[11]}(ds)\phi_{[11]}(uc)]. \quad \text{(C11)}$$

The vector $[211]_{F_1}$ of Ref. [24] can be rewritten as

$$|[211]_F 3211\rangle = \sqrt{\frac{1}{24}}[\phi_{[2]}(uc)\phi_{[11]}(sd) + \phi_{[2]}(cd)\phi_{[11]}(us) + 2\phi_{[2]}(cs)\phi_{[11]}(ud) - 3\phi_{[2]}(sd)\phi_{[11]}(uc) - 3\phi_{[2]}(us)\phi_{[11]}(cd)]. \quad \text{(C12)}$$

For the irrep $[211]'_F$ the Young-Yamanouchi-Rutherford basis vectors are defined like in Eqs. (C8) and (C9) but on the right-hand side one must use the vectors $[211]'_{F_i}$ instead of $[211]_{F_i}$, i.e., Eqs. (A.23) and (A.22) of Ref. [24]. One obtains

$$|[211]'_F\overline{1321}\rangle = \sqrt{\frac{1}{3}}[\phi_{[11]}(ud)\phi_{[2]}(sc) - \phi_{[11]}(us)\phi_{[2]}(cd) + \phi_{[11]}(ds)\phi_{[2]}(uc)], \quad \text{(C13)}$$

and

$$|[211]'_F\widetilde{1321}\rangle = \sqrt{\frac{1}{6}}[\phi_{[11]}(ud)\phi_{[11]}(sc) + \phi_{[11]}(us)\phi_{[11]}(cd) + \phi_{[11]}(ds)\phi_{[11]}(uc) - \phi_{[11]}(cd)\phi_{[11]}(us) - \phi_{[11]}(sc)\phi_{[11]}(ud) - \phi_{[11]}(uc)\phi_{[11]}(ds)]. \quad \text{(C14)}$$

They obviously have $I = 0$. The third basis vector $[211]'_{F_1}$ can be rewritten in the convenient form

$$|[211]'_F 3211\rangle = \sqrt{\frac{1}{3}}[\phi_{[2]}(sc)\phi_{[11]}(ud) - \phi_{[2]}(cd)\phi_{[11]}(us) - \phi_{[2]}(uc)\phi_{[11]}(sd)] \quad \text{(C15)}$$

which also has $I = 0$.

[1] R. Aaij *et al.* (LHCb Collaboration), Observation of a Narrow Pentaquark State, $P_c(4312)^+$, and of Two-Peak Structure of the $P_c(4450)^+$, Phys. Rev. Lett. **122**, 222001 (2019).

[2] Z. H. Guo and J. A. Oller, Anatomy of the newly observed hidden-charm pentaquark states: $P_c(4312)$, $P_c(4440)$ and $P_c(4457)$, Phys. Lett. B **793**, 144 (2019).

[3] F. K. Guo, H. J. Jing, U. G. Meißner, and S. Sakai, Isospin breaking decays as a diagnosis of the hadronic molecular structure of the $P_c(4457)$, Phys. Rev. D **99**, 091501 (2019).

[4] C. J. Xiao, Y. Huang, Y. B. Dong, L. S. Geng, and D. Y. Chen, Exploring the molecular scenario of $P_c(4312)$, $P_c(4440)$, and $P_c(4457)$, Phys. Rev. D **100**, 014022 (2019).

[5] C. W. Xiao, J. Nieves, and E. Oset, Heavy quark spin symmetric molecular states from $\bar{D}^{(*)}\Sigma_c^{(*)}$ and other coupled channels in the light of the recent LHCb pentaquarks, Phys. Rev. D **100**, 014021 (2019).

[6] Y. Shimizu, Y. Yamaguchi, and M. Harada, Heavy quark spin multiplet structure of $P_c(4312)$, $P_c(4440)$, and $P_c(4457)$, arXiv:1904.00587.

[7] Y. H. Lin and B. S. Zou, Strong decays of the latest LHCb pentaquark candidates in hadronic molecule pictures, Phys. Rev. D **100**, 056005 (2019).
[8] M. Z. Liu, Y. W. Pan, F. Z. Peng, M. S. Sánchez, L. S. Geng, A. Hosaka, and M. P. Valderrama, Emergence of a Complete Heavy-Quark Spin Symmetry Multiplet: Seven Molecular Pentaquarks in Light of the Latest LHCb Analysis, Phys. Rev. Lett. **122**, 242001 (2019).
[9] L. Meng, B. Wang, G. J. Wang, and S. L. Zhu, The hidden charm pentaquark states and $\Sigma_c\bar{D}^{(*)}$ interaction in chiral perturbation theory, Phys. Rev. D **100**, 014031 (2019).
[10] Q. Wu and D. Y. Chen, Production of P_c states from Λ_b decay, Phys. Rev. D **100**, 114002 (2019).
[11] M. Pavon Valderrama, One pion exchange and the quantum numbers of the $P_c(4440)$ and $P_c(4457)$ pentaquarks, Phys. Rev. D **100**, 094028 (2019).
[12] M. L. Du, V. Baru, F. K. Guo, C. Hanhart, U. G. Meißner, J. A. Oller, and Q. Wang, Interpretation of the LHCb P_c States as Hadronic Molecules and Hints of a Narrow $P_c(4380)$, Phys. Rev. Lett. **124**, 072001 (2020).
[13] G. J. Wang, L. Y. Xiao, R. Chen, X. H. Liu, X. Liu, and S. L. Zhu, Probing hidden-charm decay properties of P_c states in a molecular scenario, arXiv:1911.09613.
[14] H. Xu, Q. Li, C. H. Chang, and G. L. Wang, Recently observed P_c as molecular states and possible mixture of $P_c(4457)$, Phys. Rev. D **101**, 054037 (2020).
[15] H. X. Chen, Decay properties of P_c states through the Fierz rearrangement, arXiv:2001.09563.
[16] T. J. Burns and E. S. Swanson, Molecular interpretation of the $P_c(4440)$ and $P_c(4457)$ states, Phys. Rev. D **100**, 114033 (2019).
[17] C. Fernandez-Ramirez, A. Pilloni, M. Albaladejo, A. Jackura, V. Mathieu, M. Mikhasenko, J. A. Silva-Castro, and A. P. Szczepaniak (JPAC Collaboration), Interpretation of the LHCb $P_c(4312)^+$ Signal, Phys. Rev. Lett. **123**, 092001 (2019).
[18] A. Ali and A. Y. Parkhomenko, Interpretation of the narrow $J/\psi p$ Peaks in $\Lambda_b \to J/\psi p K^-$ decay in the compact diquark model, Phys. Lett. B **793**, 365 (2019).
[19] X. Z. Weng, X. L. Chen, W. Z. Deng, and S. L. Zhu, Hidden-charm pentaquarks and P_c states, Phys. Rev. D **100**, 016014 (2019).
[20] J. B. Cheng and Y. R. Liu, $P_c(4457)^+$, $P_c(4440)^+$, and $P_c(4312)^+$: Molecules or compact pentaquarks?, Phys. Rev. D **100**, 054002 (2019).
[21] C. W. Xiao, J. Nieves, and E. Oset, Prediction of hidden charm strange molecular baryon states with heavy quark spin symmetry, Phys. Lett. B **799**, 135051 (2019).
[22] B. Wang, L. Meng, and S. L. Zhu, Spectrum of the strange hidden charm molecular pentaquarks in chiral effective field theory, Phys. Rev. D **101**, 034018 (2020).
[23] J. Ferretti and E. Santopinto, Hidden-charm and bottom tetra- and pentaquarks with strangeness in the hadro-quarkonium and compact tetraquark models, arXiv:2001.01067.
[24] S. G. Yuan, K. W. Wei, J. He, H. S. Xu, and B. S. Zou, Study of $qqqc\bar{c}$ five quark system with three kinds of quark-quark hyperfine interaction, Eur. Phys. J. A **48**, 61 (2012).
[25] W. Park, S. Cho, and S. H. Lee, Where is the stable pentaquark?, Phys. Rev. D **99**, 094023 (2019).
[26] B. Wu and B. Q. Ma, Exotic baryons with charm number +-1 from Skyrme model, Phys. Rev. D **70**, 034025 (2004).
[27] E. Ortiz-Pacheco, R. Bijker, and C. Fernandez-Ramirez, Hidden charm pentaquarks: Mass spectrum, magnetic moments, and photocouplings, J. Phys. G **46**, 065104 (2019).
[28] J. X. Lu, E. Wang, J. J. Xie, L. S. Geng, and E. Oset, The $\Lambda_b \to J/\psi K^0 \Lambda$ reaction and a hidden-charm pentaquark state with strangeness, Phys. Rev. D **93**, 094009 (2016).
[29] A. Ali, I. Ahmed, M. J. Aslam, A. Y. Parkhomenko, and A. Rehman, Mass spectrum of the hidden-charm pentaquarks in the compact diquark model, J. High Energy Phys. 10 (2019) 256.
[30] L. Y. Glozman and D. O. Riska, The spectrum of the nucleons and the strange hyperons and chiral dynamics, Phys. Rep. **268**, 263 (1996).
[31] L. Y. Glozman, Z. Papp, and W. Plessas, Light baryons in a constituent quark model with chiral dynamics, Phys. Lett. B **381**, 311 (1996).
[32] L. Y. Glozman, Z. Papp, W. Plessas, K. Varga, and R. F. Wagenbrunn, Light and strange baryons in a chiral constituent-quark model, Nucl. Phys. **A623**, 90C (1997).
[33] F. Stancu, Spectrum of the $uudc\bar{c}$ hidden charm pentaquark with an SU(4) flavor-spin hyperfine interaction, Eur. Phys. J. C **79**, 957 (2019).
[34] L. Y. Glozman and D. O. Riska, The charm and bottom hyperons and chiral dynamics, Nucl. Phys. **A603**, 326 (1996); Erratum, Nucl. Phys. **A620**, 510 (1997).
[35] M. Tanabashi *et al.* (Particle Data Group), Review of particle physics, Phys. Rev. D **98**, 030001 (2018).
[36] S. Pepin, F. Stancu, M. Genovese, and J. M. Richard, Tetraquarks with color blind forces in chiral quark models, Phys. Lett. B **393**, 119 (1997).
[37] M. Genovese, J. M. Richard, F. Stancu, and S. Pepin, Heavy flavor pentaquarks in a chiral constituent quark model, Phys. Lett. B **425**, 171 (1998).
[38] F. Stancu, Positive parity pentaquarks in a Goldstone boson exchange model, Phys. Rev. D **58**, 111501 (1998).
[39] F. Stancu and S. Pepin, Isoscalar factors of the permutation group, Few Body Syst. **26**, 113 (1999).
[40] M. I. Eides, V. Y. Petrov, and M. V. Polyakov, New LHCb pentaquarks as hadrocharmonium states, arXiv:1904.11616.
[41] F. Stancu, S. Pepin, and L. Y. Glozman, The nucleon-nucleon interaction in a chiral constituent quark model, Phys. Rev. C **56**, 2779 (1997).
[42] F. Stancu, Group theory in subnuclear physics, Oxford Stud. Nucl. Phys. **19**, 1 (1996).
[43] M. Harvey, On the fractional parentage expansions of color singlet six quark states in a cluster model, Nucl. Phys. **A352**, 301 (1981); Erratum, Nucl. Phys. **A481**, 834 (1988).
[44] M. Harvey, Effective nuclear forces in the quark model with Delta and hidden color channel coupling, Nucl. Phys. **A352**, 326 (1981).