Avoiding 5/4-powers on the alphabet of nonnegative integers (extended abstract)*

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Abstract. We identify the structure of the lexicographically least word avoiding 5/4-powers on the alphabet of nonnegative integers.

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1 Introduction

For any (finite or infinite) alphabet Σ, we let Σ^* denote the set of finite words on Σ. We start indexing (finite and infinite) words at position 0.

A *morphism* on an alphabet Σ is a map $\varphi\colon \Sigma \to \Sigma^*$. It extends naturally to finite and infinite words by concatenation. We say that a morphism φ on Σ is *k-uniform* if $|\varphi(c)| = k$ for all $c \in \Sigma$. A 1-uniform morphism is also called a *coding*. If there exists a letter $c \in \Sigma$ such that $\varphi(c)$ starts with c, then iterating φ on c gives a word $\varphi^\omega(c)$, which is a fixed point of φ beginning with c.

A fractional power is a partial repetition, defined as follows. Let a and b be relatively prime positive integers. If $v = v_0 v_1 \cdots v_{\ell-1}$ is a nonempty word whose length ℓ is divisible by b, the *a/b-power* of v is the word

$$v^{a/b} := v^{\lfloor a/b \rfloor} v_0 v_1 \cdots v_{\ell\cdot\{a/b\}-1},$$

where $\{a/b\} = a/b - \lfloor a/b \rfloor$ is the fractional part of a/b. Note that $|v^{a/b}| = \frac{a}{b}|v|$. If $a/b > 1$, then a word w is an a/b-power if and only if w can be written $v^e u$ where e is a positive integer, u is a prefix of v, and $\frac{|w|}{|v|} = \frac{a}{b}$.

Example 1. The 3/2-power of the word 0111 is $(0111)^{3/2} = 011101$.

In general, a 5/4-power is a word of the form $(xy)^{5/4} = xyx$, where $|xy| = 4\ell$ and $|xyx| = 5\ell$ for some $\ell \geq 1$. It follows that $|x| = \ell$ and $|y| = 3\ell$.

Elsewhere in the literature, researchers have been interested in words with no α-power factors for all $\alpha \geq a/b$. In this paper, we consider a slightly different notion, and we say that a word is *a/b-power-free* if none of its factors is an (exact) a/b-power.

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Notation 1. Let a and b be relatively prime positive integers such that $a/b > 1$. Define $\mathbf{w}_{a/b}$ to be the lexicographically least infinite word on $\mathbb{Z}_{\geq 0}$ avoiding a/b-powers.

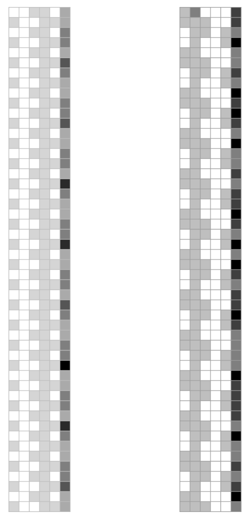


Fig. 1. Portions of $\mathbf{w}_{3/2}$ (left) and $\mathbf{w}_{5/4}$ (right), partitioned into rows of width 6. The letter 0 is represented by white cells, 1 by slightly darker cells, and so on. The word $\mathbf{w}_{3/2}$ is shown from the beginning. The word $\mathbf{w}_{5/4} = w(0)w(1)\cdots$ is shown beginning from $w(i)_{i\geq 6758}$; the term $w(6759)$ (top row, second column) is the last entry in $w(6i+3)_{i\geq 0}$ that is not 1.

Guay-Paquet and Shallit [1] identified the structure of $\mathbf{w}_a$ for each integer $a \geq 2$. In particular,

$$\mathbf{w}_2 = 01020103010201040102010301020105\cdots$$

is the fixed point of the 2-uniform morphism μ on the alphabet of nonnegative integers defined by $\mu(n) = 0(n+1)$ for all $n \geq 0$. The first-named author and Shallit [3] studied the structure of

$$\mathbf{w}_{3/2} = 001102100112001103100113001102100114001103 1\cdots,$$

which is the image under a coding of a fixed point of a 6-uniform morphism. A prefix of this word appears in Figure 1. Pudwell and the first-named author [2] undertook a large study of $\mathbf{w}_{a/b}$ for rational numbers in the range $1 < \frac{a}{b} < 2$, and identified many of these words as images under codings of fixed points of morphisms.

The simplest number $\frac{a}{b}$ in this range for which the structure of $\mathbf{w}_{a/b}$ was not known is $\frac{5}{4}$. In this paper we give a morphic description for $\mathbf{w}_{5/4}$. Let $w(i)$ be the ith letter of $\mathbf{w}_{5/4}$. For the morphic description of $\mathbf{w}_{5/4}$, we need 8 types of letters, $n_0, n_1, \ldots, n_7$ for each integer $n \in \mathbb{Z}$. For example, $0_0, 0_1, \ldots, 0_7$ are the 8 different letters of the form 0_j. The subscript j of the letter n_j will determine the first 5 letters of $\varphi(n_j)$, which correspond to the first 5 columns in Figure 1. The definition of φ in Notation 2 implies that these columns are eventually periodic with period length 1 or 4.

Notation 2. Let Σ_8 be the alphabet $\{n_j \mid n \in \mathbb{Z}, 0 \le j \le 7\}$. Let φ be the 6-uniform morphism defined on Σ_8 by

$$\begin{aligned}
\varphi(n_0) &= 0_0 1_1 0_2 0_3 1_4 (n+3)_5 \\
\varphi(n_1) &= 1_6 1_7 0_0 0_1 0_2 (n+2)_3 \\
\varphi(n_2) &= 1_4 1_5 1_6 0_7 0_0 (n+3)_1 \\
\varphi(n_3) &= 0_2 1_3 1_4 0_5 1_6 (n+2)_7 \\
\varphi(n_4) &= 0_0 1_1 0_2 0_3 1_4 (n+1)_5 \\
\varphi(n_5) &= 1_6 1_7 0_0 0_1 0_2 (n+2)_3 \\
\varphi(n_6) &= 1_4 1_5 1_6 0_7 0_0 (n+1)_1 \\
\varphi(n_7) &= 0_2 1_3 1_4 0_5 1_6 (n+2)_7.
\end{aligned}$$

We also define the coding $\tau(n_j) = n$ for all $n_j \in \Sigma_8$. In the rest of the paper, we think about the definitions of φ and $\tau \circ \varphi$ as 8×6 arrays of their letters. In particular, we will refer to letters in images of φ and $\tau \circ \varphi$ by their columns (first through sixth).

The following gives the structure of $\mathbf{w}_{5/4}$.

Theorem 1. *There exist words* $\mathbf{p}, \mathbf{z}$ *of lengths* $|\mathbf{p}| = 6764$ *and* $|\mathbf{z}| = 20226$ *such that* $\mathbf{w}_{5/4} = \mathbf{p}\,\tau(\varphi(\mathbf{z})\varphi^2(\mathbf{z})\cdots)$.

This word is more complicated than previously studied words $\mathbf{w}_{a/b}$ in two major ways. Unlike all words $\mathbf{w}_{a/b}$ whose structures were previously known, the natural description of $\mathbf{w}_{5/4}$ is not as an image under a coding of a fixed point of a morphism, since $\tau(\mathbf{z})$ is not a factor of $\mathbf{w}_{5/4}$. Additionally, the value of d in the image $\varphi(n_j) = u\,(n+d)_i$ varies with j. The sequence $3, 2, 3, 2, 1, 2, 1, 2, \ldots$ of d values is periodic with period length 8, hence the 8 types of letters. These properties make the proofs significantly more intricate.

We will define $\mathbf{s} = \mathbf{z}\varphi(\mathbf{z})\varphi^2(\mathbf{z})\cdots$. To prove Theorem 1, we must show that

1. $\mathbf{p}\,\tau(\varphi(\mathbf{s}))$ is 5/4-power-free, and
2. $\mathbf{p}\,\tau(\varphi(\mathbf{s}))$ is lexicographically least (by showing that decreasing any letter introduces a 5/4-power ending in that position).

We use *Mathematica* to carry out several computations required in the proofs.

Theorem 1 implies the following recurrence for letters sufficiently far out in $\mathbf{w}_{5/4}$ with positions $\equiv 1 \mod 6$.

Corollary 1. *Let $w(i)$ be the ith letter of $\mathbf{w}_{5/4}$. Then, for all $i \geq 0$,*

$$w(6i+123061) = w(i+5920) + \begin{cases} 3 & \text{if } i \equiv 0,2 \mod 8 \\ 1 & \text{if } i \equiv 4,6 \mod 8 \\ 2 & \text{if } i \equiv 1 \mod 2. \end{cases}$$

This paper is organized as follows. Section 2 gives preliminary properties about φ, $\mathbf{p}$, and $\mathbf{z}$. In Section 3, we introduce the concept of pre-5/4-power-freeness. We show that $\mathbf{w}_{5/4} = \mathbf{p}\,\tau(\varphi(\mathbf{s}))$ in two steps. First, in Section 4, we show that $\mathbf{p}\,\tau(\varphi(\mathbf{s}))$ is 5/4-power-free using the fact that $\mathbf{s}$ is pre-5/4-power-free. Second, in Section 5, we show that $\mathbf{p}\,\tau(\varphi(\mathbf{s}))$ is lexicographically least. Proofs of some supporting lemmas and propositions are omitted in this extended abstract due to space constraints and appear in the full version of the paper [4].

2 Preliminary properties

Note that in Notation 2 the subscripts in each image under φ increase by 1 modulo 8 from one letter to the next and also from the end of each image to the beginning of the next.

Definition 1. A (finite or infinite) word w on Σ_8 is *subscript-increasing* if the subscripts of the letters of w increase by 1 modulo 8 from one letter to the next.

If w is a subscript-increasing word on Σ_8, then so is $\varphi(w)$. For every subscript-increasing word w on Σ_8, it follows from Notation 2 that the subsequence of letters with even subscripts in $\varphi(w)$ is a factor of $(0_0 0_2 1_4 1_6)^\omega$.

Iterating φ on any word on Σ_8 will eventually give a word containing letters n_j with arbitrarily large n. Indeed, after one iteration, we see a letter with subscript 3 or 7, so after two iterations we see a letter with subscript 7. Since $\varphi(n_7)$ contains $(n+2)_7$, the alphabet is growing.

Before position 6764, we cannot expect the prefix of $\mathbf{w}_{5/4}$ to be the image of another word under the morphism φ because the five columns have not become periodic yet (recall Figure 1 where $w(6759)$ is the last term of $w(6i+3)$ before a periodic pattern appears).

One checks programmatically that there are two subscript-increasing pre-images of $w(6764)w(6765)\cdots$ under $\tau \circ \varphi$. We choose the following for $\mathbf{s}$.

Definition 2. Let $\mathbf{p}$ denote the length-6764 prefix of $\mathbf{w}_{5/4}$. We define

$$\mathbf{z} = 0_2 0_3 3_4 0_5 1_6 1_7 (-1_0) 2_1 0_2 2_3 2_4 0_5 \cdots 0_0 1_1 0_2 0_3 1_4 2_5 1_6 2_7 0_0 0_1 0_2 3_3$$

to be the length-20226 subscript-increasing word on Σ_8 starting with 0_2 and satisfying

$$\tau(\varphi(\mathbf{z})) = w(6764)w(6765)\cdots w(6764+6|\mathbf{z}|-1).$$

We define $\mathbf{s} = \mathbf{z}\varphi(\mathbf{z})\varphi^2(\mathbf{z})\cdots$ as before.

Lemma 1. *Let $\Gamma \subset \Sigma_8$ be the finite alphabet*

$$\{-3_0, -3_2, -2_0, -2_1, -2_2, -2_3, -2_5, -2_7, -1_1, -1_3, -1_4, -1_5, -1_6, -1_7, 0_4, 0_6\}.$$

We have the following properties.

1. *The length-*844 *suffixes of* $\mathbf{p}$ *and* $\tau(\mathbf{z})$ *are equal.*
2. *The word* $\mathbf{z}$ *is a subscript-increasing finite word whose alphabet is the* 32*-letter set*

$$\mathrm{Alph}(\mathbf{z}) = \{-1_0, -1_2, 0_0, 0_1, 0_2, 0_3, 0_5, 0_7, 1_0, 1_1, 1_2, 1_3, 1_4, 1_5, 1_6, 1_7, 2_1, 2_3, 2_4, 2_5, 2_6, 2_7, 3_1, 3_3, 3_4, 3_5, 3_6, 3_7, 4_1, 4_3, 4_5, 4_7\}.$$

   In particular, $\mathbf{z}$ *contains no letters in* Γ*. The last letter of the form* -1_j *appears at position* 80*, and the subsequence of letters with even subscripts starting at position* 86 *is a finite prefix of* $(0_0 0_2 1_4 1_6)^\omega$*.*
3. *The word* $\mathbf{s}$ *is a subscript-increasing infinite word whose alphabet is* $\mathrm{Alph}(\mathbf{z}) \cup \{5_3, 6_3\} \cup \{n_7 \mid n \geq 5\}$*. In particular,* $\mathbf{s}$ *contains no letters in* Γ*.*
4. *For all words* w *on* Σ_8*, the only letters of* $\varphi(w)$ *of the form* n_j *with even* j *are* 0_j *and* 1_j*. More precisely, the only letter with subscript* 0 *(resp.,* 2*; resp.,* 4*; resp.,* 6*) in* $\varphi(w)$ *is* 0_0 *(resp.,* 0_2*; resp.,* 1_4*; resp.,* 1_6*).*
5. *As long as* $n_j \in (\Sigma_8 \setminus \Gamma)$ *with* $0 \leq j \leq 7$*, the last letter of* $\varphi(n_j)$ *is not of the form* 0_i *or* 1_i*. In particular, for all letters* n_j *of* $\mathbf{s}$*, the last letter of* $\varphi(n_j)$ *is of the form* n_i *where* $n \geq 2$*.*

3 Pre-5/4-power-freeness

A morphism μ on Σ is *a/b-power-free* if μ preserves a/b-power-freeness, that is, for all a/b-power-free words w on Σ, $\mu(w)$ is also a/b-power-free. Previously studied a/b-power-free words [1,3,2] have all been described by a/b-power-free morphisms. However, the morphism φ defined in Notation 2 is not 5/4-power-free. Indeed for any integers $n, \bar{n} \in \mathbb{Z}$, the word $0_4 n_5 \bar{n}_6$ is 5/4-power-free, but $\varphi(0_4 n_5 \bar{n}_6)$ contains the length-10 factor $1_4 1_5 1_6 1_7 0_0 0_1 0_2 (n+2)_3 1_4 1_5$, which is a 5/4-power. Therefore, to prove that $\mathbf{w}_{5/4}$ is 5/4-power-free, we use a different approach. We still need to guarantee that there are no 5/4-powers in images of certain words under φ. Specifically, we would like all factors xyx' of a word with $|x| = \frac{1}{3}|y| = |x'|$ to satisfy $\varphi(x) \neq \varphi(x')$.

Definition 3. A *pre-*5/4*-power-free* word is a (finite or infinite) subscript-increasing word w on Σ_8 such that, for all factors xyx' of w with $|x| = \frac{1}{3}|y| = |x'|$, there exists $0 \leq m \leq |x| - 1$ such that

1. if the subscripts of $x(m)$ and $x'(m)$ are equal or odd, then $\tau(x(m)) \neq \tau(x'(m))$, and
2. if the subscripts of $x(m)$ and $x'(m)$ are even and differ by 4, then $\tau(x(m)) - \tau(x'(m)) \notin \{-2, 2\}$.

Note that if the subscripts of $x(m)$ and $x'(m)$ are not equal, then they differ by 4 since $|xy| = 4|x|$. Definition 3 involves the set $\{-2, 2\}$ because if the subscripts of $x(m)$ and $x'(m)$ are even and differ by 4 and $\tau(x(m)) - \tau(x'(m)) \in \{-2, 2\}$ then $\varphi(x(m)) = \varphi(x'(m))$.

For example, the word $0_0 n_1 \bar{n}_2 \bar{\bar{n}}_3 2_4$ is not pre-5/4-power-free because

$$\varphi(0_0 n_1 \bar{n}_2 \bar{\bar{n}}_3 2_4) = 0_0 1_1 0_2 0_3 1_4 3_5 \varphi(n_1 \bar{n}_2 \bar{\bar{n}}_3) 0_0 1_1 0_2 0_3 1_4 3_5$$

is a 5/4-power of length 30. On the other hand, the word $0_0 n_1 \bar{n}_2 \bar{\bar{n}}_3 0_4$ is pre-5/4-power-free because

$$\varphi(0_0 n_1 \bar{n}_2 \bar{\bar{n}}_3 0_4) = 0_0 1_1 0_2 0_3 1_4 3_5 \varphi(n_1 \bar{n}_2 \bar{\bar{n}}_3) 0_0 1_1 0_2 0_3 1_4 1_5$$

is not a 5/4-power. The word $0_0 n_1 \bar{n}_2 \bar{\bar{n}}_3 0_4$ is also 5/4-power-free, since 0_0 and 0_4 are different letters. The next proposition states that pre-5/4-power-freeness implies 5/4-power-freeness in general.

Proposition 1. *If a word is pre-*$5/4$*-power-free, then it is also* $5/4$*-power-free.*

One checks programmatically that the finite word $\mathbf{z}$ is pre-5/4-power-free. As a consequence, we will show that $\mathbf{p}\tau(\varphi(\mathbf{s}))$ is 5/4-power-free in Section 4.

Lemma 2. *Let* $\ell \geq 6$ *be an integer. If* w *is a pre-*$5/4$*-power-free word on* Σ_8*, then* $\varphi(w)$ *contains no* $5/4$*-power of length* 5ℓ.

Lemma 2 takes care of large 5/4-powers. Toward avoiding all 5/4-powers, the following proposition shows that φ preserves the property of pre-5/4-power-freeness.

Proposition 2. *Let* Γ *be the alphabet*

$$\{-3_0, -3_2, -2_0, -2_1, -2_2, -2_3, -2_5, -2_7, -1_1, -1_3, -1_4, -1_5, -1_6, -1_7, 0_4, 0_6\}.$$

For all pre-$5/4$*-power-free words* w *on* $\Sigma_8 \setminus \Gamma$*,* $\varphi(w)$ *is pre-*$5/4$*-power-free.*

Next we show that the word $\mathbf{s} = \mathbf{z}\varphi(\mathbf{z})\varphi^2(\mathbf{z})\cdots$ is pre-5/4-power-free. The common length-4 suffix 0003 of $\mathbf{p}$ and $\tau(\mathbf{z})$ is a possible factor of φ, which requires extra consideration in Theorems 2 and 3, and in Proposition 3.

Theorem 2. *The word* $\mathbf{s}$ *is pre-*$5/4$*-power-free.*

Proof. We show that, for all $e \geq 1$, $\mathbf{z}\varphi(\mathbf{z})\cdots\varphi^e(\mathbf{z})$ is pre-5/4-power-free. Note that $\mathbf{z}\varphi(\mathbf{z})\cdots\varphi^e(\mathbf{z}) \in (\Sigma_8 \setminus \Gamma)^*$ for all $e \geq 1$. We proceed by induction on $e \geq 1$.

Base case. Suppose that $e = 1$. We wrote code to check that $\mathbf{z}\varphi(\mathbf{z})$ is pre-5/4-power-free. The computation took about 13 hours.

Induction step. We suppose that $\mathbf{z}\varphi(\mathbf{z})\cdots\varphi^e(\mathbf{z})$ is pre-5/4-power-free for $e \geq 1$. We show that $\mathbf{z}\varphi(\mathbf{z})\cdots\varphi^{e+1}(\mathbf{z})$ is also pre-5/4-power-free. First, the word $\mathbf{z}\varphi(\mathbf{z})\cdots\varphi^{e+1}(\mathbf{z})$ is subscript-increasing. Let xyx' be a factor of $\mathbf{z}\varphi(\mathbf{z})\cdots\varphi^{e+1}(\mathbf{z})$ with $|x| = \frac{1}{3}|y| = |x'|$.

Suppose that xyx' is a factor of either $\mathbf{z}$ or $\varphi(\mathbf{z})\cdots\varphi^{e+1}(\mathbf{z})$. In the first case, Definition 3 holds since $\mathbf{z}$ is pre-5/4-power-free. In the second case, Definition 3 holds too. Indeed, it suffices to use Proposition 2 with $\mathbf{z}\varphi(\mathbf{z})\cdots\varphi^{e}(\mathbf{z}) \in (\Sigma_8\backslash\Gamma)^*$, which is pre-5/4-power-free by the induction hypothesis.

It remains to check the case where xyx' overlaps $\mathbf{z}$. If yx' overlaps $\mathbf{z}$, then x is a factor of $\mathbf{z}$ and $|xyx'| = 5|x| \leq 5|\mathbf{z}| < 6|\mathbf{z}| = |\varphi(\mathbf{z})|$. In particular, xyx' is a factor of $\mathbf{z}\varphi(\mathbf{z})$. The base case of the induction implies that Definition 3 holds in this case. Thus we may suppose that x overlaps $\mathbf{z}$.

If the overlap length is at least 5, then x contains the suffix $2_7 0_0 0_1 0_2 3_3$ of $\mathbf{z}$. If the subscripts of x and x' differ by 4, then x' being a factor of $\varphi(\mathbf{z})\cdots\varphi^{e+1}(\mathbf{z})$ implies that the factor $2_7 0_0 0_1 0_2 3_3$ of x corresponds to a factor $n_3 1_4 \bar{n}_5 1_6 \bar{\bar{n}}_7$ of x'. So Definition 3 is fulfilled. If the subscripts of x and x' line up, then x' being a factor of $\varphi(\mathbf{z})\cdots\varphi^{e+1}(\mathbf{z})$ implies that the factor $2_7 0_0 0_1 0_2 3_3$ of x corresponds to a factor $n_7 0_0 \bar{n}_1 0_2 \bar{\bar{n}}_3$ of x'. If $n_7 \neq 2_7$ or $\bar{n}_1 \neq 0_1$ or $\bar{\bar{n}}_3 \neq 3_3$, then Definition 3 holds. Otherwise, then x' contains $2_7 0_0 0_1 0_2 3_3$, which is impossible since this factor does not appear in $\varphi(\mathbf{z})\cdots\varphi^{e+1}(\mathbf{z})$.

Suppose the overlap length is less than or equal to 4. If $|x| = |x'|$ is odd, then the subscripts of x and x' differ by 4. Since x contains the factor $3_3 1_4$, then x' being a factor of $\varphi(\mathbf{z})\cdots\varphi^{e+1}(\mathbf{z})$ implies that the factor $3_3 1_4$ of x corresponds to a factor $n_7 0_0$ of x', and Definition 3 holds. If $|x| = |x'|$ is even, then the subscripts of x and x' line up. The words x and x' agree on even subscripts (because the length-4 suffix of $\mathbf{z}$ is $0_0 0_1 0_2 3_3$). For odd subscripts, if the corresponding letters of x and x' belong to different columns (that is, their positions are not congruent modulo 6), then Definition 3 holds by Part 5 of Lemma 1. If the corresponding letters belong to the same column, then x and x' agree everywhere except maybe in the sixth column. Toward a contradiction, suppose the words are equal in the sixth column, so $x = x'$. Since x' is a factor of $\varphi(\mathbf{z})\cdots\varphi^{e+1}(\mathbf{z})$, there exists a word $w \in \Sigma_8^*$ such that x' is a factor of $\varphi(w)$ and w itself is a factor of $\mathbf{z}\varphi(\mathbf{z})\cdots\varphi^{e}(\mathbf{z})$. We take w to be of minimal length.

If $|w| \leq 3$, then $|x| = |x'| \leq |\varphi(w)| \leq 18$, so $|xyx'| = 5|x| \leq 90 < |\mathbf{z}\varphi(\mathbf{z})|$, which means that xyx' is a factor of $\mathbf{z}\varphi(\mathbf{z})$. Due to the base case, we already know that xyx' satisfies the conditions of Definition 3.

If $|w| \geq 4$, then w can take two different forms:

$$w = \begin{cases} 1_1 0_2 0_3 3_4 \cdots \\ 1_5 2_6 0_7 1_0 \cdots \end{cases}$$

since $0_0 0_1 0_2 3_3 \varphi(\mathbf{z})\varphi^2(\mathbf{z})\cdots = 0_0 0_1 0_2 3_3 1_4 1_5 1_6 0_1 0_0 3_1 \cdots$. Due to the definition of the morphism φ, the letters 1_0 and 3_4 do not occur in $\varphi(\mathbf{z})\cdots\varphi^{e}(\mathbf{z})$. Since w is a factor of $\mathbf{z}\varphi(\mathbf{z})\cdots\varphi^{e}(\mathbf{z})$, they must belong to $\mathbf{z}$. But none of the words $1_1 0_2 0_3 3_4$ and $1_5 2_6 0_7 1_0$ belongs to $\mathbf{z}$. Indeed, the letter 3_4 occurs in $\mathbf{z}$ only at positions 2 and 66, and we have $z(63)z(64)z(65)z(66) = 0_1 0_2 0_3 3_4$. Similarly, the letter 1_0 only occurs in $\mathbf{z}$ only at positions 22 and 54 and 78, and we find

$$z(19)z(20)z(21)z(22) = 2_5 2_6 0_7 1_0,$$
$$z(51)z(52)z(53)z(54) = 1_5 2_6 2_7 1_0,$$
$$z(75)z(76)z(77)z(78) = 1_5 2_6 1_7 1_0.$$

This case is thus impossible.

4 5/4-power-freeness

In this section we show that $\mathbf{p}\tau(\varphi(\mathbf{s}))$ is 5/4-power-free. As a corollary of Theorem 2, we obtain the following.

Corollary 2. *The infinite word $\varphi(\mathbf{s})$ is 5/4-power-free.*

Proof. By Theorem 2, $\mathbf{s}$ is pre-5/4-power-free. Since $\mathbf{s} \in (\Sigma_8 \setminus \Gamma)^\omega$, Proposition 2 implies that $\varphi(\mathbf{s})$ is pre-5/4-power-free. By Proposition 1, $\varphi(\mathbf{s})$ is 5/4-power-free.

The following lemma asserts that applying the coding τ on $\varphi(\mathbf{s})$ preserves the 5/4-power-freeness.

Lemma 3. *The infinite word $\tau(\varphi(\mathbf{s}))$ is 5/4-power-free.*

The last step in showing 5/4-power-freeness is to prove that prepending $\mathbf{p}$ to $\tau(\varphi(\mathbf{s}))$ also yields a 5/4-power-free word. To that aim, we introduce the following notion.

Definition 4. Let N be a set of integers, and let $\alpha, \beta \in \mathbb{Z} \cup \{n+1, n+2, n+3\}$, where n is a symbol. Then α and β are *possibly equal with respect to* N if there exist $m, m' \in N$ such that $\alpha|_{n=m} = \beta|_{n=m'}$.

Two letters α, β are possibly equal with respect to N if we can make them equal by substituting integers from N for the symbol n. In particular, for every nonempty set N, two integers α, β are possibly equal if and only if $\alpha = \beta$. The definition of possibly equal letters extends to words in the natural way. The next two lemmas will be used to prove Theorem 3.

Lemma 4. *Let n be a symbol, and let $N \supseteq \{-3, -2, \ldots, 4\}$. Let α, β be elements of $\{0, 1, \ldots, 5\} \cup \{n+1, n+2, n+3\}$. If α and β are possibly equal with respect to N, then they are possibly equal with respect to $\{-3, -2, \ldots, 4\}$.*

Lemma 5. *Let n be a symbol, let $D = \{1, 2, 3\}$, and let $N = \{-3, -2, \ldots, 4\}$. Let v be the prefix of $\mathbf{p}\tau(\varphi(\mathbf{s}))$ of length $|\mathbf{p}| + 952 = 7716$. For each position $i \geq 0$ in the eventually periodic word*

$$v\tau\Big(\varphi(n_1)\varphi(n_2)\varphi(n_3)\varphi(n_4)\varphi(n_5)\varphi(n_6)\varphi(n_7)\varphi(n_0)\cdots\Big)$$

on $\{0,1,\dots,5\} \cup \{n+1, n+2, n+3\}$*, define the set*

$$X_i = \begin{cases} \{c\} & \text{if the letter in position } i \text{ is } c \in \mathbb{Z} \\ N+d & \text{if the letter in position } i \text{ is } n+d \text{ where } d \in D. \end{cases}$$

For all $i, j \geq 0$*, if the letters in positions* i, j *are possibly equal with respect to* N*, then* $X_i \cap X_j$ *is nonempty.*

Proof. The set X_i is the set of integer letters c such that the letter in position i in $v\tau(\varphi(n_1)\varphi(n_2)\varphi(n_3)\cdots)$ is possibly equal to c with respect to N. Since v is a word on $\{0,1,\dots,5\}$ and the integer letters of $\tau \circ \varphi$ are elements of $\{0,1\} \subseteq \{0,1,\dots,5\}$, the integer letters of $v\tau(\varphi(n_1)\varphi(n_2)\varphi(n_3)\cdots)$ are in $\{0,1,\dots,5\}$. There are three cases to consider depending on the nature of the letters in positions i and j.

If both letters are integers, let c be the letter in position i and c' be the letter in position j. Since they are possibly equal with respect to N, then $c = c'$. So $c \in X_i \cap X_j$.

If one letter is an integer and the other is symbolic, without loss of generality, let c be the letter in position i and $n+d$ be the letter in position j with $d \in D$. By assumption, $c = n+d$ for some $n \in N$. So $X_i \cap X_j = \{c\} \cap (N+d) = \{c\}$.

If both letters are symbolic, let $n+d$ be the letter in position i and $n+d'$ be the letter in position j with $d, d' \in D$. By assumption, $X_i = N+d$ and $X_j = N+d'$, so $0 \in X_i \cap X_j$.

Theorem 3. *The infinite word* $\mathbf{p}\tau(\varphi(\mathbf{s}))$ *is* 5/4*-power-free.*

Proof. Since $\mathbf{p}$ is the prefix of $\mathbf{w}_{5/4}$ of length 6764, $\mathbf{p}$ is 5/4-power-free. By Lemma 3, $\tau(\varphi(\mathbf{s}))$ is also 5/4-power-free. So if $\mathbf{p}\tau(\varphi(\mathbf{s}))$ contains a 5/4-power, then it must overlap $\mathbf{p}$ and $\tau(\varphi(\mathbf{s}))$. We will show that there are no 5/4-powers xyx starting in $\mathbf{p}$.

For factors xyx with $|x| < 952$ starting in $\mathbf{p}$, note that $|x| < 952$ implies $|xyx| < 5 \cdot 952$, so it is enough to look for 5/4-powers in $\mathbf{p}\tau(\varphi(\mathbf{z}))$ — as opposed to $\mathbf{p}\tau(\varphi(\mathbf{s}))$ — starting in $\mathbf{p}$. We check programmatically that there is no such 5/4-power xyx. The computation took less than a minute.

For long factors, we show that each length-952 factor x starting in $\mathbf{p}$ only occurs once in $\mathbf{p}\tau(\varphi(\mathbf{s}))$. This will imply that there is no 5/4-power xyx in $\mathbf{p}\tau(\varphi(\mathbf{s}))$ starting in $\mathbf{p}$ such that $|x| \geq 952$. Since $\mathbf{s} = 0_2 0_3 3_4 \cdots$, the word $\mathbf{p}\tau(\varphi(\mathbf{s}))$ is of the form

$$\mathbf{p}\tau\Big(\varphi(n_2)\varphi(n_3)\varphi(n_4)\varphi(n_5)\varphi(n_6)\varphi(n_7)\varphi(n_0)\varphi(n_1)\cdots\Big).$$

Here we abuse notation; namely, the n's are not necessarily equal. Observe that $|\varphi(n_2)\varphi(n_3)\cdots\varphi(n_0)\varphi(n_1)| = 48$. We use a method based on [2, Section 6]. However, this is another situation in which the structure of the word $\mathbf{w}_{5/4}$ is more complex. The sequence $3,2,1,2,1,2,3,2,\dots$ of increments d in Notation 2 is periodic with period length 8, but each of the first five columns is periodic with period length at most 4. In other words, the factors starting at positions

i and $i+24$ are possibly equal when i is sufficiently large. Therefore we need to run the following procedure for two different sets of positions instead of one. These two sets are

$$S_1 = \{0, 1, \ldots, |\mathbf{p}| - 1\} \cup \{|\mathbf{p}|, |\mathbf{p}| + 1, \ldots, |\mathbf{p}| + 23\}$$

and

$$S_2 = \{0, 1, \ldots, |\mathbf{p}| - 1\} \cup \{|\mathbf{p}| + 24, |\mathbf{p}| + 25, \ldots, |\mathbf{p}| + 47\}.$$

The positions in $\{0, 1, \ldots, |\mathbf{p}| - 1\}$ represent factors starting in the prefix $\mathbf{p}$ of $\mathbf{p}\tau(\varphi(\mathbf{s}))$, while the other positions are representatives of general positions modulo 48 in the suffix $\tau(\varphi(\mathbf{s}))$. We also need to specify a set N of integers that, roughly speaking, represent the possible values that each symbolic n can take.

Let S be a set of positions, and let N be a set of integers. As in Lemma 5, the set N represents values of n such that the last letter of $\tau(\varphi(n_j))$, namely $n + d$ for some $d \in \{1, 2, 3\}$, is a letter in $\mathbf{p}\tau(\varphi(\mathbf{s}))$. We maintain classes of positions corresponding to possibly equal factors starting at those positions. Start with $\ell = 0$, for which all length-0 factors are equal. Then all positions belong to the same class S. At each step, we increase ℓ by 1, and for each position i we consider the factor of length ℓ starting at position i, extended from the previous step by one letter to the right. We break each class into new classes according to the last letter of each extended factor, as described in the following paragraph. We stop once each class contains exactly one position, because then each factor occurs at most once. Note that this procedure does not necessarily terminate, depending on the inputs. If it terminates, then the output is ℓ.

We define the *value* of each letter in Σ_8 to be its image under τ. For each class $\mathcal{I}$, we build subclasses $\mathcal{I}_c$ indexed by integers c. For each position $i \in \mathcal{I}$, we consider the extended factor of length ℓ starting at position i. If $0 \leq i \leq |\mathbf{p}| - 1$, then the new letter is in $\mathbb{Z}$ because $i + \ell - 1$ represents a particular position in $\mathbf{p}\tau(\varphi(\mathbf{s}))$. If $i \geq |\mathbf{p}|$, then the new letter is either in $\mathbb{Z}$ or symbolic in n because $i + \ell - 1$ represents all sufficiently large positions congruent to $i + \ell - 1$ modulo 48. Now there are two cases. If the new letter is an integer c, we add the position i to the class $\mathcal{I}_c$. If the new letter is $n + d$ where $d \in \{1, 2, 3\}$, then we add the position i to the class $\mathcal{I}_{n'+d}$ for each $n' \in N$. We do this for all classes $\mathcal{I}$ and we use the union

$$\bigcup_{\mathcal{I}} \{\mathcal{I}_c : c \in \mathbb{Z}\}$$

as our new set of classes in the next step.

For the sets S_1 and S_2, we use $N = \{0, 1, 2, 3, 4\}$. The prefix of $\mathbf{p}\tau(\varphi(\mathbf{s}))$ of length $|\mathbf{p}| + 952$ is a word on the alphabet $\{0, 1, \ldots, 5\}$. Therefore, since $d \in \{1, 2, 3\}$, at most the eight classes $\mathcal{I}_0, \mathcal{I}_1, \ldots, \mathcal{I}_7$ arise in each step of the procedure, since $\{0, 1, \ldots, 5\} \cup (N + \{1, 2, 3\}) = \{0, 1, \ldots, 7\}$. We wrote code that implements this procedure. For both sets, it terminates and gives $\ell = 952$. Our implementation took about 20 seconds for each set.

It remains to show that using the set $N = \{0, 1, 2, 3, 4\}$ is sufficient to guarantee that if the procedure terminates then each length-952 factor x starting in

$\mathbf{p}$ only occurs once in $\mathbf{p}\tau(\varphi(\mathbf{s}))$. Since $\mathbf{p}\tau(\varphi(\mathbf{s}))$ is a word on the alphabet $\mathbb{N}$, it suffices to choose a subset N of $\mathbb{N} - \{1, 2, 3\} = \{-3, -2, \dots\}$.

There exist letters in $\mathbf{p}\tau(\varphi(\mathbf{s}))$ that arise as the last letter of $\tau(\varphi(n_j))$ for arbitrarily large n, but the procedure cannot use an infinite set N. We use Lemmas 4 and 5 to show that N need not contain any integer greater than 4. Let v be the prefix of $\mathbf{p}\tau(\varphi(\mathbf{s}))$ of length $|\mathbf{p}|+952$. The procedure examines factors of $v\tau(\varphi(n_1)\varphi(n_2)\varphi(n_3)\cdots)$, which is a word on the alphabet $\{0, 1, \dots, 5\} \cup \{n+1, n+2, n+3\}$. Let $N \supseteq \{0, 1, \dots, 5\} - \{1, 2, 3\} = \{-3, -2, \dots, 4\}$. On the step corresponding to length ℓ in the procedure, suppose the length-ℓ factors of $v\tau(\varphi(n_1)\varphi(n_2)\varphi(n_3)\cdots)$ starting at positions i and j are possibly equal with respect to N. By Lemma 4, the two factors are possibly equal with respect to $\{-3, -2, \dots, 4\}$. In particular, the last two letters, which have positions $i+\ell-1$ and $j+\ell-1$, are possibly equal with respect to $\{-3, -2, \dots, 4\}$. By Lemma 5, there exists $c \in X_{i+\ell-1} \cap X_{j+\ell-1}$. Therefore the letters in positions $i+\ell-1$ and $j+\ell-1$ are both possibly equal to c with respect to $\{-3, -2, \dots, 4\}$, so i and j are both added to $\mathcal{I}_c$.

We have shown that $N \subseteq \{-3, -2, \dots, 4\}$ suffices. Next we remove -3 and -2. We continue to consider letters in $\mathbf{p}\tau(\varphi(\mathbf{s}))$ that arise as the last letter of $\tau(\varphi(n_j))$ for $n \in N$. Since $\tau(\mathbf{s})$ does not contain the letter -3, this implies $n+3$ and 0 are not possibly equal with respect to the alphabet of $\tau(\mathbf{s})$, so N need not contain -3. Similarly, $\tau(\mathbf{s})$ does not contain the letter -2; therefore $n+2$ and 0 are not possibly equal, and $n+3$ and 1 are not possibly equal, so N need not contain -2. Therefore $N \subseteq \{-1, 0, 1, 2, 3, 4\}$ suffices.

To remove -1, we run the procedure on both sets S_1 and S_2 again but with the set $\{-1, 0, 1, 2, 3, 4\}$. However, we artificially stop the procedure at $\ell = 952$.

For the set S_1, there are 4 nonempty classes of positions remaining, namely $\{6760, 6784\}$, $\{6761, 6785\}$, $\{6762, 6786\}$, $\{6763, 6787\}$. The smallest position in each class is one of the last 4 positions in $\mathbf{p}$. As length-952 factors of $\mathbf{p}\tau(\varphi(n_2)\varphi(n_3)\cdots)$, each pair of factors starting at those positions are possibly equal with respect to N. For instance, consider the two factors

$$\begin{array}{l} 000\qquad 3\;11100\qquad 3\;01101\qquad 2\;01001\qquad 4\;11000\qquad 2\;11100\qquad 2\;\cdots, \\ 000(n+2)11100(n+1)01101(n+2)01001(n+3)11000(n+2)11100(n+3)\cdots \end{array}$$

starting at positions $6760, 6784$. The first factor is $0003\tau(\varphi(\mathbf{s}))$ and the second is $000(n+2)\tau(\varphi(n_6)\varphi(n_7)\cdots)$, which occurs every 48 positions in the repetition period $\tau(\varphi(n_2)\varphi(n_3)\cdots)$. For these two factors to be equal, the pair of letters 2 and $n+3$ have to be equal, and solving $2 = n+3$ gives $n = -1$. Similarly, the other three pairs of factors are only equal if the same pair of letters are equal, which again gives $n = -1$. But letters -1 only appear in $\mathbf{s}$ in its prefix $\mathbf{z}$ and only with subscripts 0 and 2 and only at the nine positions $6, 14, 16, 32, 40, 48, 56, 70, 80$. A finite check shows that the factors in each pair are different.

For S_2, there are also 4 nonempty classes of positions remaining. To show that the factors in each pair are different, we use slightly longer prefixes of $003\tau(\varphi(\mathbf{s}))$ and $000(n+2)\tau(\varphi(n_2)\varphi(n_3)\cdots)$ than we used for S_1, and we again find a pair of letters 2 and $n+3$. This again implies $n = -1$ for each class.

Therefore we can remove -1 from N. We are left with $N \subseteq \{0, 1, 2, 3, 4\}$, so $N = \{0, 1, 2, 3, 4\}$ suffices.

5 Lexicographic-leastness

In this section we show that $\mathbf{p}\tau(\varphi(\mathbf{s}))$ is lexicographically least by showing that decreasing any nonzero letter introduces a 5/4-power ending in that position.

Recall that, for all $e \geq 1$, the subsequence of letters with even subscripts in $\varphi^e(\mathbf{z})$ is a factor of $(0_0 0_2 1_4 1_6)^\omega$ by Lemma 1.

Lemma 6. *If* 0_0 *(resp.,* 0_2*; resp.,* 1_4*; resp.,* 1_6*) appears in* $\mathbf{s}$ *at a position* $i \geq 90$*, then the letter at position* $i - 4$ *in* $\mathbf{s}$ *is* 1_4 *(resp.,* 1_6*; resp.,* 0_0*; resp.,* 0_2*).*

Proposition 3. *Decreasing any nonzero letter of* $\mathbf{p}\tau(\varphi(\mathbf{s}))$ *introduces a 5/4-power ending in that position.*

Proof. We proceed by induction on the positions i of letters in $\mathbf{p}\tau(\varphi(\mathbf{s}))$. As a base case, since we have computed a long enough common prefix of $\mathbf{w}_{5/4}$ and $\mathbf{p}\tau(\varphi(\mathbf{s}))$, decreasing any nonzero letter of $\mathbf{p}\tau(\varphi(\mathbf{s}))$ at position $i \in \{0, 1, \ldots, 331039\}$ introduces a 5/4-power in $\mathbf{p}\tau(\varphi(\mathbf{s}))$ ending in that position.

Now suppose that $i \geq 331040$ and assume that decreasing any nonzero letter in any position $< i$ in $\mathbf{p}\tau(\varphi(\mathbf{s}))$ introduces a 5/4-power in $\mathbf{p}\tau(\varphi(\mathbf{s}))$ ending in that position. We will show that decreasing the letter in position i in $\mathbf{p}\tau(\varphi(\mathbf{s}))$ introduces a 5/4-power in $\mathbf{p}\tau(\varphi(\mathbf{s}))$ ending in that position. Since $|\mathbf{p}| = 6764$ and $|\mathbf{z}| = 20226$, observe that the letter in position $i \geq 331040$ actually belongs to the suffix $\tau(\varphi^2(\mathbf{s}))$, and $i - |\mathbf{p}\tau(\varphi(\mathbf{z}))| = i - |\mathbf{p}| - 6|\mathbf{z}|$ gives its position in $\tau(\varphi^2(\mathbf{s}))$. Recall that every such letter is a factor of $\tau(\varphi(n_j))$ for some $n \in \mathbb{N}$ and $j \in \{0, 1, \ldots, 7\}$. We make use of the array in Notation 2 of letters of φ.

If $i - |\mathbf{p}| - 6|\mathbf{z}|$ is not congruent to 5 modulo 6, then the letter in position i belongs to one of the first five columns. Any 0 letters cannot be decreased. Observe that the fourth column is made of letters 0. Since the second column contains only letters 1, decreasing any letter 1 to 0 in the second column produces a new 5/4-power of length 5 between the fourth and second columns. Since the even-subscript letters in $\varphi(\mathbf{s})$ form the word $(1_4 1_6 0_0 0_2)^\omega$, then decreasing any letter 1 to 0 in the first, third or fifth column introduces a 5/4-power of length 5 of the form $0y0$.

It remains to consider positions i such that $i - |\mathbf{p}| - 6|\mathbf{z}| \equiv 5 \mod 6$, that is, letters in the sixth column. By Part 5 of Lemma 1, note that the letters in the sixth column belong to $\{n_j \mid n \geq 2, 0 \leq j \leq 7\}$, that is, their values are greater than or equal to 2. If we decrease the letter in position i to 0, then we create one of the following 5/4-powers of length 10:

$$\begin{array}{ll}
10 \cdot 1(n+2)0100 \cdot 10, & 10 \cdot 1(n+2)0100 \cdot 10, \\
00 \cdot 1(n+3)1100 \cdot 00, & 00 \cdot 1(n+1)1100 \cdot 00, \\
00 \cdot 0(n+2)1110 \cdot 00, & 00 \cdot 0(n+2)1110 \cdot 00, \\
10 \cdot 0(n+3)0110 \cdot 10, & 10 \cdot 0(n+1)0110 \cdot 10.
\end{array}$$

If we decrease the letter in position i to 1, then we create a new 5/4-power of length 5 because each letter in the second column is 1.

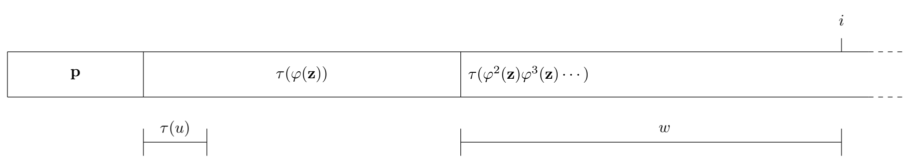
It remains to show that decreasing the letter in position i to some letter c with $c \geq 2$ introduces a 5/4-power ending in that position. Intuitively, this operation corresponds, under $\tau \circ \varphi$, to decreasing a letter n_j of $\varphi(\mathbf{s})$ to $(c-d)_j$ with $0 \leq j \leq 7$ and $d \in \{1, 2, 3\}$. To be more precise, the last letter of $\tau(\varphi((c-d)_j))$ is c. We examine three cases according to the value of d.

Case 1. If $d = 1$, then the corresponding letter in position i in $\varphi^2(\mathbf{s})$ is of the form $(n+1)_1$ or $(n+1)_5$. From Notation 2, we see that $(n+1)_1$ and $(n+1)_5$ appear in the images of n_4 and n_6 under φ. By Part 4 of Lemma 1, the only letters with subscripts 4 and 6 in $\varphi(\mathbf{s})$ are 1_4 and 1_6. The images of these letters under $\tau \circ \varphi$ contain $\tau((n+1)_1) = 2$ and $\tau((n+1)_5) = 2$. So there is nothing to check since the value of the letters is too small.

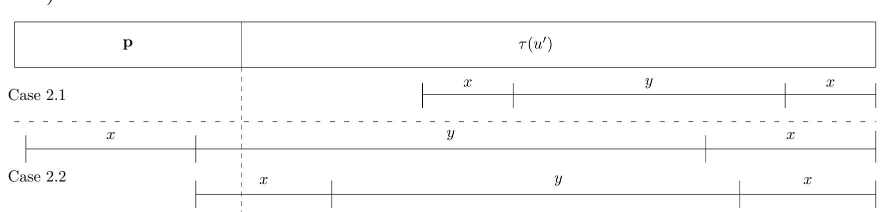
Case 3. If $d = 3$, then the corresponding letter in position i in $\varphi^2(\mathbf{s})$ is of the form $(n+3)_1$ or $(n+3)_5$, which appear in the images of n_0 and n_2 under φ. From Part 4 of Lemma 1, the only letters with subscripts 0 or 2 in $\varphi(\mathbf{s})$ are 0_0 and 0_2, so the only relevant value of c is 2. Lemma 6 tells us that decreasing the value of a letter $(n+3)_1$ or $(n+3)_5$ to $c = 2$ introduces one of the following 5/4-powers of length 30:

$$\tau(\varphi(1_4 n_5 \bar{n}_6 \bar{\bar{n}}_7))01001(3-1) = 010012\tau(\varphi(n_5 \bar{n}_6 \bar{\bar{n}}_7))010012,$$
$$\tau(\varphi(1_6 n_7 \bar{n}_0 \bar{\bar{n}}_1))11100(3-1) = 111002\tau(\varphi(n_7 \bar{n}_0 \bar{\bar{n}}_1))111002.$$

Case 2. If $d = 2$, then the corresponding letter in position i in $\varphi^2(\mathbf{s})$ is of the form $(n+2)_3$ or $(n+2)_7$, which appear in the images of n_1, n_3, n_5, and n_7 under φ. Let w be the length-$(i - |\mathbf{p}| - 6|\mathbf{z}| + 1)$ prefix of $\tau(\varphi^2(\mathbf{s}))$ with last letter $n+2$. Since $i - |\mathbf{p}| - 6|\mathbf{z}| \equiv 5 \mod 6$, let u be the prefix of $\varphi(\mathbf{s})$ of length $\frac{i-|\mathbf{p}|-6|\mathbf{z}|+1}{6}$. Then $\tau(\varphi(u)) = w$ and $\tau(u)$ ends with n as pictured.



Let w' be the word obtained by decreasing the last letter $n+2$ to c in w, and let u' be the word obtained by decreasing the last letter with value n of u to $c-2$. Then $\tau(\varphi(u')) = w'$. By the induction hypothesis, $\mathbf{p}\tau(u')$ contains a 5/4-power suffix xyx. Now we consider two subcases, depending on where xyx starts in $\mathbf{p}\tau(u')$ as depicted below. (We show that the middle case does not actually occur.)



Case 2.1. Suppose that xyx starts after $\mathbf{p}$ in $\mathbf{p}\tau(u')$, that is, xyx is a suffix of $\tau(u')$. Write $x = \tau(x') = \tau(x'')$ and $y = \tau(y')$ where $x'y'x''$ is the corresponding subscript-increasing suffix of u'. Since $|xy|$ is divisible by 4, the subscripts of x' and x'' are either equal or differ by 4. Since $d = 2$, the subscripts of the last letters of x' and x'' are odd. If $|x| = 1$, then $x = c - 2$ and $\varphi(x') = \varphi(x'')$ by definition of φ. Now if $|x| \geq 2$, the subscripts of the penultimate letters of x' and x'' are even and either equal each other or differ by 4. They cannot differ by 4; otherwise $\tau(x') \neq \tau(x'')$ since the subsequence of letters with even subscripts in $\varphi(\mathbf{s})$ is a factor of $(0_0 0_2 1_4 1_6)^\omega$. So they must be equal and then $\varphi(x') = \varphi(x'')$. Then $\mathbf{p}\tau(\varphi(\mathbf{z}))w' = \mathbf{p}\tau(\varphi(\mathbf{z}))\tau(\varphi(u'))$ contains the 5/4-power $\tau(\varphi(x'y'x''))$ as a suffix. Therefore, decreasing that letter $n + 2$ to c introduces a 5/4-power in $\mathbf{p}\tau(\varphi(\mathbf{s}))$ ending in position i.

Case 2.2. Suppose that xyx starts before $\tau(u')$ in $\mathbf{p}\tau(u')$. In particular, $\tau(u')$ is a suffix of xyx. Since $i \geq 331040$, the first x is not a factor of $\mathbf{p}$. Otherwise, we have

$$\frac{i - |\mathbf{p}| - 6|\mathbf{z}| + 1}{6} = |u| = |\tau(u')| \leq |xyx| = 5|x| \leq 5|\mathbf{p}|,$$

which implies $i \leq 31|\mathbf{p}| + 6|\mathbf{z}| - 1 = 331039$. Therefore, the first x overlaps $\mathbf{p}$ but is not a factor of $\mathbf{p}$. Suppose the overlap length is at least 5. Then the first x contains 20003 as a factor. But since 20003 is never a factor of $\tau \circ \varphi$, this case does not happen. Now suppose that the overlap length is less than or equal to 4. Then the first factor $x = sv$ is made of a nonempty suffix s of 0003 followed by a nonempty prefix v of $\tau(u')$ such that $vyx = \tau(u')$. Write $v = \tau(v')$, $x = \tau(x'')$ and $y = \tau(y')$ where $v'y'x'' = u'$. To get around the fact that $\mathbf{p}$ does not have subscripts, we use $\mathbf{z}$ instead. Recall that 0003 is a common suffix of $\mathbf{p}$ and $\tau(\mathbf{z})$, and the corresponding suffix in $\mathbf{z}$ is $0_0 0_1 0_2 3_3$. So let s' be the suffix of $0_0 0_1 0_2 3_3$ such that $s = \tau(s')$. Now observe that $x' = s'v'$ is a subscript-increasing factor of $\mathbf{z}u'$, overlapping $\mathbf{z}$. Thus $\varphi(x'y'x'')$ is a subscript-increasing suffix of $\varphi(\mathbf{z}u')$, overlapping $\varphi(\mathbf{z})$. Similarly to Case 2.1, since $\tau(x') = \tau(s'v') = sv = x = \tau(x'')$ and $|x| \geq 2$, the subscripts of x' and x'' are equal. Consequently, $x' = x''$ and $\varphi(x') = \varphi(x'')$. Then $\mathbf{p}\tau(\varphi(\mathbf{z}))w' = \mathbf{p}\tau(\varphi(\mathbf{z}u'))$ contains the 5/4-power $\tau(\varphi(x'y'x''))$ as a suffix. Therefore, decreasing that letter $n + 2$ to c introduces a 5/4-power in $\mathbf{p}\tau(\varphi(\mathbf{s}))$ ending in that position i.

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