MATH0488 – Elements of stochastic processes

Kalman filter for sensor fusion in orientation estimation with application to 9-DOF Inertial Measurement Unit (IMU)

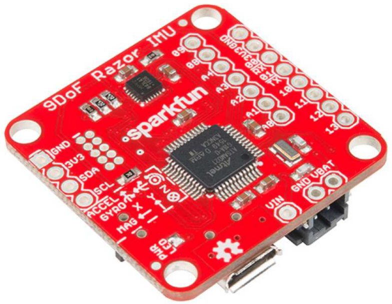
Maarten Arnst, Cedric Laruelle, Pablo Alarcón, and Colin Stoquart

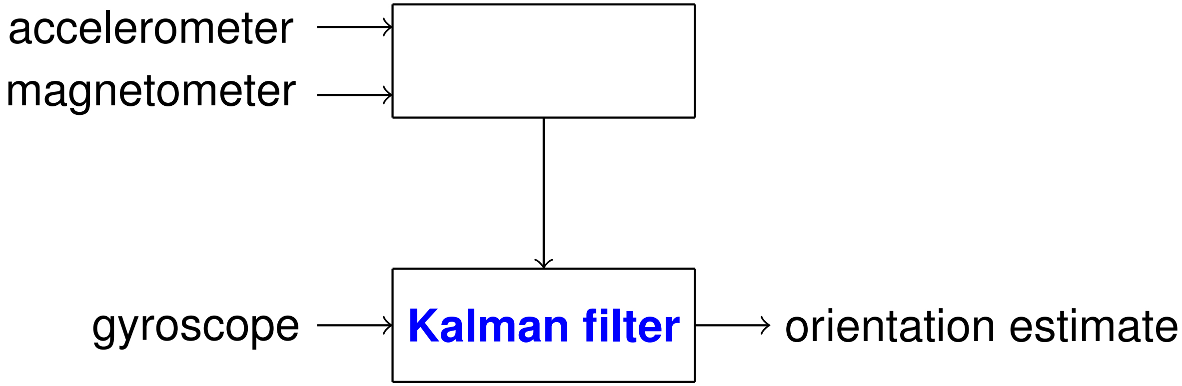
March 10, 2020

Motivation

SpaceX's self-landing boosters require position and orientation estimation.

Motivation



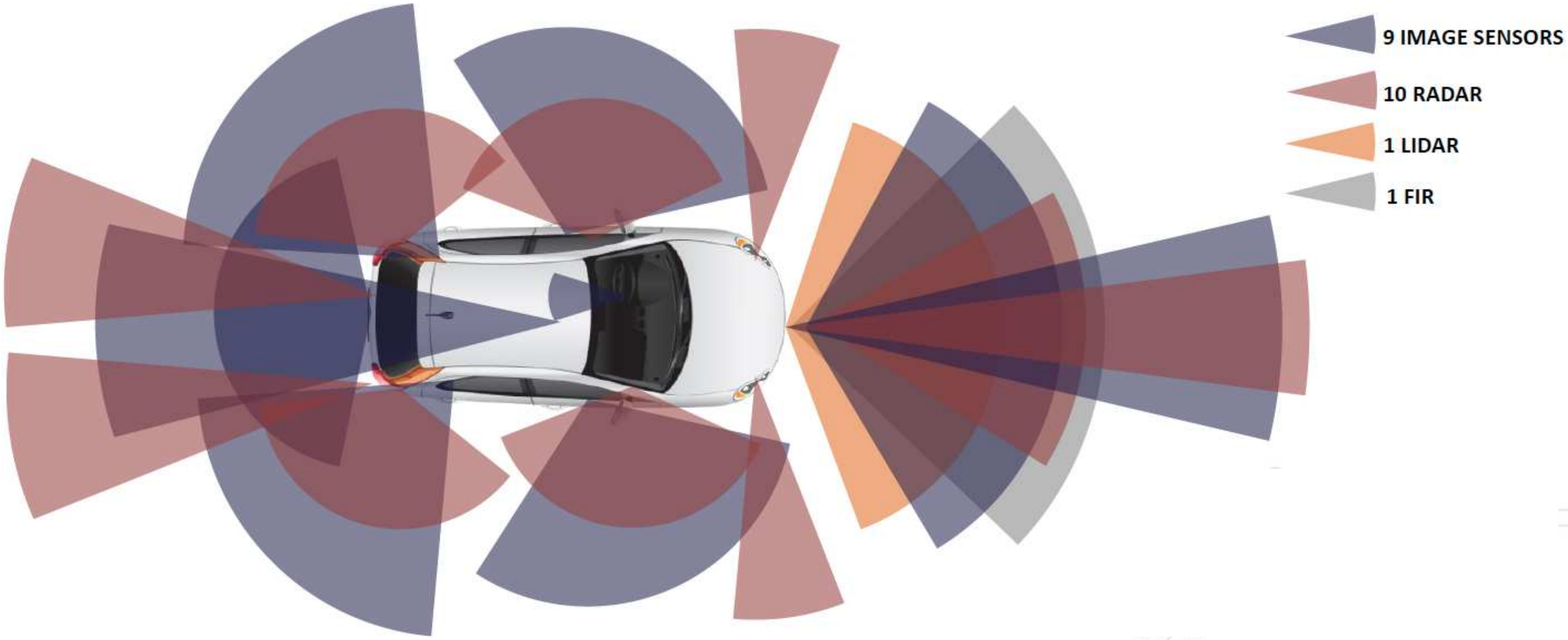


- An **inertial measurement unit (IMU)** is a sensor that features a **three-axis accelerometer**, a **three-axis gyroscope**, and possibly other sensors. While inertial sensors can also be used to obtain a position estimation, we direct our interest to its use to obtain an orientation estimation.

- A **gyroscope** measures **angular velocity**, that is, the rate of change of orientation. Gyroscope measurement data can be **integrated with respect to time** to obtain an **orientation estimate**. While estimates thus obtained are accurate on short time scales, they loose accuracy over time due to **observational noise** and possibly drift.

- To overcome this issue and improve accuracy, orientation information from gyroscopes is combined with information from other sensors $\Longrightarrow$ **sensor fusion**.

Motivation



Sensor fusion is relevant to many other present-day engineering problems, e.g., autonomous driving.

Organisation

- We will be meeting in room B37 S39 from 10h00 to 12h00 at the following dates:

1	2	3	4	5	6	7
10/03	17/03	24/03	31/3	21/04	28/04	05/05
lecture	Q&A	Q&A	Q&A	Q&A	Q&A	Q&A

- Your presence is strongly recommended for the lecture:
  - Tuesday March 10, 10h30–12h30.

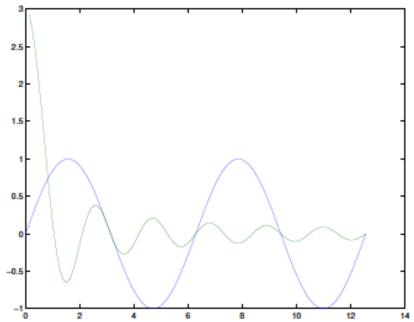
- If you should need some help, please attend the Q&A sessions or contact C. Laruelle, P. Alarcón, or M. Arnst by email to ask a question by email or schedule an appointment.

- Please work in groups of 3 people (groups of less than 2 or more than 3 people not permitted).

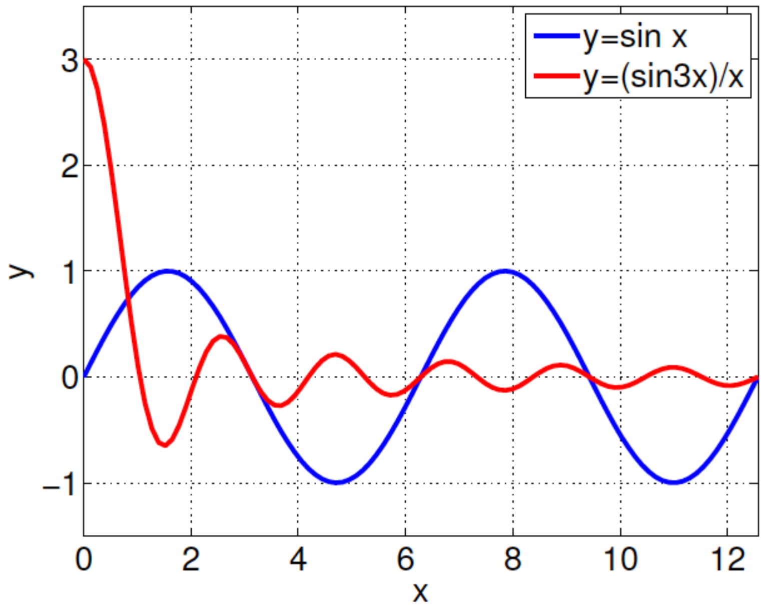
- The report must be sent in PDF format by email to M. Arnst before/on Tuesday May 5 at 22h00.

Organisation

- One report per group is required. The group is responsible for ensuring that work is fairly distributed among group members and that a high-quality report is written.
- The report must be neat, well organized, and professionally presented. All graphs must be computer plots. Label all graph axes and include proper units.
- Please include a list of all the references that you will have consulted.
- Length of 15 to 30 pages (including figs. and list of refs., single spacing, font size of 12 pt).
- The report must be sent in PDF format by email to M. Arnst before/on Tuesday May 5 at 22h00.
  - Please attach to your email (a) file(s) with any code that you will have written.
  - Please attach to your email a document that states how the work was distributed among group members. For each exercise, state which group member(s) worked on the equations (if any), the coding (if any), the analysis of the results (if any), and the writing. Each group member must sign this document, and it is a scan or a photo of this signed document that must be sent.



Bad figure.



Good figure.

Organisation

- Maarten Arnst
  Professeur
  Aérospatiale et Mécanique
  Office: B52 0/419
  Email: maarten.arnst@uliege.be

- Cédric Laruelle
  Chercheur
  Aérospatiale et Mécanique
  Office: B52 +2/541
  Email: cedric.laruelle@uliege.be

- Pablo Alarcón
  Chercheur
  Aérospatiale et Mécanique
  Office: B52 0/512
  Email: palarcon@uliege.be

Outline

- Motivation.
- Organisation.
- Plan.
- Kalman filter.
- Quaternions.
- Orientation estimation.
- Assignment.

Kalman filter

Kalman filter

- **Filtering** involves combining two sources of information:
  - a model of the time-dependent physical or engineered system under study,
  - a sequence of observations of that system.

  The goal is to deduce from these two sources of information estimates of the state of the system that are more accurate than those based on a single source of information alone.

- There can be sources of **inaccuracy in the model and the observations**, for example, due to **observational noise**. As a result, the **estimates of the state** can be **uncertain**.

- **Stochastic methods for filtering** seek to take into account such sources of inaccuracy in the model and the observations and to quantify the uncertainty in the estimates of the state. Stochastic methods for filtering use the probability theory, namely, **stochastic processes**.

- The **Kalman filter** is a stochastic method for filtering in which a **linear model** and **Gaussian probability distributions** are used.

- In the Kalman filter, the system is assumed to evolve in **discrete time steps**

$$t_0 < t_1 < \ldots < t_k < \ldots$$

- The **state** $\boldsymbol{x}_{k-1}$ at time t_{k-1} is assumed to evolve into the state $\boldsymbol{x}_k$ at time t_k according to

$$\boldsymbol{x}_k = \boldsymbol{F}_k(\boldsymbol{x}_{k-1}) + \boldsymbol{\xi}_k,$$

  a linear model in which

  - $\boldsymbol{F}_k$ is the **state transition model**,
  - $\boldsymbol{\xi}_k$ is the **process noise**, a random variable with mean $\mathbf{0}$ and covariance matrix $\boldsymbol{Q}_k$.

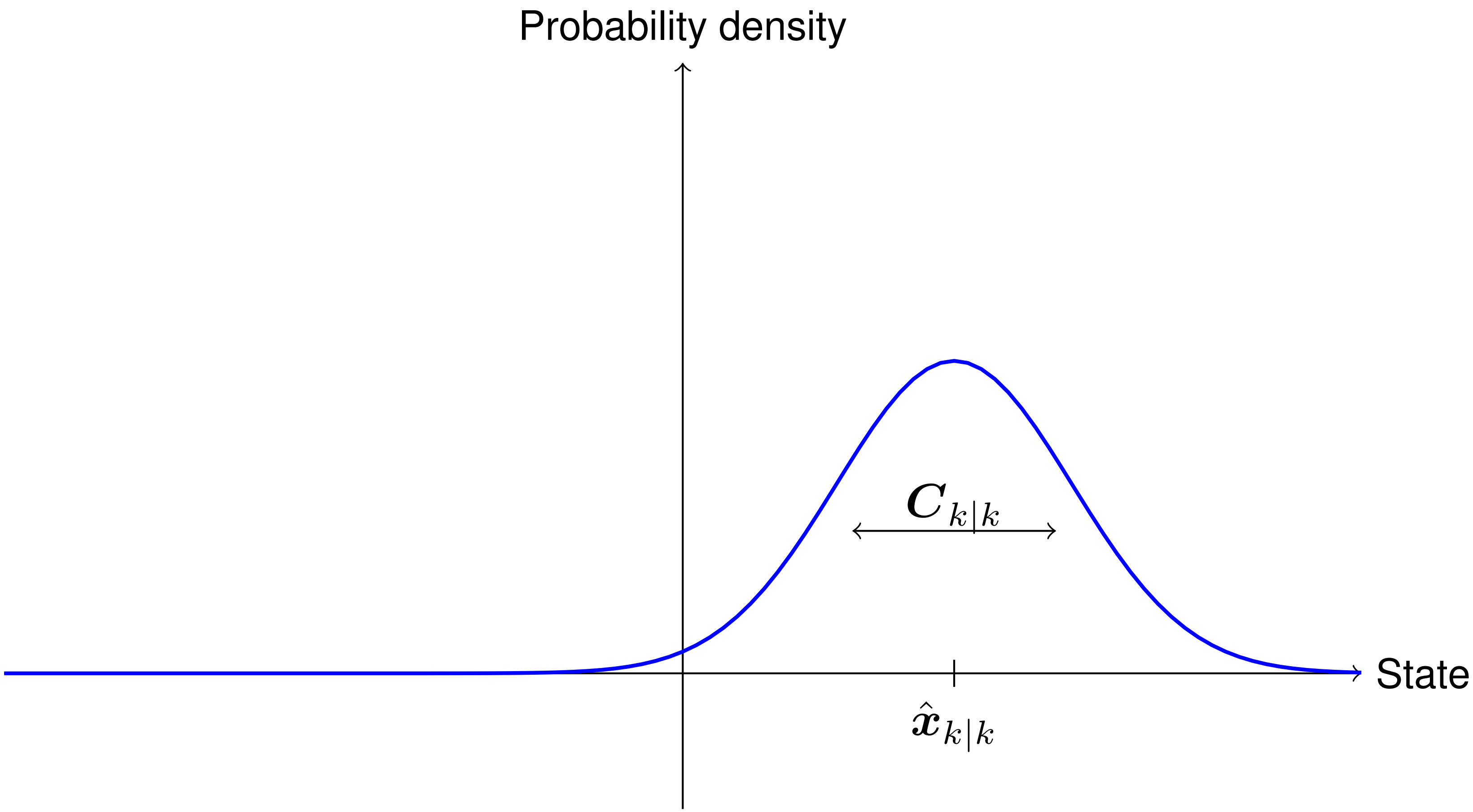
- At time t_k, an **observable** $\boldsymbol{y}_k$ is assumed to be related to the state $\boldsymbol{x}_k$ according to

$$\boldsymbol{y}_k = \boldsymbol{H}_k(\boldsymbol{x}_k) + \boldsymbol{\eta}_k,$$

  a linear model in which

  - $\boldsymbol{H}_k$ is the **observation model**,
  - $\boldsymbol{\eta}_k$ is the **observation noise**, a random variable with mean $\mathbf{0}$ and covariance matrix $\boldsymbol{R}_k$.

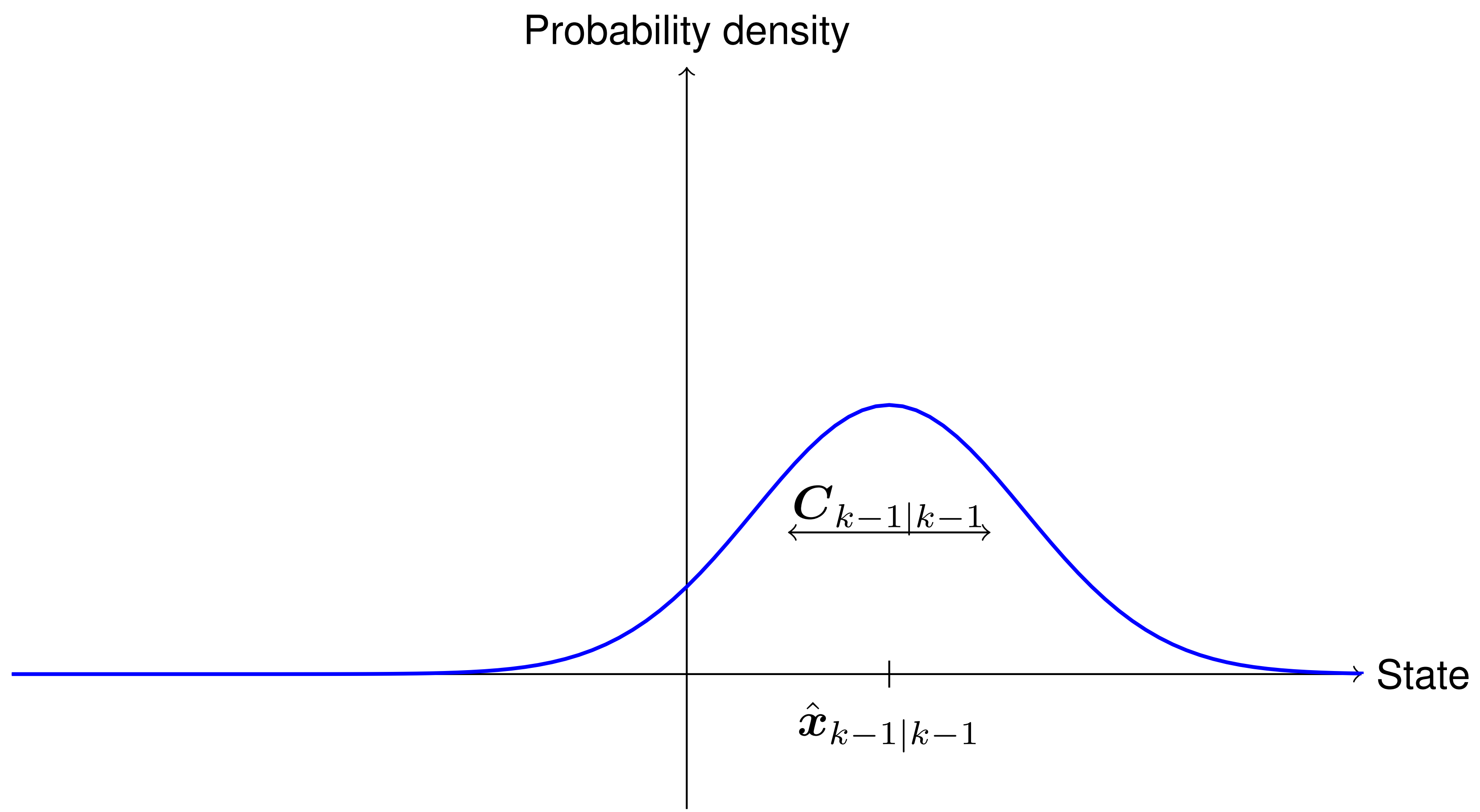
- All **noise random variables** are assumed to be **mutually independent**.



- The **state** at time t_k is represented by a **Gaussian probability density function** (PDF) with **mean vector** $\hat{\boldsymbol{x}}_{k|k}$ (**best estimate**) and **covariance matrix** $\boldsymbol{C}_{k|k}$ (**quantification of uncertainty**):

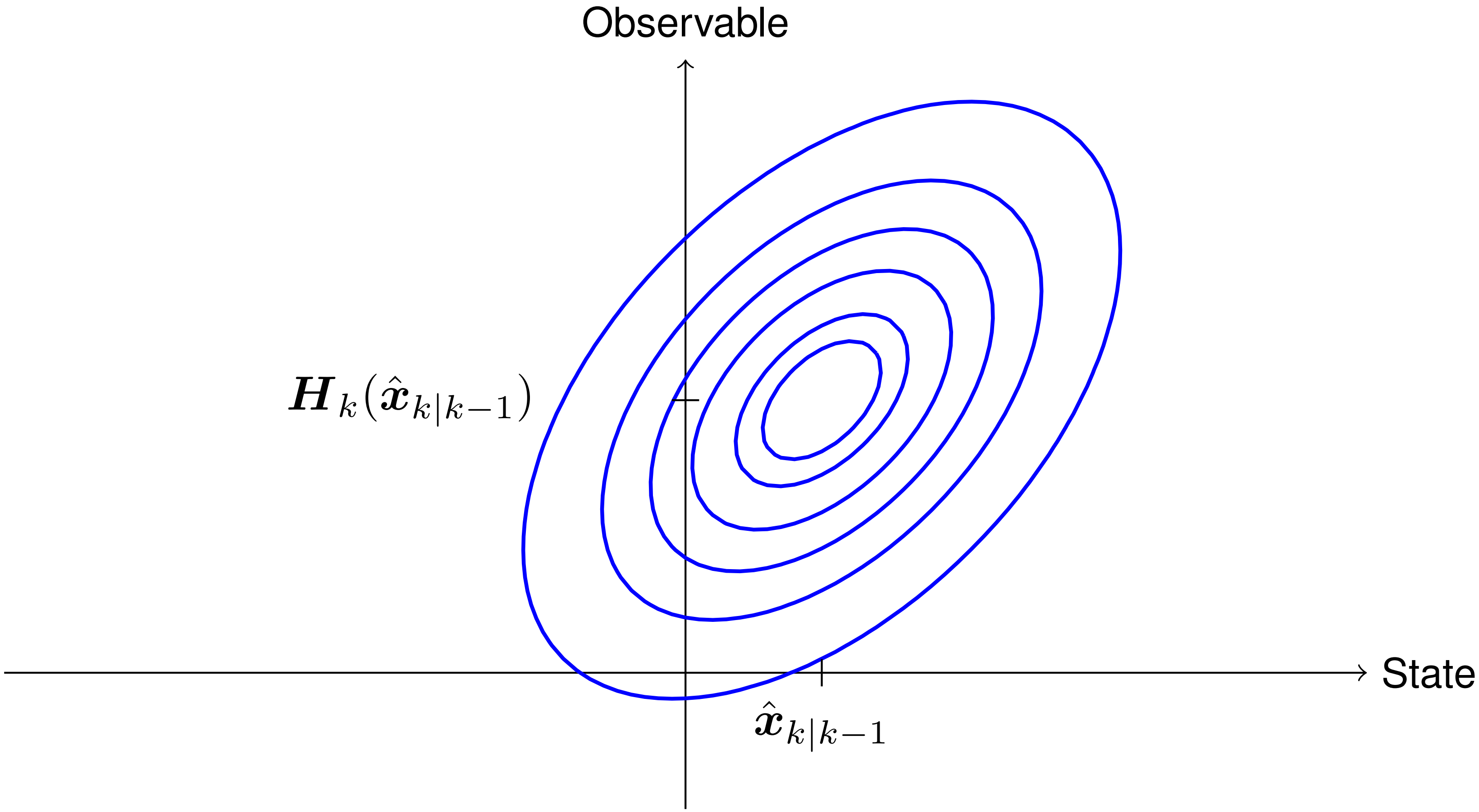
$$\mathcal{N}(\hat{\boldsymbol{x}}_{k|k}, \boldsymbol{C}_{k|k}).$$

- In the Kalman filter, $\hat{\boldsymbol{x}}_{k|k}$ and $\boldsymbol{C}_{k|k}$ at time t_k are deduced from $\hat{\boldsymbol{x}}_{k-1|k-1}$ and $\boldsymbol{C}_{k-1|k-1}$ at time t_{k-1} and the observation y_k in **two steps**.



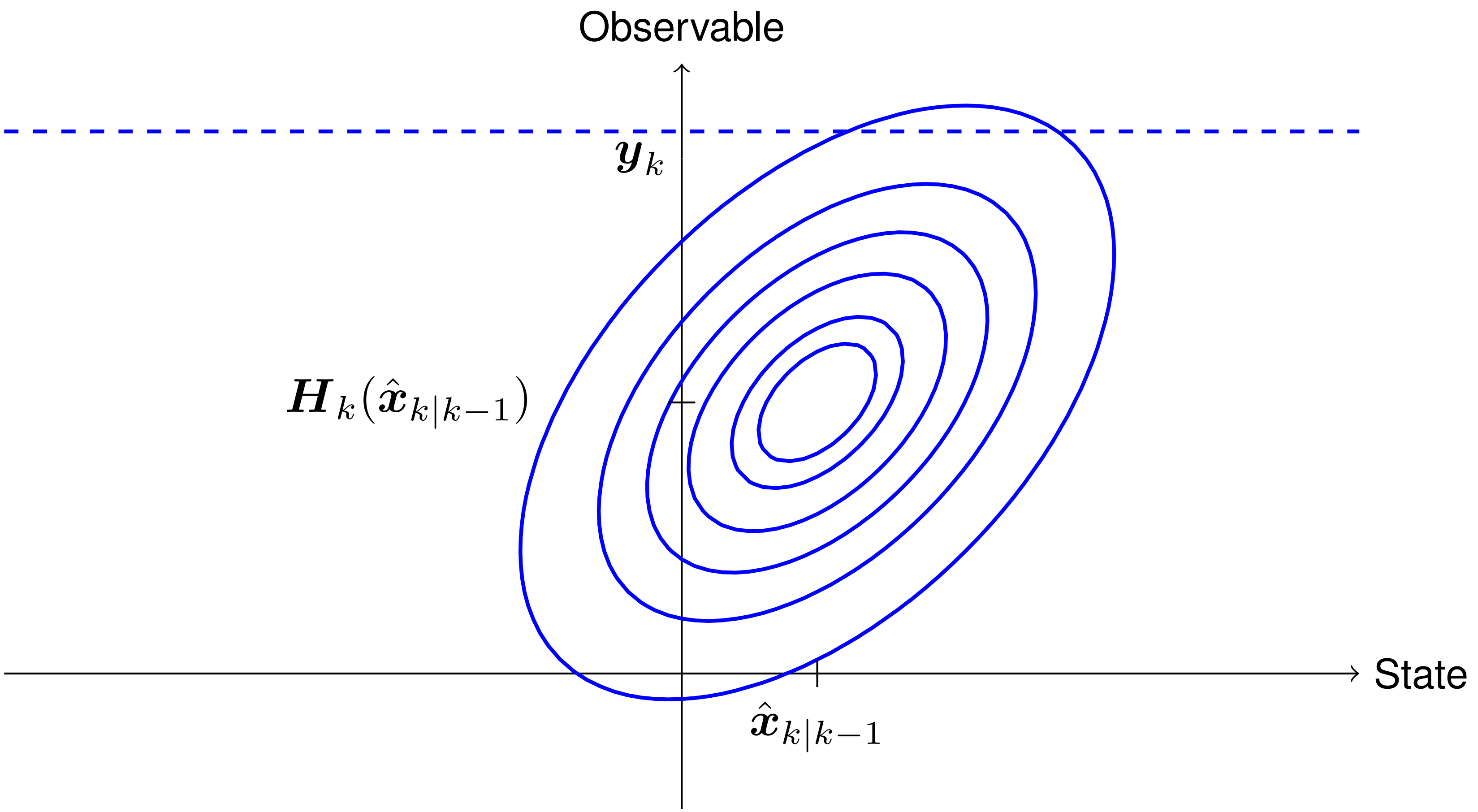
- **Step 1** (**prediction step**): The Gaussian PDF representing the state at time t_{k-1} is mapped through the **state transition model** and the **observation model** to obtain a Gaussian PDF representing a **joint prediction of the state and the observable** at time t_k:

$$\mathcal{N}\left(\begin{bmatrix}\hat{\boldsymbol{x}}_{k|k-1}\\ \boldsymbol{H}_k(\hat{\boldsymbol{x}}_{k|k-1})\end{bmatrix}, \begin{bmatrix}\boldsymbol{C}_{k|k-1} & \boldsymbol{C}_{k|k-1}\boldsymbol{H}_k^{\mathrm{T}}\\ \boldsymbol{H}_k\boldsymbol{C}_{k|k-1} & \boldsymbol{H}_k\boldsymbol{C}_{k|k-1}\boldsymbol{H}_k^{\mathrm{T}}+\boldsymbol{R}_k\end{bmatrix}\right) \text{ with } \begin{cases}\hat{\boldsymbol{x}}_{k|k-1} = \boldsymbol{F}_k(\hat{\boldsymbol{x}}_{k-1|k-1}),\\ \boldsymbol{C}_{k|k-1} = \boldsymbol{F}_k\boldsymbol{C}_{k-1|k-1}\boldsymbol{F}_k^{\mathrm{T}}+\boldsymbol{Q}_k.\end{cases}$$



- **Step 1** (**prediction step**): The Gaussian PDF representing the state at time t_{k-1} is mapped through the **state transition model** and the **observation model** to obtain a Gaussian PDF representing a **joint prediction of the state and the observable** at time t_k:

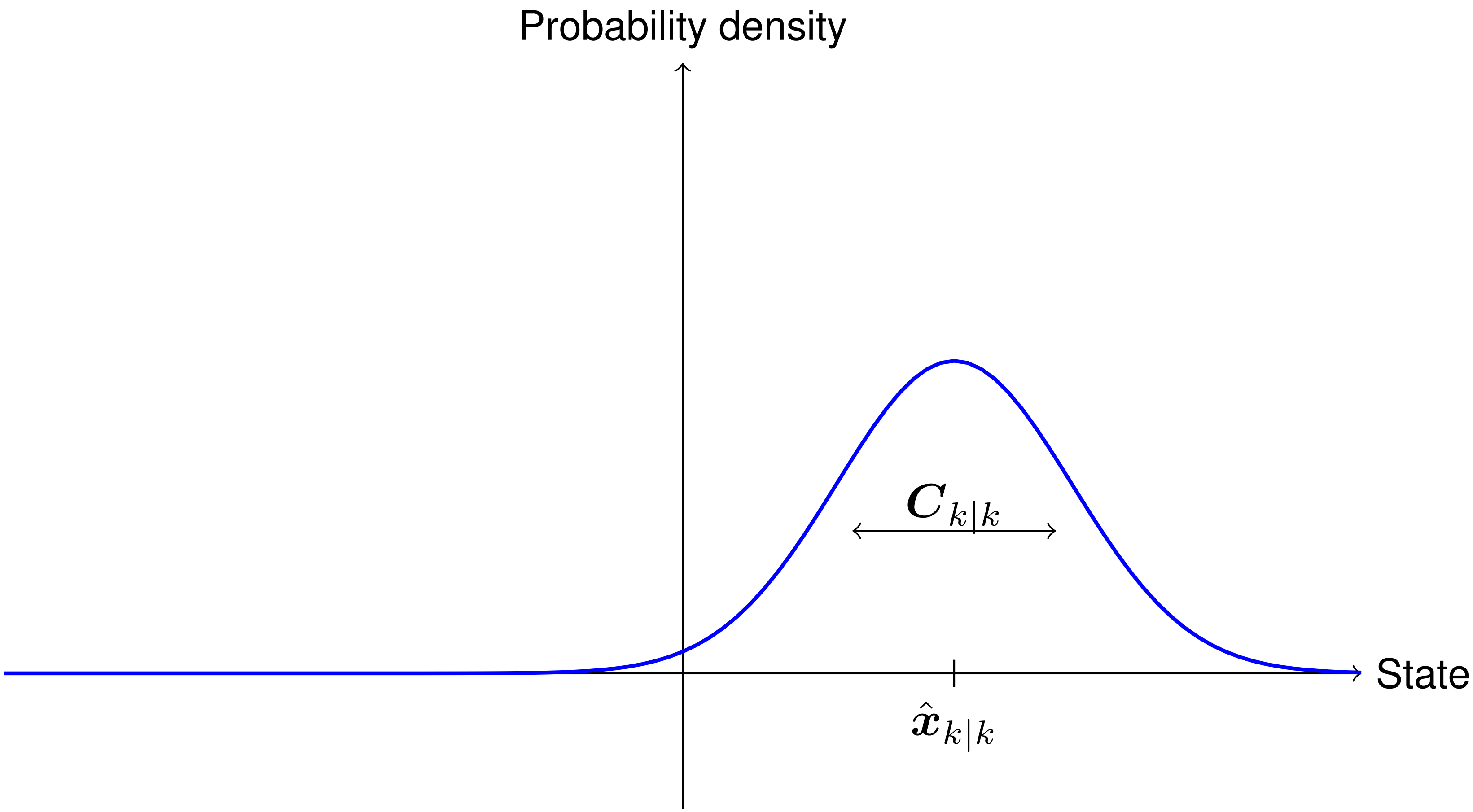
$$\mathcal{N}\left(\begin{bmatrix}\hat{\boldsymbol{x}}_{k|k-1}\\ \boldsymbol{H}_k(\hat{\boldsymbol{x}}_{k|k-1})\end{bmatrix},\begin{bmatrix}\boldsymbol{C}_{k|k-1} & \boldsymbol{C}_{k|k-1}\boldsymbol{H}_k^{\mathrm{T}}\\ \boldsymbol{H}_k\boldsymbol{C}_{k|k-1} & \boldsymbol{H}_k\boldsymbol{C}_{k|k-1}\boldsymbol{H}_k^{\mathrm{T}}+\boldsymbol{R}_k\end{bmatrix}\right) \text{ with } \begin{cases}\hat{\boldsymbol{x}}_{k|k-1}=\boldsymbol{F}_k(\hat{\boldsymbol{x}}_{k-1|k-1}),\\ \boldsymbol{C}_{k|k-1}=\boldsymbol{F}_k\boldsymbol{C}_{k-1|k-1}\boldsymbol{F}_k^{\mathrm{T}}+\boldsymbol{Q}_k.\end{cases}$$



- **Step 2** (**correction step**): The estimate $\hat{\boldsymbol{x}}_{k|k-1}$ is updated to obtain the estimate $\hat{\boldsymbol{x}}_{k|k}$ with covariance matrix $\boldsymbol{C}_{k|k}$ by **conditioning** on the **observation** $\boldsymbol{y}_k$:

$$\mathcal{N}(\hat{\boldsymbol{x}}_{k|k}, \boldsymbol{C}_{k|k}) = \mathcal{N}\Big(\hat{\boldsymbol{x}}_{k|k-1} + \boldsymbol{C}_{k|k-1}\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{S}_k^{-1}(\boldsymbol{y}_k - \boldsymbol{H}_k(\hat{\boldsymbol{x}}_{k|k-1})), \boldsymbol{C}_{k|k-1} - \boldsymbol{C}_{k|k-1}\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{S}_k^{-1}\boldsymbol{H}_k\boldsymbol{C}_{k|k-1}\Big),$$

in which $\boldsymbol{S}_k = \boldsymbol{H}_k\boldsymbol{C}_{k|k-1}\boldsymbol{H}_k^{\mathrm{T}} + \boldsymbol{R}_k$.



- **Step 2** (**correction step**): The estimate $\hat{\boldsymbol{x}}_{k|k-1}$ is updated to obtain the estimate $\hat{\boldsymbol{x}}_{k|k}$ with covariance matrix $\boldsymbol{C}_{k|k}$ by **conditioning** on the **observation** $\boldsymbol{y}_k$:

$$\mathcal{N}(\hat{\boldsymbol{x}}_{k|k}, \boldsymbol{C}_{k|k}) = \mathcal{N}\Big(\hat{\boldsymbol{x}}_{k|k-1} + \boldsymbol{C}_{k|k-1}\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{S}_k^{-1}(\boldsymbol{y}_k - \boldsymbol{H}_k(\hat{\boldsymbol{x}}_{k|k-1})), \boldsymbol{C}_{k|k-1} - \boldsymbol{C}_{k|k-1}\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{S}_k^{-1}\boldsymbol{H}_k\boldsymbol{C}_{k|k-1}\Big),$$

in which $\boldsymbol{S}_k = \boldsymbol{H}_k\boldsymbol{C}_{k|k-1}\boldsymbol{H}_k^{\mathrm{T}} + \boldsymbol{R}_k$.

- Thus, by putting things together, the Kalman filter can be expressed as follows:
  - **Inititialization**:

$$(\hat{\boldsymbol{x}}_{0|0}, \boldsymbol{C}_{0|0}) = (\boldsymbol{m}_0, \boldsymbol{Q}_0).$$

  - For $k = 1, 2, \ldots$:
    - **Step 1** (**prediction step**):

$$\hat{\boldsymbol{x}}_{k|k-1} = \boldsymbol{F}_k(\hat{\boldsymbol{x}}_{k-1|k-1}),$$
$$\boldsymbol{C}_{k|k-1} = \boldsymbol{F}_k \boldsymbol{C}_{k-1|k-1} \boldsymbol{F}_k^{\mathrm{T}} + \boldsymbol{Q}_k.$$

    - **Step 2** (**correction step**):

$$\hat{\boldsymbol{x}}_{k|k} = \hat{\boldsymbol{x}}_{k|k-1} + \boldsymbol{C}_{k|k-1} \boldsymbol{H}_k^{\mathrm{T}} \boldsymbol{S}_k^{-1} \big(\boldsymbol{y}_k - \boldsymbol{H}_k(\hat{\boldsymbol{x}}_{k|k-1})\big),$$
$$\boldsymbol{C}_{k|k} = \boldsymbol{C}_{k|k-1} - \boldsymbol{C}_{k|k-1} \boldsymbol{H}_k^{\mathrm{T}} \boldsymbol{S}_k^{-1} \boldsymbol{H}_k \boldsymbol{C}_{k|k-1},$$

    in which

$$\boldsymbol{S}_k = \boldsymbol{H}_k \boldsymbol{C}_{k|k-1} \boldsymbol{H}_k^{\mathrm{T}} + \boldsymbol{R}_k.$$

- The previous equations follow from fundamental results from linear algebra and probability theory, notably, fundamental results about affine transformations and conditioning of random variables.
- An affine transformation $\boldsymbol{v} = \boldsymbol{A}(\boldsymbol{u}) + \boldsymbol{b}$ of a random variable $\boldsymbol{U}$ with mean vector $\underline{\boldsymbol{u}}$ and covariance matrix $\boldsymbol{C}_{\boldsymbol{U}}$ is a random variable $\boldsymbol{V}$ with mean vector $\underline{\boldsymbol{v}}$ and covariance matrix $\boldsymbol{C}_{\boldsymbol{V}}$ with

$$\underline{\boldsymbol{v}} = E\{\boldsymbol{V}\} = \{\boldsymbol{A}(\boldsymbol{U}) + \boldsymbol{b}\} = \boldsymbol{A}(E\{\boldsymbol{U}\}) + \boldsymbol{b} = \boldsymbol{A}(\underline{\boldsymbol{u}}) + \boldsymbol{b},$$
$$\boldsymbol{C}_{\boldsymbol{V}} = E\{(\boldsymbol{V} - \underline{\boldsymbol{v}})(\boldsymbol{V} - \underline{\boldsymbol{v}})^{\mathrm{T}}\} = \boldsymbol{A}E\{(\boldsymbol{U} - \underline{\boldsymbol{u}})(\boldsymbol{U} - \underline{\boldsymbol{u}})^{\mathrm{T}}\}\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}\boldsymbol{C}_{\boldsymbol{U}}\boldsymbol{A}^{\mathrm{T}}.$$

- An affine transformation of a Gaussian random variable is a Gaussian transformation.
- Let $\boldsymbol{U}$ be a Gaussian random variable with mean vector $\underline{\boldsymbol{u}}$ and covariance matrix $\boldsymbol{C}$ so that $\boldsymbol{U}$, $\underline{\boldsymbol{u}}$, and $\boldsymbol{C}$ can be decomposed in block form as follows

$$\boldsymbol{U} = \begin{bmatrix} \boldsymbol{U}_1 \\ \boldsymbol{U}_2 \end{bmatrix}, \quad \underline{\boldsymbol{u}} = \begin{bmatrix} \underline{\boldsymbol{u}}_1 \\ \underline{\boldsymbol{u}}_2 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} \boldsymbol{C}_{11} & \boldsymbol{C}_{21}^{\mathrm{T}} \\ \boldsymbol{C}_{21} & \boldsymbol{C}_{22} \end{bmatrix};$$

thus, $\boldsymbol{U}$ is distributed according to $\mathcal{N}(\underline{\boldsymbol{u}}, \boldsymbol{C})$, which can be written in block form as follows:

$$\begin{bmatrix} \boldsymbol{U}_1 \\ \boldsymbol{U}_2 \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \underline{\boldsymbol{u}}_1 \\ \underline{\boldsymbol{u}}_2 \end{bmatrix}, \begin{bmatrix} \boldsymbol{C}_{11} & \boldsymbol{C}_{21}^{\mathrm{T}} \\ \boldsymbol{C}_{21} & \boldsymbol{C}_{22} \end{bmatrix}\right).$$

Then, the conditional probability distribution of $\boldsymbol{U}_1$ given $\boldsymbol{U}_2 = \boldsymbol{u}_2$ is given by

$$(\boldsymbol{U}_1 | \boldsymbol{U}_2 = \boldsymbol{u}_2) \sim \mathcal{N}\big(\underline{\boldsymbol{u}}_1 + \boldsymbol{C}_{21}^{\mathrm{T}}\boldsymbol{C}_{22}^{-1}(\boldsymbol{u}_2 - \underline{\boldsymbol{u}}_2), \boldsymbol{C}_{11} - \boldsymbol{C}_{21}^{\mathrm{T}}\boldsymbol{C}_{22}^{-1}\boldsymbol{C}_{21}\big).$$

- Inverse of a blocked square matrix with symmetric diagonal blocks:

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B}^{\mathrm{T}} \\ \boldsymbol{B} & \boldsymbol{C} \end{bmatrix}^{-1} = \begin{bmatrix} (\boldsymbol{A} - \boldsymbol{B}^{\mathrm{T}}\boldsymbol{C}^{-1}\boldsymbol{B})^{-1} & (-(\boldsymbol{C} - \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}^{\mathrm{T}})^{-1}\boldsymbol{B}\boldsymbol{A}^{-1})^{\mathrm{T}} \\ -(\boldsymbol{C} - \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}^{\mathrm{T}})^{-1}\boldsymbol{B}\boldsymbol{A}^{-1} & (\boldsymbol{C} - \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}^{\mathrm{T}})^{-1} \end{bmatrix}.$$

- Factorization of a blocked square matrix:

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B}^{\mathrm{T}} \\ \boldsymbol{B} & \boldsymbol{C} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{B}^{\mathrm{T}}\boldsymbol{C}^{-1} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{A} - \boldsymbol{B}^{\mathrm{T}}\boldsymbol{C}^{-1}\boldsymbol{B} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{C}^{-1}\boldsymbol{B} & \boldsymbol{I} \end{bmatrix},$$

  in which $\boldsymbol{I}$ is the (appropriately sized) identity matrix.

- Determinant of a blocked square matrix:

$$\det \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B}^{\mathrm{T}} \\ \boldsymbol{B} & \boldsymbol{C} \end{bmatrix} = \det(\boldsymbol{C}) \det(\boldsymbol{A} - \boldsymbol{B}^{\mathrm{T}}\boldsymbol{C}^{-1}\boldsymbol{B}).$$

- Matrix identity:

$$\boldsymbol{A}^{-1}\boldsymbol{B}^{\mathrm{T}}(\boldsymbol{C} - \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}^{\mathrm{T}})^{-1} = (\boldsymbol{A} - \boldsymbol{B}^{\mathrm{T}}\boldsymbol{C}^{-1}\boldsymbol{B})^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{C}^{-1};$$

  indeed,

$$(\boldsymbol{A} - \boldsymbol{B}^{\mathrm{T}}\boldsymbol{C}^{-1}\boldsymbol{B})\boldsymbol{A}^{-1}\boldsymbol{B}^{\mathrm{T}} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{C}^{-1}(\boldsymbol{C} - \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}^{\mathrm{T}})$$

$$\boldsymbol{B}^{\mathrm{T}} - \boldsymbol{B}^{\mathrm{T}}\boldsymbol{C}^{-1}\boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}^{\mathrm{T}} \stackrel{!}{=} \boldsymbol{B}^{\mathrm{T}} - \boldsymbol{B}^{\mathrm{T}}\boldsymbol{C}^{-1}\boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}^{\mathrm{T}}.$$

- The aforementioned fundamental result about conditioning of random variables follows from the aforementioned fundamental results from linear algebra:

$$\rho_{\boldsymbol{U}_1|\boldsymbol{U}_2}(\boldsymbol{u}_1|\boldsymbol{u}_2) = \frac{\rho_{(\boldsymbol{U}_1,\boldsymbol{U}_2)}(\boldsymbol{u}_1,\boldsymbol{u}_2)}{\rho_{\boldsymbol{U}_2}(\boldsymbol{u}_2)}$$

$$= \frac{\frac{1}{\sqrt{(2\pi)^{n_1+n_2}\det\begin{bmatrix}\boldsymbol{C}_{11} & \boldsymbol{C}_{21}^{\mathrm{T}}\\ \boldsymbol{C}_{21} & \boldsymbol{C}_{22}\end{bmatrix}}}\exp\left(-\frac{1}{2}\left(\begin{bmatrix}\boldsymbol{u}_1\\ \boldsymbol{u}_2\end{bmatrix}-\begin{bmatrix}\underline{\boldsymbol{u}}_1\\ \underline{\boldsymbol{u}}_2\end{bmatrix}\right)^{\mathrm{T}}\begin{bmatrix}\boldsymbol{C}_{11} & \boldsymbol{C}_{21}^{\mathrm{T}}\\ \boldsymbol{C}_{21} & \boldsymbol{C}_{22}\end{bmatrix}^{-1}\left(\begin{bmatrix}\boldsymbol{u}_1\\ \boldsymbol{u}_2\end{bmatrix}-\begin{bmatrix}\underline{\boldsymbol{u}}_1\\ \underline{\boldsymbol{u}}_2\end{bmatrix}\right)\right)}{\frac{1}{\sqrt{(2\pi)^{n_2}\det(\boldsymbol{C}_{22})}}\exp\left(-\frac{1}{2}(\boldsymbol{u}_2-\underline{\boldsymbol{u}}_2)^{\mathrm{T}}\boldsymbol{C}_{22}^{-1}(\boldsymbol{u}_2-\underline{\boldsymbol{u}}_2)\right)}$$

$$= \frac{1}{\sqrt{(2\pi)^{n_1}\det(\boldsymbol{C}_{11}-\boldsymbol{C}_{21}^{\mathrm{T}}\boldsymbol{C}_{22}^{-1}\boldsymbol{C}_{21})}}\exp\Bigg(-\frac{1}{2}\left(\boldsymbol{u}_1-\underline{\boldsymbol{u}}_1-\boldsymbol{C}_{21}^{\mathrm{T}}\boldsymbol{C}_{22}^{-1}(\boldsymbol{u}_2-\underline{\boldsymbol{u}}_2)\right)^{\mathrm{T}}$$
$$(\boldsymbol{C}_{11}-\boldsymbol{C}_{21}^{\mathrm{T}}\boldsymbol{C}_{22}^{-1}\boldsymbol{C}_{21})^{-1}\left(\boldsymbol{u}_1-\underline{\boldsymbol{u}}_1-\boldsymbol{C}_{21}^{\mathrm{T}}\boldsymbol{C}_{22}^{-1}(\boldsymbol{u}_2-\underline{\boldsymbol{u}}_2)\right)\Bigg);$$

Indeed, with the aforementioned matrix identity:

$$\begin{aligned}
&(\boldsymbol{u}_1-\underline{\boldsymbol{u}}_1)^{\mathrm{T}}(\boldsymbol{C}_{11}-\boldsymbol{C}_{21}^{\mathrm{T}}\boldsymbol{C}_{22}^{-1}\boldsymbol{C}_{21})^{-1}(\boldsymbol{u}_1-\underline{\boldsymbol{u}}_1)\\
&\quad -2(\boldsymbol{u}_1-\underline{\boldsymbol{u}}_1)^{\mathrm{T}}\left((\boldsymbol{C}_{22}-\boldsymbol{C}_{21}\boldsymbol{C}_{11}^{-1}\boldsymbol{C}_{21}^{\mathrm{T}})^{-1}\boldsymbol{C}_{21}\boldsymbol{C}_{11}^{-1}\right)^{\mathrm{T}}(\boldsymbol{u}_2-\underline{\boldsymbol{u}}_2)\\
&\quad +(\boldsymbol{u}_2-\underline{\boldsymbol{u}}_2)^{\mathrm{T}}(\boldsymbol{C}_{22}-\boldsymbol{C}_{21}\boldsymbol{C}_{11}^{-1}\boldsymbol{C}_{21}^{\mathrm{T}})^{-1}(\boldsymbol{u}_2-\underline{\boldsymbol{u}}_2)\\
&\quad -(\boldsymbol{u}_2-\underline{\boldsymbol{u}}_2)^{\mathrm{T}}\boldsymbol{C}_{22}^{-1}(\boldsymbol{u}_2-\underline{\boldsymbol{u}}_2)\\
&= \left(\boldsymbol{u}_1-\underline{\boldsymbol{u}}_1-\boldsymbol{C}_{21}^{\mathrm{T}}\boldsymbol{C}_{22}^{-1}(\boldsymbol{u}_2-\underline{\boldsymbol{u}}_2)\right)^{\mathrm{T}}(\boldsymbol{C}_{11}-\boldsymbol{C}_{21}^{\mathrm{T}}\boldsymbol{C}_{22}^{-1}\boldsymbol{C}_{21})^{-1}\left(\boldsymbol{u}_1-\underline{\boldsymbol{u}}_1-\boldsymbol{C}_{21}^{\mathrm{T}}\boldsymbol{C}_{22}^{-1}(\boldsymbol{u}_2-\underline{\boldsymbol{u}}_2)\right).
\end{aligned}$$

Quaternions

Not all the material to follow is required to work on the assignment for this project. However, if you are interested in knowing where the equations to be used come from, you will find that insight in the following. Also, bringing the material together in a coherent manner was needed because different references from the literature use different conventions, such as directions in which orientations are considered positive. Be mindful of such different conventions if you consult references yourself.

- Let us consider the m-dimensional **Euclidean vector space** $\mathbb{R}^m$.
- For two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{R}^m$, the (Euclidean) **inner product** is the scalar denoted by $\boldsymbol{a} \cdot \boldsymbol{b}$.
- We denote by $\{\boldsymbol{i}_1, \ldots, \boldsymbol{i}_m\}$ an **orthonormal basis** for $\mathbb{R}^m$, that is, a basis such that $\boldsymbol{i}_k \cdot \boldsymbol{i}_\ell = \delta_{k\ell}$, $1 \leq k, \ell \leq m$, where $\delta_{k\ell}$ is the **Kronecker delta** equal to 1 if $k = \ell$ and 0 otherwise.
- Given an orthonormal basis $\{\boldsymbol{i}_1, \ldots, \boldsymbol{i}_m\}$ for $\mathbb{R}^m$, we have that any **vector** $\boldsymbol{a}$ in $\mathbb{R}^m$ can be **represented** by a **column matrix**

$$\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$$

  of its **components** a_k such that $\boldsymbol{a} = \sum_{k=1}^m a_k \boldsymbol{i}_k$ with $a_k = \boldsymbol{a} \cdot \boldsymbol{i}_k,\ 1 \leq k \leq m$.
- For two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, the inner product $\boldsymbol{a} \cdot \boldsymbol{b}$ is the scalar $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{k=1}^m a_k b_k$.
- If $m = 3$, for two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{R}^3$, the **vector product** $\boldsymbol{a} \times \boldsymbol{b}$ is the vector $\boldsymbol{a} \times \boldsymbol{b}$ in $\mathbb{R}^3$ such that $\boldsymbol{a} \times \boldsymbol{b} = (a_2 b_3 - a_3 b_2)\boldsymbol{i}_1 + (a_3 b_1 - a_1 b_3)\boldsymbol{i}_2 + (a_1 b_2 - a_2 b_1)\boldsymbol{i}_3$.
- If $m = 3$, for three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ in $\mathbb{R}^3$, we have the **vector triple product identity**:

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}.$$

- A **linear mapping** $\boldsymbol{A}$ from $\mathbb{R}^m$ into $\mathbb{R}^m$ is a function that maps any vector $\boldsymbol{a}$ in $\mathbb{R}^m$ onto a vector $\boldsymbol{b} = \boldsymbol{A}(\boldsymbol{a})$ in $\mathbb{R}^m$ in a manner that satisfies additivity ($\boldsymbol{A}(\boldsymbol{a}_1 + \boldsymbol{a}_2) = \boldsymbol{A}(\boldsymbol{a}_1) + \boldsymbol{A}(\boldsymbol{a}_2)$, $\forall \boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathbb{R}^m$) and homogeneity ($\boldsymbol{A}(\alpha \boldsymbol{a}) = \alpha \boldsymbol{A}(\boldsymbol{a}), \forall \alpha \in \mathbb{R},\ \forall \boldsymbol{a} \in \mathbb{R}^m$) properties.

- Given an orthonormal basis $\{\boldsymbol{i}_1, \ldots, \boldsymbol{i}_m\}$ for $\mathbb{R}^m$, we have that any **linear mapping** $\boldsymbol{A}$ from $\mathbb{R}^m$ into $\mathbb{R}^m$ can be **represented** by a **matrix**

$$\begin{bmatrix} a_{11} & \ldots & a_{1m} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mm} \end{bmatrix}$$

  of its **components** $a_{k\ell}$ such that $a_{k\ell} = \boldsymbol{i}_k \cdot \boldsymbol{A}(\boldsymbol{i}_\ell),\ 1 \leq k, \ell \leq m$.

- We have for two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ and a linear mapping $\boldsymbol{A}$ with $\boldsymbol{b} = \boldsymbol{A}(\boldsymbol{a})$ that

$$\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} a_{11} & \ldots & a_{1m} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mm} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}.$$

- The **identity linear mapping** from $\mathbb{R}^m$ into $\mathbb{R}^m$ is the linear mapping $\boldsymbol{I}$ from $\mathbb{R}^m$ into $\mathbb{R}^m$ such that

$$\boldsymbol{I}(\boldsymbol{a}) = \boldsymbol{a}, \quad \forall \boldsymbol{a} \in \mathbb{R}^m,$$

which corresponds to the matrix-vector representation

$$\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}.$$

- The **transpose** of a linear mapping $\boldsymbol{A}$ from $\mathbb{R}^m$ into $\mathbb{R}^m$ is the linear mapping $\boldsymbol{A}^{\mathrm{T}}$ from $\mathbb{R}^m$ into $\mathbb{R}^m$ such that

$$\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{A}(\boldsymbol{b}), \quad \forall \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^m.$$

- A linear mapping $\boldsymbol{A}$ from $\mathbb{R}^m$ into $\mathbb{R}^m$ is **symmetric** if $\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}$, and it is **skew-symmetric** if $\boldsymbol{A}^{\mathrm{T}} = -\boldsymbol{A}$.

- The **trace** of a linear mapping tr$(\boldsymbol{A})$ is defined by tr$(\boldsymbol{A}) = \sum_{k=1}^{m} a_{kk}$.

- For two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{R}^m$, the **tensor product** is the linear mapping denoted by $\boldsymbol{a} \otimes \boldsymbol{b}$ that maps any vector $\boldsymbol{c}$ in $\mathbb{R}^m$ onto a vector $\boldsymbol{d} = (\boldsymbol{a} \otimes \boldsymbol{b})(\boldsymbol{c})$ in $\mathbb{R}^m$ such that

$$\boldsymbol{d} = (\boldsymbol{a} \otimes \boldsymbol{b})(\boldsymbol{c}) = \boldsymbol{a}(\boldsymbol{b} \cdot \boldsymbol{c}),$$

which corresponds to the matrix-vector representation

$$\begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix} = \begin{bmatrix} a_1 b_1 & \dots & a_1 b_m \\ \vdots & & \vdots \\ a_m b_1 & \dots & a_m b_m \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}.$$

- We have that $\boldsymbol{I} = \sum_{k=1} \boldsymbol{i}_k \otimes \boldsymbol{i}_k$. And we have the identities

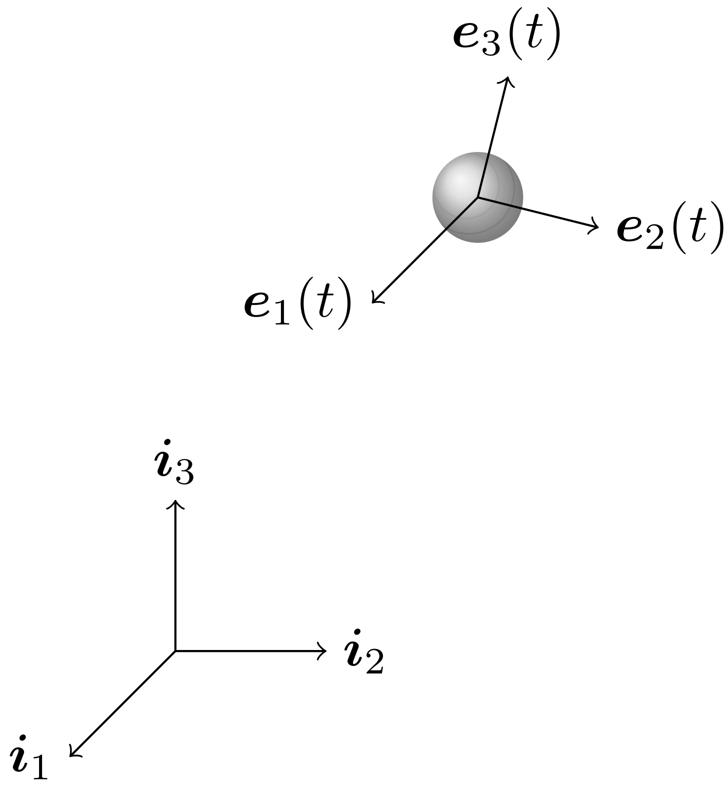
$$(\boldsymbol{a} \otimes \boldsymbol{b})(\boldsymbol{c} \otimes \boldsymbol{d}) = (\boldsymbol{b} \cdot \boldsymbol{c})(\boldsymbol{a} \otimes \boldsymbol{d}), \quad (\boldsymbol{a} \otimes \boldsymbol{b})^{\mathrm{T}} = \boldsymbol{b} \otimes \boldsymbol{a}, \quad \mathrm{tr}(\boldsymbol{a} \otimes \boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{b}.$$

- If $m = 3$, the **vector product** of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{R}^3$ is the vector $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$ in $\mathbb{R}^3$ defined previously. The mapping that, for a given $\boldsymbol{a}$, maps $\boldsymbol{b}$ onto $\boldsymbol{c}$ is linear, so that it can be **represented** by a **linear mapping** from $\mathbb{R}^3$ into $\mathbb{R}^3$ denoted by $\widehat{\boldsymbol{A}}$ such that

$$\widehat{\boldsymbol{A}}(\boldsymbol{b}) = \boldsymbol{a} \times \boldsymbol{b}, \quad \forall \boldsymbol{b} \in \mathbb{R}^3.$$

We say that $\boldsymbol{a}$ is the **axial vector** of $\widehat{\boldsymbol{A}}$, which corresponds to the matrix-vector representation

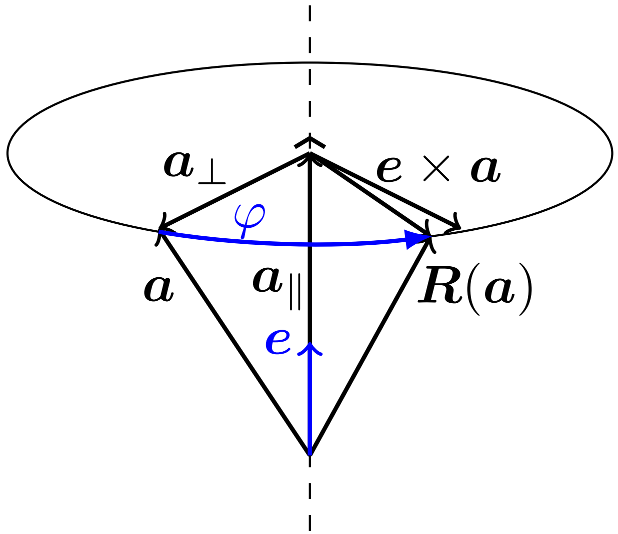
$$\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}.$$



- We assume that there are two reference frames.
- The **inertial frame** is fixed and equipped with basis vectors $\boldsymbol{i}_1$, $\boldsymbol{i}_2$, and $\boldsymbol{i}_3$.
- The **body frame** is moving and equipped with basis vectors $\boldsymbol{e}_1(t)$, $\boldsymbol{e}_2(t)$, and $\boldsymbol{e}_3(t)$.
- We denote by $\boldsymbol{R}(t)$ the linear mapping, namely, the **rotation**, that maps the basis vectors $\boldsymbol{i}_1$, $\boldsymbol{i}_2$, and $\boldsymbol{i}_3$ onto the basis vectors $\boldsymbol{e}_1(t)$, $\boldsymbol{e}_2(t)$, and $\boldsymbol{e}_3(t)$:

$$\boldsymbol{e}_k(t) = \boldsymbol{R}(t)(\boldsymbol{i}_k), \quad 1 \leq k \leq 3.$$

Linear mapping representation of rotation



- Let us construct the rotation $\boldsymbol{R}$ that rotates a vector $\boldsymbol{a}$ in $\mathbb{R}^3$ about an axis specified by a unit vector $\boldsymbol{e}$ in $\mathbb{R}^3$ (not to be confused with moving basis vectors) with an angle of φ onto a vector $\boldsymbol{R}(\boldsymbol{a})$ in $\mathbb{R}^3$.
- Decomposing $\boldsymbol{a}$ into a component $\boldsymbol{a}_{||}$ along the axis and a component $\boldsymbol{a}_\perp$ perpendicular to it,

$$\boldsymbol{a}_{||} = (\boldsymbol{a}\cdot\boldsymbol{e})\boldsymbol{e} = (\boldsymbol{e}\otimes\boldsymbol{e})(\boldsymbol{a}), \quad \boldsymbol{a}_\perp = \boldsymbol{a} - \boldsymbol{a}_{||},$$

  we can write

$$\boldsymbol{R}(\boldsymbol{a}) = \boldsymbol{a}_{||} + \cos(\varphi)\boldsymbol{a}_\perp + \sin(\varphi)(\boldsymbol{e}\times\boldsymbol{a}) = \cos(\varphi)\boldsymbol{a} + \big(1-\cos(\varphi)\big)(\boldsymbol{e}\otimes\boldsymbol{e})(\boldsymbol{a}) + \sin(\varphi)\boldsymbol{e}\times\boldsymbol{a}.$$

- Thus, a rotation about an axis $\boldsymbol{e}$ with an angle of φ admits the **axis-angle representation**

$$\boldsymbol{R} = \cos(\varphi)\boldsymbol{I} + \big(1-\cos(\varphi)\big)(\boldsymbol{e}\otimes\boldsymbol{e}) + \sin(\varphi)\widehat{\boldsymbol{E}}, \quad \|\boldsymbol{e}\| = 1 \quad \text{(Euler).}$$

Linear mapping representation of rotation

- With

$$\widehat{\boldsymbol{E}}(\boldsymbol{a}) = \boldsymbol{e} \times \boldsymbol{a},$$
$$\big(\widehat{\boldsymbol{E}}\big)^2(\boldsymbol{a}) = \boldsymbol{e} \times (\boldsymbol{e} \times \boldsymbol{a}) = (\boldsymbol{e} \cdot \boldsymbol{a})\boldsymbol{e} - (\boldsymbol{e} \cdot \boldsymbol{e})\boldsymbol{a} = (\boldsymbol{e} \cdot \boldsymbol{a})\boldsymbol{e} - \boldsymbol{a},$$
$$\big(\widehat{\boldsymbol{E}}\big)^3(\boldsymbol{a}) = \boldsymbol{e} \times \big((\boldsymbol{e} \cdot \boldsymbol{a})\boldsymbol{e} - \boldsymbol{a}\big) = -\boldsymbol{e} \times \boldsymbol{a} = -\widehat{\boldsymbol{E}}(\boldsymbol{a}),$$
$$\ldots$$

the axis-angle representation can be written equivalently as

$$\begin{aligned}\boldsymbol{R} &= \cos(\varphi)\boldsymbol{I} + \big(1 - \cos(\varphi)\big)\Big(\big(\widehat{\boldsymbol{E}}\big)^2 + \boldsymbol{I}\Big) + \sin(\varphi)\widehat{\boldsymbol{E}} \\ &= \boldsymbol{I} + \big(1 - \cos(\varphi)\big)\big(\widehat{\boldsymbol{E}}\big)^2 + \sin(\varphi)\widehat{\boldsymbol{E}} \quad \text{(Rodrigues)},\end{aligned}$$

as well as equivalently as

$$\boldsymbol{R} = \exp\big(\varphi\widehat{\boldsymbol{E}}\big) \equiv \sum_{k=0}^{+\infty} \frac{1}{k!}\big(\varphi\widehat{\boldsymbol{E}}\big)^k \quad \text{(note: this is not a component-wise exponential);}$$

indeed, with $\sin(\varphi) = \varphi - \frac{1}{3!}\varphi^3 + \frac{1}{5!}\varphi^5 - \ldots$ and $\cos(\varphi) = 1 - \frac{1}{2!}\varphi^2 + \frac{1}{4!}\varphi^4 - \ldots$, we have

$$\begin{aligned}\boldsymbol{R} &= \boldsymbol{I} + \varphi\widehat{\boldsymbol{E}} + \frac{1}{2!}\varphi^2\big(\widehat{\boldsymbol{E}}\big)^2 + \frac{1}{3!}\varphi^3\big(\widehat{\boldsymbol{E}}\big)^3 + \frac{1}{4!}\big(\widehat{\boldsymbol{E}}\big)^4 + \ldots \\ &= \boldsymbol{I} + \varphi\widehat{\boldsymbol{E}} + \frac{1}{2!}\varphi^2\big(\widehat{\boldsymbol{E}}\big)^2 - \frac{1}{3!}\varphi^3\widehat{\boldsymbol{E}} - \frac{1}{4!}\big(\widehat{\boldsymbol{E}}\big)^2 + \ldots \\ &= \boldsymbol{I} + \left(\frac{1}{2!}\varphi^2 - \frac{1}{4!}\varphi^4 + \ldots\right)\big(\widehat{\boldsymbol{E}}\big)^2 + \left(\varphi - \frac{1}{3!}\varphi^3 + \ldots\right)\widehat{\boldsymbol{E}} \\ &= \boldsymbol{I} + \big(1 - \cos(\varphi)\big)\big(\widehat{\boldsymbol{E}}\big)^2 + \sin(\varphi)\widehat{\boldsymbol{E}}.\end{aligned}$$

Linear mapping representation of rotation

- A rotation is **orthogonal**:

$$\boldsymbol{R}\boldsymbol{R}^{\mathrm{T}} = \boldsymbol{R}^{\mathrm{T}}\boldsymbol{R} = \boldsymbol{I};$$

  Indeed, completing the axis $\boldsymbol{e}$ with vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ in $\mathbb{R}^3$ such that $\boldsymbol{e}$, $\boldsymbol{p}$, and $\boldsymbol{q}$ form on orthonormal basis and $\boldsymbol{e} = \boldsymbol{p} \times \boldsymbol{q}$, we have

$$\widehat{\boldsymbol{E}}(\boldsymbol{a}) = \boldsymbol{e}\times\boldsymbol{a} = (\boldsymbol{p}\times\boldsymbol{q})\times\boldsymbol{a} = -\boldsymbol{a}\times(\boldsymbol{p}\times\boldsymbol{q}) = -(\boldsymbol{a}\cdot\boldsymbol{q})\boldsymbol{p}+(\boldsymbol{a}\cdot\boldsymbol{p})\boldsymbol{q} = -(\boldsymbol{p}\otimes\boldsymbol{q})(\boldsymbol{a})+(\boldsymbol{q}\otimes\boldsymbol{p})(\boldsymbol{a}),$$

  and hence

$$\widehat{\boldsymbol{E}} = \boldsymbol{q} \otimes \boldsymbol{p} - \boldsymbol{p} \otimes \boldsymbol{q},$$

  so that

$$\boldsymbol{R} = \cos(\varphi)\boldsymbol{I} + \big(1 - \cos(\varphi)\big)(\boldsymbol{e} \otimes \boldsymbol{e}) + \sin(\varphi)(\boldsymbol{q} \otimes \boldsymbol{p} - \boldsymbol{p} \otimes \boldsymbol{q}),$$

  and thus

$$\boldsymbol{R}\boldsymbol{R}^{\mathrm{T}} = \boldsymbol{R}^{\mathrm{T}}\boldsymbol{R} = \cos^2(\varphi)\boldsymbol{I} + \sin^2(\varphi)(\boldsymbol{e} \otimes \boldsymbol{e}) + \sin^2(\varphi)(\boldsymbol{p} \otimes \boldsymbol{p} + \boldsymbol{q} \otimes \boldsymbol{q}) = \boldsymbol{I}.$$

- The **axis** is an **eigenvector** corresponding to a **unit eigenvalue** of a rotation:

$$\boldsymbol{R}(\boldsymbol{e}) = \boldsymbol{e},$$

  and the **trace** of a rotation satisfies

$$\mathrm{tr}(\boldsymbol{R}) = 1 + 2\cos(\varphi).$$

Linear mapping representation of rotation

- We now let the rotation be a function of time: $\boldsymbol{R} = \boldsymbol{R}(t)$. The time derivative $\dot{\boldsymbol{R}}$ of the rotation $\boldsymbol{R}$ is then the product of a skew-symmetric linear mapping with this rotation:

$$\dot{\boldsymbol{R}} = \widehat{\boldsymbol{\Omega}}\boldsymbol{R}, \quad \text{(Poisson)};$$

indeed, by differentiating the expression $\boldsymbol{R}\boldsymbol{R}^{\mathrm{T}} = \boldsymbol{I}$ of orthogonality, we have

$$\dot{\boldsymbol{R}}\boldsymbol{R}^{\mathrm{T}} + \boldsymbol{R}\dot{\boldsymbol{R}}^{\mathrm{T}} = \boldsymbol{0} \implies \dot{\boldsymbol{R}} = -\boldsymbol{R}\dot{\boldsymbol{R}}^{\mathrm{T}}\boldsymbol{R} = \underbrace{\dot{\boldsymbol{R}}\boldsymbol{R}^{\mathrm{T}}}_{\equiv\widehat{\boldsymbol{\Omega}}}\boldsymbol{R}.$$

- The axial vector of $\widehat{\boldsymbol{\Omega}}$ is the **angular velocity vector** $\boldsymbol{\omega}$:

$$\boldsymbol{\omega} \times \boldsymbol{a} = \widehat{\boldsymbol{\Omega}}(\boldsymbol{a}), \quad \forall \boldsymbol{a} \in \mathbb{R}^3.$$

- The **angular velocity vector** $\boldsymbol{\omega}$ has the following **axis-angle representation**:

$$\boldsymbol{\omega} = \underbrace{\big(1 - \cos(\varphi)\big)\boldsymbol{e} \times \dot{\boldsymbol{e}} + \sin(\varphi)\dot{\boldsymbol{e}}}_{\text{change in axis}} + \underbrace{\dot{\varphi}\boldsymbol{e}}_{\text{change in angle}};$$

indeed, this representation follows from differentiating $\mathrm{tr}(\boldsymbol{R}) = 1 + 2\cos(\varphi)$ and $\boldsymbol{R}(\boldsymbol{e}) = \boldsymbol{e}$:

$$\mathrm{tr}\big(\widehat{\boldsymbol{\Omega}}\boldsymbol{R}\big) = -2\sin(\varphi)\dot{\varphi} \implies \mathrm{tr}\left(\widehat{\boldsymbol{\Omega}}\frac{\boldsymbol{R} - \boldsymbol{R}^{\mathrm{T}}}{2}\right) = -2\sin(\varphi)\dot{\varphi} \implies \mathrm{tr}\big(\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{E}}\big) = -2\dot{\varphi} \implies \boldsymbol{\omega}\cdot\boldsymbol{e} = \dot{\varphi},$$

$$(\boldsymbol{I} - \boldsymbol{R})(\dot{\boldsymbol{e}}) - \widehat{\boldsymbol{\Omega}}(\boldsymbol{e}) = \boldsymbol{0} \implies \boldsymbol{e} \times (\boldsymbol{\omega} \times \boldsymbol{e}) = \boldsymbol{e} \times \big((\boldsymbol{I} - \boldsymbol{R})(\dot{\boldsymbol{e}})\big) \implies \boldsymbol{\omega} = \boldsymbol{e} \times \big((\boldsymbol{I} - \boldsymbol{R})(\dot{\boldsymbol{e}})\big) + \dot{\varphi}\boldsymbol{e}.$$

Linear mapping representation of rotation

- The Poisson equation can be written equivalently as follows:

$$\dot{\boldsymbol{R}} = \boldsymbol{R}\widehat{\widehat{\boldsymbol{\Omega}}} \quad \text{with } \widehat{\widehat{\boldsymbol{\Omega}}} \text{ such that } \widehat{\boldsymbol{\Omega}} = \boldsymbol{R}\widehat{\widehat{\boldsymbol{\Omega}}}\boldsymbol{R}^{\mathrm{T}};$$

indeed, by differentiating the expression $\boldsymbol{R}^{\mathrm{T}}\boldsymbol{R} = \boldsymbol{I}$ of orthogonality, we have

$$\dot{\boldsymbol{R}}^{\mathrm{T}}\boldsymbol{R} + \boldsymbol{R}^{\mathrm{T}}\dot{\boldsymbol{R}} = \boldsymbol{0} \implies \dot{\boldsymbol{R}} = -\boldsymbol{R}\dot{\boldsymbol{R}}^{\mathrm{T}}\boldsymbol{R} = \boldsymbol{R}\underbrace{\boldsymbol{R}^{\mathrm{T}}\dot{\boldsymbol{R}}}_{\equiv\widehat{\widehat{\boldsymbol{\Omega}}}},$$

$$\widehat{\boldsymbol{\Omega}} = \dot{\boldsymbol{R}}\boldsymbol{R}^{\mathrm{T}} = \boldsymbol{R}\underbrace{\boldsymbol{R}^{\mathrm{T}}\dot{\boldsymbol{R}}}_{\equiv\widehat{\widehat{\boldsymbol{\Omega}}}}\boldsymbol{R}^{\mathrm{T}}.$$

- The axial vector of $\widehat{\widehat{\boldsymbol{\Omega}}}$ is the vector $\tilde{\boldsymbol{\omega}}$ such that:

$$\tilde{\boldsymbol{\omega}} \times \boldsymbol{a} = \widehat{\widehat{\boldsymbol{\Omega}}}(\boldsymbol{a}), \quad \forall \boldsymbol{a} \in \mathbb{R}^3.$$

- The vectors $\boldsymbol{\omega}$ and $\tilde{\boldsymbol{\omega}}$ are related as follows:

$$\boldsymbol{\omega} = \boldsymbol{R}(\tilde{\boldsymbol{\omega}});$$

indeed,

$$\boldsymbol{R}\widehat{\widehat{\boldsymbol{\Omega}}}\boldsymbol{R}^{\mathrm{T}}(\boldsymbol{a}) = \boldsymbol{R}\Big(\tilde{\boldsymbol{\omega}} \times \big(\boldsymbol{R}^{\mathrm{T}}(\boldsymbol{a})\big)\Big) \overset{\text{(Piola)}}{=} \boldsymbol{R}(\tilde{\boldsymbol{\omega}}) \times \big(\boldsymbol{R}\boldsymbol{R}^{\mathrm{T}}(\boldsymbol{a})\big) = \boldsymbol{R}(\tilde{\boldsymbol{\omega}}) \times \boldsymbol{a}.$$

- Let us begin with considering the **space of complex numbers**

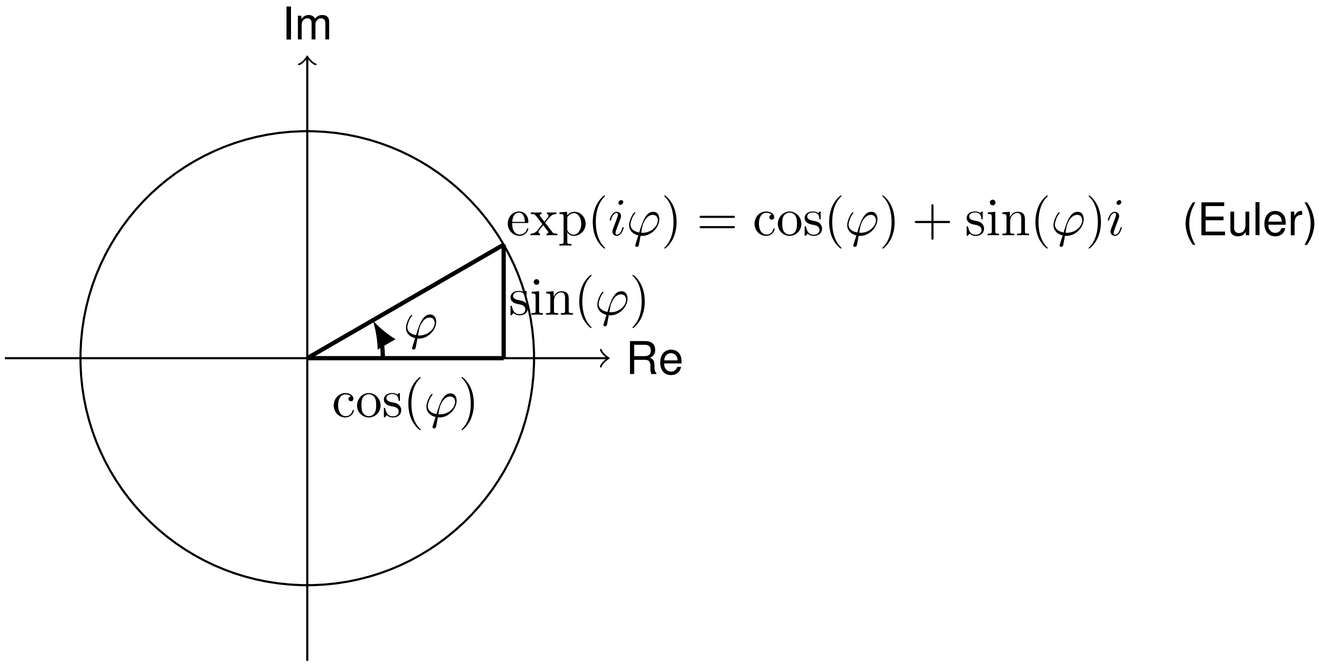
$$\mathbb{C} = \{a + bi \ : \ a, b \in \mathbb{R}, \ i^2 = -1\}.$$

- We can **add** and **multiply** in $\mathbb{C}$, for example,

$$(a + bi) + (\tilde{a} + \tilde{b}i) = (a + \tilde{a}) + (b + \tilde{b})i,$$
$$(a + bi)(\tilde{a} + \tilde{b}i) = (a\tilde{a} - b\tilde{b}) + (a\tilde{b} + \tilde{a}b)i.$$

- These **operations on complex numbers** can be **related to operations on matrices**. Indeed, by associating any complex number $a + bi$ with a corresponding matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, we have

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} \tilde{a} & -\tilde{b} \\ \tilde{b} & \tilde{a} \end{bmatrix} = \begin{bmatrix} (a + \tilde{a}) & -(b + \tilde{b}) \\ (b + \tilde{b}) & (a + \tilde{a}) \end{bmatrix},$$
$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} \tilde{a} & -\tilde{b} \\ \tilde{b} & \tilde{a} \end{bmatrix} = \begin{bmatrix} (a\tilde{a} - b\tilde{b}) & -(a\tilde{b} + \tilde{a}b) \\ (a\tilde{b} + \tilde{a}b) & (a\tilde{a} - b\tilde{b}) \end{bmatrix}.$$



- **Operations on complex numbers** are related to **2D geometry**.
- We can factor any complex number in polar coordinates:

$$a + bi = r\exp(i\varphi), \quad \tilde{a} + \tilde{b}i = \tilde{r}\exp(i\tilde{\varphi}).$$

- Multiplying two complex numbers corresponds to multiplying their lengths and their angles:

$$(a + bi)(\tilde{a} + \tilde{b}i) = r\tilde{r}\exp\big(i(\varphi + \tilde{\varphi})\big).$$

- Thus, **in 2D**, a **rotation** by an angle of φ can be **represented** by a **multiplication** by a **complex number** $\exp(i\varphi)$ of **unit magnitude**. Quaternions extend this concept to 3D.

- **Quaternions** are an **extension to complex numbers**.

- While operations on complex numbers are related to 2D geometry, **operations on quaternions** are related to **3D geometry**.

- While a complex number $a + bi$ can be associated with an ordered pair (a, b) of real numbers, a **quaternion** q can be associated with an **ordered quadruple** (q_0, q_1, q_2, q_3) **of real numbers**. An alternative notation is $q = (q_0, \boldsymbol{q}_\mathsf{v})$ with $\boldsymbol{q}_\mathsf{v} = (q_1, q_2, q_3)$.

- Special sets of quaternions are

$$\mathcal{Q}_\mathsf{v} = \{\overline{\boldsymbol{\eta}} \in \mathbb{R}^4 : \overline{\boldsymbol{\eta}} = (0, \boldsymbol{\eta}),\ \boldsymbol{\eta} \in \mathbb{R}^3\} \qquad \text{(**vectors**)},$$

$$\mathcal{Q}_1 = \{q \in \mathbb{R}^4 : \|q\|^2 = q_0^2 + \boldsymbol{q}_\mathsf{v} \cdot \boldsymbol{q}_\mathsf{v} = 1\} \qquad \text{(**unit quaternions**)}.$$

- While operations on complex numbers can be related to operations on matrices, we can **define operations on quaternions by relating them to operations on matrices**, namely, by associating any quaternion $q = (q_0, q_1, q_2, q_3)$ with a corresponding matrix $\begin{bmatrix} q_0 + q_3 i & -q_1 - q_2 i \\ q_1 - q_2 i & q_0 - q_3 i \end{bmatrix}$.
- For example, addition:

$$\begin{bmatrix} q_0 + q_3 i & -q_1 - q_2 i \\ q_1 - q_2 i & q_0 - q_3 i \end{bmatrix} + \begin{bmatrix} \tilde{q}_0 + \tilde{q}_3 i & -\tilde{q}_1 - \tilde{q}_2 i \\ \tilde{q}_1 - \tilde{q}_2 i & \tilde{q}_0 - \tilde{q}_3 i \end{bmatrix}$$
$$= \begin{bmatrix} (q_0 + \tilde{q}_0) + (q_3 + \tilde{q}_3) i & -(q_1 + \tilde{q}_1) - (q_2 + \tilde{q}_2) i \\ (q_1 + \tilde{q}_1) - (q_2 + \tilde{q}_2) i & (q_0 + \tilde{q}_0) - (q_3 + \tilde{q}_3) i \end{bmatrix}.$$

- For example, multiplication:

$$\begin{bmatrix} q_0 + q_3 i & -q_1 - q_2 i \\ q_1 - q_2 i & q_0 - q_3 i \end{bmatrix} \begin{bmatrix} \tilde{q}_0 + \tilde{q}_3 i & -\tilde{q}_1 - \tilde{q}_2 i \\ \tilde{q}_1 - \tilde{q}_2 i & \tilde{q}_0 - \tilde{q}_3 i \end{bmatrix}$$
$$= \begin{bmatrix} q_0\tilde{q}_0 - q_3\tilde{q}_3 + (q_0\tilde{q}_3 + \tilde{q}_0 q_3) i & -q_0\tilde{q}_1 + \tilde{q}_2 q_3 + (-q_0\tilde{q}_2 - \tilde{q}_1 q_3) i \\ \quad -q_1\tilde{q}_1 - q_2\tilde{q}_2 + (q_1\tilde{q}_2 - \tilde{q}_1 q_2) i & \quad -\tilde{q}_0 q_1 - q_2\tilde{q}_3 + (q_1\tilde{q}_3 - \tilde{q}_0 q_2) i \\ \tilde{q}_0 q_1 + q_2\tilde{q}_3 + (q_1\tilde{q}_3 - \tilde{q}_0 q_2) i & -q_1\tilde{q}_1 - q_2\tilde{q}_2 + (-q_1\tilde{q}_2 + \tilde{q}_1 q_2) i \\ \quad q_0\tilde{q}_1 - \tilde{q}_2 q_3 - (-q_0\tilde{q}_2 - \tilde{q}_1 q_3) i & \quad q_0\tilde{q}_0 - q_3\tilde{q}_3 + (-q_0\tilde{q}_3 - \tilde{q}_0 q_3) i \end{bmatrix}.$$

- For quaternions, addition, multiplication, and other operations are defined as follows:

(**addition**)	$q + \tilde{q} = (q_0 + \tilde{q}_0, \boldsymbol{q}_\mathsf{v} + \tilde{\boldsymbol{q}}_\mathsf{v}),$
(**multiplication**)	$q \odot \tilde{q} = (q_0 \tilde{q}_0 - \boldsymbol{q}_\mathsf{v} \cdot \tilde{\boldsymbol{q}}_\mathsf{v}, q_0 \tilde{\boldsymbol{q}}_\mathsf{v} + \tilde{q}_0 \boldsymbol{q}_\mathsf{v} + \boldsymbol{q}_\mathsf{v} \times \tilde{\boldsymbol{q}}_\mathsf{v}),$
(**conjugation**)	$q^\mathsf{c} = (q_0, -\boldsymbol{q}_\mathsf{v}),$
(**norm**)	$\lVert q \rVert^2 = q_0^2 + \boldsymbol{q}_\mathsf{v} \cdot \boldsymbol{q}_\mathsf{v},$
(**inverse**)	$q^{-1} = \lVert q \rVert^{-2} q^\mathsf{c}.$

- The following **identities** hold:

$$(q \odot \tilde{q})^\mathsf{c} = \tilde{q}^\mathsf{c} \odot q^\mathsf{c}, \quad (q \odot \tilde{q})^{-1} = \tilde{q}^{-1} \odot q^{-1}, \quad \lVert q \odot \tilde{q} \rVert = \lVert q \rVert \, \lVert \tilde{q} \rVert.$$

- The **multiplication** can also be written in terms of **linear mappings** as

$$q \odot \tilde{q} = \underbrace{\begin{bmatrix} q_0 & -\boldsymbol{q}_\mathsf{v}^\mathrm{T} \\ \boldsymbol{q}_\mathsf{v} & q_0 \boldsymbol{I} + \widehat{\boldsymbol{Q}}_\mathsf{v} \end{bmatrix}}_{\equiv \boldsymbol{Q}_\mathsf{L}} \begin{bmatrix} \tilde{q}_0 \\ \tilde{\boldsymbol{q}}_\mathsf{v} \end{bmatrix} = \underbrace{\begin{bmatrix} \tilde{q}_0 & -\tilde{\boldsymbol{q}}_\mathsf{v}^\mathrm{T} \\ \tilde{\boldsymbol{q}}_\mathsf{v} & \tilde{q}_0 \boldsymbol{I} - \widehat{\widetilde{\boldsymbol{Q}}}_\mathsf{v} \end{bmatrix}}_{\equiv \widetilde{\boldsymbol{Q}}_\mathsf{R}} \begin{bmatrix} q_0 \\ \boldsymbol{q}_\mathsf{v} \end{bmatrix}.$$

- For quaternions, the **exponential** is defined as follows:

$$\exp(q) = \sum_{k=0}^{+\infty} \frac{1}{k!} q^k \quad \text{(note: this is not a component-wise exponential)},$$

in which the quaternion power is defined recursively as follows:

$$q^k = q \odot q^{k-1} = q^{k-1} \odot q, \quad q^0 = (1, \mathbf{0}).$$

- The following **properties** hold:

$$\frac{d}{dt} \exp(tq) = q \odot \exp(tq) = \exp(tq) \odot q;$$

indeed:

$$\frac{d}{dt} \exp(tq) = \sum_{k=1}^{+\infty} \frac{t^{k-1} q^k}{(k-1)!} = q \odot \sum_{k=0}^{+\infty} \frac{1}{k!} (tq)^k = q \odot \exp(tq),$$

$$\frac{d}{dt} \exp(tq) = \sum_{k=1}^{+\infty} \frac{t^{k-1} q^k}{(k-1)!} = \sum_{k=0}^{+\infty} \frac{1}{k!} (tq)^k \odot q = \exp(tq) \odot q.$$

- The **exponential of a vector** $\overline{\boldsymbol{\eta}} = (0, \boldsymbol{\eta})$ in $\mathcal{Q}_{\mathsf{v}}$ returns a unit quaternion $\exp(\overline{\boldsymbol{\eta}})$ in $\mathcal{Q}_1$:

$$\exp(\overline{\boldsymbol{\eta}}) = \left(\cos(\|\boldsymbol{\eta}\|), \sin(\|\boldsymbol{\eta}\|)\frac{\boldsymbol{\eta}}{\|\boldsymbol{\eta}\|}\right) \in \mathcal{Q}_1;$$

indeed, with

$$\begin{aligned}
\overline{\boldsymbol{\eta}}^0 &= (1, \mathbf{0}),\\
\overline{\boldsymbol{\eta}}^1 &= (0, \boldsymbol{\eta}),\\
\overline{\boldsymbol{\eta}}^2 &= (-\boldsymbol{\eta}\cdot\boldsymbol{\eta}, \mathbf{0}) = \big(-\|\boldsymbol{\eta}\|^2, \mathbf{0}\big),\\
\overline{\boldsymbol{\eta}}^3 &= \big(0, -\|\boldsymbol{\eta}\|^2\boldsymbol{\eta}\big),\\
&\dots,
\end{aligned}$$

we have

$$\exp(\overline{\boldsymbol{\eta}}) = \sum_{k=0}^{+\infty}\frac{1}{k!}\overline{\boldsymbol{\eta}}^k = \left(1 - \frac{\|\boldsymbol{\eta}\|^2}{2!} + \dots, \left(\|\boldsymbol{\eta}\| - \frac{\|\boldsymbol{\eta}\|^3}{3!} + \dots\right)\frac{\boldsymbol{\eta}}{\|\boldsymbol{\eta}\|}\right).$$

- The **logarithm of a unit quaternion** $q = (q_0, \boldsymbol{q}_{\mathsf{v}})$ in $\mathcal{Q}_1$ can be defined as

$$\log(q) = \arcsin(\|\boldsymbol{q}_{\mathsf{v}}\|)\frac{\boldsymbol{q}_{\mathsf{v}}}{\|\boldsymbol{q}_{\mathsf{v}}\|}$$

for sufficiently small $\|\boldsymbol{q}_{\mathsf{v}}\|$, so that $\log\big(\exp(\overline{\boldsymbol{\eta}})\big) = \boldsymbol{\eta}$ behaves as expected. Thus, the logarithm of a unit quaternion returns a vector.

- For **small vectors and unit quaternions**, we have the following **approximations**:

$$\exp(\overline{\boldsymbol{\eta}}) \approx (1, \boldsymbol{\eta}), \quad \log(q) \approx \boldsymbol{q}_{\mathsf{v}}.$$

- A **rotation** of φ about a unit axis $\boldsymbol{e}$ is **represented** by a **unit quaternion** as

$$q = \left(\cos\left(\frac{\varphi}{2}\right), \sin\left(\frac{\varphi}{2}\right)\boldsymbol{e}\right).$$

The **rotation of a vector** $\boldsymbol{a}$ in $\mathbb{R}^3$ with an angle of φ about a unit axis $\boldsymbol{e}$ into a vector $\tilde{\boldsymbol{a}}$ in $\mathbb{R}^3$ is then **represented** as a **quaternion triple product** as

$$\bar{\tilde{\boldsymbol{a}}} = q \odot \bar{\boldsymbol{a}} \odot q^{\mathsf{c}};$$

indeed:

$$\begin{aligned}
(0, \tilde{\boldsymbol{a}}) &= (q_0, \boldsymbol{q}_{\mathsf{v}}) \odot (0, \boldsymbol{a}) \odot (q_0, -\boldsymbol{q}_{\mathsf{v}}) \\
&= \begin{bmatrix} q_0 & -\boldsymbol{q}_{\mathsf{v}}^{\mathrm{T}} \\ \boldsymbol{q}_{\mathsf{v}} & q_0 \boldsymbol{I} + \widehat{\boldsymbol{Q}}_{\mathsf{v}} \end{bmatrix} \begin{bmatrix} q_0 & \boldsymbol{q}_{\mathsf{v}}^{\mathrm{T}} \\ -\boldsymbol{q}_{\mathsf{v}} & q_0 \boldsymbol{I} + \widehat{\boldsymbol{Q}}_{\mathsf{v}} \end{bmatrix} \begin{bmatrix} 0 \\ \boldsymbol{a} \end{bmatrix} \\
&= \begin{bmatrix} 1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{q}_{\mathsf{v}}\boldsymbol{q}_{\mathsf{v}}^{\mathrm{T}} + q_0^2 \boldsymbol{I} + 2q_0 \widehat{\boldsymbol{Q}}_{\mathsf{v}} + (\widehat{\boldsymbol{Q}}_{\mathsf{v}})^2 \end{bmatrix} \begin{bmatrix} 0 \\ \boldsymbol{a} \end{bmatrix} \\
&= \begin{bmatrix} 1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R} \end{bmatrix} \begin{bmatrix} 0 \\ \boldsymbol{a} \end{bmatrix},
\end{aligned}$$

since

$$\begin{aligned}
&\sin^2(\varphi/2)\boldsymbol{e}\boldsymbol{e}^{\mathrm{T}} + \cos^2(\varphi/2)\boldsymbol{I} + 2\cos(\varphi/2)\sin(\varphi/2)\widehat{\boldsymbol{E}} + \sin^2(\varphi/2)(\widehat{\boldsymbol{E}})^2 \\
&\quad = \boldsymbol{I} + \sin^2(\varphi/2)\big(\boldsymbol{e}\boldsymbol{e}^{\mathrm{T}} - \boldsymbol{I} + (\widehat{\boldsymbol{E}})^2\big) + \sin(\varphi)\widehat{\boldsymbol{E}} \\
&\quad = \boldsymbol{I} + \big(1 - \cos(\varphi)\big)\big(\widehat{\boldsymbol{E}}\big)^2 + \sin(\varphi)\widehat{\boldsymbol{E}}.
\end{aligned}$$

Quaternion representation of rotation

- The quaternion representation can be written equivalently as follows:

$$q = \exp\left(\frac{\varphi}{2}\overline{\boldsymbol{e}}\right);$$

indeed, with

$$\begin{aligned}
\overline{\boldsymbol{e}}^0 &= (1, \mathbf{0}), \\
\overline{\boldsymbol{e}}^1 &= (0, \boldsymbol{e}), \\
\overline{\boldsymbol{e}}^2 &= (-1, \mathbf{0}), \\
\overline{\boldsymbol{e}}^3 &= (0, -\boldsymbol{e}), \\
&\dots,
\end{aligned}$$

we have

$$\exp\left(\frac{\varphi}{2}\overline{\boldsymbol{e}}\right) = \sum_{k=0}^{+\infty} \frac{1}{k!}\left(\frac{\varphi}{2}\overline{\boldsymbol{e}}\right)^k = \left(1 - \frac{1}{2!}\frac{\varphi^2}{4} + \dots, \left(\frac{\varphi}{2} - \frac{1}{3!}\frac{\varphi^3}{8} + \dots\right)\boldsymbol{e}\right).$$

- Let us now let the rotation be a function of time again: $q = q(t)$. The time derivative of the quaternion representation then satisfies:

$$\dot{q} = \frac{1}{2}\overline{\boldsymbol{\omega}} \odot q \quad \text{(Poisson)};$$

indeed,

$$\begin{aligned}
\dot{\overline{\boldsymbol{a}}} &= \dot{q} \odot \overline{\boldsymbol{a}} \odot q^{\mathrm{c}} + q \odot \overline{\boldsymbol{a}} \odot \dot{q}^{\mathrm{c}} \\
&= \frac{1}{2}\begin{bmatrix} \mathbf{0} & -\boldsymbol{\omega}^{\mathrm{T}} \\ \boldsymbol{\omega} & \widehat{\boldsymbol{\Omega}} \end{bmatrix} \begin{bmatrix} 0 \\ \boldsymbol{R}(\boldsymbol{a}) \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ \boldsymbol{R}(\boldsymbol{a}) \end{bmatrix} \odot \begin{bmatrix} 0 \\ -\boldsymbol{\omega} \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} -\boldsymbol{\omega}^{\mathrm{T}}\boldsymbol{R}(\boldsymbol{a}) + \boldsymbol{R}(\boldsymbol{a}) \cdot \boldsymbol{\omega} \\ \widehat{\boldsymbol{\Omega}}\boldsymbol{R}(\boldsymbol{a}) - \boldsymbol{R}(\boldsymbol{a}) \times \boldsymbol{\omega} \end{bmatrix} \\
&\overset{!}{=} \begin{bmatrix} 0 \\ \widehat{\boldsymbol{\Omega}}\boldsymbol{R}(\boldsymbol{a}) \end{bmatrix}.
\end{aligned}$$

- For constant $\boldsymbol{\omega}$, the solution to the Poisson equation is given by

$$q(t) = \exp\left(\frac{\overline{\boldsymbol{\omega}}}{2}t\right) \odot q(0);$$

indeed,

$$\dot{q}(t) = \frac{\overline{\boldsymbol{\omega}}}{2} \odot \exp\left(\frac{\overline{\boldsymbol{\omega}}}{2}t\right) \odot q(0) \overset{!}{=} \frac{1}{2}\overline{\boldsymbol{\omega}} \odot q(t).$$

Orientation estimation

- The **evolution of the quaternion representation of the rotation of the IMU** is described by

$$\dot{q} = \frac{1}{2}\overline{\boldsymbol{\omega}} \odot q \quad \text{(Poisson)}.$$

For $\boldsymbol{\omega}$ constant on $[t, t + \Delta t]$, the solution at time $t + \Delta t$ is related to the solution at time t as

$$q(t + \Delta t) = \exp\left(\frac{\overline{\boldsymbol{\omega}}}{2}\Delta t\right) \odot q(t);$$

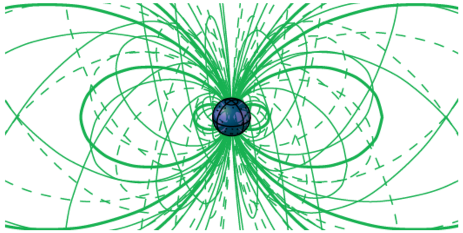
- The **gyroscope** measures the **angular velocity of the IMU expressed in its moving frame**, namely, the vector $\tilde{\boldsymbol{\omega}}$ defined previously, at successive time instants $t_0, t_1, t_2, \ldots$, with time step Δt, that is, $t_k = k\Delta t$ with $k = 0, 1, 2, \ldots$. The measurements are corrupted by **noise**.

- Denoting by q_k and $\boldsymbol{R}_k$ the quaternion and linear mapping representations of the rotation of the IMU at t_k, by $\boldsymbol{y}_{\tilde{\boldsymbol{\omega}},k}$ the gyroscope measurement at t_k, and by $\boldsymbol{\xi}_{\tilde{\boldsymbol{\omega}},k}$ the noise at t_k and assuming the angular velocity is approximately constant in each time step, the **state** evolves as

$$q_{k+1} = \exp\left(\frac{\overline{\boldsymbol{R}_k(\boldsymbol{y}_{\tilde{\boldsymbol{\omega}},k} - \boldsymbol{\xi}_{\tilde{\boldsymbol{\omega}},k})}}{2}\Delta t\right) \odot q_k,$$

in which $\boldsymbol{R}_k$ serves to convert between the inertial and the moving frame and $\boldsymbol{\xi}_{\tilde{\boldsymbol{\omega}},k}$ is a Gaussian random variable with a mean vector of $\mathbf{0}$ and a covariance matrix that we denote by $\boldsymbol{\Sigma}_{\tilde{\boldsymbol{\omega}}}$.

Orientation estimation



- Assuming that the acceleration of the IMU is negligible, the **accelerometer** measures the **local gravity vector in the IMU's moving frame**, at the time instants $t_0, t_1, t_2, \ldots$. The measurements are corrupted by **noise**. The evolution of the observable of the accelerometer is written as

$$\boldsymbol{y}_{\boldsymbol{a},k} = \boldsymbol{R}_k^{\mathrm{T}}(\boldsymbol{g}) + \boldsymbol{\xi}_{\boldsymbol{a},k},$$

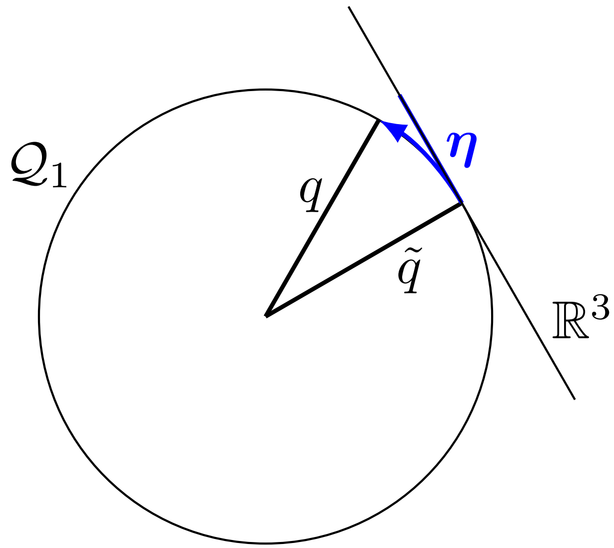
  in which $\boldsymbol{g}$ is the local gravity vector in the inertial frame, $\boldsymbol{R}_k$ serves to convert between the inertial and the moving frame, and $\boldsymbol{\xi}_{\boldsymbol{a},k}$ is a Gaussian random variable with a mean vector of $\mathbf{0}$ and a covariance matrix that we denote by $\boldsymbol{\Sigma}_{\boldsymbol{a}}$.

- The **magnetometer** measures the **local magnetic field**, induced by earth and the presence of magnetic material, **in the IMU's moving frame**, at the time instants $t_0, t_1, t_2, \ldots$. The measurements are corrupted by **noise**. The evolution of the observable of the magnetometer is written as

$$\boldsymbol{y}_{\boldsymbol{m},k} = \boldsymbol{R}_k^{\mathrm{T}}(\boldsymbol{m}) + \boldsymbol{\xi}_{\boldsymbol{m},k},$$

  in which $\boldsymbol{m}$ is the local magnetic field in the inertial frame, $\boldsymbol{R}_k$ serves to convert between the inertial and the moving frame, and $\boldsymbol{\xi}_{\boldsymbol{m},k}$ is a Gaussian random variable with a mean vector of $\mathbf{0}$ and a covariance matrix that we denote by $\boldsymbol{\Sigma}_{\boldsymbol{m}}$.

Orientation estimation



- Uncertainty in a unit quaternion representing a rotation cannot be directly represented as a 4-dimensional Gaussian random variable because the realizations of a 4-dimensional Gaussian random variable do not in general satisfy the unit norm constraint and thus are not valid rotations.
- Instead, we represent an **uncertain unit quaternion** representing an **uncertain rotation** as a **composition** of a deterministic unit quaternion representing a **reference rotation** and a random unit quaternion representing an **uncertain rotation deviation**:

$$q = \exp\left(\frac{\overline{\boldsymbol{\eta}}}{2}\right) \odot \tilde{q},$$

  in which $\boldsymbol{\eta}$ is a 3-dimensional centered Gaussian random variable; please note that this is not a component-wise exponential; it is the quaternion exponential.

- The Kalman filter assumes the state observation model and the observation model to be linear. However, in the previous equations, the state observation model and the observation model are nonlinear. In order to overcome this issue, we linearize the state observation model and the observation model at each time instant. This yields the so-called **extended Kalman filter**.
- The **linearization** of the **state observation model** is obtained as follows:

$$\begin{aligned}\boldsymbol{\eta}_{k+1} &= 2\log\left(\exp\left(\frac{\overline{\boldsymbol{R}_k(\boldsymbol{y}_{\tilde{\boldsymbol{\omega}},k}-\boldsymbol{\xi}_{\tilde{\boldsymbol{\omega}},k})}}{2}\Delta t\right)\odot\exp\left(\frac{\overline{\boldsymbol{\eta}_k}}{2}\right)\odot\tilde{q}_k\odot\tilde{q}^{\mathsf{c}}_{k+1}\right)\\ &= 2\log\left(\exp\left(\frac{\overline{\boldsymbol{R}_k(\boldsymbol{y}_{\tilde{\boldsymbol{\omega}},k}-\boldsymbol{\xi}_{\tilde{\boldsymbol{\omega}},k})}}{2}\Delta t\right)\odot\exp\left(\frac{\overline{\boldsymbol{\eta}_k}}{2}\right)\odot\exp\left(-\frac{\overline{\boldsymbol{R}_k(\boldsymbol{y}_{\tilde{\boldsymbol{\omega}},k})}}{2}\Delta t\right)\right);\end{aligned}$$

by differentiating with respect to $\boldsymbol{\eta}_k$ and $\boldsymbol{\xi}_{\tilde{\boldsymbol{\omega}},k}$ at $\boldsymbol{\eta}_k=\mathbf{0}$ and $\boldsymbol{\xi}_{\tilde{\boldsymbol{\omega}},k}=\mathbf{0}$, we obtain

$$\mathsf{D}_{\boldsymbol{\eta}_k}\boldsymbol{\eta}_{k+1} = 2\underbrace{\mathsf{D}_q\log(q)}_{\approx\boldsymbol{I}}\underbrace{\left[\exp\left(\frac{\overline{\boldsymbol{R}_k(\boldsymbol{y}_{\tilde{\boldsymbol{\omega}},k})}}{2}\Delta t\right)\right]_{\mathsf{L}}\left[\exp\left(-\frac{\overline{\boldsymbol{R}_k(\boldsymbol{y}_{\tilde{\boldsymbol{\omega}},k})}}{2}\Delta t\right)\right]_{\mathsf{R}}}_{\approx\boldsymbol{I}}\underbrace{\mathsf{D}_{\boldsymbol{\eta}_k}\exp\left(\frac{\overline{\boldsymbol{\eta}_k}}{2}\right)}_{\approx\frac{1}{2}\begin{bmatrix}\mathbf{0} & \boldsymbol{I}\end{bmatrix}^{\mathrm{T}}}\approx\boldsymbol{I},$$

$$\mathsf{D}_{\boldsymbol{\xi}_{\tilde{\boldsymbol{\omega}},k}}\boldsymbol{\eta}_{k+1} = 2\underbrace{\mathsf{D}_q\log(q)}_{\approx\boldsymbol{I}}\underbrace{\left[\exp\left(-\frac{\overline{\boldsymbol{R}_k(\boldsymbol{y}_{\tilde{\boldsymbol{\omega}},k})}}{2}\Delta t\right)\right]_{\mathsf{R}}}_{\approx\boldsymbol{I}}\underbrace{\mathsf{D}_{\boldsymbol{\xi}_{\tilde{\boldsymbol{\omega}},k}}\exp\left(\frac{\overline{\boldsymbol{R}_k(\boldsymbol{y}_{\tilde{\boldsymbol{\omega}},k}-\boldsymbol{\xi}_{\tilde{\boldsymbol{\omega}},k})}}{2}\Delta t\right)}_{\approx\begin{bmatrix}\mathbf{0} & -\frac{\Delta t}{2}\boldsymbol{R}_k^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}}\approx-\Delta t\boldsymbol{R}_k;$$

hence,

$$\boldsymbol{\eta}_{k+1}\approx\boldsymbol{\eta}_k-\Delta t\boldsymbol{R}_k(\boldsymbol{\xi}_{\tilde{\boldsymbol{\omega}},k}).$$

- The **linearization** of the **observation model** is obtained as follows:

$$
\begin{aligned}
\boldsymbol{y}_{\boldsymbol{a},k} &= \boldsymbol{R}_k^{\mathrm{T}}(\boldsymbol{g}) + \boldsymbol{\xi}_{\boldsymbol{a},k} \\
&= \widetilde{\boldsymbol{R}}_k^{\mathrm{T}} \exp\big(\widehat{\boldsymbol{H}}_k\big)^{\mathrm{T}}(\boldsymbol{g}) + \boldsymbol{\xi}_{\boldsymbol{a},k} \\
&\approx \widetilde{\boldsymbol{R}}_k^{\mathrm{T}} \big(\boldsymbol{I} + \widehat{\boldsymbol{H}}_k\big)^{\mathrm{T}}(\boldsymbol{g}) + \boldsymbol{\xi}_{\boldsymbol{a},k} \\
&= \widetilde{\boldsymbol{R}}_k^{\mathrm{T}} \big(\boldsymbol{I} - \widehat{\boldsymbol{H}}_k\big)(\boldsymbol{g}) + \boldsymbol{\xi}_{\boldsymbol{a},k} \\
&= \widetilde{\boldsymbol{R}}_k^{\mathrm{T}}(\boldsymbol{g}) + \widetilde{\boldsymbol{R}}_k^{\mathrm{T}} \widehat{\boldsymbol{G}}(\boldsymbol{\eta}_k) + \boldsymbol{\xi}_{\boldsymbol{a},k}, \\
\boldsymbol{y}_{\boldsymbol{m},k} &= \boldsymbol{R}_k^{\mathrm{T}}(\boldsymbol{m}) + \boldsymbol{\xi}_{\boldsymbol{m},k} \\
&= \widetilde{\boldsymbol{R}}_k^{\mathrm{T}} \exp\big(\widehat{\boldsymbol{H}}_k\big)^{\mathrm{T}}(\boldsymbol{m}) + \boldsymbol{\xi}_{\boldsymbol{m},k} \\
&\approx \widetilde{\boldsymbol{R}}_k^{\mathrm{T}} \big(\boldsymbol{I} + \widehat{\boldsymbol{H}}_k\big)^{\mathrm{T}}(\boldsymbol{m}) + \boldsymbol{\xi}_{\boldsymbol{m},k} \\
&= \widetilde{\boldsymbol{R}}_k^{\mathrm{T}} \big(\boldsymbol{I} - \widehat{\boldsymbol{H}}_k\big)(\boldsymbol{m}) + \boldsymbol{\xi}_{\boldsymbol{m},k} \\
&= \widetilde{\boldsymbol{R}}_k^{\mathrm{T}}(\boldsymbol{m}) + \widetilde{\boldsymbol{R}}_k^{\mathrm{T}} \widehat{\boldsymbol{M}}(\boldsymbol{\eta}_k) + \boldsymbol{\xi}_{\boldsymbol{m},k}.
\end{aligned}
$$

- In these equations, $\widehat{\boldsymbol{H}}_k$ is the linear mapping such that $\boldsymbol{\eta}_k$ is the axial vector of $\widehat{\boldsymbol{H}}_k$, and $\widetilde{\boldsymbol{R}}_k$ is the linear mapping representation corresponding to the quaternion representation $\tilde{q}_k$, so that $q_k = \exp\left(\frac{\boldsymbol{\eta}_k}{2}\right) \odot \tilde{q}_k$ corresponds to $\boldsymbol{R}_k = \exp\big(\widehat{\boldsymbol{H}}_k\big)\widetilde{\boldsymbol{R}}_k$.

- The extended Kalman filter for orientation estimation can be expressed as follows:
  - **Inititialization**:

$$(\tilde{q}_{0|0}, \boldsymbol{C}_{0|0}).$$

  - For $k = 1, 2, \ldots$:
    - **Step 1** (**prediction step**):

$$\tilde{q}_{k|k-1} = \exp\left(\frac{\overline{\widetilde{\boldsymbol{R}}_{k-1|k-1}(\boldsymbol{y}_{\tilde{\boldsymbol{\omega}},k})}}{2}\Delta t\right) \odot \tilde{q}_{k-1|k-1},$$

$$\boldsymbol{C}_{k|k-1} = \boldsymbol{C}_{k-1|k-1} + (\Delta t)^2 \widetilde{\boldsymbol{R}}_{k|k-1} \boldsymbol{\Sigma}_{\tilde{\boldsymbol{\omega}}} \widetilde{\boldsymbol{R}}^{\mathrm{T}}_{k|k-1}.$$

    - **Step 2** (**correction step**):

$$\hat{\boldsymbol{\eta}}_{k|k} = \boldsymbol{C}_{k|k-1} \boldsymbol{H}_k^{\mathrm{T}} \boldsymbol{S}_k^{-1} (\boldsymbol{y}_k - \boldsymbol{y}_{k|k-1}),$$

$$\boldsymbol{C}_{k|k} = \boldsymbol{C}_{k|k-1} - \boldsymbol{C}_{k|k-1} \boldsymbol{H}_k^{\mathrm{T}} \boldsymbol{S}_k^{-1} \boldsymbol{H}_k \boldsymbol{C}_{k|k-1},$$

    in which

$$\boldsymbol{H}_k = \begin{bmatrix} \widetilde{\boldsymbol{R}}^{\mathrm{T}}_{k|k-1} \widehat{\boldsymbol{G}} \\ \widetilde{\boldsymbol{R}}^{\mathrm{T}}_{k|k-1} \widehat{\boldsymbol{M}} \end{bmatrix}, \; \boldsymbol{S}_k = \boldsymbol{H}_k \boldsymbol{C}_{k|k-1} \boldsymbol{H}_k^{\mathrm{T}} + \begin{bmatrix} \boldsymbol{\Sigma}_{\boldsymbol{a}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Sigma}_{\boldsymbol{m}} \end{bmatrix}, \; \boldsymbol{y}_k = \begin{bmatrix} \boldsymbol{y}_{\boldsymbol{a},k} \\ \boldsymbol{y}_{\boldsymbol{m},k} \end{bmatrix}, \; \boldsymbol{y}_{k|k-1} = \begin{bmatrix} \widetilde{\boldsymbol{R}}^{\mathrm{T}}_{k|k-1}(\boldsymbol{g}) \\ \widetilde{\boldsymbol{R}}^{\mathrm{T}}_{k|k-1}(\boldsymbol{m}) \end{bmatrix}.$$

    - $\tilde{q}_{k|k} = \exp\left(\frac{\overline{\hat{\boldsymbol{\eta}}_{k|k}}}{2}\right) \odot \tilde{q}_{k|k-1}.$

Assignment

1. We wrote the equations involved in the correction step of the Kalman filter as follows:

$$\hat{\boldsymbol{x}}_{k|k} = \hat{\boldsymbol{x}}_{k|k-1} + \boldsymbol{C}_{k|k-1}\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{S}_k^{-1}\big(\boldsymbol{y}_k - \boldsymbol{H}_k(\hat{\boldsymbol{x}}_{k|k-1})\big),$$
$$\boldsymbol{C}_{k|k} = \boldsymbol{C}_{k|k-1} - \boldsymbol{C}_{k|k-1}\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{S}_k^{-1}\boldsymbol{H}_k\boldsymbol{C}_{k|k-1},$$

in which

$$\boldsymbol{S}_k = \boldsymbol{H}_k\boldsymbol{C}_{k|k-1}\boldsymbol{H}_k^{\mathrm{T}} + \boldsymbol{R}_k.$$

(a) Show that the update of the covariance matrix may be written equivalently as follows:

$$\boldsymbol{C}_{k|k} = (\boldsymbol{C}_{k|k-1}^{-1} + \boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{R}_k^{-1}\boldsymbol{H}_k)^{-1}.$$

Hint: Similarly to the matrix identity on Slide 18, show that $\boldsymbol{A}^{-1}\boldsymbol{B}^{\mathrm{T}}(\boldsymbol{C} + \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}^{\mathrm{T}})^{-1}$ $= (\boldsymbol{A} + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{C}^{-1}\boldsymbol{B})^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{C}^{-1}$. After inserting the expression of $\boldsymbol{S}_k$ into the expression of $\boldsymbol{C}_{k|k}$, use this matrix identity and conclude.

(b) Show that the update of the best estimate of the state may be written equivalently as follows:

$$\hat{\boldsymbol{x}}_{k|k} = \hat{\boldsymbol{x}}_{k|k-1} + \boldsymbol{C}_{k|k}\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{R}_k^{-1}\big(\boldsymbol{y}_k - \boldsymbol{H}_k(\hat{\boldsymbol{x}}_{k|k-1})\big).$$

Hint: Use the aforementioned matrix identity and the aforementioned equivalent expression of the update of the covariance matrix.

Assignment

2. Write a small library of functions to carry out computations with quaternions. Please represent quaternions as 4-by-1 vectors and linear mappings representing rotations as 3-by-3 matrices.

   (a) Write for each one of the addition, multiplication, conjugation, norm, and inverse operations defined on Slide 35 a function that implements this operation. For example, write a function named `quaternConj` that takes as input a quaternion q and returns as output its conjugate q^{c}, write a function named `quaternProd` that takes as input two quaternions q and $\tilde{q}$ and returns as output their quaternion product $q \odot \tilde{q}$, and so forth.

   (b) Write for the exponential defined on Slide 37 a function named `vector2unitQuatern` that takes as input a 3-by-1 vector $\boldsymbol{\eta}$ and returns as output the unit quaternion $\exp(\overline{\boldsymbol{\eta}})$.

   (c) Write a function named `quatern2rotMat` that takes as input a unit quaternion q and returns as output the corresponding rotation $\boldsymbol{R}$. You may use $\boldsymbol{R} = \boldsymbol{q}_{\mathsf{v}}\boldsymbol{q}_{\mathsf{v}}^{\mathrm{T}} + q_0^2\boldsymbol{I} + 2q_0\widehat{\boldsymbol{Q}}_{\mathsf{v}} + (\widehat{\boldsymbol{Q}}_{\mathsf{v}})^2$. As part of your work, include in your report a proof of this formula based on the axis-angle representation of the rotation on Slide 27 and that of the quaternion on Slide 38. Provide a detailed justification of each step.

   (d) Perform a few checks to verify whether you implemented everything correctly. As part of your work, describe in your report the checks that you performed.

3. Let us build some further understanding of the equations involved in the extended Kalman filter.

   Let us begin with taking a closer look at the first equation involved in the prediction step. As explained on Slides 40 and 42, this equation follows from the fact that for $\boldsymbol{\omega}$ constant on $[t, t+\Delta t]$, the solution to the Poisson equation at time $t+\Delta t$ is related to the solution at time t as

$$q(t+\Delta t) = \exp\left(\frac{\overline{\boldsymbol{\omega}}}{2}\Delta t\right) \odot q(t);$$

   indeed, $\tilde{q}_{k-1|k-1}$ and $\tilde{q}_{k|k-1}$ may be associated with $q(t)$ and $q(t+\Delta t)$, respectively; and the observed value $\boldsymbol{y}_{\tilde{\boldsymbol{\omega}},k}$ is a noisy perturbation of the angular velocity vector expressed in the IMU's moving frame, which the multiplication with $\widetilde{\boldsymbol{R}}_{k-1|k-1}$ transports to the inertial frame. As stated on Slide 37, for small values of $\frac{\boldsymbol{\omega}}{2}\Delta t$, the equation above may be approximated as

$$q(t+\Delta t) \approx \left(1, \frac{\boldsymbol{\omega}}{2}\Delta t\right) \odot q(t) = \begin{bmatrix} q_0(t) - \frac{\boldsymbol{\omega}}{2}\Delta t \cdot \boldsymbol{q}_{\mathsf{v}}(t) \\ \boldsymbol{q}_{\mathsf{v}}(t) + \frac{\boldsymbol{\omega}}{2}\Delta t q_0(t) + \frac{\boldsymbol{\omega}}{2}\Delta t \times \boldsymbol{q}_{\mathsf{v}}(t) \end{bmatrix}.$$

   The question is then as follows. Please insert the axis-angle representation of the angular velocity vector of Slide 29 and that of the quaternion of Slide 38 into this approximation. Simplify the resulting expression (Hint: use trigonometric angle sum identities) (Hint: it follows from $\boldsymbol{e} \cdot \boldsymbol{e} = 1$ that $\boldsymbol{e} \cdot \dot{\boldsymbol{e}} = 0$). And interpret the end result.

4. Let us continue to build understanding of the equations involved in the extended Kalman filter.

The second equation in the prediction step and the second equation in the correction step serve to obtain a quantification of the uncertainty in the estimates of the quaternions obtained to represent the orientation of the IMU in these steps. However, the covariance matrices $\boldsymbol{C}_{k|k-1}$ and $\boldsymbol{C}_{k|k}$ are *not* 4-by-4 covariance matrices that provide a direct quantification of the uncertainty in the quaternions. Instead, as explained on Slide 44, the uncertainty quantification follows from

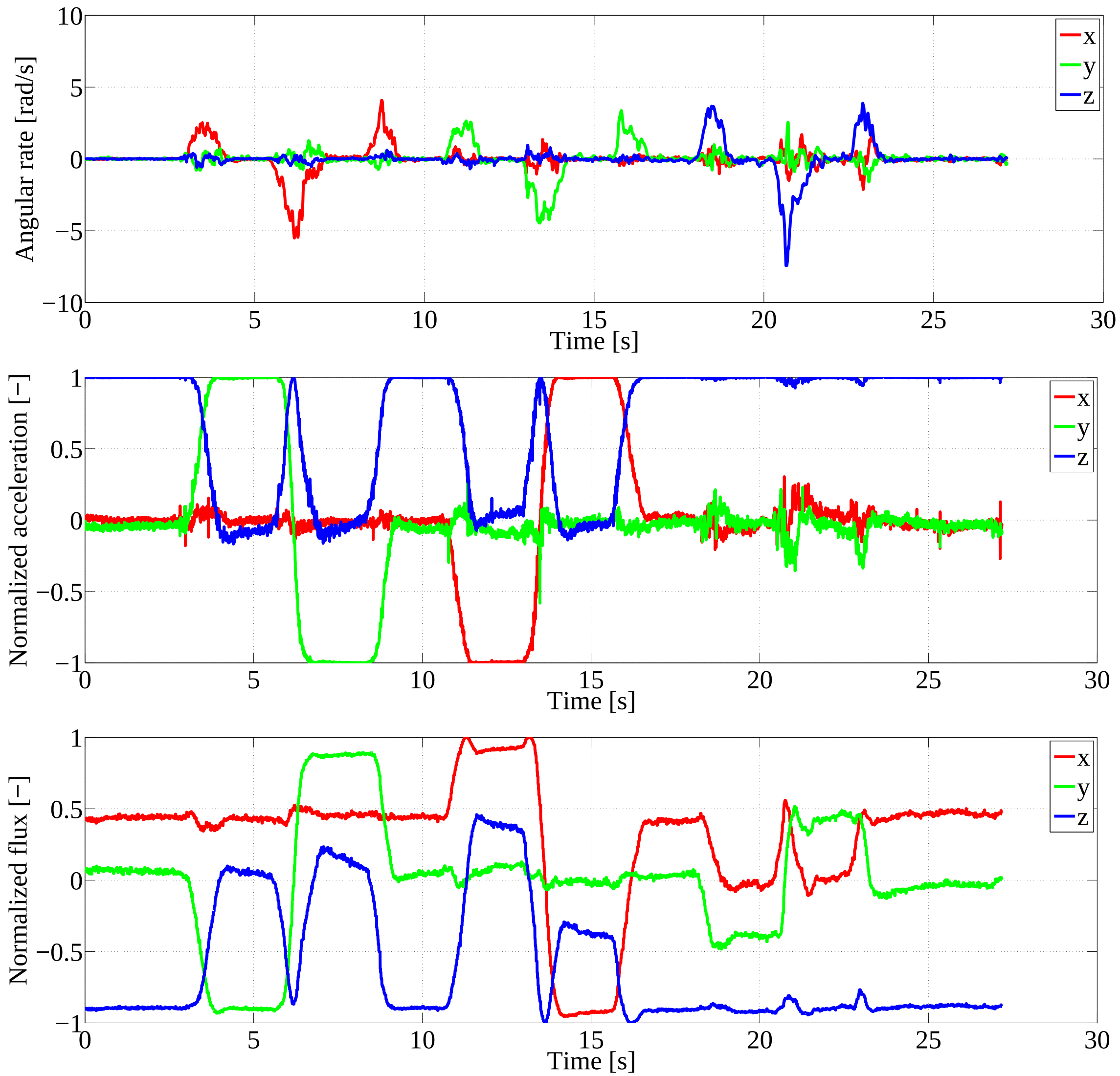
$$q = \exp\left(\frac{\overline{\boldsymbol{\eta}}}{2}\right) \odot \tilde{q},$$

and $\boldsymbol{C}_{k|k-1}$ and $\boldsymbol{C}_{k|k}$ are 3-by-3 covariance matrices for uncertain vectors $\boldsymbol{\eta}_{k|k-1} = \boldsymbol{0}$ and $\boldsymbol{\eta}_{k|k}$ in representations of the uncertain quaternions as uncertain rotation deviations about the best estimates of the quaternions obtained to represent the orientation of the IMU in the prediction and correction steps. The questions are then as follow:

(a) Please interpret why the second equation in the prediction step involves the addition of one term to another, and hence increased uncertainty, and the correction step involves the subtraction of one term from another, and hence decreased uncertainty.

(b) As stated on Slide 37, for small values of $\frac{\boldsymbol{\eta}}{2}$, $\exp\left(\frac{\overline{\boldsymbol{\eta}}}{2}\right)$ may be approximated as $\exp\left(\frac{\overline{\boldsymbol{\eta}}}{2}\right) \approx \left(1, \frac{\boldsymbol{\eta}}{2}\right)$. Insert this approximation into the equation stated above, and deduce from the approximate transformation thus obtained an approximate covariance matrix for the uncertain quaternion as a function of the covariance matrix of the uncertain vector.

5. Let us apply the extended Kalman filter to the following data set:



This data set is made available to you in the file data.mat. This data set was taken from the literature, and it was *not* generated by means of the `sparkfun 9DoF Razor IMU`.

5. You may use the following values for the parameters: $\Delta t = 1/256\,$s, $\boldsymbol{\Sigma}_{\tilde{\boldsymbol{\omega}}} = 0.007305\,rad^2/s^2\boldsymbol{I}$, $\boldsymbol{\Sigma}_{\boldsymbol{a}} = 0.0002001\boldsymbol{I}$, $\boldsymbol{\Sigma}_{\boldsymbol{m}} = 0.0001680\boldsymbol{I}$, $\boldsymbol{g} = (0, 0, 1)$, and $\boldsymbol{m} = (0.4230, 0.0630, -0.9040)$. Please note that since we direct our interest to only the orientation, and not the position, we work with normalized, hence unitless, data for the accelerometer and the magnetometer.

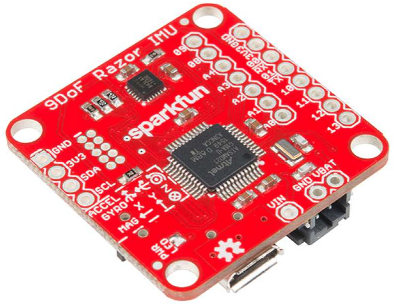
   (a) Please implement the extended Kalman filter. Perform a few checks to verify whether you implemented everything correctly. Describe in your report the checks that you performed.

   (b) Apply the extended Kalman filter to the data set. Plot as a function of time the best estimate of the quaternion representing the orientation of the IMU, that is, $\tilde{q}_{k|k}$ as a function of t_k. As the quaternion has 4 components, you should plot 4 curves. Interpret the results.

   (c) Use the formula that you established under Question 4(b) to deduce at each time instant from the covariance matrix $\boldsymbol{C}_{k|k}$ an approximate covariance matrix for the uncertain quaternion. The diagonal elements of this approximate covariance matrix are squares of approximate standard deviations. Use your solution to Question 5(b) and these approximate standard deviations to plot as a function of time "plus and minus 3 sigma" uncertainty ranges for the estimate of the quaternion representing the orientation of the IMU (Matlab: `fill`).

   (d) Use your implementation to provide some insight into the effect of the sensor fusion. As part of your work, you could consider perturbing the data for the gyroscope by additional noise (Matlab: `randn`) (adjust $\boldsymbol{\Sigma}_{\tilde{\boldsymbol{\omega}}}$ accordingly). And you could consider carrying out a comparison with a case in which the sensor fusion is disabled by replacing the correction step and the last step in the extended Kalman filter with $\tilde{q}_{k|k} = \tilde{q}_{k|k-1}$ and $\boldsymbol{C}_{k|k} = \boldsymbol{C}_{k|k-1}$.

6. Apply the extended Kalman filter to a real data set:



Use one of the `sparkfun 9DoF Razor IMUs` made available to you to collect a data set for a sequence of rotational movements of the IMU of your choice. Apply the extended Kalman filter to the data set thus obtained. As part of your work, think carefully about how to set up the experiment and about how to choose good values for the parameters, such as the covariance matrices describing the significance of the observational noise. Describe your approach in your report.

(Matlab: `s=serial('COM1','Baudrate',115200); fopen(s);`
`fscanf(s,'%f,%f,%f,%f,%f,%f,%f,%f,%f',[9 1]);`)

List of references

- D. Aubry. Mécanique des milieux continus. Ecole Centrale Paris. Lecture notes.
- D. Aubry. Dynamique des systèmes de corps rigides. Ecole Centrale Paris. Lecture notes.
- J. Hol. Sensor fusion and calibration of inertial sensors, vision, and ultra-wideband and GPS. PhD Thesis, Linköping University, Linköping, Sweden, 2011.
- M. Kok, J. Hol, and T. Schon. Using inertial sensors for position and orientation estimation. *Foundations and Trends in Signal Processing*, 11:1–153, 2017.
- S. Madgwick. An efficient orientation filter for inertial and inertial/magnetic sensor arrays. Technical Report, University of Bristol, Bristol, UK, 2010.
- T. Sullivan. Introduction to uncertainty quantification. Springer, 2015.
- N. Deom. Extended Kalman filter for IMUs. Project Report, MECA0010 Reliability and stochastic modeling of engineered systems, ULiège, 2017–2018.