ON THE IDENTIFICATION OF AXIAL FORCE IN STAY CABLES ANCHORED TO FLEXIBLE SUPPORTS

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Abstract

This paper analytically investigates the effect of flexible restraints on the modal properties of a shallow cable with small bending stiffness. An asymptotic second-order accurate closed form equation for the natural frequencies of the cable is presented. This expression generalizes a well-known result of the literature, valid in the particular case of perfectly clamped end sections, and allows one to define a single parameter that globally describes the flexibility of the cable restraints. The identifiability of this parameter within a frequency-based identification procedure is then critically discussed and a simple yet effective procedure to simultaneously identify the axial force and the bending stiffness of the cable is presented. A numerical example illustrates the performance of the proposed identification procedure in the case of frequencies contaminated by measurement errors.

INTRODUCTION

Identification of the axial force in stay cables is of paramount importance for health monitoring and safety assessment purposes. Vibration based testing techniques provide the ground for quick and cheap identification strategies, based on the knowledge of (a) a set of identified natural frequencies, and (b) a structural model that relates natural frequencies to the axial force value. Reliability of results, hence, is inherently related to the predictive capabilities of the underlying structural model. Errors may arise, in particular, from the modeling of cable bending stiffness and boundary conditions.

Different modeling strategies have been proposed and thouroghly investigated in the literature, including: (a) the classic taut-string model [1, 2], (b) cable models accounting for the bending stiffness, but neglecting the effects due to the sagging and to the axial extensibility [3, 4], (c) cable models accounting for bending stiffness, cable sagging and axial extensibility effects [5]. In most cases, stay cables are characterized by small values of the sag-to-span ratio (in the order of few percents) and of the Irvine parameter λ^2 [6, 7]. As a consequence, they can be modeled as straight and unstreachable one-dimensional structural elements by neglecting sagging and axial extensibility effects. On the other hand, neglecting bending stiffness can lead to oversimplified structural schemes and unacceptable inaccuracies on the results of axial force identification procedures [3]. Very often, the bending stiffness values can only be roughly estimated through approximate expressions, due to the complex internal geometry (e.g. bundles of parallel or helical strands with or without cement grouting) of stay cables. On a practical ground, hence, bending stiffness should be added to the unknowns of the structural identification problem.

Boundary conditions are often assumed in the form of either perfectly hinged or perfectly clamped cable end sections, to simplify the analytical treatment of the problem. More realistic structural schemes, however, could be defined by introducing equivalent translational and rotational springs at the cable end sections to model the flexibility of both the anchoring devices and the support structures.

Definition of equivalent springs is strongly related to the particular technological solutions adopted to realized the cable anchorages and is inherently related to several different sources of uncertainties, such as those related to geometric imperfections and aging of the anchoring devices [3]. As a consequence, the flexibility of the supports should be included as well among the unknowns of the structural identification problem.
The latter remark naturally yields one to consider the problem of judging about the identifiability of support flexibility parameters within the context of an axial force identification procedure. This topic has been recently addresses by the authors in a series of papers [8-11] aiming at defining a minimal set of unknown structural parameters along with some guidelines to judge about their identifiability for two different classes problems: (a) identification procedures based on the knowledge of a set of natural frequencies of the cable, and (b) identification procedures combining informations related to both the natural frequencies and mode shapes of the cable.

In the present paper, some of the main findings of the aforementioned authors' works will be reviewed and applied to assess the effect of flexible end restraints on the modal properties of a taut cable with small bending stiffness. Within this context, an asymptotic second-order accurate closed form equation for the natural frequencies of the cable is first presented. This expression generalizes a well-known result of the literature, valid in the particular case of perfectly clamped end sections [12], and allows one to define a single parameter that globally describes the flexibility of the cable restraints. The identifiability of this parameter within a frequency-based identification procedure is then critically discussed and a simple yet effective procedure to simultaneously identify the axial force and the bending stiffness of the cable is presented. The proposed procedure allows to generalize the current practice aiming at fitting the frequency vs. mode number relation obtained with simpler structural models (e.g. the taut string model). Moreover it paves the way to a simple assessment of the effects of the uncertainties related to the definition of the flexibility of the cable restraints on the identified structural parameters. A numerical example is presented to illustrate the performance of the proposed identification procedure in the case of frequencies contaminated by measurement errors.

THE CABLE MODEL

Let us consider a stay cable of length l, with constant bending stiffness (EI) and mass per unit of length (m), subject to an axial force T (see Fig. 1). By neglecting sag-extensibility and shear deformability effects, undamped planar flexural vibrations are governed by the partial differential equation:

$$EI\, \partial_x^4 y - T\, \partial_x^2 y + m\, \partial_t^2 y = 0 \tag{1}$$

where $y(x, t)$ is the transverse displacement of the cable centerline, $x \in [0, l]$ is a spatial coordinate running over the chord of the element and t is the time. By introducing the characteristic frequency $\Omega_0 = \sqrt{(T/ml^2)}$ and the non-dimensional bending stiffness $\varepsilon = \sqrt{(EI/Tl^2)}$, Eq. (1) can be re-written in the non-dimensional form:

$$\varepsilon^2\, \partial_\xi^4 \upsilon - \partial_\xi^2 \upsilon + \partial_\tau^2 \upsilon = 0 \tag{2}$$

where $\xi = x/l$, $\tau = w_0 t$ and $\upsilon(x, t) = y(x(\xi), t(\tau))/l$. General solutions of Eq. (2) can be expressed as $\upsilon = \phi(\xi) \sin(\omega\tau - \theta)$, where ω is a non-dimensional vibration frequency, θ is a phase angle depending on initial conditions and $\phi(\xi)$ is a mode shape function. The vibration frequencies ω and shape functions $\phi(x)$ are the eigensolutions of a fourth order Sturm-Liouville problem defined by the ordinary differential equation:

$$\varepsilon^2 \phi'''' - \phi'' - \omega^2 \phi = 0 \qquad (3)$$

along with suitable boundary conditions modeling the cable restraints. Please notice that a prime is adopted to denote differentiation with respect to ξ.

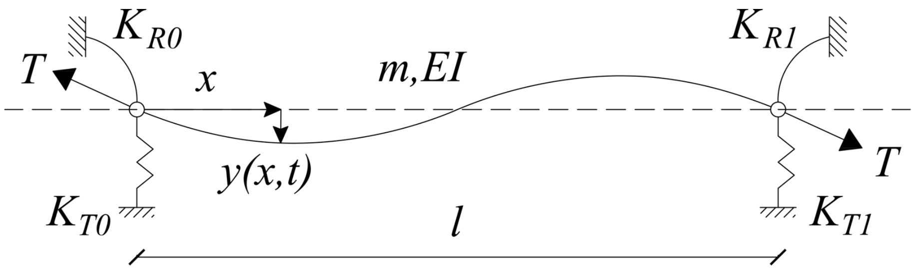

Figure 2. Schematic representation of a stay cable on flexible supports subject to a tensile load T.

By taking into account both the rotational and translational flexibility of the end restraints, the boundary conditions can be introduced as [10]:

$$\begin{gathered}(1-\rho_{T0})\,[\varepsilon^3\,\phi'''(0) - \varepsilon\,\phi'(0)] + \rho_{T0}\,\phi(0) = 0\\ (1-\rho_{T1})\,[\varepsilon^3\,\phi'''(1) - \varepsilon\,\phi'(1)] + \rho_{T1}\,\phi(1) = 0\\ (1-\rho_{R0})\,\varepsilon^2\,\phi''(0) - \rho_{R0}\,\varepsilon\phi'(0) = 0\\ (1-\rho_{R1})\,\varepsilon^2\,\phi''(1) + \rho_{R1}\,\varepsilon\phi'(1) = 0\end{gathered} \qquad (4)$$

where ρ_{Ti} and ρ_{Ri} (i=0,1) are, respectively, the translational and rotational degree-of-fixity parameters:

$$\begin{gathered}\rho_{Ti} = \varepsilon\, k_{Ti} / (1+\varepsilon\, k_{Ti})\text{ , with: } k_{Ti} = K_{Ti}\, l/T\\ \rho_{Ri} = k_{Ri} / (\varepsilon + k_{Ri})\text{ , with: } k_{Ri} = K_{Ri}\, /(Tl)\end{gathered} \qquad (5)$$

As it can be easily appreciated through an inspection of Eq. (5), each degree-of-fixity parameter is bounded within the closed unit interval [0,1], with lower and upper bound values corresponding, respectively, to the ideal cases of free and perfectly restrained degree-of-freedom.

The general solution of Eq. (3) can be expressed as:

$$\begin{gathered}\phi(\xi;\,\omega) = A_1\,\sin(z_1(\omega)\,\xi) + A_2\,\cos(z_1(\omega)\,\xi) + \\ A_3\,\exp(-z_2(\omega)\,\xi) + A_4\,\exp(-\,(1-z_2(\omega))\,\xi)\end{gathered} \qquad (6)$$

where A_i (i=1,…,4) are integration constants, while z_1 and z_2 are defined as:

$$\varepsilon\sqrt{2}\, z_j(\omega) = \sqrt{[(-1)^j + \sqrt{(1+4\varepsilon\omega^2)}]},\ \ j=1,2 \qquad (7)$$

Substitution of Eqs. (6) and (7) in (4) yields the algebraic eigenvalue problem:

$$\mathbf{B}(\omega\,;\,\varepsilon,\,\Pi)\,\mathbf{a} = \mathbf{0} \qquad (8)$$

where **a** is a column vector listing the integration constant A_i (i=1,…,4) and **B** is a 4×4 matrix whose entries depend on the non-dimensional frequency ω, the non-dimensional bending stiffness ε and the degree of fixity parameters $\Pi=\{\rho_{T0}, \rho_{T1}, \rho_{TR0}, \rho_{R1}\}$.
It is worth noting that values of ε typical of stay cables are in the order of 1%-2% or lower [7]. This makes the boundary value problem (3)-(4) and its algebraic counterpart (8) singularly perturbed problems, hinting at the same time the existence of boundary layers in the mode shapes and possible numerical difficulties if appropriate solution strategies are not used.

Due to the smallness of the bending stiffness parameter ε, however, approximate solutions of the eigenvalue problem (8) can be conveniently obtained through standard perturbation techniques. The authors have recently derived in [10] the following second-order accurate asymptotic expression for the non-dimensional natural frequencies of the cable:

$$\omega_k = k\pi \ \{1+ 2\, p\varepsilon\ + [(k\pi)^2/2 + 4p^2]\ \varepsilon^2\} + o(\varepsilon^3)\ \ , k=1, 2, 3,... \tag{9}$$

where p is a single non-dimensional parameter that globally takes into account the flexibility of the restraints:

$$p = 1 + \rho_R - 1/\rho_T \tag{10}$$

with

$$\rho_R = (\rho_{R0} + \rho_{R1})/2 \quad \text{and} \quad \rho_T = 2\rho_{T0}\rho_{T1}/(\rho_{T0}+\rho_{T1}) \tag{11}$$

The Eq. (9) generalizes to the case of cables on flexible translational and rotational supports the asymptotic solution derived by Morse and Ingard [12] for a doubly-clamped cable. As expected, at the leading order term, the asymptotic solution in Eq. (9) delivers the non-dimensional natural frequencies of the taut string model, i.e. $k\pi$. The cable bending stiffness and the flexibility of the restraints affect the first and second order corrections through the non-dimensional variables ε and p.

Once the non-dimensional natural frequencies are known, a simple re-scaling operation yields the natural frequencies Ω_k of the cable as:

$$\Omega_k = \Omega_0\ \omega_k = \Omega_0\ k\pi\ \{1+ 2\, p\varepsilon\ + [(k\pi)^2/2 + 4p^2]\ \varepsilon^2\} + o(\varepsilon^3)\ \ , k=1, 2, 3,... \tag{12}$$

The Eq. (12) allows one to appreciate how, the progression of the natural frequencies reported to the mode rank) is a quadratic function of the mode number k with only two terms: the intercept and the second degree coefficient. As a consequence, only two out of the three independent parameters $\{\Omega_0, \varepsilon, p\}$ can be identified on the basis of the knowledge of a set of natural frequencies of a stay cable characterized, as it is usual, for small values of the non-dimensional bending stiffness parameter ε.

THE PARAMETER IDENTIFICATION PROBLEM

The second-order accurate asymptotic equation (12) can be used to set up a simple but effective procedure to simultaneously identify the parameters Ω_0 and ε. Once these parameters are known, the cable axial force and bending stiffness can be simply evaluated as:

$$T = ml\Omega_0^2 \quad \text{and} \quad EI = Tl^2\varepsilon^2 \tag{13}$$

The proposed identification procedure, firstly developed by the authors in [10], is based on a standard linear regression analysis in a transformed coordinate system. The proposed method is as simple as the classic identification strategies relying on the taut string model but additionally allows one to (a) get an estimate of the cable bending stiffness, and (b) explicitly accounting for the flexibility of the restraints through the parameter p. Although this parameter is generally unknown and cannot be estimated on the sole knowledge of a set of measured frequencies (as long as typical stay cables with ε<<1 are considered), as it has been theoretically shown in the previous Section, the proposed identification strategy allows one to simply assess the influence of the flexibility of the cable restraints on the outcomes of the identification procedure.

Let us consider an ordered set of M natural frequencies $\{\Omega_{k1}, \Omega_{k2}, \ldots, \Omega_{kM}\}$, with $k_j \in \mathrm{N}^+$ for any $j\in[1,M]$ and $k_i<k_j$ if and only if $i<j$. The points of the set can be re-ordered by introducing a pair of integral coordinates (η_m, γ_m) defined as:

$$\eta_m = 1/m \cdot \Sigma^m{}_{j=1,m}\,[\Omega_{kj}/(k_j\,\pi)] \quad \text{and} \quad \gamma_m = 1/m \cdot \Sigma^m{}_{j=1,m}\, k^2{}_j\,,\ m=1, 2,\ldots, M \tag{14}$$

Substitution of Eq. (12) in the above definitions of (η_m, γ_m) yields the linear relationship:

$$\eta_m = \beta_0 + \beta_1\,\gamma_m \tag{15}$$

with intercept β_0 and slope coefficient β_1, respectively defined as:

$$\beta_0 = \Omega_0 \cdot (1 + 2p\varepsilon + 4p^2\varepsilon^2) \quad \text{and} \quad \beta_1 = \Omega_0/2 \cdot \pi^2\varepsilon^2 \tag{16}$$

Whenever a set of M frequencies $\{\Omega_{k1}{}^*, \Omega_{k2}{}^*, \ldots, \Omega_{kM}{}^*\}$ is known from tests, the definitions in Eq. (14) can be used to plot the experimental data in the plane (η_m, γ_m). Standard linear fitting techniques, then, can be adopted to get estimates $\beta_0{}^*$ and $\beta_1{}^*$ of the intercept and slope coefficient of the linear equation (15). Substitution of $\beta_0{}^*$ and $\beta_1{}^*$ in Eq. (16) yields of non-linear algebraic equations that can be solved to get estimates $\Omega_0{}^*$ and ε^* of the unknown parameters Ω_0 and ε. By noticing that β_0 is of the same order of magnitude of Ω_0, β_1 is of the same order of magnitude of ε^2 and $\varepsilon<<\Omega_0$ for typical stay cables, first-order accurate approximate expressions for Ω_0 and ε read:

$$\Omega_0{}^* = \beta_0{}^*\,(1 - 2p\varepsilon^*) \tag{17}$$

and

$$\varepsilon^* = \sqrt{[(2\beta_1{}^*)/(\pi\beta_0)]} \tag{18}$$

A simple inspection of Eqs. (17) and (18) reveal that the flexibility of the restraints doesn't affect the estimate of the non-dimensional bending coefficient, but introduces a bias on the estimator $\Omega_0{}^*$:

$$\text{bias}[\Omega_0^*]=2(p^*-p)\ \varepsilon^*\Omega_0^* \tag{19}$$

where p^* and p denote, respectively, an assumed value and the unknown true value of the parameter modeling the restraints.

Upper and lower bounds of the bias can be easily determined in the special, but practically important, case of negligible translational flexibility of the cable restraints. In this case the parameter p turns out to be bounded in the unit interval [0,1], and the maximum absolute value of the difference (p^*-p) is equal to one. Whenever information on the rotational stiffness of the support is not available, as it is often the case, a pragmatic choice leading in general to a minimization of the bias (19) is to set p^*=0.5. Any other user-defined choice is also possible, including an interval analysis.

APPLICATION EXAMPLE

The performance of the proposed parameter identification procedure is illustrated in this Section with reference to a typical stay cable, characterized by Ω_0=5.66 rad/s, ε=0.01, p=0.75 and T=4000 kN. In order to simulate experimental input data, the eigenvalue problem (8) has been numerically solved to get the first five natural frequencies of the cable. These frequencies, then, have been corrupted by multiplying their nominal values by a unit-mean and low intensity Gaussian noise, to account for the effects of measurement errors. Different values of noise intensity, ranging from 0 to 2.5%, have been considered. For each noise intensity value, a sample of 5,000 sets of noisy natural frequencies has been randomly generated.

Each set of noisy natural frequencies has been first represented in the plane (η_m, γ_m), through an application of the transformation of coordinates (14), as it is shown in Fig. 2. For each set of simulated experimental data, then, the coefficients β_0 and β_1, have been evaluated through an application of the ordinary least squares method. The corresponding values of Ω_0 and ε have been calculated through Eqs. (17) and (18) under two different assumptions on the restraint parameter p, namely: p=0 and p=1. These two values, as already noticed in the previous section, correspond to the upper and lower theoretical values of p whenever the translational flexibility of the cable restraints can be considered as negligible. The outcomes of the proposed identification procedure (curves labeled as 'LR, p=0' and 'LR, p=1 ' in Fig. 3) have been compared to those obtained through a simple inversion of the formula for the fundamental frequencies of the taut string model (curves labeled as 'TS' in Fig. 3).

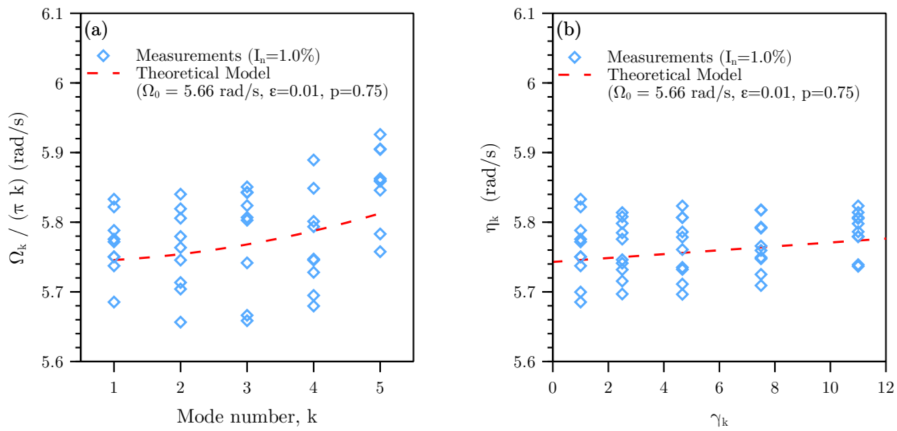

Figure 2. First five natural frequencies of the stay cable. Measured results are simulated by corrupting the solutions of the eigenvalue problem (8) through a zero-mean Gaussian noise with intensity I_n = 1.0%. (a) Representation in the plane (Ω_k /$k\pi$, k). (b) Representation in the plane (η_m, γ_m). A set of ten simulated measurement results is shown in the figures.

As it can be appreciated from Fig. 3, the proposed identification strategy delivers estimates of the characteristic frequency Ω_0 (directly related to the cable axial force through Eq. (15)) as good as those obtained through the classic taut-string formula over the whole range of noise intensity values herein considered. Differently than the taut-string formula, however, the proposed identification strategies allows to easily get upper and lower bound estimates for Ω_0 along with an estimate of the non-dimensional bending stiffness ε. It is worth noting that the estimate of ε does not depend on the assumed value of the restraint parameter p and is characterized by a coefficient of variation significantly higher than the one associated to the estimates of Ω_0 (cf. Figs. 3(b) and 3(c)). This latter remark allows one to conclude in order to obtain reliable estimates of the non-dimensional bending stiffness ε, a large number of experimental data, such as the one that can be obtained e.g. through a continuous monitoring strategy, is needed.

CONCLUSIONS

The modal properties of a stay cable anchored to flexible supports have been analytically investigated, leading to the definition of a single parameter that globally describes the flexibility of the restraints. The identifiability of this parameter within a frequency-based identification procedure has been critically discussed and a simple yet effective procedure to simultaneously identify the axial force and the bending stiffness of the cable has been presented. The proposed procedure allows to generalize the current practice aiming at fitting the frequency vs. mode number relation obtained with simpler structural models (e.g. the taut string model). Moreover it paves the way to a simple assessment of the effects of the uncertainties related to the definition of the flexibility of the cable restraints on the identified structural parameters.

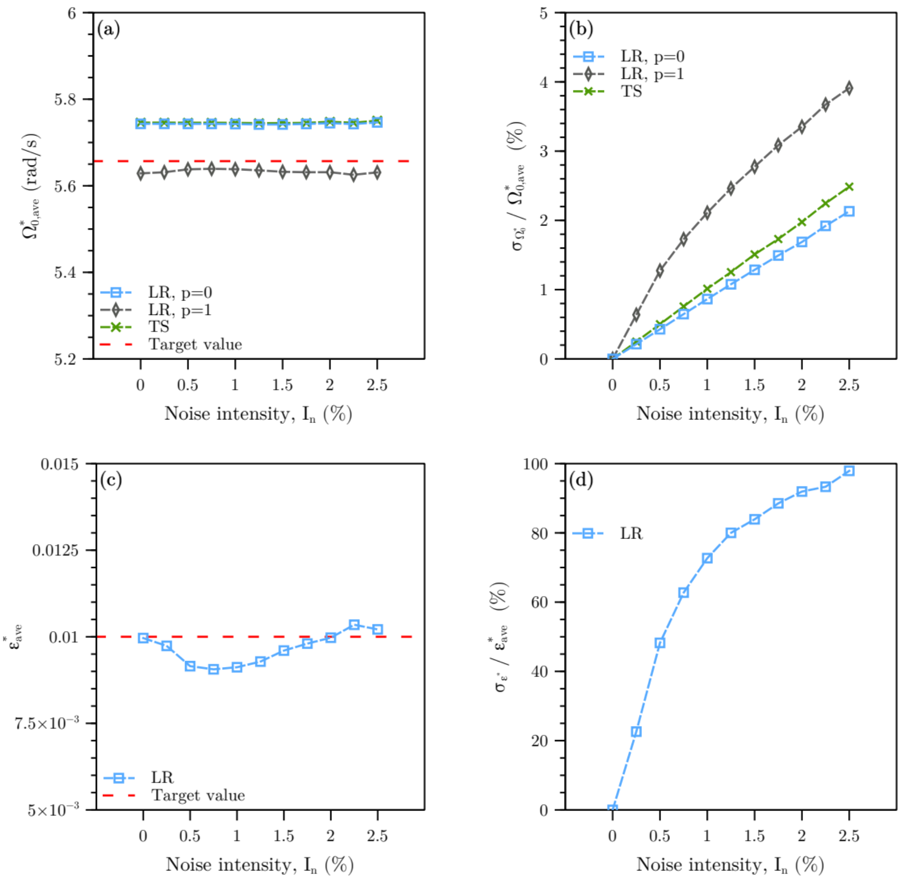

Figure 3. Results of the proposed identification strategy, averaged over five-thousand runs, as a function of the noise intensity for a stay cable anchored to flexible restraints characterized by a theoretical value of the restraint parameter equal

to p=0.75. (a) Characteristic frequency Ω_0 (target value: Ω_0=5.66 rad/s). (b) Coefficient of variation of the estimated value of Ω_0. (c) Non-dimensional bending stiffness Σ (target value: 0.01). (d) Coefficient of variation of the estimated value of ε. The results shown in Fig. 13(a) and (b) have been obtained by setting the restraint parameter p equal to: p = 0 (LR, p = 0), and p = 1 (LR, p = 1). Figures (a) and (b) also show a comparison with the outcomes of a simple identification procedure based on the taut string model (TS). Notice that a single curve LR is shown in Figures (c) and (d), since estimates of Σ are independent of the restraint parameter p.

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