ANGSIZ User's Guide

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Note: For more information, see the article 'A general and practical method for calculating cosmological distances' by Rainer Kayser, Phillip Helbig and Thomas Schramm in *Astronomy and Astrophysics*, **1996.**

a. General description

The **ANGSIZ** routine calculates the angular size distance between two objects as a function of their redshifts. (In world models which have no big bang, but contract from infinity to a finite size before expanding, there are two or four independent distances. In these cases, the first distance is returned by **ANGSIZ** and the rest by the routine **BNGSIZ**.) The distance also depends on the cosmological parameters λ_0, Ω_0 and $\eta(z)$. These are not passed to **ANGSIZ** but rather to an auxiliary routine, **INICOS**, which calculates all z-independent information for a given world model.

The result is in units of cH_0^{-1}. Multiply the result by 3×10^3 (actually 2.99792458×10^3) to get the distance in Mpc for $H_0 = 100\,\mathrm{km \cdot s^{-1} \cdot Mpc^{-1}}$. One can define h to be H_0 in units of $H_0 = 100\,\mathrm{km \cdot s^{-1} Mpc^{-1}}$. Thus, multiplying by 3×10^9 corresponds to an h of 1. For other values of h, simply *divide* the result by h, for example by 0.5 for $H_0 = 50\,\mathrm{km \cdot s^{-1} Mpc^{-1}}$. (Be aware that h can be and sometimes is defined as 1 for other values of H_0, usually 50. *Sometimes* this is indicated by a subscript, *i.e.* h_{50}, h_{100} and so on.)

The calling sequence is described below in section b.

All routines are in absolutely standard **FORTRAN77** and have been tested on a number of different combinations of compiler/hardware/OS so essentially the same numerical results should be obtained everywhere.

i. INICOS

The routine **INICOS** takes the cosmological parameters and 'user wishes' as input and calculates quantities needed by **ANGSIZ**—which are relayed internally to **ANGSIZ**—and also returns these to the calling routine. In addition, as in **ANGSIZ** itself, an error message is returned. **INICOS** can be used independently of **ANGSIZ** of course, but not the other way around.

INICOS returns the **INTEGER** variable **WMTYPE**, world model type, which gives a qualitative classification of the cosmological model. The table shows the correspondence between the value of **WMTYPE** and some standard classifications from the literature, and also gives some brief information on the temporal and spatial properties of the corresponding world model. The figure shows the location of the various world model types in the λ_0-Ω_0-plane.

WMTYPE	$t=\infty$	$R=\infty$	$\exists z_{\max}$	bounce	name	λ_0	k	Ω_0	σ_0	q_0	SR	HB
1	no	yes	no	no		<0	-1	0	0	>0	$O(0)$	1(iii)
2	no	yes	no	no		<0	-1	>0	>0	>0	$O(1)$	1(iii)
3	no	yes	no	no		<0	0	>0	>0	$>\frac{1}{2}$	$O(2)$	1(ii)
4	no	no	no	no		<0	$+1$	>0	>0	$>\frac{1}{2}$	$O(3)$	3(iv)
5	no	no	no	no	MTW	0	$+1$	>1	>0	$>\frac{1}{2}$	$O(4)$	3(iv)
6	no	no	no	no		>0	$+1$	$>\Omega_{0,\mathrm{c}}$	$>\sigma_{0,\mathrm{c}}$	$>\frac{1}{2}$	$O(5)$	3(iii)(a)
7	yes	yes	no	no	Milne	0	-1	0	0	0	$M_1(1)$	1(ii)
8	yes	yes	no	no	'LCDM'	0	-1	>0	>0	$0<q_0<\frac{1}{2}$	$M_1(2)$	1(ii)
9	(yes)	yes	no	no	Einstein-de Sitter	0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$M_1(3)$	2(ii)
10	yes	yes	no	no		>0	-1	0	0	$-1<q_0<0$	$M_1(0)$	1(i)
11	yes	yes	no	no		>0	-1	>0	>0	$-1<q_0<\frac{1}{2}$	$M_1(4)$	1(i)
12	yes	yes	no	no	de Sitter	1	0	0	0	-1	S	2(i)
13	yes	yes	no	no	'ΛCDM'	>0	0	>0	>0	$-1<q_0<\frac{1}{2}$	$M_1(5)$	2(i)
14	yes	no	no	no	Lemaître	>0	$+1$	note	note	note	$M_1(6)$	3(1)
15	(yes)	no	no	no		>0	$+1$	$\Omega_{0,\mathrm{c}}$	$\sigma_{0,\mathrm{c}}$	$>\frac{1}{2}$	A_1	3(ii)(b)
16	(yes)	no	(yes)	(no)	Einstein	>0	$+1$	$\Omega_{0,\mathrm{c}}=\infty$	$\Omega_{0,\mathrm{c}}=\infty$	∞	E	3(ii)(a)
17	yes	no	yes	no	Eddington	>0	$+1$	$\Omega_{0,\mathrm{c}}$	$\sigma_{0,\mathrm{c}}$	<-1	A_2	3(ii)(c)
18	yes	no	yes	yes	bounce	>0	$+1$	$0<\Omega_0<\Omega_{0,\mathrm{c}}$	$0<\sigma_0<\sigma_{0,\mathrm{c}}$	<-1	$M_2(1)$	3(iii)(b)
19	yes	no	yes	yes	Lanczos	>0	$+1$	0	0	<-1	$M_2(0)$	3(iii)(b)

Table 1. Classification of cosmological models as done by INICOS. Further information is provided in the text, particularly concerning parenthetical expressions and when 'note' appears instead of the expression in question.

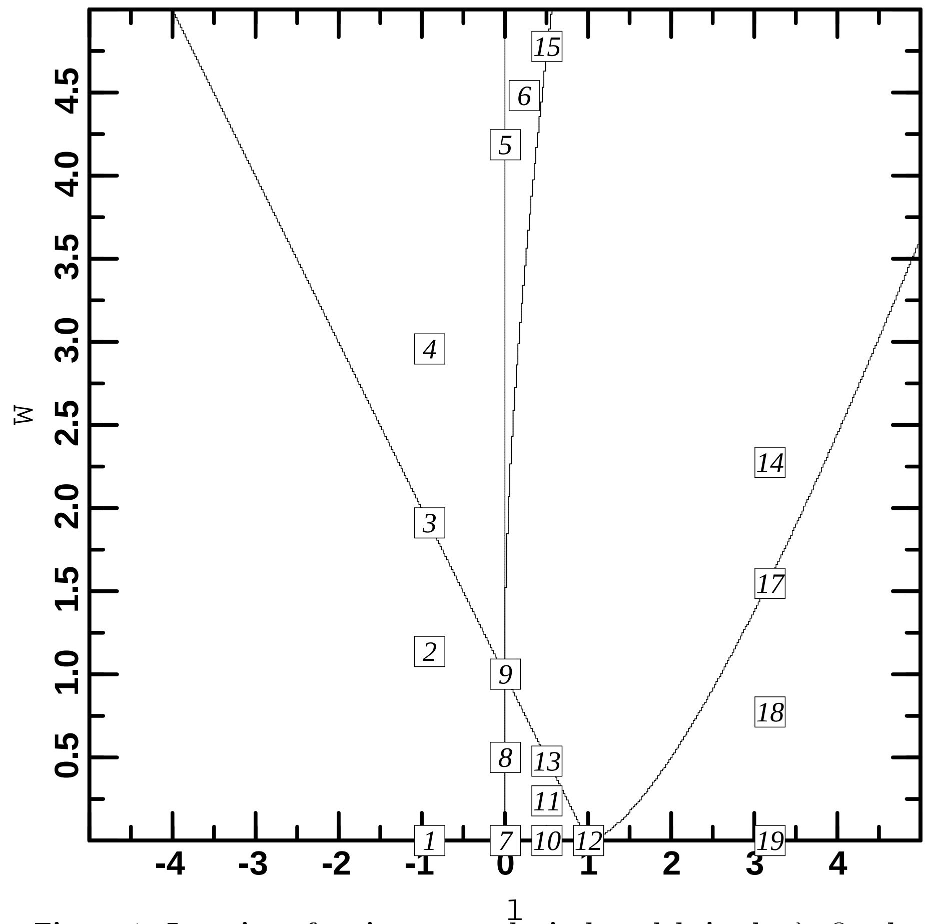

Figure 1. Location of various cosmological models in the λ_0-Ω_0-plane. A given world model is either on a line or curve segment (bounded by at least one vertex), on a vertex or in the space between the various lines and curves. The diagonal line corresponds to $k = 0$, the vertical line to $\lambda_0 = 0$, the curve immediately to the right of this line divides models which will eventually collapse (to the left) to those which will not (to the right) and the curve at lower right divides world models with a big bang (to the left) from those with no big bang (to the right).

`WMTYPE` refers to the classification returned by `INICOS` as the value of the variable `WMTYPE`. The following cosmological models are *definitively* ruled out by observations: 1, 7, 10, 12, 16, 17, 18, 19. 16 is ruled out because it has no expansion; the rest which aren't ruled out by the fact that they are empty are ruled out by the fact that a z_{max} which is at least as large as the largest observed redshift implies a value for Ω_0 which is so low as to be definitively ruled out.

$t = \infty$ answers the question whether the universe will expand forever or, more precisely, is the opposite of the answer to the question 'will the universe collapse to $R = 0$ in the future'. The subtlety arises because of the fact that `WMTYPE`s 9, 15 and 16, while not collapsing, don't necessarily 'expand forever' (perhaps depending on how this is precisely defined) although they might 'exist forever'. In the Einstein-de Sitter model `WMTYPE 9`, the expansion rate asymtotically approaches 0; the value of the scale factor R at $t = \infty$ is, however, $R = \infty$. Thus, the universe expands forever, although at $t = \infty$ the value of $\dot{R}$ is 0. The Eddington universe, `WMTYPE 15`, expands forever, as well, but at $t = \infty$ not only is the value of $\dot{R}$ 0 but R reaches a finite maximum as well. The Einstein universe, `WMTYPE 16`, is static. It exists forever, but doesn't really expand forever, since it doesn't expand at all. R is fixed at a finite value and $\dot{R}$ is 0 at all times.

$R = \infty$ answers the question whether the 3-dimensional space is infinite in extent. It is infinite for $k = -1$ or $k = 0$ and finite for $k = +1$ (assuming a simple topology).

$\exists z_{\max}$ If a $z_{\max}$ exists, then there was no big bang. (In the de Sitter model the big bang occurs at $t = -\infty$.) Rather, the universe expands from a minimum value of R. The static Einstein universe (`WMTYPE 16`) has a redshift of 0 for all objects, thus distance cannot be determined from redshift in this case.

bounce If there has been a bounce, the universe is expanding after contracting from $R = \infty$ to $R = R_{\min}$. This means that there are four independent distances for a given redshift, unless the dependence of η on z is the same during the collapsing and expanding phases, in which case there are only two independent distances. (In the Einstein static universe, `WMTYPE` 16, there are an infinite number of distances for the only possible redshift of 0.)

name refers to a commonly used term for the given world model. MTW refers to the fact that this world model is discussed in *Gravitation* by Misner *et al.* [1973] in §27.10 and Box 27.4 as an example of the type of cosmological model preferred by Einstein ($\lambda = 0$ because Einstein preferred a 'simpler' universe without the cosmological constant after it became clear that a) his static model cannot describe the real universe, not even as a 0^{th} approximation and b) non-static solutions with a cosmological constant exist; $k = +1$ in order to have no problems with boundary conditions at ∞). This world model is noted as having been 'investigated by Einstein' in Stabell & Refsdal [1966]; the name 'Einstein universe', although perhaps more appropriate for this cosmological model, is historically irrevocably associated with the static universe (`WMTYPE 16`). Milne's cosmological model is not really our `WMTYPE 7`, although it is equivalent; Milne had $G = 0$ in his kinematic relativity, and either $G = 0$ or $\rho = 0$ has the effect of making the expansion independent of gravitation. 'LCDM' refers to the fact that these cosmological parameters are used in the typical 'low density standard cosmological model'; here CDM is an abbreviation for 'cold dark matter', which includes details on scales small enough where homogeneity cannot be used as an approximation and doesn't concern us here. Naturally, global properties of cosmological models are independent of such local structure. Of course, the parameters are the same for LHDM (hot) and LHCDM (or LMDM: mixed). 'ΛCDM' refers to the fact that these cosmological parameters are used in the typical 'low density flat standard cosmological model with a cosmological constant'. Note: our 'Eddington model' (`WMTYPE` 17) is referred to as the Lemaître model in Stabell & Refsdal [1966] and as the Eddington-Lemaître model in Bondi [1961]. Harrison [1981] also uses our term Eddington model for this case. Similarly. our Lemaître model (`WMTYPE` 14) is called the Eddington-Lemaître in

Berry [1986]. Bondi [1961] and Harrison [1981] also use our term Lemaître model for this case.

λ_0 the range of λ_0.

k the sign of k.

Ω_0 the range of Ω_0. $\Omega_{0,\mathrm{c}} = 2\sigma_{0,\mathrm{c}}$. In `WMTYPE` 14, Ω_0, σ_0 and q_0 can take on all possible values, but not all combinations are possible. Specifically, for $-1 < q_0 < \frac{1}{2}$, $\sigma_{0,\mathrm{c}}$ ($\Omega_{0,\mathrm{c}}$) has no meaning, and the range $0 < \sigma_{0,\mathrm{c}}$ is allowed. Otherwise, we must have $\Omega_0 > \Omega_{0,\mathrm{c}}$ ($\sigma_0 > \sigma_{0,\mathrm{c}}$). (Of course, $\Omega_0 = 2\sigma_0$.)

σ_0 the range of σ_0; $\sigma = 0.5 \times \Omega_0$. $\sigma_{0,\mathrm{c}}$ is defined as

$$\sigma_{0,\mathrm{c}} = \frac{1}{6}(q_0+1)\left(q_0+1 \pm \sqrt{(q_0+1)\left(1_0 - \frac{1}{3}\right)}\right) \text{ (SR Eq. 12)}$$

In `WMTYPE` 14, Ω_0, σ_0 and q_0 can take on all possible values, but not all combinations are possible. Specifically, for $-1 < q_0 < \frac{1}{2}$, $\sigma_{0,\mathrm{c}}$ ($\Omega_{0,\mathrm{c}}$) has no meaning, and the range $0 < \sigma_{0,\mathrm{c}}$ is allowed. Otherwise, we must have $\sigma_0 > \sigma_{0,\mathrm{c}}$ ($\Omega_0 > \Omega_{0,\mathrm{c}}$).

q_0 the sign of q_0; $q_0 = \sigma_0 - \lambda_0$. In `WMTYPE` 14, Ω_0, σ_0 and q_0 can take on all possible values, but not all combinations are possible. Specifically, for $-1 < q_0 < \frac{1}{2}$, $\sigma_{0,\mathrm{c}}$ ($\Omega_{0,\mathrm{c}}$) has no meaning, and the range $0 < \sigma_{0,\mathrm{c}}$ is allowed. Otherwise, we must have $\sigma_0 > \sigma_{0,\mathrm{c}}$ ($\Omega_0 > \Omega_{0,\mathrm{c}}$).

SR refers to the description of cosmological models in Stabell & Refsdal [1966, p. 383] where models are classified with a scheme similar to that used by `INICOS`.

HB Refers to the classification in chapter IX of Bondi [1961]. Since Bondi implicitly assumes $\Omega_0 > 0$, this is not entirely correct in the cases where there is a qualitative difference for $\Omega_0 = 0$, as in `WMTYPE`s 7, 10 and 12. In this case, q_0 is never positive, so the qualitative curves given by Bondi should be extrapolated back to $R = 0$ without the negative curvature part at the beginning.

ii. ANGSIZ

This routine calculates the angular size distance between two redshifts (just one of the possible distances in the bounce models). This is done by the numerical integration of a second order differential equation. (For details see Kayser, Helbig & Schramm [1996].) As far as we know, no analytic solution for the general case exists. An advantage of using `ANGSIZ` even when an analytic solution for a special case exists is that `ANGSIZ` *always* gives a valid result, *i.e.*, the equation holds for *all* cosmological models and no cases must be distinguished. (`ANGSIZ` is not recommended for use in the Eddington model, `WMTYPE` 17. If the user specifically wants to use the routines for this cosmological model, it is possible to allow this by declaring the `COMMON` block `ARTHUR` in the calling routine, containing a logical variable, which should be set to `.TRUE.` before calling `ANGSIZ`. This will override the default behaviour.)

iii. ETAZ

The variable η gives the fraction of homogeneously distributed matter in the cosmological model. On large scales, homogeneity is assumed, in accordance with the Robertson-Walker metric, the Cosmological Principle, and so on. However, on smaller scales, matter can of course be clumped. The fraction $1-\eta$ of matter is in these clumps, which means that they are outside (and sufficiently far from) the cone of light rays between source and observer. (If a clump of matter is near or in the beam itself, one must take account of this explicitly as a gravitational lens effect.)

It is important to remember that the value of η—for a fixed real situation—depends on the angular scale involved. For example, a halo of compact MACHO type objects around a galaxy in a distant cluster would be counted among the homogeneously distributed matter if one were concerned with the angular size distance to background galaxies further away, but would be considered clumped on scales such as those important when considering microlensing by the compact objects themselves. Thus, the clumps must have a scale comparable to the separation of light rays from an extragalactic object or larger.

Since we don't know exactly how dark matter is distributed, different η values can be examined to get an ideas as to how this uncertainty affects whatever it is one is interested in.

Of course, η can be a function of z. If the user wishes η to be constant for all z, then the variable `VARETA` should be set to `.FALSE.` when calling `INICOS`. In this case, the value of `ETA` passed to `INICOS` determines η. If `VARETA` ist `.TRUE.`, then the value of `ETA` is irrelevant and is given by the `REAL FUNCTION ETA(Z)`, which the user can modify as needed. If a constant value of `ETA` is desired, this can of course be achieved through defining the `FUNCTION ETA(Z)` appropriately, but this will typically make the computing time 2–3 times longer than setting `VARETA` to `.FALSE.` and using `ETA` to determine the value of η.

iv. BNGSIZ

This `SUBROUTINE BNGSIZ` is essentially the same as `ANGSIZ` except that it calculates the other three distances in bounce models. As long as `ETAZ` doesn't make the value of η depend on whether the universe is expanding or contracting, there is only one additional independent distance, `D14`; `D34` is the same as `D12` and `D32` is the same is `D14`. `D12` is the distance returned by `ANGSIZ`, where both the starting point and the end point of the integration are in the expansion phase. `D14` has its starting point `Z1` in the expansion phase and its end point `Z2` in the contraction phase, thus the angular size distance is found by integrating from `Z1` to `ZMAX` and back to `Z2`. `D34` has both boundaries in the contraction phase, and `D32` its starting point in the contraction phase and its end point in the expansion phase.

If one is not interested in the bounce models, this routine isn't needed. If one is only interested in the 'primary' distance in bounce models, this is returned by `ANGSIZ`, so also in this case `BNGSIZ` isn't needed.

`BNGSIZ` must be called after calling `ANGSIZ`: there are no input parameters; these and other necessary information are obtained by `BNGSIZ` from `ANGSIZ`.

b. Use as a black box: what the user needs to know

In the descriptions of the input variables in the parameter lists to the routines called by the user we also indicate a range of suggested values in square brackets. Of course, others are possible, but the ones we suggest correspond to cosmological models which are not firmly ruled out by other arguments (and some which are) and have been tested. Physically meaningless values should result in an error message. We didn't think it meaningful to check the routines for correctness for values outside these ranges. For output variables, the range indicates possible values. The mathematical and physical meaning of the variables can be found in Kayser *et al.* [1996].

i. INICOS

The interface to `INICOS` is

```
      SUBROUTINE INICOS(LAMBDA,OMEGA,ETA,VARETA,DEBUG,
C                       ^^^^^^^^^^^^^^^^^^^^^^^^^^^^^
C                                    in
C
     $                  WMTYPE,MAXZ,ZMAX,BOUNCE,ERROR)
C                       ^^^^^^^^^^^^^^^^^^^^^^^^^^^^^
C                                    out
```

The input variables are:

`REAL LAMBDA` is the normalised cosmological constant. $[-10,\ldots,+10]$

`REAL OMEGA` is the density parameter. $[0,\ldots,10]$

`REAL ETA` is the homogeneity parameter. $[0,\ldots,1]$

`LOGICAL VARETA` if `.TRUE.` means that η is not given by `ETA` but by the value returned by the `REAL FUNCTION ETAZ`.

`LOGICAL DEBUG` if `.TRUE.` means that error messages should be written to standard output.

The output variables are:

`INTEGER WMTYPE` gives some information about the cosmological model (see the table). $[1,\ldots,19]$

`LOGICAL MAXZ` if `.TRUE.` indicates that a maximum redshift exists; only in this case is the variable `ZMAX` correctly defined and only in this case should the variable `ZMAX` be used.

`REAL ZMAX` gives the maximum redshift in the cosmological model, if `LOGICAL MAXZ` is `.TRUE.`; otherwise, the maximum redshift is infinite, and for convenience set to zero.

`LOGICAL BOUNCE` if `.TRUE.` indicates that we have a 'bounce' model (see the table); only in this case can the other distances be computed by `BNGSIZ`.

`INTEGER ERROR` indicates if an error has occurred and what this means (see section vi.). $[0,\ldots,13]$

ii. ANGSIZ

The interface to ANGSIZ is:

```
      SUBROUTINE ANGSIZ(Z1,Z2, D12,ERROR)
C                       ^^^^^  ^^^^^^^^^
C                        in       out
```

The input variables are:

REAL Z1 is a redshift; set Z1 to 0 for the common case of the 'distance from a normal observer'. [0,...,5]

REAL Z2 is a redshift; set Z2 to the redshift of the object if one is interested in the angular size distance from a 'normal observer' to the object and correspondingly Z1 has been set to 0. [0,...,5]

The output variables are:

REAL D12 Is the angular size distance between Z1 and Z2. In bounce models (and of course in big bang models) this corresponds to the distance such that the universe has always been in a state of expansion between the times of light emission and reception.

INTEGER ERROR indicates if an error has occurred and what this means (see section vi.). [0,...,13]

ANGSIZ is not recommended for WMTYPE 17, since distances can become arbitrarily large, and the necessary overhead would complicate the routines to such a degree that 'normal' performance would be significantly hampered. If the user specifically wants to use the routines for this cosmological model, it is possible to allow this by declaring the COMMON block ARTHUR with a logical variable, which should be set to .TRUE. in the calling routine. This will override the default behaviour.

iii. BNGSIZ

The interface to BNGSIZ is:

```
      SUBROUTINE BNGSIZ(D14,D34,D32,ERROR)
C                       ^^^^^^^^^^^^^^^^^
C                             out
```

REAL D14 is the angular size distance between Z1 and Z2. This corresponds to the distance such that the universe is *not* in a state of *contraction* at Z1 and *not* in state of *expansion* at Z2.

REAL D34 is the angular size distance between Z1 and Z2. This corresponds to the distance such that the universe is in a state of *expansion neither* at Z1 *nor* at Z2.

REAL D32 is the angular size distance between Z1 and Z2. This corresponds to the distance such that the universe is *not* in a state of *expansion* at Z1 and *not* in a state of *contraction* at Z2.

INTEGER ERROR indicates if an error has occurred and what this means (see section vi.). [0,...,13]

iv. ETAZ

The interface to `ETAZ` is:

```
REAL FUNCTION ETAZ(Z)
```

which, being a function with no side effects, has no output variables. The one input variable is:

`REAL Z` this is the redshift; η is a function of z and is given by this function and *not* by the value of `ETA` supplied to `INICOS` if `VARETA` is `.TRUE.`; only in this case will `ETAZ` be used.

The `COMMON` block `WHICH` contains the `INTEGER` variable `CHOICE`. If this `COMMON` block is also in the calling routine, then a decision can be made there as to which dependency of η on z is to be used; see the use of `CHOICE` in our example `ETAZ`. This is initialised in our `BLOCK DATA COSANG` and so should not be initialised by the user.

For bounce models, it is of course possible that the dependence of η on z depends on whether the universe is expanding or contracting. Since `VALUE2` is set to `.TRUE.` during the phase of *contraction* and otherwise to `.FALSE.` by `ANGSIZ` and `BNGSIZ`, which can access `VALUE2` through the `COMMON` block `CNTRCT`, this variable can be used in `ETAZ` order to have two different dependencies, as in our example. *Additionally, in this case, as in our example, the value of* `CHOICE` *must be* negative *and otherwise must be positive.* This distinction is needed by `BNGSIZ` in order to avoid calculating `D34` and `D32` if these are not independent of the other distances.

v. Calling sequence

Thus, one should call `INICOS` anew for each cosmological model tested, indicating the cosmological model, whether or not η is to be a constant or is to be calculated by `ETAZ` and whether or not error messages are to be displayed. (`ERROR` is of course always appropriately set.) If one wants to calculate distances, then `ANGSIZ` should be called with two redshifts, and the angular size distance is returned. One should use the output variables of `INICOS` to determine whether or not to call `ANGSIZ`. If desired, `ETAZ` can be used; it can also be modified by the user. The calling routine can also make use of the `COMMON` block `WHICH` to enable different z dependencies of η to be calculated without having to change the code, recompile and relink for each case. `BNGSIZ` can be called after calling `ANGSIZ` if one is interested in the additional distances in the case of the bounce models.

vi. Error messages

There are fourteen possible 'error states', namely the values [0,...,13] of the `INTEGER` variable `ERROR`. Errors 1–3 can occur in `INICOS`, 4–12 in `ANGSIZ`, 12–13 in `BNGSIZ`. (Thus, only one error variable is needed in the calling routine.) `ERROR` is *always* appropriately set; the variable `DEBUG` merely controls whether or not messages are written to standard output. With the exception of `ERROR .EQ. 12`, the message is just the information contained in the following overview.

0 No error was detected.

1 `(OMEGA .LT. 0.0)`.

2 `((ETA .LT. 0.0) .OR. (ETA .GT. 1.0))` (disabled if `VARETA` is `.TRUE.`, in which case the value of `ETA` is irrelevant.) `ETAZ`, however, checks to see if `ETA` is within the allowed range.

3 The world model is so close to the A2 curve that the calculation of `ZMAX` is numerically unstable. This usually happens—when it does—very near the de Sitter model. However, it should be rare in practice, as noted below. Even if one were able to calculate a good value for `ZMAX`, this would probably lead to an error in `ANGSIZ` if `Z1` or `Z2` is near `ZMAX`.

4 The world model is `WMTYPE` 17. In this world model, the universe is infinitely old, and there is a finite `ZMAX`. This means that distances (especially for $\eta = 0$) can become arbitrarily large. The routines are not recommended for calculation in this cosmological model, since enabling the detection of overflow would unnecessarily complicate the routines. Also, the practical value of `ZMAX`, used internally, would have to be appreciable less than the real `ZMAX`. If the user specifically wants to use the routines for this cosmological model, it is possible to allow this by declaring the `COMMON` block `ARTHUR` with a logical variable, which should be set to `.TRUE.` in the calling routine. This will override the default behaviour.

5 `(Z1 .LT. 0.0)`

6 `(Z2 .LT. 0.0)`

7 `(Z1 .GT. ZMAX)`

8 The value of `ZMAX` returned by `INICOS` is the calculated value; however, internally a somewhat smaller `ZMAX` is sometimes necessary for numerical stability. This internal value is close enough to the real `ZMAX` for all practical purposes (at worst it has an error of about 10^{-3}). This message means that `Z1` is too large for numerical stability, being very near the real `ZMAX` but slightly smaller.

9 `(Z2 .GT. ZMAX)`

10 The value of `ZMAX` returned by `INICOS` is the calculated value; however, internally a somewhat smaller `ZMAX` is sometimes necessary for numerical stability. This internal value is close enough to the real `ZMAX` for all practical purposes (at worst it has an error of about 10^{-3}). This message means that `Z2` is too large for numerical stability, being very near the real `ZMAX` but slightly smaller.

11 Overflow error is possible. Calculation can be continued but the result might not be as exact as usual or subsequent calculation with the same cosmological parameters might lead to an 'unhandled exception'.

12 Very rare. This means an error has occurred deep down in the numerical integration. If `DEBUG` is `.TRUE.` then a message will be printed identifying where.

13 This means that `BNGSIZ` was called for a non-bounce world model.

When an error is returned, with the exception of 11, then *none* of the other output variables of the routine should be used and all distances are set to 0.

c. Inside the black box: description of all routines

Here is a short description of all of the `SUBROUTINE`s and `FUNCTION`s:

`BLOCK DATA COSANG` initialises `COMMON` block variables.

`SUBROUTINE INICOS` is described above.

`REAL FUNCTION QQ` calculates $Q^2(z)$.

`SUBROUTINE ANGSIZ` is described above.

`SUBROUTINE BNGSIZ` is described above.

`SUBROUTINE LOWLEV` Performs the low-level integration, mainly calling `ODEINT`.

`SUBROUTINE ASDRHS` calculates the right hand sides of the differential equation for the angular size distance

`SUBROUTINE ODEINT` is a 'driver' routine for the numerical integration.

`SUBROUTINE MMM` is for the modified midpoint method used by `ODEINT`.

`SUBROUTINE POLEX` performs polynomial extrapolation used by `ODEINT`.

`SUBROUTINE BSSTEP` performs one Bulirsch-Stoer integration step.

`REAL FUNCTION ETAZ` is described above.

i. Call tree

```
[COSANG]
[INICOS]
|  [QQ]
[ANGSIZ]
|  [LOWLEV]
|  |  [ODEINT]
|  |  |  [ASDRHS]
|  |  |  |  [ETAZ]
|  |  |  [BSSTEP]
|  |  |  |  [MMM]
|  |  |  |  |  [ASDRHS]
|  |  |  |  |  |  [ETAZ]
|  |  |  |  [POLEX]
|  [QQ]
[BNGSIZ]
|  [LOWLEV]
|  |  [ODEINT]
|  |  |  [ASDRHS]
|  |  |  |  [ETAZ]
|  |  |  [BSSTEP]
|  |  |  |  [MMM]
|  |  |  |  |  [ASDRHS]
|  |  |  |  |  |  [ETAZ]
|  |  |  |  [POLEX]
|  [QQ]
```

d. Caveats

The following `COMMON` block names are used and, since these are global names, should not be used in a conflicting way in any other routines: `COSMOL`, `ANGINI`, `WHICH`, `BNCSIZ`, `LOWSIZ`, `CNTRCT`, `ARTHUR`, `MERROR` and `PATH`. With the exception of the optional `WHICH`, all of these are used internally and need not further concern the user. The same is true of the `SUBROUTINE` and `FUNCTION` names listed above (except for `INICOS`, `ANGSIZ`, `BNGSIZ` and `ETAZ`, of course.)

The rest of this section should be of absolutely no concern to almost all users, since 'incorrect' results due to the reasons discussed below should only occur due to roundoff error corresponding to a ridiculous accuracy in the cosmological parameters; in most of these cases this would only be in world models which are uninteresting because they are ruled out conclusively (see the table).

The value of `WMTYPE` can differ from the 'real' value as calculated analytically from λ_0 and Ω_0 but only if one is so close to the boundary between two regions of parameter space that this happens due to ordinary inexactness in the internal representation of 'real' numbers. The same goes for the values of `MAXZ` and `BOUNCE`, since these are trivially related to `WMTYPE`. The A1 and A2 curves (see Stabell & Refsdal [1966]), corresponding to `WMTYPE`s 15 and 17, have been 'drawn with a numerically thick pencil' but this should only be noticeable for `LAMBDA` and `OMEGA` values precise to 10^{-5}. This is also true of the endpoints of the curves, `WMTYPE`s 9 and 12. `WMTYPE` 16 cannot be returned by `INICOS`, since this corresponds to the static Einstein cosmological model, in which λ_0 and Ω_0 are both $= \infty$. The value of `ZMAX` is probably exact in the third figure after the decimal point for world models *extremely near* the A1 or A2 curve; otherwise, it is as good as any numerically calculated quantity.

The values of `D12` and `D34` returned should be correct in the third digit after the decimal point and shouldn't be off by more than one digit in the fourth. The values of `D14` and `D32` are probably somewhat less precise, correct in the second decimal digit.

e. Development and tests

The development and most testing of the routines was done on a Digital VAXStation 3100 Model 76 running VMS 5.5-2 and with DEC FORTRAN. To make sure that numerical accuracy is monitored in a robust way, comparisons were made to the following systems:

- DEC ALPHA 4000/710, VMS, DEC FORTRAN (`FORTRAN 77`)
- DEC ALPHA 4000/710, VMS, DEC FORTRAN 90
- Cray C916, UNICOS 8, native `FORTRAN 77` compiler
- Convex C3840 and C220, Convex OS 11.0, native `FORTRAN 77`
- DSM Infinity 8000, SCO-UNIX, Green Hills Fortran Version 1.8.5
- Fujitsu/Siemens S100, UXP/M, UXP/M Fortran 77 EX/VP (V12)
- Hewlett-Packard 9000/735, HPUX, native `FORTRAN 77` and Fortran90 compilers
- IBM RS/6000, AIX, xlf90
- IBM RS/6000, AIX, xlf90, optimisation -qarch=pwr

- IBM RS/6000, AIX, xlf90, optimisation -qarch=ppc
- IBM RS/6000, AIX, xlf90, optimisation -qarch=pwr2
- 'generic InTel PC', Linux, f2c+gcc
- Intel Pentium, MS-DOS, Lahey Fortran90
- Apple PowerMac 6100/60, Macintosh OS D 7.5, Language Systems Fortran PPC
- IBM 9121-440, MVS, VS FORTRAN

f. Updates

Corrections, updates and so on will be posted under the subject 'ANGSIZ' in the newsgroup sci.astro.research. Comments, bug reports, *etc.* should be sent by email to `phelbig@hs.uni-hamburg.de`. Please put 'ANGSIZ' in the subject line.

g. Disclaimer

We've tested the routines to a greater degree than usual for serious scientific work, though of course not as extensively as for (serious) commercial software. To the best of our knowledge, they work as described, but of course we can take no responsibility for the consequences of any errors in the code. As far as possible we will correct any errors; in any case we will make them known in the newsgroup posting.

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