YAAI: Yet another age inequality

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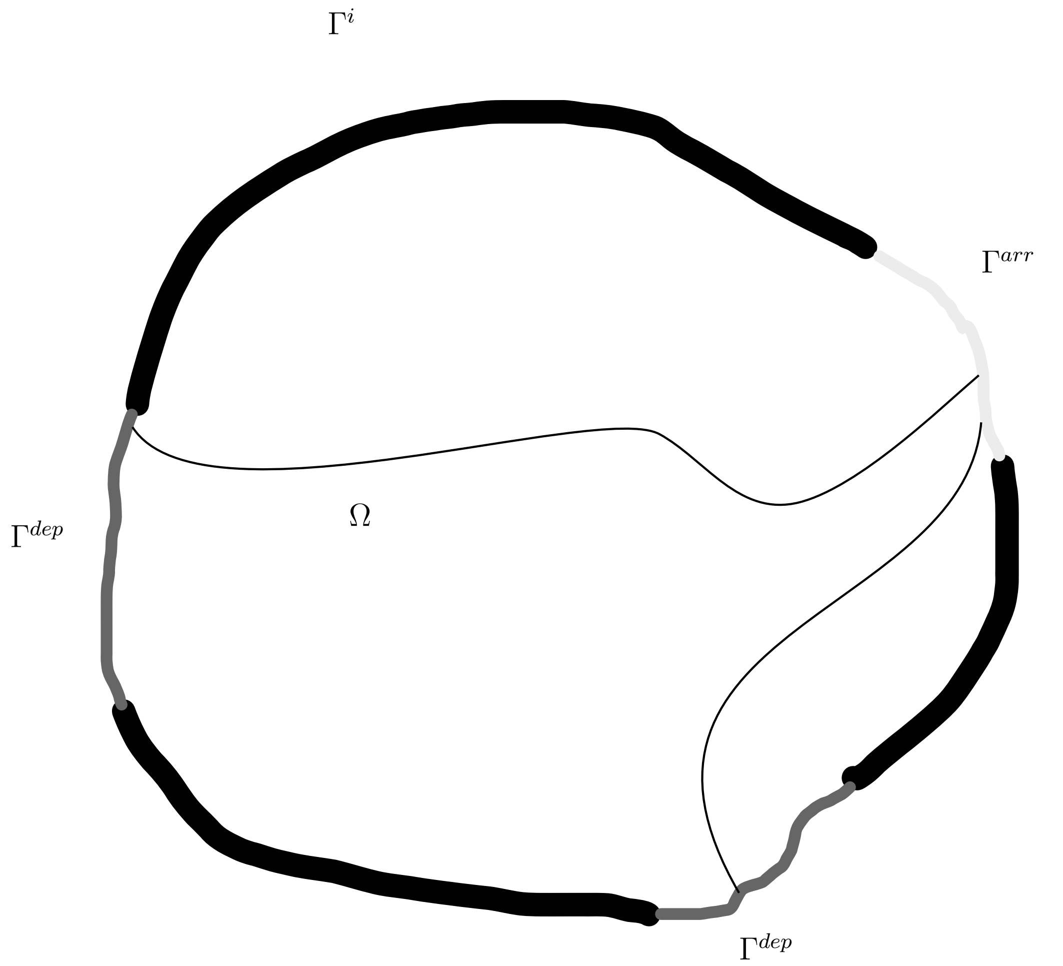
Abstract

A proof of a conjecture advanced in [1] is presented.



In [1], a system of equations similar to the following was looked at

$$\frac{\partial C}{\partial t} = -\gamma C - \boldsymbol{\nabla}\boldsymbol{\cdot}\left(\boldsymbol{u} C - \boldsymbol{K}\boldsymbol{\cdot}\boldsymbol{\nabla}\, C\right) \tag{1}$$

$$\frac{\partial \alpha}{\partial t} = -\gamma \alpha + C - \boldsymbol{\nabla}\boldsymbol{\cdot}\left(\boldsymbol{u} \alpha - \boldsymbol{K}\cdot\boldsymbol{\nabla}\, \alpha\right) \tag{2}$$

Equation (1) can be written in compact notation as

$$\mathcal{L}\left[C\right] = 0 \tag{3}$$

with the linear operator $[L]$ being

$$\mathcal{L}[\cdot] = \frac{\partial \cdot}{\partial t} + \gamma \cdot + \boldsymbol{\nabla}\boldsymbol{\cdot}\left(\boldsymbol{u}\cdot - \boldsymbol{K}\boldsymbol{\cdot}\boldsymbol{\nabla}\cdot\right) \tag{4}$$

Equation (1) describe thus how a tracer concentration C is transported in a domain Ω by a divergence-free velocity field $\boldsymbol{u}$ over time t with a diffusion modelled using a positive-defined symmetric diffusivity tensor $\boldsymbol{K}$. A linear decay term with decay rate $\gamma \geq 0$ is also included. All parameters might change with time t and position $\mathbf{x}$ as long as the governing equations remain linear for C and α.

Usual boundary conditions are prescribed with zero fluxes in the normal direction $\boldsymbol{n}$ to impermeable boundaries Γ^i:

$$[\boldsymbol{K} \cdot \boldsymbol{\nabla} C] \cdot \boldsymbol{n} = 0 \quad \mathbf{x} \text{ on } \Gamma^i \tag{5}$$

$$[\boldsymbol{K} \cdot \boldsymbol{\nabla} \alpha] \cdot \boldsymbol{n} = 0 \quad \mathbf{x} \text{ on } \Gamma^i \tag{6}$$

At the departure boundary Γ^{dep}, zero values for α and unit values for concentration C where used in the original problem, but we replace it by using a Robin condition with a piston velocity χ^{dep} assumed to be a constant here:

$$[\boldsymbol{K} \cdot \boldsymbol{\nabla} C] \cdot \boldsymbol{n} = \chi^{dep} \left(C^{dep} - C\right) \qquad \text{on } \Gamma^{dep} \tag{7}$$

$$[\boldsymbol{K} \cdot \boldsymbol{\nabla} \alpha] \cdot \boldsymbol{n} = \chi^{dep} \left(0 - \alpha\right) \qquad \text{on } \Gamma^{dep} \tag{8}$$

We can recover the original case by using $\chi^{dep} \to \infty$ and $C^{dep} = 1$.

At the arrival boundary Γ^{arr} we also use a general Robin condition but nudging the concentration and age concentration towards zero with constant piston velocity χ^{arr}

$$[\boldsymbol{K} \cdot \boldsymbol{\nabla} C] \cdot \boldsymbol{n} = \chi^{arr} \left(0 - C\right) \qquad \text{on } \Gamma^{arr} \tag{9}$$

$$[\boldsymbol{K} \cdot \boldsymbol{\nabla} \alpha] \cdot \boldsymbol{n} = \chi^{arr} \left(0 - \alpha\right) \qquad \text{on } \Gamma^{arr} \tag{10}$$

The present set of equations is a slight generalization of the problem posed in [1] in the sense that a linear decay term was added (so we can retrieve the original set by letting $\gamma = 0$) and that the Dirichlet conditions or Neumann conditions are replaced by a Robin condition. We recover the Dirichlet condition if $\chi^{arr} \to \infty$. For a zero gradient condition $\chi^{arr} = 0$. As in [1], zero initial conditions on C and α are applied.

We take the usual definitions of age a, concentration C and age concentration α as presented in the CART framework (`www.climate.be/cart`) such that $\alpha = aC$.

The conjecture to be proven is that the age for Dirichlet conditions at the arrival boundary Γ^{arr} is lower than the one for zero gradient conditions, a conjecture which is generalized here to

$$\frac{\partial a}{\partial \chi^{arr}} \leq 0 \tag{11}$$

1 Green-function approach

$\mathcal{G}(t, \mathbf{x}; t', \mathbf{x}')$ is Green function defined by the solution of the following problem:

$$\mathcal{L}\left[\mathcal{G}(t, \mathbf{x}; t', \mathbf{x}')\right] = \delta(t - t')\delta(\mathbf{x} - \mathbf{x}') \tag{12}$$

$$[\boldsymbol{K} \cdot \boldsymbol{\nabla} \mathcal{G}(t, \mathbf{x}; t', \mathbf{x}')] \cdot \boldsymbol{n} = 0 \quad \mathbf{x} \text{ on } \Gamma^i \tag{13}$$

$$[\boldsymbol{K}\cdot\boldsymbol{\nabla}\,\mathcal{G}(t,\mathbf{x};t',\mathbf{x}')]\cdot\boldsymbol{n} = -\chi^{arr}\mathcal{G}(t,\mathbf{x};t',\mathbf{x}') \quad \mathbf{x} \text{ on } \Gamma^{arr} \tag{14}$$

$$[\boldsymbol{K}\cdot\boldsymbol{\nabla}\,\mathcal{G}(t,\mathbf{x};t',\mathbf{x}')]\cdot\boldsymbol{n} = -\chi^{dep}\mathcal{G}(t,\mathbf{x};t',\mathbf{x}') \quad \mathbf{x} \text{ on } \Gamma^{dep} \tag{15}$$

$$\mathcal{G}(0,\mathbf{x};t',\mathbf{x}') = 0 \tag{16}$$

where the Dirac function is noted $\delta()$. It can be shown that (for a positive-defined diffusivity tensor as we assumed in the problem definition) $\mathcal{G}(t,\mathbf{x};t',\mathbf{x}') \geq 0$ and that for zero initial concentration C we have

$$C(t,\mathbf{x}) \;=\; \int_0^t \int_{\Gamma^{dep}} \mathcal{G}(t,\mathbf{x};t',\mathbf{x}'_{\mathbf{s}})\chi^{dep}C^{dep}(t',\mathbf{x}'_{\mathbf{s}})\mathrm{d}\mathbf{x}'_{\mathbf{s}}\mathrm{d}t' \tag{17}$$

Note that the Green's function defined also allows for contributions from inside the domain and therefore allows to calculate age concentration α. The latter indeed satisfies the same equation as C, except that it has no non-homogeneous part in the boundary condition but has a "source" term C in the domain Ω. Hence the solution is simply the superposition of all contributions of these "sources" propagated by the Green's function and reads

$$\alpha(t,\mathbf{x}) \;=\; \int_0^t \int_{\Omega} \mathcal{G}(t,\mathbf{x};t',\mathbf{x}')\,C(t',\mathbf{x}')\,\mathrm{d}\mathbf{x}'\mathrm{d}t' \tag{18}$$

$$\;=\; \int_0^t \int_0^{t'} \int_{\Omega} \int_{\Gamma^{dep}} \mathcal{G}(t,\mathbf{x};t',\mathbf{x}')\,\mathcal{G}(t',\mathbf{x}';t'',\mathbf{x}''_{\mathbf{s}})\chi^{dep}C^{dep}(t'',\mathbf{x}''_{\mathbf{s}})\,\mathrm{d}\mathbf{x}''_{\mathbf{s}}\mathrm{d}\mathbf{x}'\mathrm{d}t''\mathrm{d}t' \tag{19}$$

Later in the analysis we would like to check how the age-concentration changes when the piston velocity χ^{arr} is changed. Therefore, following the idea of [2], it is interesting to calculate the changes in the Green's function $\frac{\partial\mathcal{G}}{\partial\chi^{arr}}$ noted $\mathcal{G}_\chi(t,\mathbf{x}\,;t',\mathbf{x}')$. Deriving all equations for the Green's function with respect to this piston velocity, we see that it must obey

$$\mathcal{L}\,[\mathcal{G}_\chi(t,\mathbf{x}\,;t',\mathbf{x}')] = \delta(t-t')\delta(\mathbf{x}-\mathbf{x}') \tag{20}$$

$$[\boldsymbol{K}\cdot\boldsymbol{\nabla}\,\mathcal{G}_\chi(t,\mathbf{x}\,;t',\mathbf{x}')]\cdot\boldsymbol{n} = 0 \quad \mathbf{x} \text{ on } \Gamma^{i} \tag{21}$$

$$[\boldsymbol{K}\cdot\boldsymbol{\nabla}\,\mathcal{G}_\chi(t,\mathbf{x}\,;t',\mathbf{x}')]\cdot\boldsymbol{n} = -\chi^{dep}\mathcal{G}_\chi(t,\mathbf{x}\,;t',\mathbf{x}') \quad \mathbf{x} \text{ on } \Gamma^{dep} \tag{22}$$

$$[\boldsymbol{K}\cdot\boldsymbol{\nabla}\,\mathcal{G}_\chi(t,\mathbf{x}\,;t',\mathbf{x}')]\cdot\boldsymbol{n} = -\chi^{arr}\mathcal{G}_\chi(t,\mathbf{x}\,;t',\mathbf{x}') - \mathcal{G}(t,\mathbf{x};t',\mathbf{x}') \quad \mathbf{x} \text{ on } \Gamma^{arr} \tag{23}$$

$$\mathcal{G}_\chi(t,\mathrm{x}\,;t',\mathbf{x}') = 0 \tag{24}$$

i.e. again the same set of equations as the problem on C, except for a "source" term $-\mathcal{G}(t,\mathbf{x};t',\mathbf{x}')$ on the boundary Γ^{arr} instead of the departure boundary. Hence the solution has the same structure as (17) but where $\chi^{dep}C^{dep}$ is replaced by $-\mathcal{G}(t,\mathbf{x};t',\mathbf{x}')$ and integration is done on the arrival boundary

$$\frac{\partial \mathcal{G}}{\partial \chi} \;=\; \mathcal{G}_\chi(t,\mathrm{x}\,;t',\mathbf{x}') \;=\; -\int_0^t \int_{\Gamma^{arr}} \mathcal{G}(t,\mathbf{x};t'',\mathbf{x}_\mathbf{s}'')\,\mathcal{G}(t'',\mathbf{x}_\mathbf{s}'';t',\mathbf{x}')\,\mathrm{d}\mathbf{x}_\mathbf{s}''\mathrm{d}t'' \quad \leq 0 \tag{25}$$

We are now ready to proceed to the demonstration. First we rewrite the age a as

$$a = \frac{\alpha}{C} = \frac{\alpha/\chi^{dep}}{C/\chi^{dep}} = \frac{\tilde{\alpha}}{\tilde{C}} \tag{26}$$

with

$$\tilde{C}(t,\mathbf{x}) \;=\; \int_0^t \int_{\Gamma^{dep}} \mathcal{G}(t,\mathbf{x};t',\mathbf{x}_\mathbf{s}')\,C^{dep}(t',\mathbf{x}_\mathbf{s}')\,\mathrm{d}\mathbf{x}_\mathbf{s}'\mathrm{d}t' \tag{27}$$

$$\tilde{\alpha}(t,\mathbf{x}) \;=\; \int_0^t \int_\Omega \mathcal{G}(t,\mathbf{x};t',\mathbf{x}')\,\tilde{C}(t',\mathbf{x}')\,\mathrm{d}\mathbf{x}'\mathrm{d}t' \tag{28}$$

$$\;=\; \int_0^t \int_0^{t'} \int_\Omega \int_{\Gamma^{dep}} \mathcal{G}(t,\mathbf{x};t',\mathbf{x}')\,\mathcal{G}(t',\mathbf{x}';t'',\mathbf{x}_\mathbf{s}'')\,C^{dep}(t'',\mathbf{x}_\mathbf{s}'')\,\mathrm{d}\mathbf{x}_\mathbf{s}''\mathrm{d}\mathbf{x}'\mathrm{d}t''\mathrm{d}t' \tag{29}$$

The conjecture $\frac{\partial a}{\partial \chi^{arr}} \leq 0$ is true if

$$\tilde{C}\frac{\partial \tilde{\alpha}}{\partial \chi^{arr}} - \tilde{\alpha}\frac{\partial \tilde{C}}{\partial \chi^{arr}} \leq 0 \tag{30}$$

or

$$\int_0^t \int_\Omega \tilde{C}(t,\mathbf{x})\,\mathcal{G}_\chi(t,\mathbf{x};t',\mathbf{x}')\,\tilde{C}(t',\mathbf{x}')\,\mathrm{d}\mathbf{x}'\mathrm{d}t' \;+\; \int_0^t \int_\Omega \tilde{C}(t,\mathbf{x})\,\mathcal{G}(t,\mathbf{x};t',\mathbf{x}')\,\frac{\partial \tilde{C}(t',\mathbf{x}')}{\partial \chi}\,\mathrm{d}\mathbf{x}'\mathrm{d}t'$$
$$\leq\; \int_0^t \int_\Omega \frac{\partial \tilde{C}(t,\mathbf{x})}{\partial \chi}\mathcal{G}(t,\mathbf{x};t',\mathbf{x}')\,\tilde{C}(t',\mathbf{x}')\,\mathrm{d}\mathbf{x}'\mathrm{d}t' \tag{31}$$

All terms are negative, but there is a strong symmetry in terms 2 and term 3 which will be shown to cancel each other. Exploiting the expression (17) and (25) we can rewrite the inequality to prove as

$$\int_0^t \int_{\Gamma^{dep}} \int_0^t \int_0^{t'} \int_\Omega \int_{\Gamma^{dep}} (\mathcal{H}_1 + \mathcal{H}_2 - \mathcal{H}_3)\; C^{dep}(t''',\mathbf{x}_\mathbf{s}''')C^{dep}(t'',\mathbf{x}_\mathbf{s}'')\mathrm{d}\mathbf{x}_\mathbf{s}''\mathrm{d}\mathbf{x}'\mathrm{d}t''\mathrm{d}t'\mathrm{d}\mathbf{x}_\mathbf{s}'''\mathrm{d}t''' \leq 0 \tag{32}$$

$$\mathcal{H}_1 \;=\; \mathcal{G}(t,\mathbf{x};t''',\mathbf{x}_\mathbf{s}''')\,\mathcal{G}_\chi(t,\mathbf{x};t',\mathbf{x}')\,\mathcal{G}(t',\mathbf{x}';t'',\mathbf{x}_\mathbf{s}'') \tag{33}$$
$$\mathcal{H}_2 \;=\; \mathcal{G}(t,\mathbf{x};t''',\mathbf{x}_\mathbf{s}''')\,\mathcal{G}(t,\mathbf{x};t',\mathbf{x}')\,\mathcal{G}_\chi(t',\mathbf{x}';t'',\mathbf{x}_\mathbf{s}'') \tag{34}$$
$$\mathcal{H}_3 \;=\; \mathcal{G}_\chi(t,\mathbf{x};t''',\mathbf{x}_\mathbf{s}''')\,\mathcal{G}(t,\mathbf{x};t',\mathbf{x}')\,\mathcal{G}(t',\mathbf{x}';t'',\mathbf{x}_\mathbf{s}'') \tag{35}$$

and finally

$$\int_0^t \int_{\Gamma^{dep}} \int_0^t \int_0^{t'} \int_\Omega \int_{\Gamma^{dep}} \int_0^t \int_{\Gamma^{arr}} (\mathcal{F}_1 + \mathcal{F}_2 - \mathcal{F}_3)\ C^{dep}(t''', \mathbf{x}_\mathbf{s}''') C^{dep}(t'', \mathbf{x}_\mathbf{s}'') \mathrm{d}\mathbf{x}_\mathbf{s}'''' \mathrm{d}t'''' \mathrm{d}\mathbf{x}_\mathbf{s}'' \mathrm{d}\mathbf{x}' \mathrm{d}t'' \mathrm{d}t' \mathrm{d}\mathbf{x}_\mathbf{s}''' \mathrm{d}t''' \geq 0 \tag{36}$$

$$\mathcal{F}_1 = \mathcal{G}(t, \mathbf{x}; t''', \mathbf{x}_\mathbf{s}''')\, \mathcal{G}(t, \mathbf{x}; t'''', \mathbf{x}_\mathbf{s}'''')\, \mathcal{G}(t'''', \mathbf{x}_\mathbf{s}''''; t', \mathbf{x}')\, \mathcal{G}(t', \mathbf{x}'; t'', \mathbf{x}_\mathbf{s}'') \tag{37}$$

$$\mathcal{F}_2 = \mathcal{G}(t, \mathbf{x}; t''', \mathbf{x}_\mathbf{s}''')\, \mathcal{G}(t, \mathbf{x}; t', \mathbf{x}')\, \mathcal{G}(t', \mathbf{x}'; t'''', \mathbf{x}_\mathbf{s}'''')\, \mathcal{G}(t'''', \mathbf{x}_\mathbf{s}''''; t'', \mathbf{x}_\mathbf{s}'') \tag{38}$$

$$\mathcal{F}_3 = \mathcal{G}(t, \mathbf{x}; t'''', \mathbf{x}_\mathbf{s}'''')\, \mathcal{G}(t'''', \mathbf{x}_\mathbf{s}''''; t''', \mathbf{x}_\mathbf{s}''')\, \mathcal{G}(t, \mathbf{x}; t', \mathbf{x}')\, \mathcal{G}(t', \mathbf{x}'; t'', \mathbf{x}_\mathbf{s}'') \tag{39}$$

For the integration of $\mathcal{F}_2$ we can exploit

$$\int_\Omega \mathcal{G}(t, \mathbf{x}; t', \mathbf{x}')\, \mathcal{G}(t', \mathbf{x}'; t'''', \mathbf{x}_\mathbf{s}'''') \mathrm{d}\mathbf{x}' = \mathcal{G}(t, \mathbf{x}; t'''', \mathbf{x}_\mathbf{s}'''') \tag{40}$$

meaning that the distribution at time t and position $\mathbf{x}$ due to a Dirac at moment t'''' $\mathbf{x}_\mathbf{s}''''$ can be obtained by looking what looks the solution at a later moment t' in any place of the domain $\mathbf{x}'$, i.e. $\mathcal{G}(t', \mathbf{x}'; t'''', \mathbf{x}_\mathbf{s}'''')$ and calculate how this new "initial condition" would propagate to time t. Similarly for integration of $\mathcal{F}_3$ we can use

$$\int_\Omega \mathcal{G}(t, \mathbf{x}; t', \mathbf{x}')\, \mathcal{G}(t', \mathbf{x}'; t'', \mathbf{x}_\mathbf{s}'') \mathrm{d}\mathbf{x}' = \mathcal{G}(t, \mathbf{x}; t'', \mathbf{x}_\mathbf{s}'') \tag{41}$$

But then the remaining integrals of $\mathcal{F}_2$ and $\mathcal{F}_3$ are the same as the running parameters $t'', \mathbf{x}_\mathbf{s}''$ and $t''', \mathbf{x}_\mathbf{s}'''$ (over the same boundary Γ^{dep}) can be interchanged in one of them, yielding then the same expression for the integrals of $\mathcal{F}_2$ and $\mathcal{F}_3$.

Q.E.D.

References

[1] E. Deleersnijder, *A conjecture about age inequalities*, http://hdl.handle.net/2078.1/227647, (2020), p. 7.

[2] E. Delhez, *Influence of the piston velocity on the age*, personal communication, (2014).