Is there a flatness problem in classical cosmology?

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Abstract. I discuss various definitions of the flatness problem and previous claims that it does not exist. I also present a new quantitative argument which shows that it does not exist in cosmological models which collapse in the future.

1. Introduction

Questioning the existence of the flatness problem might seem to some like questioning the existence of the expansion of the universe. The flatness problem (e.g. [2]) and the fact that inflation can solve it (e.g. [4]) have become part of standard cosmology, at least for many definitions of 'standard'. How can something so fundamental not exist? My own view is that the emphasis has been so much on the *solution* of the flatness problem through inflation that the flatness problem itself has been rather neglected and its existence just assumed without being investigated in detail.

2. Basic cosmology

I assume that, at the level of detail necessary, the universe can be described by the Friedmann–Lemaître equation

$$\dot{R}^2 = \frac{8\pi G\rho R^2}{3} + \frac{\Lambda R^2}{3} - kc^2 \tag{1}$$

where the symbols have their usual meaning (e.g. [6]). It can be useful to express Eq. (1) with the values of the dimensionless parameters as observed now, denoted by the suffix 0. This leads to

$$\dot{R}^2 = \dot{R}_0^2 \left(\frac{\Omega_0 R_0}{R} + \frac{\lambda_0 R^2}{R_0^2} - K_0 \right) . \tag{2}$$

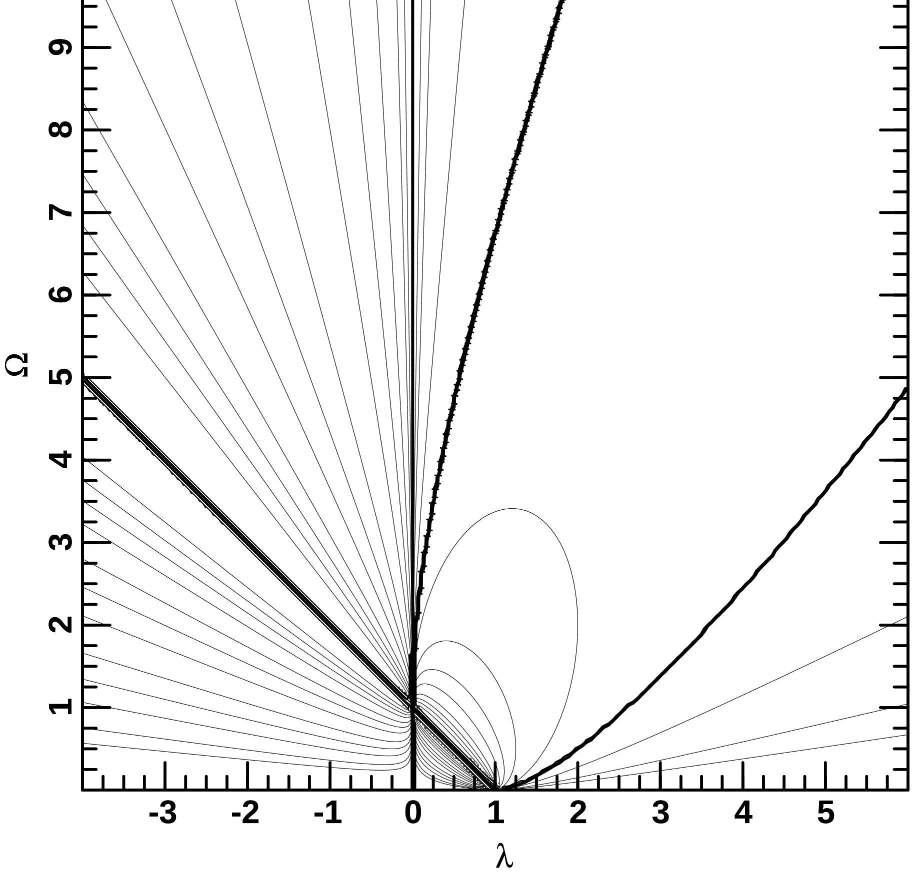

Figure 1. Evolutionary trajectories in the λ-Ω plane.

Note that, at any time,

$$R = \frac{c}{H} \frac{\mathrm{sign}(K)}{\sqrt{|K|}} \tag{3}$$

In general, H, λ and Ω all change with time. (See [9] for an excellent discussion of the evolution of λ and Ω.) The change in λ with time is due entirely to the change in H with time, since Λ is constant; the variation in Ω is due both to variation in H and to the decrease in density as the universe expands. For the present discussion, the basic information needed can be seen in Figure 1, referring at the moment only to the thick lines and curves. The vertical line it and $k = +1$ above it. The curve near the vertical line (corresponding to the A1 curve in [9]) separates models which will collapse (to the left) from those which will expand forever (to the right). Models on the curve start arbitrarily close to the Einstein–de Sitter model (like all non-empty big-bang models) and asymptotically approach the static Einstein model which has $\lambda = \Omega = \infty$ (since $H = 0$; Λ and ρ have finite values). The other curve (corresponding to the A2 curve in [9]) separates big-bang models (to the left) from non–big-bang models (to the right); the latter contract from an infinite to a finite size then expand forever. Models on the curve start at the static Einstein model and asymptotically approach the de Sitter model (the latter feature is shared with all models which expand forever and have $\lambda > 0$). The thin curves show some sample trajectories in the λ-Ω parameter space. (Note that all the thick lines, curves and points of intersection in Figure 1 are also trajectories.) Also, note that the trajectories do not cross; this means that the history of a cosmological model (i.e. the way λ and Ω change with time) is completely determined by the values at any point on it (in practice, by measuring the values at the present time, λ_0 and Ω_0).

3. A very brief history of the flatness problem

The flatness problem appears in two forms. One states that if $\Omega \approx 1$ today, then in the early universe it was arbitrarily close to 1; the assumption is that some 'mechanism' is needed to explain this 'fine-tuning' (e.g. [4]). (It is usually not stated but almost always assumed that no fine-tuning would be necessary if Ω were not ≈ 1 today.) The other states that if Ω changes with time, then we should be surprised that Ω is (still) ≈ 1 today [7].[1] Coles & Ellis [1] discuss three 'solutions' to the flatness problem – $\Omega \equiv 1$ (and $\lambda \equiv 0$), $k = 0$, anthropically selected special time – which, however, are ultimately unsatisfactory. Are there any satisfactory ones?

The flatness problem is often presented as a fine-tuning problem (e.g. [4]): if Ω is near 1 to day, then at some time t_{fine} in the past it must have been 1 to a very high accuracy. I refer to this sense of the flatness problem as the 'qualitative flatness problem'. This argument is completely bogus, as has been pointed out by many authors [1, 7]: *all* non-empty models begin their evolution at the Einstein–de Sitter model, so of course the further back in

[1] Historically, the flatness problem was first discussed during a time when λ was thought to be zero. If λ is not constrained to be zero, then the flatness problem should be re-phrased as the Einstein–de Sitter problem, i.e. the question is why the universe is (in some sense) close to the Einstein–de Sitter model (which is an unstable fixed point and a repulsor) today when $|\lambda|$ and Ω can take on values between 0 and ∞. However, for brevity I will continue to use the term 'flatness problem' even for the more general case and sometimes mention only the change in Ω with time.

classical cosmology, there is nothing special about a time t_{fine} chosen so that Ω is very close to 1 at that point.

Evrard & Coles [3] (see also Coles & Ellis [1]) also point out that the assumption implicit in the qualitative flatness problem, namely that some wide range of Ω values are a priori equally likely at some early time, constitutes a prior which is incompatible with the assumption of minimal information. This can be regarded as a quantitative solution to the qualitative flatness problem (or, perhaps, an argument against its existence).

The qualitative flatness problem thus does not exist; it is merely a consequence of the way in which a universe, described by the Friedmann–Lemaître equation, evolves and how dimensionless observable quantities such as Ω are defined. Nevertheless, even if it is not a puzzle why $\Omega = 1$ at early times, one can still ask whether we should be surprised that $\Omega \approx 1$ today. The rest of this article is concerned mainly with the second form: should we be surprised that $\Omega \approx 1$ today? This 'quantitative flatness problem' is more subtle, but also has solutions within the context of classical cosmology.

4. Cosmological models which collapse in the future

All cosmological models (assumed to be expanding now) with $\lambda < 0$ will collapse in the future: $\ddot{R}$ is negative for all values of R and for large R is proportional to R. Models with $\lambda = 0$ will collapse for $\Omega > 1$. In addition, models with $\lambda > 0$ will collapse provided that $\Omega > 1$ (which in this case implies $K > 0$, i.e. $k = +1$), $q > 0$ and $\alpha < 1$, where

$$\alpha = \operatorname{sign}(K)\frac{27\Omega^2\lambda}{4K^3} \tag{4}$$

[9, 7]. (The A1 and A2 curves mentioned above have $\alpha = 1$.) In Figure 1, these are in the area between $\lambda = 0$ and the A1 curve. Empty big-bang models start arbitrarily close to the Milne model with (λ,Ω) values of $(0,0)$; non-empty big-bang models start arbitrarily close to the Einstein–de Sitter model with (λ,Ω) values of $(0,1)$. The evolution of λ and Ω can be viewed as trajectories in the parameter space: λ and Ω evolve from the starting point to infinity and return along the same trajectory. (For the definitive discussion, see [9]; a very useful visualization can be found at [8].) The interesting question with regard to the flatness problem is the amount of time spent in various parts of parameter space. To quantify this, I have calculated the quotient of the age of the universe now and at the time of maximum expansion as a function of λ and Ω. The age of the universe is given by

$$t = \int_0^{R(t)} \frac{\mathrm{d}R}{\sqrt{\dot{R}_0^2\left(\frac{\Omega_0 R_0}{R} + \frac{\lambda_0 R^2}{R_0^2} - K_0\right)}} \tag{5}$$

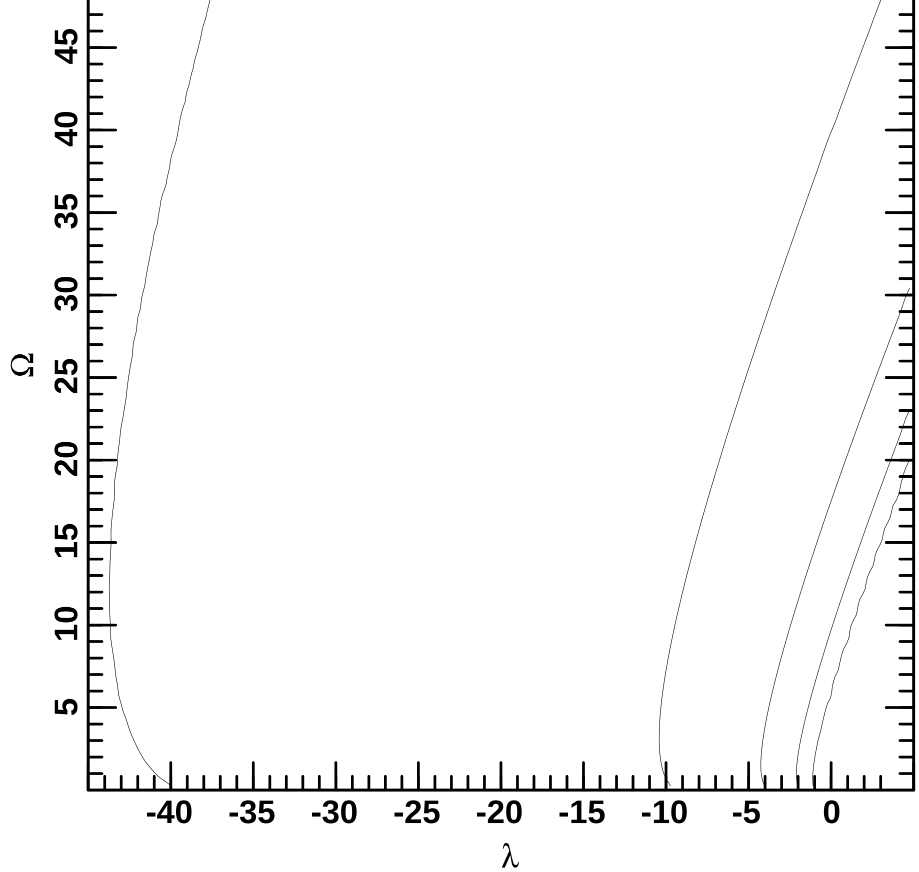

Figure 2. The age of the universe as a fraction of the time between the big bang and maximum expansion. Contours, from right to left, are at 0.5, 0.6, 0.7, 0.8 and 0.9.

which follows from Eq. (2). For the current age, the upper limit is given by Eq. (3); at the time of maximum expansion it is found by calculating the (smallest) zero of $\dot{R}^2$ (since $\dot{R}^2$ cannot be negative). This is shown in Figure (2). It is clear that large values of λ and Ω occur only during a relatively short time in the history of the universe, near the time of maximum expansion (at the precise time of maximum expansion, λ and Ω are infinite since $H = 0$). Note that this argument is completely independent of H_0.

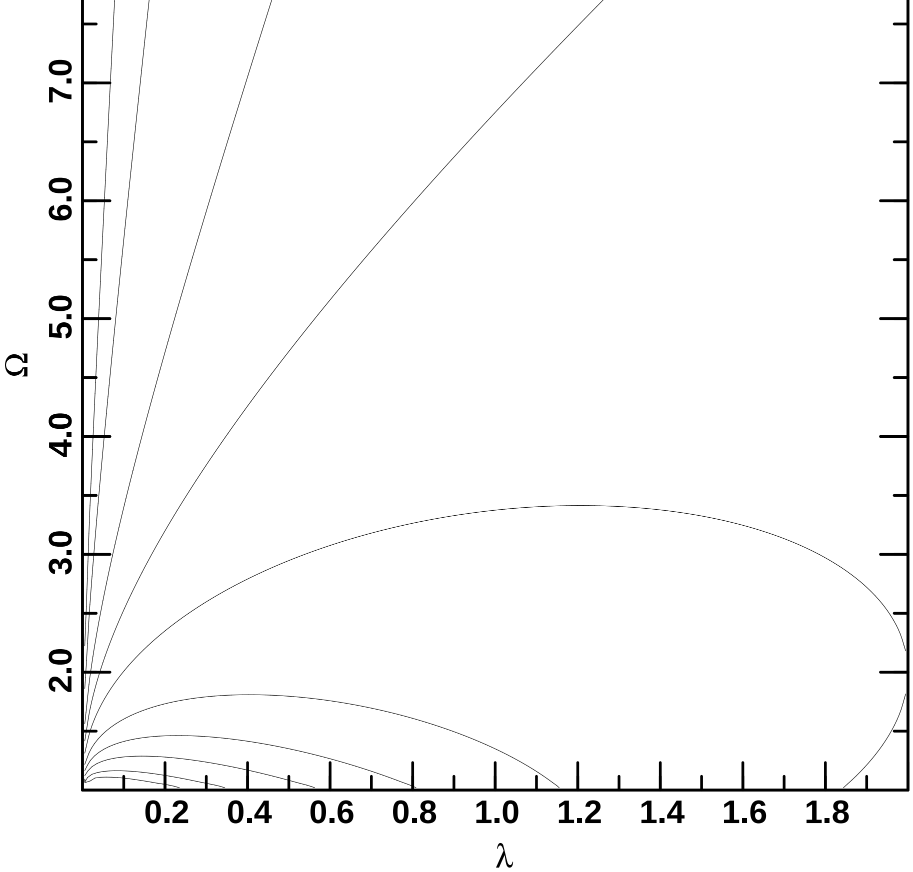

Figure 3. The constant of motion α (see Eq. (4)). From upper left to lower left, contours are at 0.1, 0.2, 0.5, 1, 2, 5, 10, 20, 50 and 100.

5. Cosmological models which expand forever

Lake [7] has presented a solution which solves the flatness problem as well for models with $k = +1$ which will expand forever. (For non-collapsing models, large values of λ and Ω are possible only for $k = +1$.) Trajectories in the λ-Ω plane have a constant of motion given by Eq. (4).It seems natural to distinguish cosmological models on the basis of their value of α. Large values of λ and Ω are possible only for $\alpha \lesssim 1$. This is shown in Figure 3. (Note that, for clarity, only $\Omega > 1$ is shown!) It is obvious that $\alpha \leq 1$ is a necessary condition for of λ is just 2 (for $\Omega = 2$) and the maximum value of Ω is ≈ 3.5 (for $\lambda = 1.25$).

In this case, the fine-tuning argument is reversed; only in the case of fine-tuning do λ and Ω become arbitrarily large. This demonstrates quantitatively that there is no quantitative flatness problem regarding arbitrarily large values of λ or Ω for models which expand forever. This argument is also independent of the value of H_0. However, all non-empty models which expand forever asymptotically approach the de Sitter model at $(\lambda, \Omega) = (1, 0)$. Thus, one final aspect of the quantitative flatness problem remains: Ω can become arbitrarily small. This is investigated in the next section.

6. Cosmological models which expand forever

I have now covered the entire λ-Ω parameter space except for big-bang models with (a) $q < 0$ (which implies $\lambda > 0$) and (b) Ω less than ≈ 2 (all three values of k are possible) and shown that in all cases there is no flatness problem. What about this remaining portion of parameter space? Models here all have $K \approx 0$ and approach the de Sitter model asymptotically. This means that there is no flatness problem in the restricted sense, as pointed out by Lake [7]. However, Ω becomes arbitrarily small (and λ arbitrarily close to 1). Thus, there is still a problem in that we do not observe such values, even though they exist for almost the entire (infinite) lifetime of the universe. This is essentially the question 'if the universe lasts forever, then why are we near the beginning?' Note that this question could be asked at any time. One could leave it at that and say that since any finite age is arbitrarily close to the beginning, there is nothing special about our time and thus no flatness problem in the time-scale sense (i.e. the quantitative flatness problem, why is Ω not arbitrarily small today). This is discussed in more detail in [5].

7. Summary

The qualitative flatness problem, i.e. the puzzle why the universe was arbitrarily close to the Einstein–de Sitter model (or, for an empty universe, the Milne or de Sitter model) at early times, does not exist. It is merely a consequence of the way λ and Ω are defined. Neither does the quantitative flatness problem exist: although the cosmological parameters in general evolve with time, it is not puzzling that we don't observe extreme values for them today. In the case of models which will collapse in the future this is because large (absolute) values of λ and Ω occur only during a relatively short time in the lifetime of such a universe, namely near the time of maximum expansion. λ and Ω can become large only when H becomes small, and this happens only during the time when the universe is at or near its maximum size. (Arbitrarily small (absolute) values, if they occur at all, also occur for only a relatively short time). For models which will expand forever, large values are possible tuning argument is reversed; only in the case of fine-tuning do λ and Ω become arbitrarily large. Since all models which will expand forever asymptotically approach $\Omega = 0$, arbitrarily small values of Ω can occur. Those with $\lambda = 0$ (and hence $k = -1$) approach the Milne model with $\Omega = 0$; models with $\lambda > 0$, whatever the value of k, approach the de Sitter model with $\lambda = 1$ (the Milne and de Sitter models themselves are of course stationary points). (If $\lambda = 0$ at any time then $\lambda = 0$ at all times. Otherwise, arbitrarily small values of λ, if they occur at all, occur only for a relatively short time.) However, if H_0 has a value similar to or smaller than the observed value, small values of Ω will occur only in the far future when anthropic arguments probably make the observation of such a low value of Ω unlikely. While (for $\lambda > 0$) a higher value of H_0 would allow a low value of Ω even for an age near the observed age, such a universe would have spent only a very short time during which Ω was not very small, so structure formation would have been strongly suppressed.

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