CORRESPONDENCE

To the Editors of 'The Observatory'
A Formula for Confusion

That Virginia Trimble found a mistake in a book[1] which she reviewed[2] for this *Magazine* is not surprising, and in itself not disturbing: mistakes are made, and it is difficult to avoid them completely. In this particular case, however, four things make it worth writing about: it is a common mistake, made by a well-known astronomer (though, fortunately, not by someone who works in cosmology), in a book from a major scientific publisher, and has been pointed out in the literature decades ago. To recap: the book contains a formula for cosmological "redshifts larger than one", namely the relativistic Doppler formula*. It should be obvious that that formula cannot be correct, because it gives the velocity as a function of just the redshift; there are no cosmological parameters! Even the Hubble constant is absent. This implies that, even at arbitrarily high redshift, the velocity is independent of the cosmological model, and does not depend on even the Hubble constant (despite the fact that the latter is the proportionality constant in the velocity-distance relation!) while at the same time it is common knowledge that, except in the limit of small redshifts, the relation between observed and 'absolute' or 'physical' quantities depends on the cosmological model. Indeed, the inverse process — fitting for the cosmological parameters by comparing observations to theoretical predictions for various cosmological models — is the basis of classical observational cosmology.

We shouldn't blame the author too much, though; no-one can be an expert in all areas covered by a wide-ranging book such as the one reviewed. Such authors usually can't rely on the primary literature, but must turn to secondary (or farther-removed) sources. The bigger problem is thus that this mistake is common in the secondary — and not unheard of even in the primary — literature, and as the current example indicates, is propagated via the tried and tested practice of copying mistakes coupled with insufficient diligence on the part of those involved.

This entire issue should have disappeared after Harrison's article[3], which specifically addresses not just this confusion, but also the fact that even by the time it was written there was a history of misconceptions. He also devotes much space to this in his classic textbook[4]; both the book and the article are essential reading for anyone even remotely interested in cosmology. In particular, this is not a case of genuine confusion, or an undecided issue: Harrison got it right, and many others got it wrong. The facts that he explicitly pointed out that others got it wrong and that no-one has rebutted him should make it clear that he is right. Probably the only way to eradicate this misconception is to point it out — as Trimble did in her review — whenever and wherever it occurs, regardless of who makes the mistake. (Even a third of my previous correspondence in this *Magazine*[5,6] has touched on this misconception.)

In addition to the luminosity and angular-diameter distances mentioned by Trimble (certainly the two most important in observational cosmology), there is also the proper distance or metric distance, which is the basic distance in General Relativity and is the easiest to visualize: it is what one would measure with a rigid ruler (done during so short a time that the expansion of the Universe can be neglected). It is this distance and its derivative with respect to cosmic time which appear in the velocity-distance relation $v = HD$ (v is the velocity, H the Hubble constant, and D the proper distance), which (in a

Friedmann–Lemaître cosmological model (a homogeneous and isotropic model based on General Relativity)) is always exactly linear (otherwise a homogeneous and isotropic universe would not remain so as it expands) and valid for arbitrarily large distances and velocities. However, neither of these is 'directly' observable, even by the roundabout definition of 'directly' in observational cosmology. (One can determine H from observations in the limit of small redshift z by using z as a proxy for velocity, and luminosity distance or angular-size distance as an approximation (exactly valid in the limit of zero redshift) for proper distance. Unlike the velocity-distance relation, however, in general the redshift-distance (or redshift-apparent-magnitude) relation is valid only in the limit of small redshifts. Thus, the non-relativistic Doppler formula $v = cz$ is valid at low redshifts (whether or not one thinks of the cosmological redshift as some sort of Doppler shift[7]) essentially because many things are linear to first order; this does not imply that the relativistic Doppler formula is correct for large cosmological redshifts.)

Why does this misconception still persist? One reason might be that in practice cosmological velocities play no role in observational cosmology. Given a redshift and cosmological model, the velocity and distance in the velocity–distance relation can be calculated, but this is of little practical use. (In particular, in general the Hubble sphere — defined as c/H, the distance at which the recession velocity is equal to the speed of light — does not correspond to any sort of horizon.[8]) At the same time, despite its simple and useful true meaning — that $1 + z$ is the ratio of the scale factor of the Universe now to that at the time the radiation received now was emitted — popular-science writers like to introduce it *via* the Doppler effect, which is not really useful because if one has to explain the acoustic Doppler effect to get the analogy to work, one might as well cut to the chase and say that the wavelength is stretched via the expansion of the Universe. (Whether this completely correct in some sense is another question, but this point of view doesn't lead to misconceptions or wrong numerical results.)

Observational cosmology is based on the way distances as a function of redshift depend on the cosmological parameters. Not only are there several types of distances (not all of which have practical uses in cosmology), but the general solution involves elliptic integrals in the idealized Friedmann–Lemaître case (and is even more complicated if one takes into account the fact that small-scale inhomogeneities can appreciably affect observable distances, even if they do not appreciably affect the large-scale kinematics or dynamics of the Universe).[9] Although the relationships between the various distances are relatively simple (though their values can be very different at high redshift), the topic is perhaps too advanced for many popular-science works, so it is glossed over. Also, since light years are deemed to be easier to understand than parsecs, this unit of distance is often used. While a simple factor converts one to the other, 'light year' evokes the idea of distance measured *via* light-travel time, which is the one distance which is *not* simply related to the others. Compared to this, the velocity-distance relation might seem easier to tackle, so readers are presented with something which is neither useful in practical observational cosmology (because neither the distance nor the velocity involved is observable) nor understandable without having to first understand why Special Relativity is not applicable in this case.

In summary, popular-science writers have ignored Einstein's advice to make things as simple as possible — but not simpler. This is probably due to the fact that the details of distance calculation are too complicated for such books, leading to an over-emphasis on the velocity-distance relation and the Doppler effect (which are not actually needed at all in classical cosmology), which are then presented wrongly, probably because of confusion between the redshift–distance and velocity–distance laws[3]). Non-cosmologists (and, sadly, even some cosmologists) then uncritically repeat this mistake. Some blame must also go to publishers, since obviously not enough fact-checking is done (there are numerous examples of this), perhaps because of the apparently wrong assumption that the author has had this done before submitting the manuscript. Since even well-known authors have made this mistake, other authors and/or publishers might be sceptical of claims that so much of the popular literature is wrong in this respect. Lack of basic fact-checking is also supported by Trimble's examples of misspelled names and 'fake facts' presented as truth. (To some extent, buggy and/or wrongly used software, perhaps in conjunction with an editor whose first language is not English, might be responsible: maybe the manuscript was correct, but errors were introduced later. I know of no other way to explain how 'one' (the indefinite pronoun, and certainly written that way by the author) became '1' in the finished product in another book I reviewed in these pages.[10])

As I side-note, to some extent I can understand the frustration of non-experts trying to make sense of the primary literature on cosmological distances. Different symbols are used by different authors for the same distance and, more confusingly, the same symbol is used for different distances. (In extreme cases, a distance — unambiguously identified by a formula — is given a wrong name.) For example, just last week I was re-reading (an English translation of) a classic paper by Zel'dovich[11], where he discusses the calculation of distances in various cosmological models, but instead of the redshift z as the independent variable, he uses Δ, which one can show (this is not even mentioned in the text) to be equal to $1 - 1/(1 + z)$. This has the nice property that it ranges from 0 to 1 instead of from 1 to ∞. In the Einstein-de Sitter universe, the angular-size distance (called f by Zeldovich — more confusion), in units of the Hubble length c/H, is given by $2(1 - \Delta)(1 - \sqrt{1 - \Delta})$ while in (the relativistic equivalent of) the Milne model it is $\Delta(1 - \Delta/2)$. The equivalents are $2((1 + z)^{-1} - (1 + z)^{-2/2})$ and $1/2(1 - (1 + z)^{-2})$, respectively. As help for those interested in these matters, we collected various formulae into appendices in a paper[9] concerned with distance calculation, using modern and consistent notation in the formulae and in the definitions of all distances. (However, at the time, *A&A* insisted on some material being published only electronically, and separately from the main paper. Both are available, together as intended, at arXiv.[12])

Yours faithfully,
PHILIP HELBIG

Thomas-Mann-Straße 9
D-63477 Maintal
Germany

helbig@astro.multivax.de

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*Presumably this is supposed to be the formula correct for all redshifts, but giving significantly different results from $v = cz$ only at redshifts larger than 1, but even this is wrong: it is never the correct formula, and significant deviations exist for $z < 1$ as well.