The flatness problem and the age of the Universe

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Abstract

Several authors have made claims, none of which has been rebutted, that the flatness problem, as formulated by Dicke and Peebles, is not really a problem but rather a misunderstanding. In particular, no fine-tuning in the early Universe is needed, neither in general nor in order to explain the fact that no strong departure from flatness is observed today. Nevertheless, the flatness problem is still widely perceived to be real. Most of the arguments against the idea of a flatness problem are based on the change with time of the density parameter Ω and normalized cosmological constant λ and, since the Hubble constant H is not considered, are independent of time scale. However, it is sometimes claimed that fine-tuning is required in order to produce a Universe which neither collapsed after a short time nor expanded so quickly that no structure formation could take place. I show that these arguments also do not imply that fine-tuning of the basic cosmological parameters is necessary and discuss some pitfalls of the typical *gedankenexperiment* involved.

Introduction

Here, I consider only ideal Friedmann models, because historically fine-tuning claims have been discussed within the context of these models, and the issues remain even in more-realistic models. Note that the flatness problem is different from another problem of classical cosmology, the isotropy or horizon problem. The latter does not exist, by definition, in an ideal Friedmann universe, while the point of the former is that *even given the fact* that the Universe is described by a Friedmann model (*why* that is the case is, of course, a different question), there is something puzzling about the *values* of the cosmological parameters which are observed.

Notation

For a universe consisting of non-relativistic matter ('dust') of density ρ and the cosmological constant Λ (with dimension time^{-2}), the change in scale factor with time is described by the Friedmann equation

$$\dot{R}^2 = \frac{8\pi G\rho R^2}{3} + \frac{\Lambda R^2}{3} - kc^2 \tag{1}$$

with the dimensionless constant k equal to $-1, 0, +1$ depending on spatial curvature (negative, vanishing, or positive, respectively); R is the scale factor (with dimension length) of the universe, G the gravitational constant, and c the speed of light. It is useful to define the following terms:

$$\begin{aligned}
H &:= \tfrac{\dot{R}}{R} \\
\lambda &:= \tfrac{\Lambda}{3H^2} \\
\Omega &:= \tfrac{\rho}{\rho_{\rm crit}} && \equiv \tfrac{8\pi G\rho}{3H^2} \\
K &:= \Omega + \lambda - 1 \\
k &:= \mathrm{sign}(K) \\
q &:= \tfrac{-\ddot{R}R}{\dot{R}^2} && \equiv \tfrac{-\ddot{R}}{RH^2} \equiv \tfrac{\Omega}{2} - \lambda
\end{aligned}$$

The Hubble constant H has the dimension time^{-1}; all other quantities defined above are dimensionless: the normalized cosmological constant λ, the density parameter Ω, the curvature parameter K, and the deceleration parameter q. $\rho_{\rm crit}$ is the so-called critical density.[1] See Helbig (2012) or Kayser et al. (1997) for more details on this notation.

From the definitions above, it follows that

$$R = \frac{c}{H}\frac{\mathrm{sign}(k)}{\sqrt{|\Omega+\lambda-1|}} = \frac{c}{H}\frac{\mathrm{sign}(k)}{\sqrt{|K|}} \quad . \tag{2}$$

Note that K is positive if the curvature is positive. Often, $\Omega_{\rm k}$ is defined as $-K$, so that the Friedman equation is $\Omega_{\rm m} + \Omega_\lambda + \Omega_{\rm k} = 1$ ($\Omega_{\rm m} \equiv \Omega$, $\Omega_\lambda \equiv \lambda$).[2] Denoting the current epoch of observation with the suffix 0 and using the definitions above, one can write the Friedmann equation as

$$\dot{R}^2 = \dot{R}_0^2\left(\frac{\Omega_0 R_0}{R} + \frac{\lambda_0 R^2}{R_0^2} - K_0\right) \quad . \tag{3}$$

From this, one can calculate the scale factor as a function of time

$$R = \int_0^t \mathrm{d}t\sqrt{\dot{R}_0^2\left(\frac{\Omega_0 R_0}{R} + \frac{\lambda_0 R^2}{R_0^2} - K_0\right)} \tag{4}$$

or the age of the universe as a function of the scale factor

$$t = \int_0^R \frac{\mathrm{d}R}{\sqrt{\dot{R}_0^2\left(\frac{\Omega_0 R_0}{R} + \frac{\lambda_0 R^2}{R_0^2} - K_0\right)}} \quad . \tag{5}$$

Using the definition of H, this can be written as

$$t = \frac{1}{H_0}\int_0^R \frac{\mathrm{d}R}{\sqrt{\left(\frac{\Omega_0 R_0^3}{R} + \lambda_0 R^2 - K_0 R_0^2\right)}} \quad . \tag{6}$$

Alternatively, dividing Eq. (3) by R^2 and factoring out R_0^{-2} on the right-hand side results in

$$\left(\frac{\dot{R}}{R}\right)^2 = \left(\frac{\dot{R}_0}{R_0}\right)^2\left(\frac{\Omega_0 R_0^3}{R^3} + \lambda_0 - \frac{K_0 R_0^2}{R^2}\right) \tag{7}$$

or, due to the definition of H, in

$$H^2 = H_0^2\left(\frac{\Omega_0 R_0^3}{R^3} + \lambda_0 - \frac{K_0 R_0^2}{R^2}\right) \quad . \tag{8}$$

which expresses the Hubble constant as a function of the scale factor. From the definitions above follows the dependence of λ on H

$$\lambda = \lambda_0\left(\frac{H_0}{H}\right)^2 \quad . \tag{9}$$

Since the density depends on the scale factor,

$$\rho = \left(\frac{\rho_0 R_0}{R}\right)^3 \quad , \tag{10}$$

Ω depends on it as well as on H

$$\Omega = \Omega_0\left(\frac{H_0}{H}\right)^2\left(\frac{R_0}{R}\right)^3 \quad , \tag{11}$$

although these are related by Eq. (8). Thus, in an expanding universe, λ and Ω can increase with time only if H decreases.[3]

Previous arguments against the flatness problem

Although it had been discussed earlier (*e.g.* Dicke, 1970), most treatments of the flatness problem can be traced back to the formulation of the problem by Dicke & Peebles (1979), who pointed out that a universe with $\Omega \neq 1$[4] is inherently unstable. Many concluded from this that $\Omega = 1$ must hold exactly, which, if one assumes that $\Lambda = 0$ (which was common in the time after Dicke & Peebles (1979) until observations made it clear in the 1990s that $\Lambda > 0$), implies that our Universe must be the Einstein–de Sitter universe exactly or that some process, such as inflation, drove it very close to the Einstein–de Sitter universe.[5]

Both the fine-tuning argument ('there must be some reason why $\Omega = 1$ to very high precision in the early universe') and the instability argument ('even given that $\Omega = 1$ to very high precision in the early universe, if Ω is not exactly 1, then it would be unlikely to observe $\Omega \approx 1$') today have been shown to be wrong. The fine-tuning argument is wrong basically because Ω is not the appropriate parameter to use (*e.g.* Cho & Kantowski, 1994; Coule, 1995; Evrard & Coles, 1995; Coles & Ellis, 1997; Kirchner & Ellis, 2003; Adler & Overduin, 2005; Gibbons & Turok, 2008; Roukema & Blanlœil, 2010; Helbig, 2012); this is most easily seem by studying the evolution change in λ and Ω during the evolution of the universe as a dynamical system (*e.g.* Stabell & Refsdal, 1966; Rindler & Ehlers, 1989; Goliath & Ellis, 1999; Uzan & Lejoucq, 2001; Coley, 2003; Wainwright & Ellis, 2005), some such studies explicitly pointing out that this point of view demonstrates the lack of a flatness problem in classical cosmology (*e.g.* Kirchner & Ellis, 2003; Lake, 2005; Helbig, 2012).

Lake (2005) demonstrated that the instability argument does not hold for universes which expand forever because λ and Ω are large and the universe significantly non-flat only in the case that they are fine-tuned in the sense that $\alpha = (27\Omega^2\lambda)/(4K^3) \approx 1$. Note that this is the opposite of the claim that fine-tuning is required in order to have a flat universe (though, as noted above, that claim is false). Lake suggested that α, which has a fixed value throughout the life of the universe, is what should be use to characterize model universes. Adler & Overduin (2005) discussed various definitions of 'nearly flat', using essentially using the same parameter as α used by Lake (2005), and arriving at the same conclusion, namely that a significantly non-flat universe implies a fine-tuning in α.

Helbig (2012) showed that, while λ and Ω become arbitrarily large in a universe which collapses, this is the case only during a relatively short (and special) time in the lifetime of the universe, thus a typical observer would not measure very large values.[6] (This holds for all universes which collapse except some with $\lambda > 0$, but in those cases Lake's fine-tuning argument applies.) Of course, Ω approaches 0 for almost[7] all universes which expand forever, but the fact that Ω is not observed to be arbitrarily small is no more puzzling than the fact that we are, in some sense, infinitely close to the big bang if the Universe will expand forever.

Holman (2018) discussed in detail various questionable arguments and misconceptions regarding the flatness problem as well as different varieties of it. Although not a review *per se*, it is an excellent treatment of the flatness problem and misunderstandings of it, exploring some of the arguments against it, in particular the 'reverse–fine-tuning' argument of Lake (2005) and the relative–time-scale argument of Helbig (2012), as well as related issues in a wider context. More recently, Lewis & Barnes (2017), in a book-length discussion of fine-tuning in physics and cosmology, came to the conclusion that the flatness problem is mostly harmless.

Even though the arguments mentioned above have been around for years or even decades, the argument of Dicke & Peebles (1979) is still found in its original form in modern textbooks (*e.g.* Ryden, 2017) and review articles (*e.g.* O'Raifeartaigh et al., 2018; Adams, 2019). Even many observational astronomers are familiar with the flatness problem and see inflation as an attractive solution (*e.g.* Schmidt, 1989; Sandage, 1995).

Time-scale arguments

The first suggestion that the flatness problem could be avoided *via* a time-scale argument seems to be due to Tangherlini (1993), though not in the context of an FRW universe. Using a similar argument, as noted above, Helbig (2012), pointed out that, in a universe which will collapse, a typical observer would not observe large values of Ω and λ. The important point is the *relative* amount of time during which Ω and λ are $\gg 1$.

However, it is sometimes claimed, following Dicke (1970), 'that any deviations from flat geometry in the early universe would quickly escalate into a runaway open or closed universe, neither of which is observed' (O'Raifeartaigh et al., 2018, footnote 40, is a typical example). Claims referring to the age of the universe must involve the Hubble constant, whereas the papers cited farther above discuss only λ and Ω.

The argument is usually something like this:

> Imagine, shortly after the big bang, slightly increasing the density of the Universe; that would cause it to collapse after a very short time, perhaps only a few seconds or less.

Another version replaces 'density' by 'density parameter', *i.e.* Ω. Increasing the density while keeping the Hubble constant H fixed would also increase Ω, and *vice versa*. However, one could also increase Ω by keeping the density constant and decreasing H. This should already hint at the resolution: the Friedmann equation (Eq. (1)) is called the Friedmann equation because it is an equation; it makes no sense to imagine changing just *one* parameter. One would have to change at least two in order for the equation to remain valid. However, in general such minimal changes describe universes very different from our own, such as a closed universe with a mass of one kilogram. Yes, such a universe might collapse after a very short time, but this is irrelevant since it is not our Universe nor even a slight perturbation of it in any meaningful sense.

Since the usual objection is at best not well defined and at worse misleading or even wrong, one could leave it at that, but let us consider it more quantitatively.

Note that Eq. (5) implicitly depends on H_0, *via* R_0; Eq. (6) makes this dependency even more explicit. Thus, any discussion of the age of the universe as a function of the cosmological parameters must include the Hubble constant, explicitly or implicitly. Consider a finite universe with positive curvature[8] so that the mass of the universe is given by

$$\begin{aligned}
M &= \rho V && (12)\\
&= \rho 2\pi^2 R^3 \quad . && (13)
\end{aligned}$$

Making use of the definition of Ω and Eq. (2), it follows that

$$\begin{aligned}
M &= \frac{3H^2\Omega}{8\pi G}2\pi^2 R^3 && (14)\\
&= \frac{3H^2\Omega}{8\pi G}2\pi^2\left(\frac{c}{H}\frac{1}{\sqrt{|\Omega+\lambda-1|}}\right)^3 && (15)\\
&= \frac{3\pi c^3\Omega}{4GH\left(\sqrt{|\Omega+\lambda-1|}\right)^3} \quad . && (16)
\end{aligned}$$

The mass of the universe is constant in time and is inversely proportional to $\Omega/(H\sqrt{|K|}^3)$. Since the arguments of Lake (2005) and Helbig (2012) make it unlikely that an observer would measure values of Ω or K which are not of order 1, it is clear that a large (in terms of mass) universe implies a low Hubble constant. On the other hand, the age of the universe is also inversely proportional to H. Thus, all else being equal, a universe which collapses after a second would have a mass about that of a globular cluster, clearly very different from our Universe. That a small perturbation (of course, properly carried out, not just changing one parameter as in the typical *gedankenexperiment*) in the early Universe can result in a universe so different than ours is merely another aspect of the fine-tuning problem, or rather the lack thereof: *all* FRW models are arbitrarily close to the Einstein–de Sitter universe early on. (To be sure, one could have a highly non-flat universe today with the same age as that of our Universe, but this would imply a smaller value of H_0 and thus, in the $k = +1$ case, a more massive universe, but, due to the argument of Helbig (2012), the corresponding values of the cosmological parameters would occur only for a *relatively* short time during the lifetime of the universe. As noted above, it is impossible to have a universe which differs from ours in only *one* respect.) Lake (2005) argued that α, essentially the product of the square of the mass of the universe and Λ, should be thought of as the free parameter when 'choosing a universe'. (Since $\alpha = 0$ for $\lambda = 0$ or $\Omega = 0$, one can use the non-zero parameter as the free parameter in these cases.) It should be clear that a small perturbation to our Universe, caused by changing some parameters in the Friedmann equation at a time shortly after the big bang, should be small in terms of this parameter, which obviously does not lead to a vastly different age of the Universe.

Summary and conclusions

Since its original formulation by Dicke (1970) and the popularization by Dicke & Peebles (1979), especially after the idea of inflation became popular (*e.g.* Guth, 1981; Linde, 1982), many arguments were made, though largely ignored, which demonstrated that neither is fine-tuning in the early Universe needed in order to explain the values of $\lambda = 1$ and $\Omega = 0$ observed today, whatever they might be, nor is it puzzling that we don't observe values $\gg 1$ or $\ll 1$ for them. Also, the argument that the early Universe must have been fine-tuned in order for it to last as long as it has is wrong since it is based on the impossible idea of modifying just *one* parameter in the early Universe. Even if the early Universe is 'correctly perturbed' in the sense of retaining the validity of the Friedmann equation, this argument is wrong since it is essentially a variation of the bogus fine-tuning argument.

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[1] For $\lambda = 0$ and $k = 0$, $\rho = \rho_{\rm crit} = \frac{3H^2}{8\pi G}$. This density is 'critical' in the sense that, for $\lambda = 0$, a greater (lesser) density implies a positive (negative) curvature and a universe (assumed to be expanding now) which will collapse in the future (expand forever); similarly, for $k = 0$, a greater (lesser) density implies a negative (positive) cosmological constant and a universe (assumed to be expanding now) which will collapse in the future (expand forever). However, in the general case ($k \neq 0$ and $\lambda \neq 0$), $\rho_{\rm crit}$ doesn't have any special meaning, though Ω remains a useful parameter.

[2] I have long used K as defined above, as do Goliath & Ellis (1999), though Wainwright & Ellis (2005) define K with the opposite sign, using it as others use $\Omega_{\rm k}$. One also sees K used as I use k, (*e.g.* Ellis et al., 2012).

[3] This is an important point. Since all non-empty big-bang models begin their evolution arbitrarily close to the Einstein-de Sitter model with $\lambda = 0$ and $\Omega = 1$, large values of these parameters can be due *only* to a low value of the Hubble constant.

[4] They assumed that $\Lambda = 0$. If one replaces Ω with $\Omega + \lambda$, then some, but not all, of their arguments still hold.

[5] Of course, even if there is no flatness problem in classical cosmology, it does not follow that inflation could not have happened.

[6] Depending on how large one deems that α must be in order not to be fine-tuned, this argument is probably somewhat weaker than that of Lake (2005) in that it cannot explain why $K = 1$ to within at least a per cent or so, as is indicated by observations, but it still shows that the original flatness-problem argument (which was essentially the question why Ω is not 10^5, say) is incorrect.

[7] The exceptions are the extremely fine-tuned cases of a universe which asymptotically approaches the static Einstein universe (in this case the value of R has an upper limit which is reached after an infinite time) and the Einstein–de Sitter model, which always has $\Omega = 1$; the latter expands forever and has no upper limit on R, but $\dot{R}$, H, and ρ all approach 0. (Although, as discussed above, models near the Einstein–de Sitter model are not fine-tuned, the Einstein–de Sitter model itself is infintely fine-tuned.)

[8] I assume trivial topology. Also, for definiteness, I concentrate on the $k = 1$ case, though similar arguments are also possible for other values of k.