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A general and practical method for calculating cosmological distances

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Abstract. The calculation of distances is of fundamental importance in extragalactic astronomy and cosmology. However, no practical implementation for the general case has previously been available. We derive a second-order differential equation for the angular size distance valid not only in all *homogeneous* Friedmann-Lemaître cosmological models, parametrised by λ_0 and Ω_0, but also in *inhomogeneous* 'on-average' Friedmann-Lemaître models, where the inhomogeneity is given by the (in the general case redshift-dependent) parameter η. Since most other cosmological distances can be obtained trivially from the angular size distance, and since the differential equation can be efficiently solved numerically, this offers for the first time a practical method for calculating distances in a large class of cosmological models. We also briefly discuss our numerical implementation, which is publicly available.

Key words: cosmology: theory – methods: numerical – cosmology: distance scale – cosmology: miscellaneous – gravitational lensing

1. Introduction

The determination of distances is one of the most important problems in extragalactic astronomy and cosmology. Distances between two objects X and Y depend on their redshifts z_x and z_y, the Hubble constant H_0, the cosmological constant λ_0, the density parameter Ω_0 and the inhomogeneity parameter η.[1] Usually, smaller distances are determined by the traditional 'distance ladder' technique and larger distances are calculated from the redshift, assuming some cosmological model. Since the redshift is for most purposes exactly measurable, knowledge of or assumptions about two of the factors (a) Hubble constant, (b) other cosmological parameters and (c) 'astronomical distance' (i.e. ultimately tied in to the local distance scale) determines the third. In this paper we discuss distances given the Hubble constant H_0, the redshifts z_x and z_y and the cosmological parameters λ_0, Ω_0 and η. Traditionally, a simple cosmological model is often assumed for ease of calculation, although the distances thus obtained, and results which depend on them, might be false if the assumed cosmological model does not appropriately describe our universe. A general method allows one to look at cosmological models whether or not they are easy-to-calculate special cases and offers the possibility of determining cosmological distances which are important for other astrophysical topics once the correct cosmological model is known.

We stress the fact that the inhomogeneity can be as important as the other cosmological parameters, both in the field of more traditional cosmology and in the case of gravitational lensing, where, e.g. in the case of the time delay between the different images of a multiply imaged source, the inhomogeneity cannot be neglected in a thorough analysis (Kayser & Refsdal 1983). For an example involving a more traditional cosmological test, Perlmutter et al. (1995) (see also Goobar & Perlmutter (1995)) discuss using supernovae with $z \approx 0.25$–0.5 to determine q_0; for z near the top of this range or larger, the uncertainty due to our ignorance of η is comparable with the other uncertainties of the method.

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[1] When discussing the distance between *two* objects, one can always make a coordinate transformation such that the contribution from the θ and ϕ terms in Eq. (1) vanish. Then one simply needs the redshifts and cosmological parameters in order to determine the distance between them. When discussing the distances between several objects, for example QSOs with α, δ and z as coordinates, this is no longer possible. In many cases, however, suitable geometrical approximations can be made so that the most complicated part of the problem is essentially a determination of a distance between two objects. This point is further discussed in Sect. 5.

The plan of this paper is as follows. In Sect. 2 the basics of Friedmann-Lemaître cosmology are briefly discussed; this also serves to define our terms, which is important since various conflicting notational schemes are in use. (For a more thorough discussion using a similar notation see, e.g., Feige (1992).) Section 3 defines the various distances used in cosmology. In Sect. 4 our new differential equation is derived. Similar efforts in the literature are briefly discussed. Section 5 briefly describes our numerical implementation and gives the details on how to obtain the source code for use as a 'black box' (which however can be opened) for use in cosmology and extragalactic astronomy. The symmetry properties of the angular size distance, analytic solutions and methods of calculating the volume element are addressed in three appendices.

2. Basic theory

Considering for the moment *homogeneous* Friedmann-Lemaître cosmological models, we can write the familiar Robertson-Walker line element:

$$ds^2 = c^2\mathrm{d}t^2 - R^2(t) \times \left(\frac{\mathrm{d}\sigma^2}{(1-k\sigma^2)} + \sigma^2\mathrm{d}\theta^2 + \sigma^2\sin^2\theta\mathrm{d}\phi^2\right), \quad (1)$$

where the symbols are defined as follows (with the corresponding units):

s	4-dimensional interval	[length]
c	speed of light	[velocity]
t	time	[time]
R	scale factor	[length]
σ	radial coordinate	[dimensionless]
k	curvature constant	[dimensionless]
θ	angular coordinate	[dimensionless]
ϕ	angular coordinate	[dimensionless]

The dynamics of the universe is given by the Friedmann equations

$$\dot{R}^2(t) = \frac{8\pi G\rho(t)R^2(t)}{3} + \frac{\Lambda R^2(t)}{3} - kc^2 \quad (2)$$

and

$$\frac{\ddot{R}(t)}{R(t)} = -\frac{4\pi G\rho(t)}{3} + \frac{\Lambda}{3}, \quad (3)$$

where dots denote derivatives with respect to t, G is the gravitational constant, $\rho(t)$ the matter density (this paper assumes negligible pressure), Λ the cosmological constant and the sign of k determines the curvature of the 3-dimensional space.

Introducing the usual parameters

$$H = \frac{\dot{R}}{R} \quad \text{(Hubble parameter)}$$
$$\Omega = \frac{8\pi G\rho}{3H^2} \quad \text{(density parameter)} \quad (4)$$
$$\lambda = \frac{\Lambda}{3H^2} \quad \text{(normalised cosmological constant)}$$

(Ω and λ are dimensionless and H has the dimension t^{-1}) we can use Eq. (2) to calculate

$$kc^2 = R^2H^2(\Omega+\lambda-1), \quad (5)$$

so that

$$k = \mathrm{sign}(\Omega+\lambda-1). \quad (6)$$

Since $R>0$ we can write

$$R = \frac{c}{H}\frac{1}{\sqrt{|\Omega+\lambda-1|}}; \quad (7)$$

this is the radius of curvature of the 3-dimensional space at time t. For $k=0$ it is convenient to *define* the scale factor R to be c/H. In the following the index 0 will be used to denote the present value of a given quantity, fixed, as usual, at the time t_0 of observation.[2] The explicit dependence on t will be dropped for brevity. Taking matter conservation into account and using the present-day values, we have

$$\rho R^3 = \rho_0 R_0^3 \quad (8)$$

and so from Eqs. (2), (4), (5) and (8) follows

$$\dot{R}^2 = H_0^2R_0^2\left(\frac{\Omega_0R_0}{R} + \frac{\lambda_0R^2}{R_0^2} - (\Omega_0+\lambda_0-1)\right). \quad (9)$$

Since below we want to discuss distances as functions of the cosmological redshift z, by making use of the facts that

$$z = \frac{R_0}{R} - 1 \quad (10)$$

and that R_0 is fixed, we can use Eq. (9) to get

$$\mathrm{d}z = \frac{\mathrm{d}z}{\mathrm{d}R}\dot{R}\mathrm{d}t = -H_0(1+z)\sqrt{Q(z)}\,\mathrm{d}t, \quad (11)$$

where

$$Q(z) = \Omega_0(1+z)^3 - (\Omega_0+\lambda_0-1)(1+z)^2 + \lambda_0. \quad (12)$$

Note: *Throughout this paper, the* $\sqrt{}$ *sign should be taken to signify the* positive *solution, except that* $\mathrm{sign}\sqrt{Q(z)} = \mathrm{sign}(\dot{R})$ *always.*

3. Distance measures

3.1. Distances defined by measurement

In a static Euclidean space, one can define a variety of distances according to the method of measurement, which are all equivalent.

[2] Note that this paper is concerned with the calculation of distances from redshift. We are not concerned with a change in redshift with t_0.

3.1.1. Angular size distance

Let us consider at position y two light rays intersecting at x with angle θ. If l is the distance between these light rays, it is meaningful to define the angular size distance D_{xy} as

$$D_{xy} = \frac{l}{\theta}, \tag{13}$$

since an object of projected length l at position y will subtend an angle $\theta = l/D_{xy}$ (for small θ) at distance D_{xy}.

3.1.2. Proper motion distance

The proper motion distance is similar to the angular size distance, except that l is given by vt, where v is the tangential velocity of an object and t the time during which the proper motion is measured.

3.1.3. Parallax distance

Parallax distance is similar to the proper motion distance, except that the angle π is at y instead of x, so that we have

$$D^{\pi}_{xy} = \frac{l}{\pi}. \tag{14}$$

In the canonical case, $l = 1$ AU.

3.1.4. Luminosity distance

Since the apparent luminosity L of an object at distance D is proportional to $1/D^2$, one can define the luminosity distance as

$$D^{\mathrm{L}} = D^{\mathrm{L}}_0 \sqrt{\frac{L_0}{L}}, \tag{15}$$

where L_0 is the luminosity at some fiducial distance D^{L}_0.

3.1.5. Proper distance

By proper distance D^{P} we mean the distance measured with a rigid ruler.

3.1.6. Distance by light travel time

Finally, from the time required for light to traverse a certain distance, one can define a distance D^{c} by

$$D^{\mathrm{c}} = ct \tag{16}$$

where t is the so-called look-back time.

3.2. Cosmological distances

3.2.1. General considerations

In a static Euclidean space, which was used above when defining the distances through a measurement description, these distance measures are of course equivalent. In the general case in cosmology, where the 3-dimensional space need not be flat ($k = 0$) but can be either positively ($k = +1$) or negatively ($k = -1$) curved, and where the 3-dimensional space is scaled by $R(t)$, not only do the distances defined above differ, but also (in the general case) $D_{xy} \neq D_{yx}$. The definitions are still applicable, but different definitions will result in different distances.

In reality, of course, the universe is neither perfectly homogeneous nor perfectly isotropic, as one assumes when deriving Eq. (1). However, as far as the usefulness of the Friedmann equations in determining the global dynamics is concerned, this appears to be a good approximation. (See, for example, Longair (1993) and references therein for an interesting discussion.) The approximation is certainly too crude when using the cosmological model to determine distances as a function of redshift, since the angles involved in such cases can have a scale comparable to that of the inhomogeneities. In this paper, we assume that these inhomogeneities can be sufficiently accurately described by the parameter η, which gives the fraction of homogeneously distributed matter. The rest $(1-\eta)$ of the matter is distributed clumpily, where the scale of the clumpiness is *by definition* of the same order of magnitude as the angles involved.

For example, a halo of compact MACHO type objects around a galaxy in a distant cluster would be counted among the homogeneously distributed matter if one were concerned with the angular size distance to background galaxies further away, but would be considered clumped on scales such as those important when considering microlensing by the compact objects themselves. Since we don't know exactly how dark matter is distributed, different η values can be examined to get an idea as to how this uncertainty affects whatever it is one is interested in. If one has no selection effects, then, due to flux conservation, the 'average' distance cannot change (Weinberg 1976); η introduces an additional uncertainty when interpreting observations. It is generally not possible to estimate this scatter by comparing the cases $\eta = 0$ and $\eta = 1$, since, depending on the cosmological parameters and the cosmological mass distribution, not all combinations are self-consistent. For instance, if one looks at scales where galaxies are compact objects, and the fraction of Ω_0 due to the galaxies is x, then η *must* be $\leq (1-x)$.

We further assume that light rays from the object whose distance is to be determined propagate sufficiently far from all clumps. (See Schneider et al. (1992) – hereafter SEF – for a more thorough discussion of this point.) Compared to the perfectly homogeneous and isotropic case, the introduction of the η parameter will influence the angular size and luminosity distances (as well as the proper motion and parallax distances) since these depend on angles between light rays which are influenced by the amount of matter in the beam, but not the proper distance and only negligibly the light travel time. The last two distances are discussed briefly in Sect. 3.2.2 and in App. B.3 and B.6. Since there is a simple relation between the angular size distance and the luminosity distance (Sect. 3.2.2) which also holds for the inhomogeneous case (see App. A), for the general case it suffices to discuss the angular size distance, which we do in Sect. 4.

3.2.2. Relationships between different distances

Without derivation[3] we now discuss some important distance measures, denoting the redshifts of the objects with the indices x and y. Due to symmetry considerations (see App. A)

$$D_{yx} = D_{xy}\left(\frac{1+z_y}{1+z_x}\right), \tag{17}$$

where the term in parentheses takes account of, by way of Eq. (10), the expansion of the universe. It is convenient, in keeping with the meaning of angular size distance, to think of the expansion of the universe changing the angle θ in Eq. (13) and not l, if one identifies l as the (projected) size of an object. The angle is defined at the time when the light rays intersect the plane of the observer. Thus D_{xy} with the observer at $x = 0$ defines what one normally thinks of as an angular size distance. On the other hand, D_{xy} and D_{yx} with x in general $\neq 0$ can be important in, for example, gravitational lensing.[4]

Although the angle between the rays (at the source) at the time of reception of the light is important for the luminosity distance, this distance is *not* simply D_{yx}, since in the cosmological case the observed flux is obtained by multiplying the 'non-redshifted flux' by the factor $(1+z_x)^2/(1+z_y)^2$. One factor of $(1+z_x)/(1+z_y)$ occurs because a given wavelength is increased by $(1+z_y)/(1+z_x)$, which reduces the flux correspondingly; an additional factor of $(1+z_x)/(1+z_y)$ occurs because the arrival rate of photons is also decreased. Therefore, since $D^{\rm L}$ is inversely proportional to the square root of the (observed, 'redshifted') flux the luminosity distance is

$$D^{\rm L}_{xy} = D_{yx}\left(\frac{1+z_y}{1+z_x}\right). \tag{18}$$

From this and Eq. (17) follows the relation

$$D^{\rm L}_{xy} = D_{xy}\left(\frac{1+z_y}{1+z_x}\right)^2. \tag{19}$$

This means that the surface brightness of a 'standard candle' is $\sim (1+z)^{-4}$, a result independent of the cosmological model parameters, including η.[5] (This result also holds for the inhomogeneous case, since Eq. (17) still holds (see App. A) and the additional factor due to the expansion of the universe (given by the term in parentheses in Eq. (18)) is of course present in the inhomogeneous case as well.)

Of course, this applies only to the *bolometric* luminosity. Observing in a finite band introduces two corrections. The so-called K-correction as it is usually defined today (see, e.g., Coleman et al. (1980) or, for an interesting and thorough discussion, Sandage (1995)) takes account of these, both of which come from the fact that the observed wavelength interval is redshifted compared to the corresponding interval on emission. This means that, first, for a flat spectrum, less radiation is observed, because the bandwidth at the observer is $(1+z)$ times larger than at the source. Second, the spectrum need not be flat, in which case additional corrections based on the shape of the spectrum have to be included.[6] Thus,

$$m = M + 5\log\left(\frac{D^{\rm L}[\rm pc]}{10\,\rm pc}\right) + K \tag{20}$$

where m is the apparent magnitude, M the absolute magnitude, $D^{\rm L}$ is the luminosity distance and K is the K-correction as defined in Coleman et al. (1980). Perhaps more convenient is

$$m = M + 5\log D^{\rm L} + K + N \tag{21}$$

where N is a normalisation term: $N = -5$ for $D^{\rm L}$ in units of 1 pc, $N = 25$ for $D^{\rm L}$ in units of 1 Mpc and $N = x - 5\log h$ for $D^{\rm L}$ in units of the Hubble length[7] c/H_0, where

$$x = 5\log\left(\frac{\text{Hubble length}}{1\text{ pc}}\right) - 5 \approx 42.384$$

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[3] See, e.g., Feige (1992) Berry (1986) or Bondi (1961) for a more general discussion. What we present in the rest of this section is not new, but is important in order to clarify the notation. The results are obvious from the definitions introduced above.

[4] Although not useful in cosmology or extragalactic astronomy, for completeness we mention the fact that the proper motion distance is equivalent to D_{yx} and the parallax distance is equivalent to $D_{yx}/\sqrt{1-k\sigma^2}$.

[5] Thus, a 'surface brightness test' can in principle show that cosmological redshifts are due to the expansion of the universe and not to some other cause. See, e.g., Sect. 6 in Sandage (1995).

[6] Since the observed objects generally evolve with time, and redshifted objects are necessarily observed as they were when the radiation was emitted, some authors include an evolutionary term in the K-correction. Still other authors prefer to absorb one or more of these terms into the definition of the luminosity distance. Our luminosity distance is a bolometric distance based on the geometry and includes the unavoidable dimming due to the redshift. Our K-correction takes account of both effects of a finite bandwidth. Evolutionary effects are considered separately from distances.

[7] For example, as given by our numerical implementation; see Sect. 5

and h is the Hubble constant in units of 100 km/s/Mpc. In practice one has to add terms to correct for various sources of extinction and consider the fact that M is the absolute magnitude of the object when the light was emitted, which of course could be different from the present M of similar objects at negligible redshift.

The light travel time (or lookback time) $t_{xy} = t_x - t_y$ between z_x and z_y (where $t_x = t(z_x) > t_y = t(z_y)$) is given by the integration of the reciprocal of Eq. (11):

$$t_{xy} = \int_{z_y}^{z_x} \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^{-1} \mathrm{d}z = \frac{1}{H_0}\int_{z_x}^{z_y} \frac{\mathrm{d}z}{(1+z)\sqrt{Q(z)}}, \qquad (22)$$

where the minus sign from Eq. (11) is equivalent to the swapped limits of integration on the right-hand side so that the integral gives $t_x - t_y$ instead of $t_y - t_x$, making the light travel time increase (for $\dot{R} > 0$) with z; thus $D^{\mathrm{c}}_{xy} = ct_{xy}$.

Since the proper distance would be the same as D^{c} *were there no expansion*, the former can be calculated by multiplying the integrand in Eq. (22) by $c(1+z)$. Thus

$$D^{\mathrm{P}}_{xy} = \frac{c}{H_0}\int_{z_x}^{z_y} \frac{\mathrm{d}z}{\sqrt{Q(z)}}. \qquad (23)$$

This gives the proper distance at the present time. Since D^{P} scales linearly with the expansion of the universe, the proper distance at some other time can be obtained by dividing Eq. (23) with $(1+z_i)$, where z_i is the redshift at the corresponding time. For homogeneous ($\eta = 1$) cosmological models,[8] the propagation of light rays is determined by the global geometry, so that there is a simple relation between D^{P} and D and, thus, D^{L}. This is discussed in Sect. B.3. Although not 'directly' observable, the proper distance is nevertheless important in cosmological theory, since it is the basic distance of general relativity. Although not useful as a distance, the light travel time is of course important when considering evolutionary effects.

For inhomogeneous models, where this relation between global geometry and local light propagation does not exist, another approach must be used, which takes account of both the expansion of the universe as well as the local propagation of light, when calculating angle-defined distances such as the angular size distance.

4. The general differential equation for the angular size distance

In a series of papers Zeldovich (1964), Dashevskii and Zeldovich (1965) and Dashevskii and Slysh (1966) developed a general differential equation for the distance between two light rays on the boundary of a small light cone propagating far away from all clumps of matter in an inhomogeneous universe:

$$\ddot{l} = -4\pi G\eta\rho\, l + \frac{\dot{R}}{R}\,\dot{l} \qquad (24)$$

where η and ρ are functions of the time t (*not* the lookback time of Eq. 22). The first term can be interpreted as Ricci focusing due to the matter inside the light cone, and the second term is due to the expansion of space during the light propagation. We now have to transform this *time* dependent differential equation into a *redshift* dependent differential equation. From Eq. (11) we obtain[9]

$$\mathrm{d}t = -\left(H_0(1+z)\sqrt{Q}\right)^{-1} \mathrm{d}z, \qquad (25)$$

and thus

$$\frac{\mathrm{d}l}{\mathrm{d}t} = -H_0(1+z)\sqrt{Q}\,\frac{\mathrm{d}l}{\mathrm{d}z} \qquad (26)$$

and

$$\frac{\mathrm{d}^2 l}{\mathrm{d}t^2} = H_0^2(1+z)\sqrt{Q}\frac{\mathrm{d}}{\mathrm{d}z}\left((1+z)\sqrt{Q}\,\frac{\mathrm{d}l}{\mathrm{d}z}\right) \qquad (27)$$

$$= H_0^2\left(\left((1+z)Q + (1+z)^2\frac{1}{2}\frac{\mathrm{d}Q}{\mathrm{d}z}\right)\frac{\mathrm{d}l}{\mathrm{d}z} + (1+z)^2 Q\,\frac{\mathrm{d}^2 l}{\mathrm{d}z^2}\right) \quad . \qquad (28)$$

Furthermore, since $R = R_0/(1+z)$ (Eq. (10)), we obtain, using Eq. (25),

$$\frac{\mathrm{d}R}{\mathrm{d}t} = -H_0(1+z)\sqrt{Q}\,\frac{\mathrm{d}R}{\mathrm{d}z} \quad . \qquad (29)$$

From the definition of Ω (Eq. (4)) and matter conservation (Eq. (8)) we obtain

$$4\pi G\rho = \frac{3}{2}\,H_0^2\Omega_0(1+z)^3 \quad . \qquad (30)$$

If we now insert Eqs. (26), (28), (29) and (30) into Eq. (24), sort the terms appropriately and cancel H_0^2, which appears in all terms, we obtain

$$Q\,l'' + \left(\frac{2Q}{1+z} + \frac{1}{2}\,Q'\right)\,l' + \frac{3}{2}\,\eta\,\Omega_0(1+z)\,l = 0 \quad , \qquad (31)$$

where a prime denotes a derivative with respect to redshift and from Eq. (12) follows

$$Q'(z) = 3\Omega_0(1+z)^2 - 2(\Omega_0 + \lambda_0 - 1)(1+z) \quad . \qquad (32)$$

[8] This includes *empty* models ($\Omega_0 = 0$); although η has no meaning here, the same arguments apply.

[9] This transformation causes problems if the integration interval contains a point where $\dot{R} = 0$ and thus $\sqrt{Q}$ changes sign. In this case the integration interval (t_x, t_y) has to be transformed into *two* integration intervals, namely $(z_x, z_{\max})$ and $(z_{\max}, z_y)$, where $z_{\max}$ is the redshift at $\dot{R} = 0$, with the boundary conditions for the second integration interval chosen appropriately.

From the definition of the angular size distance (Eq. (13)) it is obvious that it follows the same differential equation as l:

$$\boxed{Q\,D'' + \left(\frac{2Q}{1+z} + \frac{1}{2}Q'\right)\,D' + \frac{3}{2}\,\eta\,\Omega_0(1+z)\,D = 0} \quad (33)$$

with special boundary conditions at the redshift z_x where the two considered light rays intersect. The first boundary condition is trivially

$$D = 0 \quad \text{for} \quad z = z_x \quad , \quad (34)$$

and the second boundary condition follows from the Euclidean approximation for small distances, i.e.

$$\left.\frac{\mathrm{d}D}{\mathrm{d}t}\right|_{z=z_x} = c\,\mathrm{sign}(t_x - t_y), \quad (35)$$

hence

$$D' = \frac{c}{H_0}\frac{1}{(1+z_x)\sqrt{Q(z_x)}}\,\mathrm{sign}(t_y - t_x) \quad \text{for} \quad z = z_x, \quad (36)$$

where the sign has been chosen such that D is always > 0 locally. We denote these special solutions of Eq. (33) with $D_x(z)$, and, following the definition (Eq. (13)), the angular size distance of an object at redshift z_y is then given as

$$D_{xy} = D_x(z_y) \quad . \quad (37)$$

Figure 1 shows the influence of z, η and λ on the angular size distance, calculated using Eq. (33) with our numerical implementation.

For completeness we note that after the original derivation by Kayser (1985) an equivalent equation was derived by Linder (1988) which, however, is difficult to implement due to the cumbersome notation.

Special mention must be made of the so-called bounce models, which expand from a finite R after having contracted from $R = \infty$. (See, e.g., Feige (1992).) A glance at Eq. (10) shows that in these cosmological models there must be *four* distances for an (ordered) pair of redshifts. If we denote the distances by D_{12}, D_{14}, D_{34} and D_{32}, where 1(2) und 3(4) refer to $z_1(z_2)$ during the expanding (contracting) phase, then symmetry considerations dictate that $D_{12} = D_{34}$ and $D_{14} = D_{32}$ *as long as the dependence of η on z is the same during both phases.* In this case, there are *two* independent distances per (ordered) pair of redshifts. If this is not the case, the degeneracy is no longer present and there are *four* independent distances per (ordered) pair of redshifts.

5. Numerics and practical considerations

For the actual numerical integration of the differential equation, we have found the Bulirsch-Stoer method to be both faster and more exact than other methods such as Runge-Kutta. However, the conventional method of rational function extrapolation is rather unstable in this particular case; fortunately, using polynomial extrapolation solves the problem. Although programming the integration is rather straightforward in theory, in numerical practice considerable effort is needed to determine combinations of free parameters which work for all cases. We have tested the finished programme intensively and extensively, for example by comparing the results of calculations for $\eta = 1$ (the value of η plays no special role in the integration of the differential equation) with those in Refsdal et al. (1967) or given by the method of elliptical integrals as outlined in Feige (1992) and have used it in Kayser (1995), Helbig (1996) and Helbig & Kayser (1996). For a general discussion of various methods of integrating second-order differential equations, see Press et al. (1992). Those interested in technical details can read the comments in our source code and the accompanying user's guide.

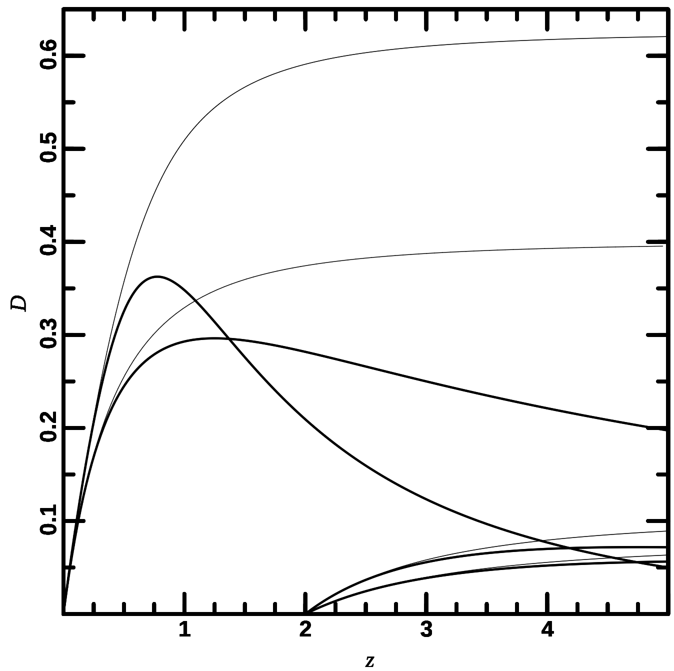

Fig. 1. The angular size distance from the observer ($z_1 = 0$) and from $z_1 = 2$ (lower right) as a function of the redshift z_2 for different cosmological models. Thin curves are for $\eta = 0$, thick for $\eta = 1$. The upper curves near $z = 0$ ($z = 2$ at lower right) are for $\lambda_0 = 2$, the lower for $\lambda_0 = 0$. $\Omega_0 = 1$ for all curves. The angular size distance D is given in units of c/H_0

Since H_0, in contrast to the other cosmological parameters, merely inversely scales the angular size distance, our routine actually calculates the angular size distance in units of c/H_0. This *dimensionless* quantity must be multiplied by c/H_0 (in whatever units are convenient) in order to obtain the actual distance. Other than reducing numerical overhead, this allows all distances to be calculated modulo c/H_0, which is convenient for expressing quantities in an H_0-independent manner. In practice, H_0 cancels out of many calculations anyway.

Apart from auxiliary routines which the user does not have to be concerned with, our implementation consists of four `FORTRAN77` subroutines. The first, `INICOS`, calculates z-independent quantities used by the other routines, some of which are returned to the calling programme. `ANGSIZ` calculates the angular size distance. Normally, η is used as a z-independent cosmological parameter, on an equal footing with λ_0 and Ω_0. If desired, however, the user can let `INICOS` know that a variable (that is, z-dependent) η is to be used; this is given by the function `VARETA`. We supply an example; the user can modify this to suit her needs. In particular, many different dependencies of η on z can be included, and a decision made in the calling programme about which one to use. This feature is also included in our example. `ANGSIZ` returns only the distance D_{12}; if one is interested in the other distances in the bounce models, our subroutine `BNGSIZ` returns all of these (though internally calculating only the independent distances, of course, depending on the dependence of η on z).

Due to the fact that not everyone has a Fortran90 compiler at his disposal, we have coded the routines in `FORTRAN77`. Only standard `FORTRAN77` features are used, and thus the routines should be able to be used on all platforms which support `FORTRAN77`. Since standard `FORTRAN77` is a subset of Fortran90, the routines can be used without change in Fortran90 as well.

With the exception of D^{c}, all distance measures can be easily transformed into one another. Thus, it suffices to calculate the angular size distance for a given case.[10]

When discussing the distance between two objects other than the observer, rather than between the observer and one object, in many cases one of two simplifying assumptions can be made:

$D(\Delta z) \ll D(\beta)$ In this case, the *proper distance* D^{P} *at the time of emission* between the two objects is $\beta D_{0x} \approx \beta D_{0y}$, where $\beta \ll 1$ is the angle in radians between the two objects on the sky.

$D(\beta) \ll D(\Delta z)$ In this case, the *angular size distance* between the two objects is D_{xy}.

$D(\Delta z)$ $(D(\beta))$ refers to the distance due to Δz (β) when setting β (Δz) equal to zero. In the first case, where the two objects are practically at the same redshift, one uses the angular size distance to this redshift to transform the observed difference in angular position on the sky into the *proper distance* between the two objects at the time of emission. This follows directly from the definition of the angular size distance. Since the distance between the objects is much less than the distance from the observer to the objects, the differently defined distances between the objects are for practical purposes degenerate. A practical example of this case would be the distance between individual galaxies in a galaxy cluster at large redshift. Naturally, one should use *one* redshift, say, of the cluster centre; the individual redshifts will in most cases be overlaid with the doppler redshift due to the velocity dispersion of the cluster, so the difference in *cosmological* redshifts is negligible. (Of course, the *present* distance would be a factor of $(1+z)$ larger, due to the expansion of the universe, were the objects comoving and not, as in a galaxy cluster, bound.) In the second case, which is typical of gravitational lensing, the angles on the sky between, for example, source and lens, are small enough to be neglected, so that the angular size distance between the objects is determined by the difference in redshift. If neither of these assumptions can be made, any sort of distance between the two objects is probably of no practical interest. (Of course, there is the trivial case where the redshifts are all $\ll 1$ in which case one can simply use α, δ and cz/H_0 as normal spherical coordinates.)

[10] The proper distance, which is η-independent, can be calculated from the angular size distance assuming $\eta = 1$, by making use of the simple relation between proper distance and angular size distance in this case. The result holds of course for all values of η.

6. Summary

After discussing cosmological distances with an emphasis on practical distance measures for general use in cosmology and extragalactic astronomy, we have obtained a new differential equation, which gives the angular size distance for a class of 'on average' Friedmann-Lemaître cosmological models, that is, models described not only by λ_0 and Ω_0 but also by $\eta(z)$, which describes the clumpiness of the distribution of matter. We have also developed a practical numerical method of solving this equation, which we have made publicly available. Since the equation is valid for *all* cases, this offers for the first time an efficient means of calculating distances in a large class of cosmological models.

The numerical implementation (in `FORTRAN77`), user's guide and a copy of the latest version of this paper can be obtained from either of the following URLs:

http://www.hs.uni-hamburg.de/english/persons/helbig/Research/Publications/Info/angsiz.html

ftp://ftp.uni-hamburg.de/pub/unihh/astro/angsiz.tar.gz

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A. Symmetry: The relation between D_{xy} and D_{yx}

The proof in this appendix follows closely the proof presented in Kayser (1985). For completeness we note that after the original derivation by Kayser (1985) an equivalent equation was derived by Linder (1988). We rewrite the differential equation, Eq. (33), for the angular size distance in the normal form:

$$a_2 D''(z) + a_1(z)\, D'(z) + a_0(z)\, D(z) = 0 \tag{A1}$$

with the coefficient functions

$$a_2(z) = Q(z) \tag{A2}$$

$$a_1(z) = \frac{2Q(z)}{1+z} + \frac{1}{2}\, Q'(z) \tag{A3}$$

$$a_0(z) = \frac{3}{2}\, \eta\, \Omega_0 (1+z) \quad . \tag{A4}$$

Now let $D^{(1)}$ and $D^{(2)}$ be two solutions of Eq. (A1) which build a fundamental system, i.e. the Wronskian for these two solutions does not vanish:

$$W(z) = \begin{vmatrix} D^{(1)} & D^{(2)} \\ \frac{\mathrm{d}D^{(1)}}{\mathrm{d}z} & \frac{\mathrm{d}D^{(2)}}{\mathrm{d}z} \end{vmatrix} \neq 0 \quad \forall z \quad . \tag{A5}$$

Every solution D_i of Eq. (A1) can then be written as a linear combination of $D^{(1)}$ and $D^{(2)}$:

$$D_i = \alpha_i\, D^{(1)} + \beta_i\, D^{(2)}; \quad \text{with } \alpha_i,\, \beta_i = \text{const} \quad . \tag{A6}$$

The angular size distances are special solutions D_x of Eq. (A1) fulfilling the following boundary conditions:

$$D_x = 0 \quad \text{for} \quad z = z_x \tag{A7}$$

and

$$\frac{\mathrm{d}D_x}{\mathrm{d}z} = b(z_x) \quad \text{for} \quad z = z_x \tag{A8}$$

with

$$b(z_x) = \frac{c}{H_0} \left((1+z_x)\sqrt{Q(z_x)} \right)^{-1} \mathrm{sign}(t_y - t_x) \quad , \tag{A9}$$

compare Eq. (36). From Eq. (A6) we obtain

$$0 = \alpha_i\, D^{(1)}(z_x) + \beta_i\, D^{(2)}(z_x) \tag{A10}$$

and

$$b(z_x) = \alpha_i \left. \frac{\mathrm{d}D^{(1)}}{\mathrm{d}z} \right|_{z=z_x} + \beta_i \left. \frac{\mathrm{d}D^{(2)}}{\mathrm{d}z} \right|_{z=z_x} \quad . \tag{A11}$$

These equations can easily be solved for α_i and β_i:

$$\alpha_i = \beta_i\, \frac{D^{(2)}(z_x)}{D^{(1)}(z_x)} \tag{A12}$$

$$\beta_i = \frac{b(z_x)\, D^{(1)}(z_x)}{W(z_x)} \tag{A13}$$

and inserting α_i and β_i back into Eq. (A6) we obtain for the special solutions D_x:

$$\begin{aligned} D_x(z) \;=\; & \frac{b(z_x)}{W(z_x)} \Big(D^{(1)}(z_x)\, D^{(2)}(z) \\ & - D^{(1)}(z)\, D^{(2)}(z_x) \Big) \quad . \end{aligned} \tag{A14}$$

If we now consider a second special solution D_y we find the relation

$$\frac{D_x(z_y)}{D_y(z_x)} = -\frac{b(z_x)}{b(z_y)}\, \frac{W(z_y)}{W(z_x)} \quad . \tag{A15}$$

The Wronskians can be calculated using Liouville's formula:

$$W(z) = W(z_0)\, \exp \int_z^{z_0} a_2(z)\, \mathrm{d}z \quad , \tag{A16}$$

where z_0 is arbitrary. Thus

$$\frac{D_x(z_y)}{D_y(z_x)} = -\frac{b(z_x)}{b(z_y)}\, \exp \int_{z_y}^{z_x} \frac{a_1(z)}{a_2(z)}\, \mathrm{d}z \tag{A17}$$

and after inserting a_0, a_1 and a_2 from Eqs. (A2), (A3) and (A4) as well as $b(z_x)$ and $b(z_y)$ from Eq. (A9) and integration we finally obtain for the angular size distances (cf. Eq. (37)) the relation

$$\frac{D_{xy}}{D_{yx}} = \frac{1+z_x}{1+z_y} \quad . \tag{A18}$$

B. Special cases

For certain special cases the differential equation can be simplified and sometimes analytically solved.

B.1. $\Omega_0 = 0$

A glance at Eq. (33) shows that for $\Omega_0 = 0$ the third term on the left hand side of Eq. (33) vanishes; one thus has a first order differential equation for D'. (Of course η has no meaning for $\Omega_0 = 0$.) Due to the fact that a vanishing Ω_0 also simplifies $Q(z)$, it is possible to calculate the angular size distance analytically. Since in this case the angular size distance is determined exclusively by global effects, one can use an approach based on global geometry.[11] Depending on the value of λ_0, one can use the

[11] See the discussion in Sect. B.3.

following expression to calculate $\chi_{xy} = \chi(y) - \chi(x)$ (Feige 1992)

$$\chi(z) = \begin{cases} \operatorname{arccosh}(\psi) & \text{for} \quad \lambda_0 < 0 \\ \ln(1+z) & \text{for} \quad \lambda_0 = 0 \\ \operatorname{arcsinh}(\psi) & \text{for} \quad 0 < \lambda_0 < 1 \\ z & \text{for} \quad \lambda_0 = 1 \\ \arcsin(\psi) & \text{for} \quad \lambda_0 > 1 \end{cases}, \tag{B1}$$

where $\psi := (1+z)\sqrt{\frac{|1-\lambda_0|}{|\lambda_0|}}$. The relationship between χ and the angular size distance D is

$$D_{xy} = \frac{R_0}{(1+z_y)} \begin{cases} \sinh\chi & \text{for } k = -1 \\ \chi & \text{for } k = 0 \\ \sin\chi & \text{for } k = +1 \end{cases}, \tag{B2}$$

as discussed below in Sect B.3.

B.2. $\eta = 0$

In the case $\eta = 0$ the third term on the left hand side of Eq. (33) vanishes; one thus has a first order differential equation for D'. Assuming $D' \neq 0$, Eq. (33) can be written as

$$\frac{D''}{D'} = -\frac{2}{1+z} - \frac{1}{2}\frac{Q'(z)}{Q(z)} \quad . \tag{B3}$$

This equation can be solved in two steps. For D' we obtain

$$D' = \frac{c_1}{\sqrt{Q(z)}\,(1+z)^2} \tag{B4}$$

and consequently for D

$$D = \int \frac{c_1}{\sqrt{Q(z)}\,(1+z)^2} + c_2 \quad . \tag{B5}$$

The constants c_1, c_2 are determined by the appropriate boundary conditions (Eqs. (34) and (35)). We then find the solution (see also SEF for an equivalent discussion with $\lambda_0 = 0$)

$$D_{xy} = \frac{\mathrm{c}}{\mathrm{H}_0}(1+z_x)(\omega(z_y) - \omega(z_x)), \tag{B6}$$

where

$$\omega(z) = \int_0^z \frac{\mathrm{d}z'}{(1+z')^2\sqrt{(1+z')^2(\Omega_0 z' + 1 - \lambda_0) + \lambda_0}}, \tag{B7}$$

or, perhaps more convenient,

$$D_{xy} = \frac{c}{H_0}(1+z_x)\int_{z_x}^{z_y} \frac{\mathrm{d}z}{(1+z)^2\sqrt{Q(z)}} \quad . \tag{B8}$$

For $\lambda_0 = 0$ there is an analytic solution (see Sect. B.4).

B.3. $\eta = 1$

The case $\eta = 1$ has all matter distributed homogeneously. Due to homogeneity, the matter *locally* affecting the propagation of light is known when the *global* geometry is known, so that the 'classical' approach of relating global geometry to observable relations is a better approach than using (the simplified form of) Eq. (33). This approach offers an analytic solution. Here, we simply sketch the most important points; the interested reader can refer to Feige (1992) for a good description of this method.

The angular size distance in this case is

$$D_{xy} = R_y\sigma_{xy} = \frac{R_0\sigma_{xy}}{(1+z_y)}, \tag{B9}$$

where σ is the radial coordinate in the Robertson-Walker metric (cf. Eq. (1)) and thus

$$D_{yx} = \frac{R_0\sigma_{xy}}{1+z_x} = D_{xy}\left(\frac{1+z_y}{1+z_x}\right), \tag{B10}$$

since this angle is inversely proportional to R for constant σ and physical size. (The value of R at the time the light rays defining the angle intersect is important.)

Since σ is given by

$$\sigma = F(\chi) = \begin{cases} \sinh\chi & \text{for } k = -1 \\ \chi & \text{for } k = 0 \\ \sin\chi & \text{for } k = +1 \end{cases}, \tag{B11}$$

an expression for $\chi(z)$ is sufficient for calculating the angular size distance D (and of course the luminosity distance D^{L} (via Eq. (19)) and the 'coordinate distance' σ (via Eq. (B11)). In general, $\sigma_{xy} \neq \sigma_y - \sigma_x$; however, $\chi_{xy} = \chi_y - \chi_x$, so that

$$\sigma_{xy} = F(\chi_{xy}) \tag{B12}$$

where F is given by Eq. (B11). Using Eq. (23) one can calculate

$$\chi_{xy} = \frac{D^P}{R_0} = \frac{c}{H_0 R_0}\int_{z_x}^{z_y} \frac{\mathrm{d}z}{\sqrt{Q(z)}}. \tag{B13}$$

In the general case, Eq. (23) can be solved by elliptic integrals, as explained in Feige (1992). For the cases $\lambda_0 = 0$ and $\Omega_0 = 0$ the formulae using elliptic integrals break down; in these cases, easier analytic formulae, which fortunately exist, can be used. The case $\Omega_0 = 0$ has been discussed above. The case $\lambda_0 = 0$ will be discussed below. *Again, we stress that the differential equation derived in Sect. 4 is completely general and can be used in* all *cases.*

B.4. $\lambda_0 = 0$

For $\lambda_0 = 0$, there is in general no simpler solution. This case has been discussed by Dyer and Roeder for $\eta = 0$ (1972) and for general η values (1973). They point out the interesting result that the maximum in the angular size distance from $z_1 = 0$ to z_2 increases monotonically from 1.25 to ∞ as η decreases from 1 to 0. See also the discussion (with a differing notation!) in Sect. 4.5.3 in SEF. However, some solutions exist for special values of Ω_0 and η. The case $\Omega_0 = 0$ has been discussed in Sect. B.1 above; the value of η is of course irrelevant in this case. With the exception of Sect B.4.3 below, in the following we simply quote results from SEF in our notation.

B.4.1. $\lambda_0 = 0$ and $\eta = 0$

As discussed above, for $\eta = 0$ Eq. (33) is effectively a first order equation for D'. For $\lambda_0 = 0$ $Q(z)$ is sufficiently simplified to allow an analytic solution. Recalling Eq. (B6),

$$D_{xy} = \frac{\mathrm{c}}{\mathrm{H}_0}(1+z_x)(\omega(z_y) - \omega(z_x)),$$

Eq. (B7) simplifies to

$$\omega(z) = \int_0^z \frac{\mathrm{d}z'}{(1+z')^3\sqrt{\Omega_0 z' + 1}}, \tag{B14}$$

which has the solution:

$$\omega(z) = \begin{cases} \frac{3\Omega_0^2}{4(\Omega_0-1)^{\frac{5}{2}}}\arctan(\psi) + \frac{3}{4(\Omega_0-1)^2} \quad \times \\ \left(\frac{\left(\Omega_0 z + \frac{5\Omega_0}{3} - \frac{2}{3}\right)\sqrt{\Omega_0 z+1}}{(1+z)^2} - \frac{5\Omega_0}{3} + \frac{2}{3}\right) & \text{(I)} \\ \frac{2}{5}\left(1 - (\Omega_0 z + 1)^{-\frac{5}{2}}\right) & \text{(II)} \\ \frac{3\Omega_0^2}{4(1-\Omega_0)^{\frac{5}{2}}}\operatorname{arctanh}(\psi) + \frac{3}{4(1-\Omega_0)^2} \quad \times \\ \left(\frac{\left(\Omega_0 z + \frac{5\Omega_0}{3} - \frac{2}{3}\right)\sqrt{\Omega_0 z+1}}{(1+z)^2} - \frac{5\Omega_0}{3} + \frac{2}{3}\right) & \text{(III)} \end{cases} \tag{B15}$$

with

$$\psi = \begin{cases} \left(\frac{\sqrt{\Omega_0-1}(1+\sqrt{\Omega_0 z+1})}{\Omega_0 - 1 + \sqrt{\Omega_0 z+1}}\right) & \text{(I)} \\ \left(\frac{\sqrt{1-\Omega_0}(1+\sqrt{\Omega_0 z+1})}{\Omega_0 - 1 + \sqrt{\Omega_0 z+1}}\right) & \text{(III)} \end{cases} \tag{B16}$$

and

$$\begin{array}{ll} \text{case I:} & \Omega_0 > 1 \\ \text{case II:} & \Omega_0 = 1\,. \\ \text{case III:} & 0 < \Omega_0 < 1 \end{array} \tag{B17}$$

Note that in SEF, the text at the top of page 137 is unclear—the expression in parentheses in the denominator of the first term $(\Omega - 1)$ for the $\Omega > 1$ case has to be replaced with $(1 - \Omega)$ as well for $\Omega < 1$. Note also that $\Omega \equiv \Omega_0$ and that after page 131 $\lambda_0 = 0$ is always assumed.

B.4.2. $\lambda_0 = 0$ and $\Omega_0 = 1$

For $\Omega_0 = 1$ and $\lambda_0 = 0$ (the Einstein-de Sitter model) we have the solution

$$D_{xy} = \frac{c}{H_0}\frac{1}{2\beta}\left(\frac{(1+z_y)^{\beta-\frac{5}{4}}}{(1+z_x)^{\beta+\frac{1}{4}}} - \frac{(1+z_x)^{\beta-\frac{1}{4}}}{(1+z_y)^{\beta+\frac{5}{4}}}\right), \tag{B18}$$

where

$$\beta := \frac{1}{4}\sqrt{25 - 24\eta}. \tag{B19}$$

B.4.3. $\lambda_0 = 0$ and $\eta = 1$

For $\eta = 1$ the special case of the expression for $\chi(z)$ for $\lambda_0 = 0$ is (Feige 1992)

$$\chi(z) = -2\begin{cases} \arcsin\left(\sqrt{\frac{\Omega_0-1}{\Omega_0(1+z)}}\right) & (\Omega_0 > 1) \\ \sqrt{\frac{1}{1+z}} & (\Omega_0 = 1)\ , \\ \operatorname{arcsinh}\left(\sqrt{\frac{1-\Omega_0}{\Omega_0(1+z)}}\right) & (0 < \Omega_0 < 1) \end{cases} \tag{B20}$$

where $\chi_{xy} = \chi(y) - \chi(x)$. (It is obvious that in the case $\lambda_0 = \Omega_0 = 0$ Eq. (B1) should be used.) From this, it is possible to obtain a general expression for the angular size distance (see, e.g., SEF):

$$D_{xy} = \frac{c}{H_0}\frac{2}{\Omega_0^2}(1+z_x)\left(R_1(z_y)R_2(z_x) - R_1(z_x)R_2(z_y)\right), \tag{B21}$$

with

$$R_1(z) = \frac{\Omega_0 z - \Omega_0 + 2}{(1+z)^2} \tag{B22}$$

and

$$R_2(z) = \frac{\sqrt{\Omega_0 z + 1}}{(1+z)^2}. \tag{B23}$$

For $z_x = 0$ and $z_y = z$ one gets for the angular size distance

$$D(z) = \frac{c}{H_0}\frac{2}{\Omega_0^2(1+z)^2} \quad \times \\ \left(\Omega_0 z - (2-\Omega_0)\left(\sqrt{\Omega_0 z + 1} - 1\right)\right). \tag{B24}$$

valid for $\Omega_0 > 0$. For $\Omega_0 = 0$ one obtains

$$D = \frac{c}{H_0}\frac{z\left(1+\frac{z}{2}\right)}{(1+z)^2} \tag{B25}$$

(Multiplying Eq. (B24) or Eq. (B25) with $(1+z)/R_0$ results in the respective expression for σ as a function of redshift as first derived by Mattig (1958). See also Sandage

(1995), Sect. 1.6.3). In this case, the volume element given by Eq. (C4) reduces to

$$\mathrm{d}V = 16\pi R_0^3 \frac{(\Omega_0 z - (2-\Omega_0)(\sqrt{\Omega_0 z+1}-1)^2)}{\Omega_0^4(1+z)^3\sqrt{\Omega_0 z+1}} \tag{B26}$$

Of course, for the physical, as opposed to comoving, density, an *additional* factor of $(1+z)^3$ must be added to the denominator.

B.4.4. $\lambda_0 = 0$ and $\eta = \frac{2}{3}$

For $\eta = \frac{2}{3}$ and $\lambda_0 = 0$ there is also an analytic solution (see SEF):

$$D_{xy} = \frac{c}{H_0}\frac{2}{3\Omega_0^2}(1+z_x) \times (R_1(z_x)R_2(z_y) - R_2(z_x)R_1(z_y)), \tag{B27}$$

with

$$R_1(z) = \frac{1}{(1+z)^2} \tag{B28}$$

and

$$R_2(z) = \frac{\sqrt{\Omega_0 z+1}(\Omega_0 z + 3\Omega_0 - 2)}{(1+z)^2}. \tag{B29}$$

B.5. Other cases

We can offer no proof that no other easier solutions, either reducing Eq. (33) to a more easily (numerically) integrated form or even to an analytic solution, exist. This is left as an exercise to the interested reader. The authors are of course interested in such solutions and are willing to verify them. As far as we know, Eq. (33) must be used except in the special cases mentioned in this appendix.

B.6. Light travel time

Feige (1992) not only gives the distance but also the light travel time by means of elliptic integrals. As for the distance, and for the same reasons, simple analytic formulae can and must be used for the special cases $\Omega_0 = 0$ and $\lambda_0 = 0$. For $k = 0$, an analytic expression for the light travel time exists, although the elliptic integrals can also be used in this case. For completeness, we give these special cases here for the light travel time $t_{xy} = t_x - t_y$.

For $\Omega_0 = 0$ we have:

$$t(z) = \frac{1}{H_0\sqrt{|\lambda_0|}} \begin{cases} \operatorname{arcsin}(\psi) & \lambda_0 < 0 \\ \frac{\sqrt{|\lambda_0|}}{(1+z)} & \lambda_0 = 0 \\ \operatorname{arcsinh}(\psi) & 0 < \lambda_0 < 1 \\ -\sqrt{|\lambda_0|}\ln(1+z) & \lambda_0 = 1 \\ \operatorname{arccosh}(\psi) & \lambda_0 > 1 \end{cases}, \tag{B30}$$

where $\psi := \frac{1}{(1+z)}\sqrt{\frac{|\lambda_0|}{|1-\lambda_0|}}$

For $\lambda_0 = 0$ we have:

$$t(z) = A \times \begin{cases} \frac{\sqrt{(\Omega_0 z+1)(\Omega_0-1)}}{\Omega_0(1+z)} - \operatorname{arcsin}\left(\sqrt{\frac{\Omega_0-1}{\Omega_0(1+z)}}\right) & \Omega_0 > 1 \\ -\frac{\sqrt{\Omega_0-1}^3}{\Omega_0}\frac{2}{3}\left(\sqrt{\frac{1}{1+z}}\right)^3 & \Omega_0 = 1 \\ \operatorname{arcsinh}\left(\sqrt{\frac{1-\Omega_0}{\Omega_0(1+z)}}\right) - \frac{\sqrt{(\Omega_0 z+1)(1-\Omega_0)}}{\Omega_0(1+z)} & 0 < \Omega_0 < 1 \end{cases}, \tag{B31}$$

where

$$A = -\frac{\Omega_0}{H_0(\sqrt{|\Omega_0 - 1|})^3}$$

(For $\Omega_0 = 0$ the appropriate case from Eq. (B30) must be used.)

For $k = 0$ we have:

$$t(z) = \frac{2}{3H_0} \times \begin{cases} \frac{1}{\sqrt{\Omega_0-1}}\operatorname{arcsin}(\psi) & \Omega_0 > 1 \\ \left(\sqrt{\frac{1}{1+z}}\right)^3 & \Omega_0 = 1 \\ \frac{1}{\sqrt{1-\Omega_0}}\operatorname{arcsinh}(\psi) & 0 < \Omega_0 < 1 \end{cases}, \tag{B32}$$

where

$$\psi = \begin{cases} \sqrt{\frac{\Omega_0-1}{\Omega_0(1+z)^3}} & \Omega_0 > 1 \\ \sqrt{\frac{1-\Omega_0}{\Omega_0(1+z)^3}} & 0 < \Omega_0 < 1 \end{cases}. \tag{B33}$$

(For $\Omega_0 = 0$ the appropriate case from Eq. (B30) must be used.)

C. Volume element

Sometimes the distance is only a means of calculating the volume element at a given redshift. In the static Euclidean case the volume element is of course

$$\mathrm{d}V = 4\pi r^2 dr. \tag{C1}$$

In the cosmological case, the volume element is, with $r = R_0\sigma$,

$$\mathrm{d}V = 4\pi r^2 \mathrm{d}D^{\mathrm{P}} = 4\pi r^2 \frac{c}{H_0}\frac{\mathrm{d}z}{\sqrt{Q(z)}}. \tag{C2}$$

$R_0\sigma_y$ is, for $\eta = 1$, simply $D_{y0} = (1+y)D_{0y}$; see Sect B.3. Thus, the distance D_{y0} is all that is needed to calculate the volume; this first can be calculated by Eq. (33) with $\eta = 1$ (This applies even if one would calculate *distances* with another value of η since the volume element is a quantity related to the global geometry of the universe—alternatively, one can use elliptic integrals, as in Sect. B.3 and Feige (1992).) If one has an expression for $\sigma(z)$, then, since

$$\mathrm{d}D^{\mathrm{P}} = R_0 \frac{\mathrm{d}\sigma}{\sqrt{1-k\sigma^2}}, \tag{C3}$$

which follows directly from Eq. (1), one can write

$$dV(\sigma) = 4\pi R_0^3 \int\limits_0^\sigma \frac{\sigma'^2 \mathrm{d}\sigma'}{\sqrt{1-k\sigma'^2}}. \tag{C4}$$

where R_0 is given by Eq. (7) for the present values:

$$R_0 = \frac{c}{H_0} \frac{1}{\sqrt{|\Omega_0 + \lambda_0 - 1|}}. \tag{C5}$$

and

$$\sigma = \frac{D_{y0}(1+z_y)}{R_0} \tag{C6}$$

Integration gives

$$V(\sigma) = \begin{cases} 2\pi r^3 \left(\frac{\sqrt{1+\sigma^2}}{\sigma^2} - \frac{\operatorname{arcsinh} \sigma}{\sigma^3} \right) & k=-1 \\ \frac{4}{3}\pi r^3 & k=0 \\ 2\pi r^3 \left(\frac{\arcsin \sigma}{\sigma^3} - \frac{\sqrt{1-\sigma^2}}{\sigma^2} \right) & k=+1 \end{cases} \tag{C7}$$

Thus, for $k=+1$, the total volume of the universe is $2\pi^2 R_0^3$. (See, e.g., Sandage (1995), Sect. 1.6.1; Sandage's d is our D^{P} and his r is our σ.) Since

$$\mathrm{d}\chi = \frac{\mathrm{d}\sigma}{\sqrt{1-k\sigma^2}}$$

Eq. (C7) can also be written as (cf. Feige (1992), Eq. (116); Feige's r is our σ)

$$V(\chi) = 2\pi R_0^3 \begin{cases} \sinh(\chi)\cosh(\chi) - \chi & k=-1 \\ \frac{2}{3}\chi^3 & k=0 \\ \chi - \sin(\chi)\cos(\chi) & k=+1 \end{cases} \tag{C8}$$

Of course, all this refers to volumes *now* at the distance corresponding to $z=y$. If the volume at another time is important, say at the time of emission of the light we see now—for instance if one is concerned with the space density of some comoving objects—then the volume element must be divided by $(1+z)^3$.

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