Computing the k-binomial complexity of the Tribonacci word

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Joint work with Michel Rigo and Matthieu Rosenfeld

k-binomial complexity of Tribonacci

Let's look at the Tribonacci word

$$\mathcal{T} = 0102010010201010201001 \cdots$$

and more precisely at its factors of a given length:

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$$\mathrm{Fac}_{\mathcal{T}}(1) = \{0, \quad \}.$$

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$$\mathcal{T} = 0{\color{red}1}0201001020101020100 1 \cdots$$

and more precisely at its factors of a given length:

$$\mathsf{Fac}_{\mathcal{T}}(1) = \{0, {\color{red}1}, \ \}.$$

k-binomial complexity of Tribonacci

Let's look at the Tribonacci word

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$$\mathsf{Fac}_{\mathcal{T}}(4) = \{0102, \qquad \}.$$

k-binomial complexity of Tribonacci

Let's look at the Tribonacci word

$$\mathcal{T} = 0{\color{red}1020}1001020101020 1001 \cdots$$

and more precisely at its factors of a given length:

$$\mathsf{Fac}_{\mathcal{T}}(4) = \{0102, {\color{red}1020}, \qquad\qquad\qquad\qquad \}.$$

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Let's look at the Tribonacci word

$$\mathcal{T} = 010\textcolor{red}{2010}010201010201001\cdots$$

and more precisely at its factors of a given length:

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The factor complexity

$$p_{\mathcal{T}} : n \mapsto \#\mathrm{Fac}_{\mathcal{T}}(n)$$

is equal to $n \mapsto 2n + 1$.

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The factor complexity

$$p_{\mathcal{T}} : n \mapsto \#\mathsf{Fac}_{\mathcal{T}}(n)$$

is equal to $n \mapsto 2n + 1$.
Changing this complexity into an other one,

$$\mathbf{b}_{\mathcal{T}}^{(k)} : n \mapsto \#(\mathsf{Fac}_{\mathcal{T}}(n)/\sim_k)$$

we want to show that it is still equal to $n \mapsto 2n + 1$.

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   - Words, factors and subwords
   - Complexity functions
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2. State of the art
3. Next result: the Tribonacci word
   - Definition
   - The theorem
   - Introduction to templates and their parents
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Factors and subwords

Let $u = u_1 u_2 \cdots u_m$ be a finite or infinite word.

Definition

A **(scattered) subword** of u is a finite subsequence of the sequence $(u_j)_{j=1}^{m}$. A **factor** of u is a contiguous subword.

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We can replace $\sim_=$ with other equivalence relations.

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We will deal with the last one.

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k-binomial equivalence

Definition (Reminder)

Let u and v be two finite words. They are **k-binomially equivalent** if

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Indeed, the words u and v are 1-abelian equivalent if

$$\binom{u}{a} = |u|_a = |v|_a = \binom{v}{a} \; \forall a \in A.$$

k-binomial complexity

Definition

If $\mathbf{w}$ is an infinite word, we can define the function

$$\mathbf{b}_{\mathbf{w}}^{(k)} : \mathbb{N} \to \mathbb{N} : n \mapsto \#(\mathrm{Fac}_{\mathbf{w}}(n)/\sim_k),$$

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We have an order relation between the different complexity functions:

$$\rho_{\mathbf{w}}^{ab}(n) \leq \mathbf{b}_{\mathbf{w}}^{(k)}(n) \leq \mathbf{b}_{\mathbf{w}}^{(k+1)}(n) \leq p_{\mathbf{w}}(n) \quad \forall n \in \mathbb{N}, k \in \mathbb{N}^+$$

where $\rho_{\mathbf{w}}^{ab}$ is the abelian complexity function of the word $\mathbf{w}$.

k-binomial complexity of Tribonacci

1. Preliminary definitions
   - Words, factors and subwords
   - Complexity functions
   - k-binomial complexity
2. State of the art
3. Next result: the Tribonacci word
   - Definition
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A famous word...

The classical **Thue–Morse word**, defined as the fixed point of the morphism

$$\varphi : \{0,1\}^* \to \{0,1\}^* : \left\{ \begin{array}{lcl} 0 & \mapsto & 01; \\ 1 & \mapsto & 10, \end{array} \right.$$

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has a bounded k-binomial complexity. The exact value is known.

Theorem (M. L., J. Leroy, M. Rigo, 2018)

Let k be a positive integer. For every $n \leq 2^k - 1$, we have

$$\mathbf{b}_{\mathbf{t}}^{(k)}(n) = p_{\mathbf{t}}(n),$$

while for every $n \geq 2^k$,

$$\mathbf{b}_{\mathbf{t}}^{(k)}(n) = \left\{ \begin{array}{ll} 3 \cdot 2^k - 3, & \text{if } n \equiv 0 \pmod{2^k}; \\ 3 \cdot 2^k - 4, & \text{otherwise.} \end{array} \right.$$

Another family

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Theorem (M. Rigo, P. Salimov, 2015)
Let $\mathbf{w}$ be a Sturmian word. We have

$$\mathbf{b}_{\mathbf{w}}^{(k)}(n) = p_{\mathbf{w}}(n) = n+1,$$

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Since $\mathbf{b}_{\mathbf{w}}^{(k)}(n) \leq \mathbf{b}_{\mathbf{w}}^{(k+1)}(n) \leq p_{\mathbf{w}}(n)$, it suffices to show that

$$\mathbf{b}_{\mathbf{w}}^{(2)}(n) = p_{\mathbf{w}}(n).$$

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1 Preliminary definitions
- Words, factors and subwords
- Complexity functions
- k-binomial complexity

2 State of the art

3 Next result: the Tribonacci word
- Definition
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k-binomial complexity of Tribonacci

1 Preliminary definitions
- Words, factors and subwords
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3 Next result: the Tribonacci word
- Definition
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The Tribonacci word

From now, let $A = \{0, 1, 2\}$. Let us define the Tribonacci word.

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Definition

The **Tribonacci word** $\mathcal{T}$ is the fixed point of the morphism

$$\tau : \{0,1,2\}^* \to \{0,1,2\}^* : \begin{cases} 0 & \mapsto & 01; \\ 1 & \mapsto & 02; \\ 2 & \mapsto & 0. \end{cases}$$

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1 Preliminary definitions
- Words, factors and subwords
- Complexity functions
- k-binomial complexity

2 State of the art

3 Next result: the Tribonacci word
- Definition
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The next result was first conjectured by Michel Rigo, and then proved.

Theorem (M. L., M. Rigo, M. Rosenfeld, 2019)
For all $k \geq 2$, the k-binomial complexity of the Tribonacci word equals its factor complexity.

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Theorem (M. L., M. Rigo, M. Rosenfeld, 2019)
For all $k \geq 2$, the k-binomial complexity of the Tribonacci word equals its factor complexity.

To prove this result, it suffices to show that, for all $n \in \mathbb{N}$,

$$\begin{cases} u, v \in \mathrm{Fac}_{\mathcal{T}}(n) \\ u \neq v \end{cases} \Rightarrow \quad u \not\sim_2 v.$$

Its k-binomial complexity

The **Parikh vector** of a word u is classicaly defined as

$$\Psi(u) := \left(\binom{u}{0} \quad \binom{u}{1} \quad \binom{u}{2}\right)^{\intercal} \in \mathbb{N}^3 \,.$$

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Let us define the **extended Parikh vector** of a word u as

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Remark

We have $u \sim_2 v \;\Leftrightarrow\; \Phi(u) = \Phi(v) \;\Leftrightarrow\; \Phi(u) - \Phi(v) = 0.$

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1 Preliminary definitions
- Words, factors and subwords
- Complexity functions
- k-binomial complexity

2 State of the art

3 Next result: the Tribonacci word
- Definition
- The theorem
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Intuitive introduction to templates

We will be interested into values of $\Phi(u) - \Phi(v)$ for $u, v \in \mathrm{Fac}_{\mathcal{T}}$. We will thus express the difference using the notion of templates.

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Informally, we will associate to every pair of words several templates, which are 5-uples:

$$A^* \times A^* \leftrightsquigarrow \mathbb{Z}^{12} \times \mathbb{Z}^3 \times \mathbb{Z}^3 \times A \times A.$$

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There exists a strong link between this notion and our thesis:

$$\mathbf{b}_{\mathcal{T}}^{(2)}(n) < p_{\mathcal{T}}(n)$$

$$\Leftrightarrow$$

$$\exists (u, v) \in \mathrm{Fac}_{\mathcal{T}} \times \mathrm{Fac}_{\mathcal{T}} \leftrightsquigarrow [\mathbf{0}, \mathbf{0}, \mathbf{0}, a, b], a \neq b$$

Templates: a formal definition

Definition

A **template** is a 5-uple of the form $t = [\mathbf{d}, \mathbf{D_b}, \mathbf{D_e}, a_1, a_2]$ where $\mathbf{d} \in \mathbb{Z}^{12}$, $\mathbf{D_b}, \mathbf{D_e} \in \mathbb{Z}^3$ and $a_1, a_2 \in A$.

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where the matrix P_3 is such that, for all $\mathbf{x} \in \mathbb{Z}^9$, $P_3 \cdot \mathbf{x} = \begin{pmatrix} 0 & 0 & 0 & \mathbf{x} \end{pmatrix}^\intercal$, and where $\otimes$ is the usual Kronecker product: if $A \in \mathbb{Z}^{m \times n}$ and $B \in \mathbb{Z}^{p \times q}$,

$$A \otimes B := \begin{pmatrix} a_{11} B & \cdots & a_{1n} B \\ \vdots & \ddots & \vdots \\ a_{m1} B & \cdots & a_{mn} B \end{pmatrix} \in \mathbb{Z}^{mp \times nq}.$$

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Why are templates useful?

Theorem

There exists $n \in \mathbb{N}$ such that

$$\mathbf{b}_{\mathcal{T}}^{(2)}(n) < p_{\mathcal{T}}(n)$$

if and only if at least one template of the form $[\mathbf{0}, \mathbf{0}, \mathbf{0}, a, b]$ with $a \neq b$ is realizable by a pair of factors of $\mathcal{T}$.

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Let $i \in \{1, \ldots, n\}$ be such that $u_i \neq v_i, u_{i+1} = v_{i+1}, \ldots, u_n = v_n$.
Then $(u_1 \cdots u_i, v_1 \cdots v_i)$ realizes $[\mathbf{0}, \mathbf{0}, \mathbf{0}, u_i, v_i]$, because $u \sim_2 v \Rightarrow u_1 \cdots u_i \sim_2 v_1 \cdots v_i$.

Using this result

We want to verify that $\forall (u,v) \in (\mathrm{Fac}_{\mathcal{T}})^2$, the pair (u,v) doesn't realize any template of the form $[\mathbf{0},\mathbf{0},\mathbf{0},a,b]$ with $a \neq b$.

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Let us suppose that there exists a pair (u, v) realizing $[\mathbf{0}, \mathbf{0}, \mathbf{0}, a, b]$ and let fix $L > 0$. Then,

- either $\min(|u|, |v|) \leq L$, or
- there exists an *ancestor template* of $[\mathbf{0}, \mathbf{0}, \mathbf{0}, a, b]$ which is realized by a pair of words $(u', v') =$ "$((\tau^{-1})^j(u), (\tau^{-1})^j(v))$" such that $L \leq \min(|u'|, |v'|) \leq 2L$.

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Preimages

Example: intuitive definition

Recall that

$$\mathcal{T} = 01 \cdot 02 \cdot 01 \cdot 0 \cdot 01 \cdot 02 \cdot 01 \cdot 01 \cdot 02 \cdot 01 \cdot 0 \cdot 01 \cdots$$

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Definition

Let u and u' be two words. The word u' is a **preimage** of u if

- u is a factor of $\tau(u')$, and
- u' is minimal: for all factors v of u', u is not a factor of $\tau(v)$.

Preimages (continued)

A word can have several preimages.

Example
Recall that

$$\mathcal{T} = 01 \cdot 02 \cdot 01 \cdot 0 \cdot 01 \cdot 02 \cdot 01 \cdot 01 \cdot 02 \cdot 01 \cdot 0 \cdot 01 \cdots$$

Take $u = 010$.

Preimages (continued)

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Take $u = 010$. It has 00, 01 and 02 as preimages.

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Templates have parents...

We will now introduce the notion of parents of a template.

Theorem

Let t be a template and let (u, v) be a pair of factors realizing t. Let u' (resp., v') be a preimage of u (resp., v).
There always exists a template t' which is realized by (u', v'). and which is, *in some way*, related to t.

$$(u, v) \quad \leftrightsquigarrow \quad [\mathbf{d}, \mathbf{D_b}, \mathbf{D_e}, a_1, a_2] = t$$

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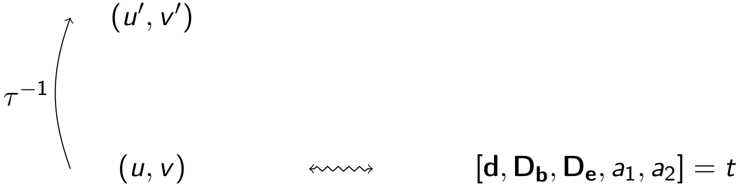

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The template t' is called a **parent template** of t.

$$(u', v') \leftrightsquigarrow [\mathbf{d}', \mathbf{D}'_{\mathbf{b}}, \mathbf{D}'_{\mathbf{e}}, a'_1, a'_2] = t'$$

$$\tau^{-1} \uparrow \qquad\qquad \uparrow \mathrm{Par}(.)$$

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- Since a word can sometimes have several preimages, a template can also have several parents.

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- Since a word can sometimes have several preimages, a template can also have several parents.
- There exists a formula allowing to compute all parents of a given template.

... and ancestors

Definition

Let t and t' be templates. We say that t' is an **(realizable) ancestor** of t if there exists a finite sequence of templates $t_0, \ldots, t_n$ such that

$$\begin{cases} t_0 = t', \\ t_n = t, \\ \forall i \in \{0, \ldots, n-1\}, t_i \text{ is a (realizable) parent of } t_{i+1}. \end{cases}$$

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- either $\min(|u|, |v|) \leq L$, or
- there exist an ancestor t' of t, and a pair (u', v') of factors realizing t', such that $L \leq \min(|u'|, |v'|) \leq 2L$.

k-binomial complexity of Tribonacci

1 Preliminary definitions
- Words, factors and subwords
- Complexity functions
- k-binomial complexity

2 State of the art

3 Next result: the Tribonacci word
- Definition
- The theorem
- Introduction to templates and their parents
- Using templates to compute $\mathbf{b}_T^{(2)}$

Verifying that no template $[\mathbf{0}, \mathbf{0}, \mathbf{0}, a, b]$ is realizable

To show that $\mathbf{b}_{\mathcal{T}}^{(2)} = p_{\mathcal{T}}$, we have to show that no template from

$$T := \{[\mathbf{0}, \mathbf{0}, \mathbf{0}, a, b] \,:\, a \neq b\}$$

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1. We check that no pair of factors (u, v) with $\min(|u|, |v|) \leq L$ realizes a template of T.
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2. We compute all the ancestors of T and we check that none of them is realized by a pair (u', v') with $L \leq \min(|u'|, |v'|) \leq 2L$.

<u>**Problem:**</u> there exists an infinite number of ancestors.

Keeping a finite number of templates

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That will leave us with a finite number of candidates.

It is then possible to verify with a computer that, in fact, none of them is realizable by a pair (u, v) with $L \leq \min(|u|, |v|) \leq 2L$.

A matrix associated to τ

Let us consider the matrix associated to τ : $\begin{cases} 0 & \mapsto & 01; \\ 1 & \mapsto & 02; \\ 2 & \mapsto & 0. \end{cases}$

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We define its extended version M_τ, such that, for all $u \in \mathrm{Fac}_{\mathcal{T}}$, we have $M_\tau \Phi(u) = \Phi(\tau(u))$.

The extended version

We have

$$M_\tau = \left(\begin{array}{ccc|ccc|ccc|ccc}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
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\hline
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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\end{array}\right)$$

For example, since $\tau(0) = 01$ and 1 is present only in $\tau(0)$ while 0 occurs once in every $\tau(a)$,

$$\binom{\tau(u)}{01} = \binom{u}{0} + \binom{u}{00} + \binom{u}{10} + \binom{u}{20}.$$

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$$\binom{\tau(u)}{01} = \binom{u}{0} + \binom{u}{00} + \binom{u}{10} + \binom{u}{20}.$$

The extended version

We have

$$M_\tau \cdot \Phi(u) = \left(\begin{array}{ccc|ccc|ccc|ccc}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
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\end{array}\right)
\begin{pmatrix}
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For example, since $\tau(0) = 01$ and 1 is present only in $\tau(0)$ while 0 occurs once in every $\tau(a)$,

$$\color{red}\binom{\tau(u)}{01}\color{black} = \binom{u}{0} + \binom{u}{00} + \color{blue}\binom{u}{10}\color{black} + \binom{u}{20}.$$

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About its eigenvalues

The Perron-Frobenius eigenvalue of M'_τ is $\theta \approx 1.839$. The matrix M_τ has

- the eigenvalue θ once;
- the eigenvalue θ^2 once;
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The bounds we will give on possibly realizable templates will concern projections of templates on the left eigenvectors associated to eigenvalues of modulus less than θ.

First restrictions

Theorem

Let λ be an eigenvalue of modulus less than 1. Let $\mathbf{r}$ be an associated eigenvector. If the template $t = [\mathbf{d}, \mathbf{D_b}, \mathbf{D_e}, a_1, a_2]$ is realizable, then

$$\min_{\boldsymbol{\delta} \in \Delta} |\mathbf{r} \cdot (\mathbf{d} + P_3 (\mathbf{D_b} \otimes \boldsymbol{\delta} + \boldsymbol{\delta} \otimes \mathbf{D_e}))| \leq 2C(\mathbf{r}),$$

where $C(\mathbf{r})$ is an easily computable constant such that, for all factors $w \in \mathrm{Fac}_{\mathcal{T}}$, we have

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$$\Delta = \left\{ \begin{pmatrix} \delta_0 \\ \delta_1 \\ \delta_2 \end{pmatrix} \in [-1.5; 1.5]^3 \, : \, \delta_0 + \delta_1 + \delta_2 = 0 \right\}.$$

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Other restrictions

Theorem

Let λ be an eigenvalue of modulus in $]1,\theta[$. Let $\mathbf{r}$ be an associated eigenvector. If the template $t = [\mathbf{d}, \mathbf{D_b}, \mathbf{D_e}, a_1, a_2]$ is realizable by a pair (u, v) with $|u| \geq L$, then

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Final algorithm

Recall that we want to compute the ancestors of T and check that none of them is realized by a pair (u, v) with

$$L \leq \min(|u|, |v|) \leq 2L. \qquad (\star)$$

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- If the program stops, `seen` contains all the possibly realizable ancestors of T, which is a finite set
- Check that none of them is realized by a pair (u, v) of factors of $\mathcal{T}$ satisfying $(\star)$

Our implementation

In our implementation, we took $L = 15$.

Computing all the possibly realizable ancestors gave us a list of 241544 elements, in more or less eight hours.

The program then checks in less than three hours that none of them is realized.

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Thus, no template from $T = \{[\mathbf{0}, \mathbf{0}, \mathbf{0}, a, b] \; : \; a \neq b\}$ is realizable.

That implies that $p_{\mathcal{T}}(n) = \mathbf{b}_{\mathcal{T}}^{(2)}(n)$ for all $n \in \mathbb{N}$.

Thank you!