On the identification of the axial force in stay cables with unknown boundary conditions

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Abstract. Identification of tensile force in axially-loaded structural elements is of paramount importance for health monitoring and safety assessment purposes. Dynamic testing techniques provide the ground for quick and cheap identification strategies, based on the knowledge of: (a) a set of identified natural frequencies, and (b) a structural model that relates natural frequencies to the axial force value. Reliability of results, hence, is inherently related to the predictive capabilities of the underlying structural model. Errors may arise, in particular, from the modeling of boundary conditions. The present paper analytically investigates the effect of unknown boundary conditions on the modal properties of a shallow cable with small bending stiffness. Starting from theoretical results obtained on this archetypal structural model, a simple but effective numerical procedure to identify the axial force in stay cables is then presented.

Keywords: Stay cables, Axial-force identification, Health-monitoring, Safety assessment, Rotational stiffness.

1 Introduction

Stay cables are lightweight and lightly damped structural elements, whose transverse vibrations can be easily set up by providing relatively small amounts of input energy. Dynamic testing techniques based on ambient vibration or hammer impact tests, hence, can be effectively used to get estimates of the lowest natural frequencies of a stay cable.

Knowledge of a set of measured natural frequencies along with a suitable structural model updating strategy, then, can be used to identify the cable axial force value (see e.g. [1]). Within this framework, reliability of the identified axial force values depends, among the other factors, on the predictive capabilities of the underlying structural model.

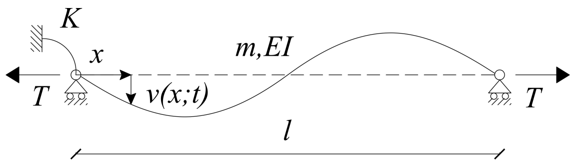

Fig. 1. Schematic representation of a stay cable subject to a tensile load *T*.

Typical models of the literature are based on the assumption of ideal boundary conditions, in the form of either perfectly hinged or perfectly clamped cable end sections (see e.g. [1, 4, 7-9]). A more realistic structural scheme could be defined, however, by considering equivalent translational and rotational springs at the cable end sections to model the flexibility of both the anchoring devices and the support structures (e.g. deck and tower for cables in stayed bridges). Proper definition of equivalent springs strongly depends on the particular technology adopted to realize the cable anchorages (see e.g. [2]) and is inherently related to several different sources of uncertainties, such as those related to geometric imperfections and aging of the anchoring devices.

The present paper presents an analytical investigation of the effects of unknown boundary conditions on the modal properties of a shallow cable with small bending stiffness. Starting from theoretical results obtained on this archetypal structural model, a simple but effective numerical procedure to identify axial force in stay cables is then presented and validated through an extensive numerical testing campaign.

2 Formulation of the problem

Let us consider a stay cable of length l, with constant bending stiffness (EI) and mass per unit of length (m), subject to an axial force T (see Fig. 1). By neglecting sag-extensibility and shear deformability effects, undamped planar flexural vibrations are governed by the partial differential equation:

$$EI\, \partial^4_x\, v - T\, \partial^2_x\, v + m\, \partial^2_t\, v = 0, \tag{1}$$

where $v(x, t)$ is the transverse displacement of the cable centerline, $x \in [0, l]$ is a spatial coordinate running over the chord of the element and t is the time.
By introducing the characteristic frequency $\omega_0 = \sqrt{(T/ml^2)}$ and the non-dimensional bending stiffness $\varepsilon = \sqrt{(EI/Tl^2)}$, Eq. (1) can be re-written in the non-dimensional form:

$$\varepsilon^2\, \partial^4_\xi\, \upsilon - \partial^2_\xi\, \upsilon + \partial^2_\tau\, \upsilon = 0, \tag{2}$$

where $\xi = x/l$, $\tau = \omega_0 t$ and $\upsilon(\xi, \tau) = v(x(\xi), t(\tau))/l$. Values of ε typical of stay cables are lower than 1%-2% [1, 8]. The small number multiplying the highest derivative hints the existence of boundary layers [3] and possible numerical difficulties if an appropriate rescaling is not used.

General solutions of Eq. (2) can be expressed as $\upsilon(\xi, \tau) = \phi(\xi) \sin(\lambda\tau - \theta)$, where λ is a non-dimensional vibration frequency, θ is a phase angle depending on initial conditions and $\phi(\xi)$ is a mode shape function. The vibration frequencies λ and shape functions $\phi(\xi)$ are the eigensolutions of a fourth order boundary value problem defined by the ordinary differential equation:

$$\varepsilon^2 \phi^{IV}(\xi) - \phi^{II}(\xi) - \lambda^2\phi(\xi) = 0, \qquad (3)$$

along with suitable boundary conditions modeling the cable restraints.

Ideal boundary conditions are often introduced, in the form of either perfectly hinged or perfectly clamped cable end sections, to simplify the analytical treatment of the problem (see e.g. [1, 4, 7-9]). As a first step towards a quantitative assessment of the effect of unknown boundary conditions on the modal properties of stay cables, a rotational spring is herein assumed to be attached to the cable end section at $x = 0$ (see Fig. 1). The degree of fixity of the rotational restraint is then defined by introducing the non-dimensional parameter: $\rho = K / (K + \varepsilon Tl)$, where K is the stiffness of the rotational spring. The parameter ρ takes values in the range: $0 \leq \rho < 1$, being strictly equal to zero for $K = 0$, i.e. for a perfectly hinged end section. The limit case $\rho = 1$, instead, is approached as $K \rightarrow \infty$, i.e. for a perfectly clamped end section.

By accounting for the definition of ρ, the boundary conditions for Eq. (3) read:

$$\phi(0) = 0,\ \phi(1) = 0,\ \ \varepsilon^2\phi^{II}(0) - (\varepsilon\rho\ / (1\text{-}\rho))\phi^{I}(0) = 0,\ \phi^{II}(1) = 0. \qquad (4)$$

The general solution of Eq. (3) can be expressed as:

$$\phi(x;\ \lambda) = A_1 \sin(z_1(\lambda)\ \xi) + A_2 \cos(z_1(\lambda)\ \xi) + \qquad (5)$$

$$A_3 \exp(-z_2(\lambda)\ \xi) + A_4 \exp(-\ (1\text{-}z_2(\lambda))\ \xi)\ ,$$

where A_i (i=1,…,4) are integration constants, while z_1 and z_2 are defined as:

$$\varepsilon\sqrt{2}\ z_j(\lambda) = \sqrt{[(-1)^j + \sqrt{(1+4\varepsilon^2\lambda^2)}]},\ j=1,2. \qquad (6)$$

Substitution of Eqs. (5) and (6) in (4) yields the algebraic eigenvalue problem:

$$\mathbf{B}(\lambda;\ \varepsilon,\ \rho)\ \mathbf{a} = \mathbf{0}, \qquad (7)$$

where **a** is a column matrix collecting the integration constant A_i (i=1,…,4) and **B** is a 4×4 matrix whose entries depend on both the non-dimensional bending stiffness ε and the degree of fixity parameter ρ.

A standard perturbation technique (see e.g. [6]) has been adopted in the present work to evaluate the eigenvalues $\lambda_k = \lambda_k(\varepsilon, \rho)$ (k = 1, 2,…) of problem (7) for small values of ε, which are typical of stay cables. Simple derivations, herein omitted for the sake of conciseness, allow to obtain the following second-order accurate asymptotic expression:

$$\lambda_k \,/\, (k\pi) = 1 + \rho\varepsilon + [(k\pi)^2/2 + \rho^2\,]\,\varepsilon^2 + O(\varepsilon^3), \quad k = 1,2,\ldots\,. \tag{8}$$

Once the eigenvalues $\lambda_k(\varepsilon,\ \rho)$ are known, multiplication by the characteristic frequency $\omega_0(T,m,l)$ gives the natural frequencies of the cable: $\omega_k = \omega_0(T,m,l)\ \lambda_k(\varepsilon,\ \rho)$, $k = 1,2,\ldots$ The closed-form Eq. (8) can be efficiently used within a frequency-based parameter identification strategy, whenever a set of measured frequencies is available from experiments. Standard vibration testing techniques (see e.g. [5]) can be applied to obtain the first M natural frequencies of a stay cable: ω_1^*, ω_2^*, …, ω_M^* (with $M \geq 1$). The difference between calculated and measured natural frequencies, then, can be quantitatively assessed by introducing the cost function:

$$F = \Sigma_{k=1,\ldots,M}\ [(\omega_k^* - \omega_k(l,\ m,\ \omega_0,\varepsilon,\ \rho))^2/\omega_k^{*2}]\ . \tag{9}$$

By assuming that the length (l) and the mass per unit of length (m) of the cable are known without uncertainties, the unknown model parameters $\mathbf{p}=(\omega_0,\varepsilon,\ \rho)^T \in P \subset \mathbb{R}^3$ can be identified by solving the optimization problem:

$$\text{Find } \mathbf{p}_{opt} \text{ such that: } F(\mathbf{p}_{opt}) = \inf\{F(\mathbf{p})\}, \tag{10}$$

where the function $F(\mathbf{p})$ is calculated through Eqs. (8)-(9). Once parameters ($\omega_0,\varepsilon,\ \rho$) are known from the solution of (10), the cable axial force can be easily calculated as $T = ml^2\omega_0^2$.

In the present work, the optimization problem in Eq. (10) has been solved through a custom implementation of the well-known Differential Evolution (DE) algorithm [10] in the MATLAB environment. DE is a heuristic gradient-free direct search algorithm for optimization over continuous parameter spaces that utilizes NP parameter vectors as a population at each iteration. The initial population is randomly chosen and offsprings are generated by perturbing trial solutions with scaled differences of randomly selected population elements.

3 Application example

The performances of the proposed parameter identification strategy have been assessed through extensive numerical testing. Results will be presented in the following for a typical stay cable characterized by: ω_0=5.66 rad/s, ε=0.01, ρ=0.75, T=4000 kN, EI=1000 kNm^2.

In order to simulate experimental input data, the eigenvalue problem (7) has been numerically solved to get the first five natural frequencies of the cable. These frequencies, then, have been corrupted by multiplying the nominal values by a unit-mean and low intensity Gaussian noise, to account for the effects of measurement errors. Different values of noise intensity, ranging from 0 to 5%, have been considered. For each noise intensity value, a sample of 5000 sets of noisy natural frequencies has been randomly generated. The optimization problem (10) has been solved for each set of numerically generated input natural frequencies by running the DE algorithm, starting from a population of *NP*=60 trial solutions randomly chosen in the parameter space $P \subset \mathbb{R}^3$:$\{(\omega_0, \varepsilon, \rho)$: $0 \le \omega_0 \le 1000$, $0 \le \varepsilon \le 1$, $0 \le \rho \le 1\}$.

Figure 2 shows the results of the identification procedure as a function of the noise intensity. Statistics of the identified parameters ω_0, ε, ρ and T are shown through box plots with whiskers corresponding to the 9th and 91st percentiles. Circles and stars denote, respectively, outlier points and mean values. The identification strategy gives fairly accurate results in terms of parameters ω_0 and ε for all values of noise intensity herein considered, but is not able to correctly identify the degree of fixity parameter ρ. More in details, the identification procedure leads, for each value of noise intensity, to a mean value of ρ equal to about 0.5. This latter value coincides with the mean value of ρ within the randomly generated population members of the DE algorithm. The degree of fixity ρ, hence, doesn't affect significantly the natural frequencies of the stay cable and cannot be accurately obtained through a frequency-based identification strategy. Direct inspection of Eq. (8) further supports the latter conclusion. Finally, it's worth noting that, in spite of an imprecise identification of the degree of fixity of the cable end sections, the proposed procedure gives a very accurate estimate of the axial force, with average errors always lower than about 2.5%.

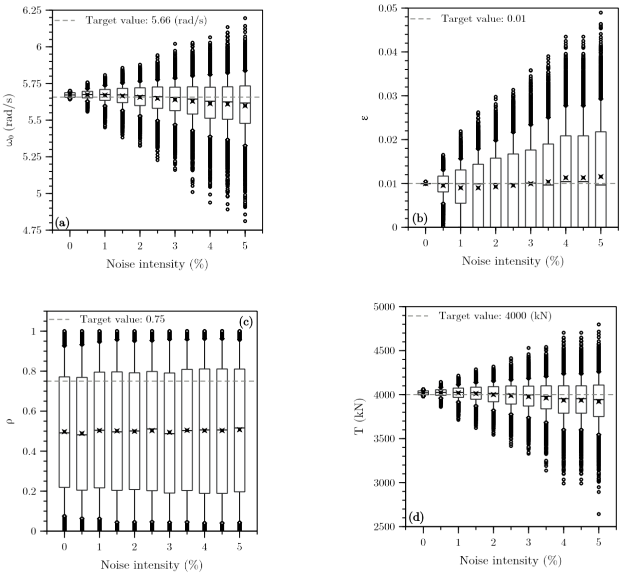

Fig. 2. Variability of identified model parameters as a function of noise intensity.

4 Conclusion

An analytical model for transverse vibrations of a shallow cable with small bending stiffness and partially restrained end conditions has been presented. A closed-form asymptotic expression for the natural frequencies of the cable has been developed and used as the basis to develop a numerical parameter identification strategy. Numerical applications have shown that in spite of a highly imprecise identification of the degree of restraint of the cable end sections, the proposed procedure gives accurate and reliable estimates of the stay cable axial force.

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