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Holomorphic Cohomological Convolution and Hadamard Product

by

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Abstract

In this article we explain the link between Pohlen's extended Hadamard product and the holomorphic cohomological convolution on $\mathbb{C}^*$. For this purpose we introduce a generalized Hadamard product, which is defined even if the holomorphic functions do not vanish at infinity, as well as a notion of strongly convolvable sets.

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§1. The extended Hadamard product

Classically, the Hadamard product of two formal power series $A(z) = \sum_{n=0}^{+\infty} a_n z^n$ and $B(z) = \sum_{n=0}^{+\infty} b_n z^n$ is defined by setting

$$(A \star B)(z) = \sum_{n=0}^{+\infty} a_n b_n z^n.$$

Using Taylor expansions, one can thus define the Hadamard product $f_1 \star f_2$ of two germs f_1 and f_2 of holomorphic functions at the origin. Exploiting the Cauchy integral representation, one obtains the formula

$$(f_1 \star f_2)(z) = \frac{1}{2i\pi} \int_{C(0,r)^+} f_1(\zeta) f_2\Big(\frac{z}{\zeta}\Big) \frac{d\zeta}{\zeta}$$

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for all z in a neighborhood of 0, $C(0,r)^+$ being a small, positively oriented circle centered at the origin (see e.g. [19, Chap. VI.3] for an introduction and [1, 8, 14] for some applications).

In his thesis [17] (see also [16]), Pohlen introduced the more general notion of a Hadamard product for holomorphic functions defined on open subsets of the Riemann sphere $\mathbb{P} = \mathbb{C} \cup \{\infty\}$ which do not necessarily contain the origin. This new definition led to interesting applications (e.g. [12] and [15]). In this introduction, we shall recall the construction and the results of Pohlen.

Definition 1.1. Let $\mathbb{P}$ be the Riemann sphere equipped with its canonical structure of a complex manifold. Let Ω be an open subset of $\mathbb{P}$. One sets

$$\mathcal{H}(\Omega) = \big\{ f \in \mathcal{O}(\Omega) : f(\infty) = 0 \big\}$$

if $\infty \in \Omega$ and $\mathcal{H}(\Omega) = \mathcal{O}(\Omega)$ otherwise.

Definition 1.2. We set $M = (\mathbb{P} \times \mathbb{P}) \setminus \{(0,\infty),(\infty,0)\}$ and extend the complex multiplication continuously as a map $\cdot : M \to \mathbb{P}$. We then have

$$\infty \cdot a = a \cdot \infty = \infty$$

if $a \in \mathbb{P}$ is not equal to zero. If A, B are subsets of $\mathbb{P}$ such that $A \times B \subset M$, one sets

$$A \cdot B = \big\{ a \cdot b : a \in A,\, b \in B \big\}.$$

One also extends the inversion $z \mapsto z^{-1}$ continuously from $\mathbb{C}^*$ to $\mathbb{P}$ by setting $0^{-1} = \infty$ and $\infty^{-1} = 0$. If $S \subset \mathbb{P}$, one sets

$$S^{-1} = \big\{ z : z^{-1} \in S \big\}.$$

For the rest of the article, we shall often drop the point and write the multiplication as a concatenation.

Definition 1.3. Two open subsets $\Omega_1, \Omega_2 \subset \mathbb{P}$ are called *star-eligible* if

(1) Ω_1 and Ω_2 are proper subsets of $\mathbb{P}$,
(2) $(\mathbb{P} \setminus \Omega_1) \times (\mathbb{P} \setminus \Omega_2) \subset M$,
(3) $(\mathbb{P} \setminus \Omega_1)(\mathbb{P} \setminus \Omega_2) \neq \mathbb{P}$.

In this case, the *star product* of Ω_1 and Ω_2, denoted $\Omega_1 \star \Omega_2$, is defined by

$$\Omega_1 \star \Omega_2 = \mathbb{P} \setminus ((\mathbb{P} \setminus \Omega_1)(\mathbb{P} \setminus \Omega_2)).$$

For the several equivalent definitions of the index/winding number of a cycle c in $\mathbb{C}$, we refer to [18]. For any cycle c in $\mathbb{C}$, one sets $\operatorname{Ind}(c,\infty) = 0$.

Definition 1.4. Let Ω be a nonempty, open subset of $\mathbb{P}$, K be a nonempty, compact subset of Ω and c be a cycle in $\Omega\backslash(K \cup \{0\} \cup \{\infty\})$. If $\infty \notin K$ and

$$\operatorname{Ind}(c, z) = \begin{cases} 1 & \text{if } z \in K, \\ 0 & \text{if } z \in \mathbb{P}\backslash\Omega, \end{cases}$$

then c is called a *Cauchy cycle* for K in Ω. If $\infty \in \Omega$ and

$$\operatorname{Ind}(c, z) = \begin{cases} 0 & \text{if } z \in K, \\ -1 & \text{if } z \in \mathbb{P}\backslash\Omega, \end{cases}$$

then c is called an *anti-Cauchy cycle* for K in Ω.

In [17, Lem. 2.3.1], Pohlen refers to ad hoc explicit constructions which ensure that Cauchy and anti-Cauchy cycles always exist for any Ω and any K. In the next section we shall see that this existence can easily be obtained by using singular homology.

Let Ω_1 and Ω_2 be two star-eligible open subsets of $\mathbb{P}$. Note that, if $z \in \Omega_1 \star \Omega_2$, then $z(\mathbb{P}\backslash\Omega_2)^{-1}$ is a closed subset of Ω_1.

Definition 1.5. Let $z \in (\Omega_1 \star \Omega_2)\backslash\{0, \infty\}$. A *Hadamard cycle* for $z(\mathbb{P}\backslash\Omega_2)^{-1}$ in Ω_1 is a cycle c in $\Omega_1\backslash(z(\mathbb{P}\backslash\Omega_2)^{-1} \cup \{0\} \cup \{\infty\})$ which satisfies the condition given in the following table.

Ω_2	Ω_1			
	$0, \infty$	∞	0	
$0, \infty$	cc$^+$ or acc$^-$	acc$^-$	cc$^+$	cc
∞	acc$^-$	acc$^-$	—	—
0	cc$^+$	—	cc$^+$	—
	acc	—	—	—

This table should be understood in the following way. The header row and the first column indicate which of these elements are in Ω_1 and Ω_2 respectively. The abbreviation cc (resp. acc) means that cycle c is a Cauchy (resp. anti-Cauchy) cycle for $z(\mathbb{P}\backslash\Omega_2)^{-1}$ in Ω_1. The abbreviation cc$^+$ (resp. acc$^-$) means that cycle c is a Cauchy (resp. anti-Cauchy) cycle with the extra condition $\operatorname{Ind}(c, 0) = 1$ (resp. $\operatorname{Ind}(c, 0) = -1$). A dash means that this case cannot occur. (See Figure 1 for an example of Hadamard cycle.)

One can now extend the standard Hadamard product.

Definition 1.6. Let $f_1 \in \mathcal{H}(\Omega_1)$ and $f_2 \in \mathcal{H}(\Omega_2)$. For each $z \in (\Omega_1 \star \Omega_2)\setminus\{0,\infty\}$ one sets

$$(f_1 \star f_2)(z) = \frac{1}{2i\pi}\int_{c_z} f_1(\zeta) f_2\Big(\frac{z}{\zeta}\Big)\frac{d\zeta}{\zeta},$$

where c_z is a Hadamard cycle for $z(\mathbb{P}\setminus\Omega_2)^{-1}$ in Ω_1. One can check that this integral does not depend on the chosen Hadamard cycle (see [17, Lem. 3.4.2]). The function $f_1 \star f_2$ is called the *Hadamard product* of f_1 and f_2.

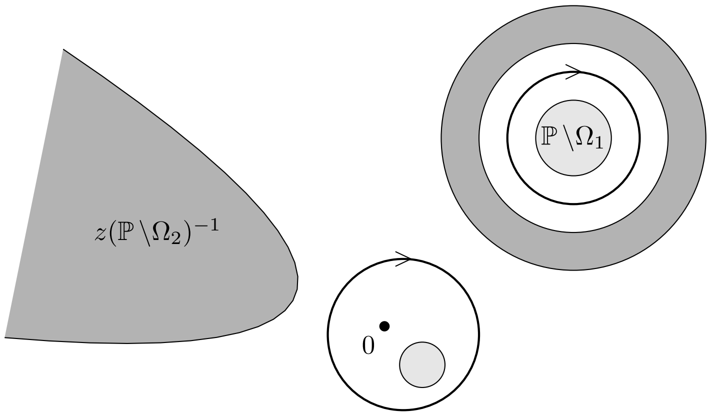

Figure 1. A Hadamard cycle for $z(\mathbb{P}\setminus\Omega_2)^{-1}$ in Ω_1, in the case where $0,\infty \in \Omega_1$ and $\infty \in \Omega_2$, $0 \notin \Omega_2$.

Proposition 1.7 ([17, Lem. 3.4.5, Prop. 3.6.4]). *The Hadamard product $f_1 \star f_2$ can be continuously extended to $\Omega_1 \star \Omega_2$. If $0 \in \Omega_1 \star \Omega_2$ (resp. $\infty \in \Omega_1 \star \Omega_2$), one has $(f_1 \star f_2)(0) = f_1(0)f_2(0)$ (resp. $(f_1 \star f_2)(\infty) = 0$). Moreover, $f_1 \star f_2$ is an element of $\mathcal{H}(\Omega_1 \star \Omega_2)$.*

Proposition 1.8 ([17, Prop. 3.6.1]). *The Hadamard product is commutative.*

In all this framework, the hypothesis $f(\infty) = 0$, when $\infty \in \Omega$, is widely used. In the next section we shall provide a more general definition of Hadamard cycles and Hadamard product, based on singular homology theory, which does not require the vanishing condition at infinity.

§2. Generalized Hadamard cycles

For classical facts about singular homology, we refer to [7] and [9]. For a general background on sheaf theory and derived functors, we refer to [10]. For a sheaf-theoretic definition of the Borel–Moore homology and the link with singular homology on HLC-spaces, we refer to [3].

Let us recall that on any topological space X, there is an orientation complex ω_X which is canonically isomorphic to $\mathbb{Z}_X[n]$ if X is an oriented, topological manifold of pure dimension n. On a topological space X, the Borel–Moore homology (resp. Borel–Moore homology with compact support) of degree k is defined by

$$^{\mathrm{BM}}H_k(X) := H^{-k}(X, \omega_X) \quad (\text{resp. } ^{\mathrm{BM}}H^c_k(X) := H^{-k}_c(X, \omega_X)).$$

Definition 2.1. Let X be an oriented, topological manifold of pure dimension n. *The orientation class* of X is the class

$$\alpha_X \in {}^{\mathrm{BM}}H_n(X) \simeq H^{-n}(X, \mathbb{Z}_X[n]) \simeq H^0(X, \mathbb{Z}_X)$$

corresponding to the constant section 1 of $\mathbb{Z}_X$.

Let X be a topological manifold X of pure dimension n. Since X is homologically locally connected, the complex $\mathrm{R}\Gamma_c(X, \omega_X)$ is canonically isomorphic to the complex of singular chains on X. Hence, $^{\mathrm{BM}}H^c_k(X)$ is isomorphic to the usual singular homology group of degree k, $H_k(X)$. Now let K be a compact subset of X and consider the two canonical excision distinguished triangles

$$\mathrm{R}\Gamma_{X\setminus K}(X, \omega_X) \to \mathrm{R}\Gamma(X, \omega_X) \to \mathrm{R}\Gamma(K, \omega_X) \xrightarrow{+}$$

and

$$\mathrm{R}\Gamma_c(X\setminus K, \omega_X) \to \mathrm{R}\Gamma_c(X, \omega_X) \to \mathrm{R}\Gamma(K, \omega_X) \xrightarrow{+} .$$

The second triangle implies that $H^{-n}(K, \omega_X)$ is canonically isomorphic to the relative singular homology group $H_n(X, X\setminus K)$. Hence, we get a sequence of morphisms

$$^{\mathrm{BM}}H_n(X) \to H^{-n}(K, \omega_X) \xrightarrow{\sim} H_n(X, X\setminus K)$$

and $\alpha_X \in {}^{\mathrm{BM}}H_n(X)$ induces a relative orientation class $\alpha_{X,K} \in H_n(X, X\setminus K)$.

Proposition 2.2. *Let Ω be a proper, open subset of $\mathbb{C}$ and let $F = \mathbb{C}\setminus\Omega$. There is a canonical isomorphism*

$$H_1(\Omega) \xrightarrow{\sim} H^0_c(F, \mathbb{Z}_F)$$

given by

$$[c] \mapsto (z \mapsto \mathrm{Ind}_z(c)).$$

Proof. Let us consider the excision distinguished triangle

$$\mathrm{R}\Gamma_c(\Omega, \omega_{\mathbb{C}}) \to \mathrm{R}\Gamma_c(\mathbb{C}, \omega_{\mathbb{C}}) \to \mathrm{R}\Gamma_c(F, \omega_{\mathbb{C}}) \xrightarrow{+1} . \tag{2.1}$$

It induces a long exact sequence

$$\cdots \longrightarrow H_2(\Omega) \longrightarrow H_2(\mathbb{C}) \longrightarrow H^{-2}\mathrm{R}\Gamma_c(F,\omega_F) \longrightarrow H_1(\Omega) \longrightarrow H_1(\mathbb{C}) \longrightarrow H^{-1}\mathrm{R}\Gamma_c(F,\omega_F) \longrightarrow \cdots.$$

Since $\mathbb{C}$ is contractible, one has $H_2(\mathbb{C}) \simeq H_1(\mathbb{C}) \simeq \{0\}$. Therefore, taking into account that $\omega_F \simeq \mathbb{Z}_F[2]$, one gets a canonical isomorphism

$$\delta\colon H^0_c(F,\mathbb{Z}_F) \xrightarrow{\sim} H_1(\Omega).$$

Let $z \in F$. Applying (2.1) with $\mathbb{C}\setminus\{z\}, \mathbb{C}$ and $\{z\}$, one gets an isomorphism

$$\delta_z\colon \mathbb{Z} \simeq H^0_c(\{z\},\mathbb{Z}_{\{z\}}) \xrightarrow{\sim} H_1(\mathbb{C}\setminus\{z\}).$$

Clearly, $\delta_z^{-1}([c]) = \mathrm{Ind}_z(c)$. Moreover, by [10, Prop. 1.3.6], there is a commutative diagram

$$\begin{tikzcd}
H^0_c(F,\mathbb{Z}_F) \arrow[r, "\delta"] \arrow[d, "i_z"'] & H_1(\Omega) \arrow[d, "j_z"] \\
H^0_c(\{z\},\mathbb{Z}_{\{z\}}) \arrow[r, "\delta_z"'] & H_1(\mathbb{C}\setminus\{z\}),
\end{tikzcd}$$

where $i_z(f) = f(z)$ and $j_z([c]) = [c]$. Hence, one sees that $\delta^{-1}([c])(z) = \mathrm{Ind}_z(c)$. Since this argument is valid for all $z \in F$, the conclusion follows. □

To introduce our definition of generalized Hadamard cycles, we have to be in the same setting as Pohlen. However, looking at Definition 1.3, we find it more natural to start with closed subsets instead of open ones.

Definition 2.3. Two closed subsets S_1 and S_2 of $\mathbb{P}$ are *star-eligible* if S_1, S_2 and S_1S_2 are proper and if $S_1 \times S_2 \subset M$.

For the rest of the section we fix S_1 and S_2, two star-eligible closed subsets of $\mathbb{P}$. If $z \in \mathbb{C}^* \setminus S_1S_2$, S_1 is a compact subset of $\mathbb{P}\setminus zS_2^{-1}$ and, thus, a compact subset of $\mathbb{P}\setminus(zS_2^{-1} \cup (\{0,\infty\}\setminus S_1))$. Moreover, one has

$$\big(\mathbb{P}\setminus(zS_2^{-1} \cup (\{0,\infty\}\setminus S_1))\big)\setminus S_1 = \mathbb{P}\setminus(S_1 \cup zS_2^{-1} \cup \{0\} \cup \{\infty\}).$$

Let $z \in \mathbb{C}^* \setminus S_1S_2$.

Definition 2.4. A *generalized Hadamard cycle* for S_1 in $\mathbb{P}\setminus(zS_2^{-1} \cup (\{0,\infty\}\setminus S_1))$ is a representative c of the class in $H_1(\mathbb{P}\setminus(S_1 \cup zS_2^{-1} \cup \{0\} \cup \{\infty\}))$ which is the image of

$$- \alpha_{\mathbb{P}\setminus(zS_2^{-1}\cup(\{0,\infty\}\setminus S_1)),S_1} \\ \in H_2\big(\mathbb{P}\setminus(zS_2^{-1}\cup(\{0,\infty\}\setminus S_1)), \mathbb{P}\setminus(S_1\cup zS_2^{-1}\cup\{0\}\cup\{\infty\})\big)$$

by the canonical map

$$\begin{array}{c} H_2\big(\mathbb{P}\setminus(zS_2^{-1}\cup(\{0,\infty\}\setminus S_1)), \mathbb{P}\setminus(S_1\cup zS_2^{-1}\cup\{0\}\cup\{\infty\})\big) \\ \downarrow \\ H_1(\mathbb{P}\setminus(S_1\cup zS_2^{-1}\cup\{0\}\cup\{\infty\})). \end{array}$$

(See Figure 2 for an example of generalized Hadamard cycle.)

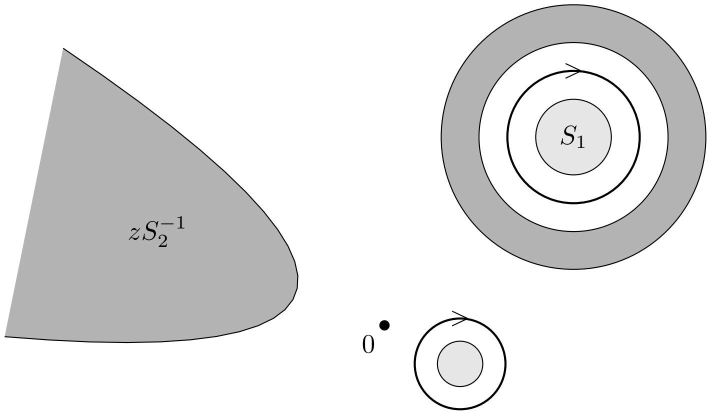

Figure 2. A generalized Hadamard cycle for S_1 in $\mathbb{P}\setminus(zS_2^{-1}\cup(\{0,\infty\}\setminus S_1))$, in the case where $0,\infty\notin S_1$ and $0\in S_2$, $\infty\notin S_2$.

Our aim is now to define a product

$$\mathcal{O}(\mathbb{P}\setminus S_1)\times\mathcal{O}(\mathbb{P}\setminus S_2)\to\mathcal{O}(\mathbb{C}^*\setminus S_1S_2)$$

which generalizes the extended Hadamard product of Pohlen.

Definition 2.5. Let $f_1\in\mathcal{O}(\mathbb{P}\setminus S_1)$ and $f_2\in\mathcal{O}(\mathbb{P}\setminus S_2)$. For each $z\in\mathbb{C}^*\setminus S_1S_2$ we set

$$(f_1\star f_2)(z)=\frac{1}{2i\pi}\int_{c_z} f_1(\zeta)f_2\Big(\frac{z}{\zeta}\Big)\frac{d\zeta}{\zeta},$$

where c_z is a generalized Hadamard cycle for S_1 in $\mathbb{P}\setminus(zS_2^{-1}\cup(\{0,\infty\}\setminus S_1))$. Since two generalized Hadamard cycles are homologous, the definition does not depend on the chosen generalized Hadamard cycle. The function $f_1 \star f_2$ is called the *generalized Hadamard product* of f_1 and f_2.

Lemma 2.6. *Let $f_1 \in \mathcal{O}(\mathbb{P} \setminus S_1)$ and $f_2 \in \mathcal{O}(\mathbb{P} \setminus S_2)$. For each compact subset K of $\mathbb{C}^* \setminus S_1 S_2$, there is a cycle c_K in $\mathbb{P} \setminus (S_1 \cup K S_2^{-1} \cup \{0\} \cup \{\infty\})$ such that*

$$(f_1 \star f_2)(z) = \frac{1}{2i\pi} \int_{c_K} f_1(\zeta) f_2\Big(\frac{z}{\zeta}\Big) \frac{d\zeta}{\zeta}$$

for all $z \in K$.

Proof. There is a relative orientation class

$$\alpha_{\mathbb{P} \setminus (K S_2^{-1} \cup (\{0,\infty\} \setminus S_1)), S_1} \in H_2\big(\mathbb{P} \setminus (K S_2^{-1} \cup (\{0,\infty\} \setminus S_1)), \mathbb{P} \setminus (S_1 \cup K S_2^{-1} \cup \{0\} \cup \{\infty\})\big).$$

We choose c_K to be a representative of the class in $H_1(\mathbb{P} \setminus (S_1 \cup K S_2^{-1} \cup \{0\} \cup \{\infty\}))$ which is the image of $-\alpha_{\mathbb{P} \setminus (K S_2^{-1} \cup (\{0,\infty\} \setminus S_1)), S_1}$ by the canonical map

$$\begin{array}{c} H_2\big((\mathbb{P} \setminus (K S_2^{-1} \cup (\{0,\infty\} \setminus S_1)), \mathbb{P} \setminus (S_1 \cup K S_2^{-1} \cup \{0\} \cup \{\infty\})\big) \\ \big\downarrow \\ H_1(\mathbb{P} \setminus (S_1 \cup K S_2^{-1} \cup \{0\} \cup \{\infty\})). \end{array}$$

For each $z \in K$, there is a canonical commutative diagram

$$\begin{tikzcd} H_2\big(\mathbb{P} \setminus (K S_2^{-1} \cup (\{0,\infty\} \setminus S_1)), \mathbb{P} \setminus (S_1 \cup K S_2^{-1} \cup \{0\} \cup \{\infty\})\big) \arrow[r] \arrow[dd] & H_1(\mathbb{P} \setminus (S_1 \cup K S_2^{-1} \cup \{0\} \cup \{\infty\})) \arrow[d] \\ & H_1(\mathbb{P} \setminus (S_1 \cup z S_2^{-1} \cup \{0\} \cup \{\infty\})) \\ H_2\big(\mathbb{P} \setminus (z S_2^{-1} \cup (\{0,\infty\} \setminus S_1)), \mathbb{P} \setminus (S_1 \cup z S_2^{-1} \cup \{0\} \cup \{\infty\})\big). \arrow[ur] & \end{tikzcd}$$

Obviously, $\alpha_{\mathbb{P} \setminus (z S_2^{-1} \cup (\{0,\infty\} \setminus S_1)), S_1}$ is the image of $\alpha_{\mathbb{P} \setminus (K S_2^{-1} \cup (\{0,\infty\} \setminus S_1)), S_1}$ by the left vertical map. Therefore, by the commutativity of the diagram, one can deduce that c_K is a generalized Hadamard cycle for S_1 in $\mathbb{P} \setminus (z S_2^{-1} \cup (\{0,\infty\} \setminus S_1))$ for all $z \in K$. Hence the conclusion. $\square$

Proposition 2.7. *The generalized Hadamard product is a well-defined map*

$$\mathcal{O}(\mathbb{P}\setminus S_1) \times \mathcal{O}(\mathbb{P}\setminus S_2) \to \mathcal{O}(\mathbb{C}^* \setminus S_1 S_2).$$

Proof. Let $f_1 \in \mathcal{O}(\mathbb{P}\setminus S_1)$ and $f_2 \in \mathcal{O}(\mathbb{P}\setminus S_2)$. We have to check that $f_1 \star f_2$ is holomorphic on $\mathbb{C}^* \setminus S_1 S_2$. Since it is a local property, it is enough to prove that $f_1 \star f_2$ is holomorphic on each small open disk $D \subset \mathbb{C}^* \setminus S_1 S_2$. Let D be such a disk. By Lemma 2.6 there is a cycle $c_{\overline{D}}$ such that

$$(f_1 \star f_2)(z) = \frac{1}{2i\pi} \int_{c_{\overline{D}}} f_1(\zeta) f_2\Big(\frac{z}{\zeta}\Big) \frac{d\zeta}{\zeta}$$

for all $z \in D$. We conclude by differentiation under the integral sign. □

We shall now prove that our product is a good generalization of the extended Hadamard product of Pohlen. By doing so, the reader shall see why we chose such a sign convention in Definition 2.4.

Proposition 2.8. *Let $f_1 \in \mathcal{H}(\mathbb{P}\setminus S_1)$ and $f_2 \in \mathcal{H}(\mathbb{P}\setminus S_2)$. Let $z \in \mathbb{C}^* \setminus S_1 S_2$. Let c_z be a generalized Hadamard cycle for S_1 in $\mathbb{P}\setminus(zS_2^{-1} \cup (\{0,\infty\}\setminus S_1))$ and d_z be a Hadamard cycle for zS_2^{-1} in $\mathbb{P}\setminus S_1$. Then*

$$\frac{1}{2i\pi} \int_{c_z} f_1(\zeta) f_2\Big(\frac{z}{\zeta}\Big) \frac{d\zeta}{\zeta} = \frac{1}{2i\pi} \int_{d_z} f_1(\zeta) f_2\Big(\frac{z}{\zeta}\Big) \frac{d\zeta}{\zeta}.$$

Proof. We treat the case where $0, \infty \notin S_1$ and $0 \in S_2$, $\infty \notin S_2$ and leave the others to the reader. By construction, it is clear that c_z satisfies

$$\mathrm{Ind}(c_z, w) = \begin{cases} 0 & \text{if } w \in zS_2^{-1} \cup \{0\}, \\ -1 & \text{if } w \in S_1. \end{cases}$$

Let c'_z be a cycle $\mathbb{P}\setminus(S_1 \cup zS_2^{-1} \cup \{0\} \cup \{\infty\})$ such that

$$\mathrm{Ind}(c'_z, w) = \begin{cases} 0 & \text{if } w \in zS_2^{-1} \cup S_1, \\ -1 & \text{if } w = 0. \end{cases}$$

Since d_z is acc$^-$, it is clear by Proposition 2.2 that d_z is homologous to $c_z + c'_z$ in $\mathbb{P}\setminus(S_1 \cup zS_2^{-1} \cup \{0\} \cup \{\infty\})$. We then have

$$\int_{c_z} f_1(\zeta) f_2\Big(\frac{z}{\zeta}\Big) \frac{d\zeta}{\zeta} = \int_{d_z} f_1(\zeta) f_2\Big(\frac{z}{\zeta}\Big) \frac{d\zeta}{\zeta} - \int_{c'_z} f_1(\zeta) f_2\Big(\frac{z}{\zeta}\Big) \frac{d\zeta}{\zeta}.$$

Moreover, by the residue theorem,

$$-\int_{c'_z} f_1(\zeta) f_2\Big(\frac{z}{\zeta}\Big)\,\frac{d\zeta}{\zeta} = 2i\pi \mathrm{Res}_{\zeta=0}\Big(\frac{f_1(\zeta)}{\zeta} f_2\Big(\frac{z}{\zeta}\Big)\Big) = 2i\pi \lim_{\zeta\to 0}\Big(f_1(\zeta) f_2\Big(\frac{z}{\zeta}\Big)\Big)$$
$$= 2i\pi f_1(0) f_2(\infty) = 0.$$

Hence the conclusion. □

Remark 2.9. Of course, the generalized Hadamard product is no longer commutative if the functions do not vanish at infinity. For example, let S_1 and S_2 be as in the proof of the previous proposition. Let $f_1 \in \mathcal{O}(\mathbb{P}\setminus S_1)$ and $f_2 \in \mathcal{O}(\mathbb{P}\setminus S_2)$. By a similar computation, one sees that

$$f_1 \star f_2 - f_2 \star f_1 = f_1(0) f_2(\infty).$$

Despite the lack of commutativity, the generalized Hadamard cycles are more symmetric with respect to 0 and ∞. In Section 5 we shall explain how one can define a convolution between 1-forms which have (not necessarily isolated) singularities at 0 and ∞. Generalized Hadamard cycles are key ingredients to compute such a convolution (see also Section 6). Moreover, the commutativity shall eventually be obtained thanks to quotient spaces that naturally occur in this context.

§3. The holomorphic integration map

Let X be a complex manifold of complex dimension d_X and $r \in \mathbb{Z}$. Recall that $\mathcal{C}^r_{\infty,X}$ admits a decomposition in bi-types

$$\mathcal{C}^r_{\infty,X} \simeq \bigoplus_{p+q=r} \mathcal{C}^{p,q}_{\infty,X}$$

which induces a decomposition of the exterior derivative d as

$$d = \partial + \bar\partial,$$

where

$$\partial\colon \mathcal{C}^{p,q}_{\infty,X} \to \mathcal{C}^{p+1,q}_{\infty,X} \quad\text{and}\quad \bar\partial\colon \mathcal{C}^{p,q}_{\infty,X} \to \mathcal{C}^{p,q+1}_{\infty,X}.$$

Similarly, $\mathcal{D}b^r_X$ admits a decomposition in bi-types

$$\mathcal{D}b^r_X \simeq \bigoplus_{p+q=r} \mathcal{D}b^{p,q}_X$$

and an associated decomposition of the distributional exterior derivative. Moreover, for any open subset U of X, we have a canonical isomorphism

$$\mathcal{D}b^r_X(U) \simeq \Gamma_c(U, \mathcal{C}^{2d_X - r}_{\infty,X})'$$

between the space of complex distributional r-forms and the topological dual of the space of infinitely differentiable complex differential $(2d_X - r)$-forms with compact support which induces the similar isomorphism

$$\mathcal{D}b_X^{p,q}(U) \simeq \Gamma_c(U, \mathcal{C}_{\infty,X}^{d_X-p,d_X-q})'.$$

In the sequel, we denote by Ω_X^p the sheaf of holomorphic differential p-forms on X and we set for short $\Omega_X = \Omega_X^{d_X}$. Of course, Ω_X^p is canonically isomorphic to both the kernel of

$$\bar{\partial}\colon\ \mathcal{C}_{\infty,X}^{p,0} \to \mathcal{C}_{\infty,X}^{p,1}$$

and the kernel of

$$\bar{\partial}\colon\ \mathcal{D}b_X^{p,0} \to \mathcal{D}b_X^{p,1}\,.$$

The double complex $\mathcal{C}_{\infty,X}^{\bullet,\bullet}$ (resp. $\mathcal{D}b_X^{\bullet,\bullet}$) is the infinitely differentiable (resp. distributional) Dolbeault complex of X. By construction, the associated simple complex is the infinitely differentiable (resp. distributional) de Rham complex $\mathcal{C}_{\infty,X}^{\bullet}$ (resp. $\mathcal{D}b_X^{\bullet}$) of X. Moreover, we have the following chains of canonical quasi-isomorphisms:

$$\mathbb{C}_X \simeq \mathcal{C}_{\infty,X}^{\bullet} \simeq \mathcal{D}b_X^{\bullet} \quad \text{and} \quad \Omega_X^p \simeq \mathcal{C}_{\infty,X}^{p,\bullet} \simeq \mathcal{D}b_X^{p,\bullet},$$

which are given by de Rham and Dolbeault lemmas.

Let $f\colon X \to Y$ be a holomorphic map from X to a complex manifold Y of complex dimension d_Y and let V be an arbitrary open subset of Y. It follows from the holomorphy of f that the pullback

$$f^*\colon\ \mathcal{C}_{\infty,Y}^{r}(V) \to \mathcal{C}_{\infty,X}^{r}(f^{-1}(V))$$

sends $\mathcal{C}_{\infty,Y}^{p,q}(V)$ into $\mathcal{C}_{\infty,X}^{p,q}(f^{-1}(V))$ if $p+q=r$. In particular,

$$\partial(f^*\omega) = f^*(\partial\omega) \quad \text{and} \quad \bar{\partial}(f^*\omega) = f^*(\bar{\partial}\omega)$$

for all $\omega \in \mathcal{C}_{\infty,Y}^{p,q}(V)$. By topological duality, it follows that there are canonical pushforward morphisms

$$\int_f \colon \Gamma_{f-\text{proper}}(f^{-1}(V), \mathcal{D}b_Y^{2d_Y-r}) \to \Gamma(V, \mathcal{D}b_Y^{2d_X-r})$$

and

$$\int_f \colon \Gamma_{f-\text{proper}}(f^{-1}(V), \mathcal{D}b_Y^{d_Y-p,d_Y-q}) \to \Gamma(V, \mathcal{D}b_Y^{d_X-p,d_X-q})$$

between distributional forms with f-proper support on $f^{-1}(V)$ and distributional forms on V and that these morphisms commute with ∂ and $\bar{\partial}$. In particular, we get a morphism of double complexes of sheaves of the form

$$\int_f : f_! \, \mathcal{D}b_X^{\bullet+d_X,\bullet+d_X} \to \mathcal{D}b_Y^{\bullet+d_Y,\bullet+d_Y} \, .$$

Moreover, if f is a surjective submersion, one can show that the pushforward of a distributional form associated with an infinitely differentiable form with f-proper support is itself associated with an infinitely differentiable form which can be computed by integration over the fibers of f. This shows that, in this case, the preceding morphism factors through a morphism of the form

$$\int_f : f_! \, \mathcal{C}_{\infty,X}^{\bullet+d_X,\bullet+d_X} \to \mathcal{C}_{\infty,Y}^{\bullet+d_Y,\bullet+d_Y} \, .$$

Thanks to the quasi-isomorphisms

$$\Omega_X^{p+d_X} \simeq \mathcal{D}b_X^{p+d_X,\bullet} \quad \text{and} \quad \Omega_Y^{p+d_Y} \simeq \mathcal{D}b_Y^{p+d_Y,\bullet},$$

this gives us a morphism

$$\int_f : \mathrm{R}f_! \Omega_X^{p+d_X}[d_X] \to \Omega_Y^{p+d_Y}[d_Y]$$

in the derived category for each $p \in \mathbb{Z}$. In the particular case where $p = 0$, we get the morphism

$$\int_f : \mathrm{R}f_! \Omega_X[d_X] \to \Omega_Y[d_Y],$$

which is usually called the *holomorphic integration map along the fibers of f* (see e.g. [10, p. 129]). Note that, if $g \colon Y \to Z$ is another holomorphic map between complex manifolds, then the well-known relation $(g \circ f)^* = f^* \circ g^*$ entails that $\int_{g\circ f} = \int_g \circ \int_f$.

§4. Holomorphic cohomological convolution

Definition 4.1. Let (G, μ) be a locally compact complex Lie group of complex dimension n. Two closed subsets S_1 and S_2 of G are said to be *convolvable* if $S_1 \times S_2$ is μ-proper, i.e. if

$$(S_1 \times S_2) \cap \mu^{-1}(K)$$

is a compact subset of $G \times G$ for any compact subset K of G.

Remark 4.2. A proper map on a locally compact topological space is universally closed, in particular closed (see e.g. [2]). Hence, if S_1 and S_2 are convolvable closed subsets of G, then $\mu|_{S_1\times S_2}$ is a proper map and $S_1+S_2 = \mu|_{S_1\times S_2}(S_1\times S_2)$ is closed.

Definition 4.3. Two distributional $2n$-forms u_1 and u_2 of G are *convolvable* if the support S_1 of u_1 and the support S_2 of u_2 are convolvable. In that case, the convolution product of u_1 and u_2 is a distributional $2n$-form on G defined by

$$u_1 \star u_2 = \int_\mu (u_1 \boxtimes u_2) := \int_\mu (p_1^* u_1 \wedge p_2^* u_2),$$

where $p_1, p_2 \colon G \times G \to G$ are the two canonical projections.

Remark 4.4. By choosing a Haar form ν on G, one can define the convolution product of two distributions by means of the isomorphism $\mathcal{D}b_G \simeq \mathcal{D}b_G^{2n}$ given by ν (see e.g. [4]).

Remark 4.5. If we define

$$\phi \colon G \times G \to G \times G \quad \text{and} \quad \psi \colon G \times G \to G \times G$$

by setting $\phi(g_1, g_2) = (g_1, \mu(g_1, g_2))$ and $\psi(g_1, g_2) = (g_1, \mu(g_1^{-1}, g_2))$, we see that ϕ and ψ are reciprocal biholomorphic bijections and that the diagram

$$\begin{tikzcd} G \times G \arrow[rr, "\sim", "\phi"'] \arrow[dr, "\mu"'] & & G \times G \arrow[dl, "p_2"] \\ & G & \end{tikzcd}$$

is commutative. This shows in particular that μ is a surjective submersion and that the preceding procedure allows us to define the convolution product of infinitely differentiable forms also.

Let S_1 and S_2 be two convolvable closed subsets of G. By construction, the convolution of distributions on G is the composition of the external product of distributions

$$\Gamma_{S_1}(G, \mathcal{D}b_G^{2n}) \otimes \Gamma_{S_2}(G, \mathcal{D}b_G^{2n}) \to \Gamma_{S_1 \times S_2}(G \times G, \mathcal{D}b_{G \times G}^{4n})$$

and the map

$$\int_\mu \colon \Gamma_{S_1 \times S_2}(G \times G, \mathcal{D}b_{G \times G}^{4n}) \to \Gamma_{\mu(S_1 \times S_2)}(G, \mathcal{D}b_G^{2n})$$

induced by the integration map along the fibers of μ,

$$\int_\mu \colon \Gamma_{\mu-\text{proper}}(G \times G, \mathcal{D}b_{G \times G}^{4n}) \to \Gamma(G, \mathcal{D}b_G^{2n})$$

and the fact that S_1 and S_2 are convolvable. It is thus natural to define the convolution of cohomology classes of holomorphic forms on G as follows.

Definition 4.6. Let S_1, S_2 be two convolvable closed subsets of G. Consider the external product morphisms

$$\mathrm{R}\Gamma_{S_1}(G,\Omega_G^{p+n})[n]\otimes \mathrm{R}\Gamma_{S_2}(G,\Omega_G^{q+n})[n]\to \mathrm{R}\Gamma_{S_1\times S_2}(G\times G,\Omega_{G\times G}^{p+q+2n})[2n]$$

and the morphisms

$$\int_\mu : \mathrm{R}\Gamma_{S_1\times S_2}(G\times G,\Omega_{G\times G}^{p+q+2n})[2n]\to \mathrm{R}\Gamma_{\mu(S_1\times S_2)}(G,\Omega_G^{p+q+n})[n],$$

induced by the holomorphic integration map and the fact that $S_1\times S_2$ is μ-proper. By composition, these morphisms give derived category morphisms

$$\star_{(G,\mu)}: \mathrm{R}\Gamma_{S_1}(G,\Omega_G^{p+n})[n]\otimes \mathrm{R}\Gamma_{S_2}(G,\Omega_G^{q+n})[n]\to \mathrm{R}\Gamma_{\mu(S_1\times S_2)}(G,\Omega_G^{p+q+n})[n],$$

that we call the *holomorphic convolution morphisms of* G. Going to cohomology groups, these morphisms give rise to the morphisms

$$\star_{(G,\mu)}: H^{r+n}_{S_1}(G,\Omega_G^{p+n})\otimes H^{s+n}_{S_2}(G,\Omega_G^{q+n})\to H^{r+s+n}_{\mu(S_1\times S_2)}(G,\Omega_G^{p+q+n}),$$

that we call the *holomorphic cohomological convolution morphisms of* G.

Remark 4.7. Consider the diagram

$$\begin{tikzcd}
H^n_{S_1}(G,\Omega_G)\otimes H^n_{S_2}(G,\Omega_G) \arrow[r] & H^n_{\mu(S_1\times S_2)}(G,\Omega_G) \\
\Gamma_{S_1}(G,\mathcal{D}b_G^{2n})\otimes \Gamma_{S_2}(G,\mathcal{D}b_G^{2n}) \arrow[r] \arrow[u] & \Gamma_{\mu(S_1\times S_2)}(G,\mathcal{D}b_G^{2n}), \arrow[u]
\end{tikzcd}$$

where the vertical arrows are given by the Dolbeault complex of Ω_G and the top (resp. the bottom) horizontal arrow is given by the holomorphic cohomological morphism of G with $p=q=r=s=0$ (resp. the convolution product of distributions). Obviously, by the definitions, this diagram is commutative. This remark will allow us to perform explicit computations in the next section.

§5. Multiplicative convolution on $\mathbb{C}^*$

In this section we will consider the case where the group G is the group $\mathbb{C}^*$ formed by the set of nonzero complex numbers endowed with complex multiplication (noted as a concatenation). We will assume that S_1, S_2 are convolvable proper closed subsets of $\mathbb{C}^*$ (note that this means that $S_1\cap KS_2^{-1}$ is compact for any compact subset K of $\mathbb{C}^*$) such that S_1S_2 is also a proper subset of $\mathbb{C}^*$ and we will show how to compute the holomorphic cohomological convolution morphism

$$\star\colon H^1_{S_1}(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \otimes H^1_{S_2}(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \to H^1_{S_1S_2}(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \tag{5.1}$$

by means of path integral formulas.

Proposition 5.1. *Let S be a proper closed subset of $\mathbb{C}^*$, then there is a canonical isomorphism*

$$H^r_S(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \simeq \begin{cases} \Omega(\mathbb{C}^* \setminus S)/\Omega(\mathbb{C}^*) & \text{if } r = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Any open subset of $\mathbb{C}$ is a Stein manifold. □

Thanks to this proposition, one can see that (5.1) can be interpreted as a bilinear map

$$\star\colon \Omega(\mathbb{C}^* \setminus S_1)/\Omega(\mathbb{C}^*) \times \Omega(\mathbb{C}^* \setminus S_2)/\Omega(\mathbb{C}^*) \to \Omega(\mathbb{C}^* \setminus S_1S_2)/\Omega(\mathbb{C}^*).$$

Now let $\omega_1 \in \Omega(\mathbb{C}^* \setminus S_1)$ and $\omega_2 \in \Omega(\mathbb{C}^* \setminus S_2)$ be two given holomorphic forms. Ideally, we would like to obtain a formula of the form

$$[\omega_1] \star [\omega_2] = [\omega],$$

where ω is a holomorphic form on $\mathbb{C}^* \setminus S_1S_2$ which can be computed from ω_1 and ω_2 by some path integral.

It is in general not possible to find such a nice formula. However, we will show that for any relatively compact open subset U of $\mathbb{C}^*$ and any open neighborhood V of S_1S_2 in $\mathbb{C}^*$, there is a holomorphic form ω on $U \setminus \overline{V}$, which can be computed from ω_1 and ω_2 by some path integral and which is such that

$$[\omega] \in \Omega(U \setminus \overline{V})/\Omega(U) \simeq H^1_{\overline{V}\cap U}(U, \Omega_{\mathbb{C}^*})$$

coincides with the image of $[\omega_1] \star [\omega_2]$ by the canonical restriction morphism

$$H^1_{S_1S_2}(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \to H^1_{\overline{V}\cap U}(U, \Omega_{\mathbb{C}^*}).$$

Thanks to the following lemma, this is in fact sufficient to completely compute $[\omega_1] \star [\omega_2]$.

Lemma 5.2. *Let S be a closed subset of $\mathbb{C}^*$. Then*

$$H^1_S(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \simeq \varprojlim_{U \in \mathcal{U}_{rc}, V \in \mathcal{V}_S} H^1_{\overline{V}\cap U}(U, \Omega_{\mathbb{C}^*}),$$

where $\mathcal{U}_{rc}$ denotes the set of relatively compact open subsets of $\mathbb{C}^*$ ordered by $\subset$ and $\mathcal{V}_S$ denotes the set of open neighborhoods of S in $\mathbb{C}^*$ ordered by $\supset$.

Proof. This follows from the Mittag-Leffler theorem for projective systems (see e.g. [10, Prop. 2.7.1]). ☐

To be able to specify the kind of path integral we need, let us first introduce the following definition.

Definition 5.3. Let F and G be two closed subsets of $\mathbb{C}^*$ which have a compact intersection and let W be an open neighborhood of $F \cap G$. A *relative Hadamard cycle for F with respect to G in W* is a relative 1-cycle

$$c \in Z_1(W \setminus F, (W \setminus F) \cap (W \setminus G))$$

such that its class

$$[c] \in H_1(W \setminus F, (W \setminus F) \cap (W \setminus G))$$

is the image of the relative orientation class

$$\alpha_{W, F\cap G} \in H_2(W, W \setminus (F \cap G))$$

by the Mayer–Vietoris morphism

$$H_2(W, W \setminus (F \cap G)) \to H_1(W \setminus F, (W \setminus F) \cap (W \setminus G))$$

associated with the decomposition

$$(W, W \setminus (F \cap G)) = ((W \setminus F) \cup W, (W \setminus F) \cup (W \setminus G)).$$

(See Figure 3 for an example of relative Hadamard cycle.)

Remark 5.4. Let $c \in Z_1(W \setminus F, (W \setminus F) \cap (W \setminus G))$ be such that the associated class $[c] \in H_1(W \setminus F, (W \setminus F) \cap (W \setminus G))$ is the image of $[W]_{F\cap G}$ by the sequence of canonical maps

$$\begin{aligned} H_2(W, W \setminus (F \cap G)) &\to H_1(W \setminus (F \cap G)) \\ &= H_1((W \setminus F) \cup (W \setminus G)) \\ &\to H_1((W \setminus F) \cup (W \setminus G), W \setminus G) \\ &\to H_1(W \setminus F, (W \setminus F) \cap (W \setminus G)). \end{aligned}$$

By construction, c is a relative Hadamard cycle for F with respect to G in W.

With this definition in hand, we can now state the main result of this section.

Theorem 5.5. *Let S_1 and S_2 be two convolvable proper closed subsets of $\mathbb{C}^*$ such that $S_1 S_2 \neq \mathbb{C}^*$ and let us assume that $\omega_1 = f_1\, dz$ and $\omega_2 = f_2\, dz$ with $f_1 \in$*

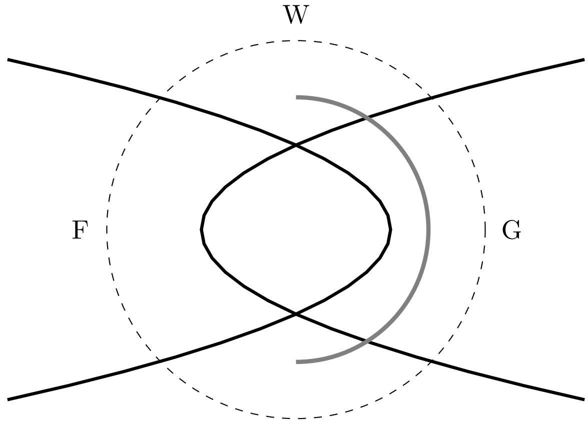

Figure 3. In grey, a relative Hadamard cycle for F with respect to G in W.

$\mathcal{O}(\mathbb{C}^*\setminus S_1)$, $f_2 \in \mathcal{O}(\mathbb{C}^*\setminus S_2)$. Fix a relatively compact open subset U of $\mathbb{C}^*$ and an open neighborhood V of S_1S_2 in $\mathbb{C}^*$. Then the image of

$$[\omega_1] \star [\omega_2] \in \Omega(\mathbb{C}^*\setminus S_1S_2)/\Omega(\mathbb{C}^*) \simeq H^1_{S_1S_2}(\mathbb{C}^*, \Omega_{\mathbb{C}^*})$$

in

$$\Omega(U \setminus \overline{V})/\Omega(U) \simeq H^1_{\overline{V}\cap U}(U, \Omega_{\mathbb{C}^*})$$

is the class of the form $\omega = f dz \in \Omega(U \setminus \overline{V})$ where

$$f(z) = \int_c f_1(\zeta) f_2\Big(\frac{z}{\zeta}\Big)\,\frac{d\zeta}{\zeta}$$

and c is a relative Hadamard cycle for S_1 with respect to $\overline{U}S_2^{-1}$ in $\mathbb{C}^*\setminus(\overline{U}\setminus V)S_2^{-1}$.

Lemma 5.6. *Let S_1 and S_2 be convolvable closed subsets of $\mathbb{C}^*$ and let $\mathcal{W}$ be a fundamental system of compact neighborhoods of 1 in $\mathbb{C}^*$. Then*

(1) *the set $S_1^W = WS_1$ (resp. $S_2^W = WS_2$, $S_1^W S_2^W = W^2 S_1 S_2$) is a closed neighborhood of S_1 (resp. S_2, S_1S_2) in $\mathbb{C}^*$ for any $W \in \mathcal{W}$;*
(2) *the closed subsets S_1^W and S_2^W are convolvable in $\mathbb{C}^*$ for any $W \in \mathcal{W}$;*
(3) *one has $\bigcap_{W\in\mathcal{W}} S_1^W = S_1$, $\bigcap_{W\in\mathcal{W}} S_2^W = S_2$ and $\bigcap_{W\in\mathcal{W}} S_1^W S_2^W = S_1S_2$;*
(4) *in particular, if S_1 and S_2 are proper convolvable closed subsets of $\mathbb{C}^*$ such that $S_1S_2 \neq \mathbb{C}^*$, if U is a relatively compact open subset of $\mathbb{C}^*$ and if V is an open neighborhood of S_1S_2 in $\mathbb{C}^*$, then there is $W \in \mathcal{W}$ such that S_1^W and S_2^W are convolvable proper closed subsets of $\mathbb{C}^*$ such that $S_1^W S_2^W \neq \mathbb{C}^*$ and $S_1^W S_2^W \cap \overline{U} \subset V$.*

Proof.

(1) This follows from the fact that FK is closed in $\mathbb{C}^*$ if F (resp. K) is closed (resp. compact) in $\mathbb{C}^*$ and from the fact that $(zW)_{W\in\mathcal{W}}$ is a fundamental system of neighborhoods of $z \in \mathbb{C}^*$.

(2) This follows from the inclusion

$$S_1^W \cap K(S_2^W)^{-1} = WS_1 \cap KW^{-1}S_2^{-1} \subset W(S_1 \cap KW^{-2}S_2^{-1})$$

which is satisfied for any compact subset K of $\mathbb{C}^*$.

(3) This is clear since for any closed subset F of $\mathbb{C}^*$ and any $z \notin F$ there is $W \in \mathcal{W}$ such that $zW^{-1} \cap F = \emptyset$.

(4) By contradiction, assume that

$$S_1^W S_2^W \cap \bar{U} \cap (\mathbb{C}^* \setminus V) \neq \emptyset$$

for all $W \in \mathcal{W}$. Then, by compactness,

$$\bigcap_{W\in\mathcal{W}} (S_1^W S_2^W \cap \bar{U} \cap (\mathbb{C}^* \setminus V)) = S_1 S_2 \cap \bar{U} \cap (\mathbb{C}^* \setminus V) \neq \emptyset,$$

but this contradicts the fact that $S_1 S_2 \cap \bar{U} \subset V$. □

Lemma 5.7. *Let S be a proper closed subset of $\mathbb{C}^*$ and let $\omega \in \Omega(\mathbb{C}^* \setminus S)$. Assume that ω admits an infinitely differentiable extension to $\mathbb{C}^*$ and denote by $\underline{\omega}$ such an extension. Then $[\omega]$, seen as an element of $H^1_S(\mathbb{C}^*, \Omega_{\mathbb{C}^*})$, is the image of*

$$[\bar{\partial}\underline{\omega}] \in H^1(\Gamma_S(\mathbb{C}^*, \mathcal{C}^{1,\bullet}_{\infty,\mathbb{C}^*}))$$

by the canonical morphism obtained by applying H^1 to the composition in the derived category of the canonical morphism

$$\Gamma_S(\mathbb{C}^*, \mathcal{C}^{1,\bullet}_{\infty,\mathbb{C}^*}) \to \mathrm{R}\Gamma_S(\mathbb{C}^*, \mathcal{C}^{1,\bullet}_{\infty,\mathbb{C}^*})$$

and the inverse of the canonical isomorphism

$$\mathrm{R}\Gamma_S(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \xrightarrow{\sim} \mathrm{R}\Gamma_S(\mathbb{C}^*, \mathcal{C}^{1,\bullet}_{\infty,\mathbb{C}^*}).$$

Proof. It follows from the distinguished triangle

$$\mathrm{R}\Gamma_S(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \to \mathrm{R}\Gamma(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \to \mathrm{R}\Gamma(\mathbb{C}^* \setminus S, \Omega_{\mathbb{C}^*}) \xrightarrow{+1}$$

that $\mathrm{R}\Gamma_S(\mathbb{C}^*, \Omega_{\mathbb{C}^*})$ is canonically isomorphic to the mapping cone $M(\rho_S)$ of the restriction morphism

$$\rho_S \colon \mathcal{C}^{1,\bullet}_{\infty,\mathbb{C}^*}(\mathbb{C}^*) \to \mathcal{C}^{1,\bullet}_{\infty,\mathbb{C}^*}(\mathbb{C}^* \setminus S)$$

shifted by -1. We know that $M[\rho_S][-1]$ is a complex concentrated in degrees 0, 1 and 2 of the form

$$\mathcal{C}^{1,0}_{\infty,\mathbb{C}^*}(\mathbb{C}^*) \to \mathcal{C}^{1,1}_{\infty,\mathbb{C}^*}(\mathbb{C}^*) \oplus \mathcal{C}^{1,0}_{\infty,\mathbb{C}^*}(\mathbb{C}^* \setminus S) \to \mathcal{C}^{1,1}_{\infty,\mathbb{C}^*}(\mathbb{C}^* \setminus S),$$

where the differentials in degrees 0 and 1 are given by the matrices

$$\begin{pmatrix} \bar\partial \\ -\rho_S \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\rho_S & -\bar\partial \end{pmatrix}.$$

What we have to show is that

$$\begin{pmatrix} \bar\partial\underline{\omega} \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ \omega \end{pmatrix}$$

are two 1-cycles of this complex which are in the same cohomology class. This is clear since

$$\begin{pmatrix} \bar\partial \\ -\rho_S \end{pmatrix} \underline{\omega} + \begin{pmatrix} 0 \\ \omega \end{pmatrix} = \begin{pmatrix} \bar\partial\underline{\omega} \\ 0 \end{pmatrix}. \qquad \square$$

Proof of Theorem 5.5. Let U and V be as in the statement of the theorem. Thanks to Lemma 5.6, we know that it is possible to find a closed neighborhood $\underline{S}_1$ of S_1 and a closed neighborhood $\underline{S}_2$ of S_2 in $\mathbb{C}^*$ such that $\underline{S}_1$ and $\underline{S}_2$ are convolvable and

$$\underline{S}_1\underline{S}_2 \cap \bar{U} \subset V.$$

Let $\underline{f}_1$ (resp. $\underline{f}_2$) be an infinitely differentiable function on $\mathbb{C}^*$ which coincides with f_1 (resp. f_2) on $\mathbb{C}^* \setminus \underline{S}_1$ (resp. $\mathbb{C}^* \setminus \underline{S}_2$) and set

$$\underline{\omega}_1 = \underline{f}_1(z)\, dz \quad \text{and} \quad \underline{\omega}_2 = \underline{f}_2(z)\, dz.$$

It follows from Lemma 5.7 that the image of

$$[\omega_1] \in \Omega(\mathbb{C}^* \setminus S_1)/\Omega(\mathbb{C}^*) \simeq H^1_{S_1}(\mathbb{C}^*, \Omega_{\mathbb{C}^*})$$

by the canonical morphism

$$H^1_{S_1}(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \to H^1_{\underline{S}_1}(\mathbb{C}^*, \Omega_{\mathbb{C}^*})$$

is the same as the image of

$$[\bar\partial\underline{\omega}_1] \in H^1(\Gamma_{\underline{S}_1}(\mathbb{C}^*, \mathcal{C}^{(1,\bullet)}_{\infty,\mathbb{C}^*}))$$

by the canonical morphism

$$H^1(\Gamma_{\underline{S}_1}(\mathbb{C}^*, \mathcal{C}^{(1,\bullet)}_{\infty,\mathbb{C}^*})) \to H^1_{\underline{S}_1}(\mathbb{C}^*, \Omega_{\mathbb{C}^*})$$

considered in this lemma. A similar conclusion is true for the image of

$$[\omega_2] \in \Omega(\mathbb{C}^* \setminus S_2)/\Omega(\mathbb{C}^*) \simeq H^1_{S_2}(\mathbb{C}^*, \Omega_{\mathbb{C}^*})$$

in $H^1_{\underline{S}_2}(\mathbb{C}^*, \Omega_{\mathbb{C}^*})$. Therefore, the image of

$$[\omega_1] \star [\omega_2] \in \Omega(\mathbb{C}^* \setminus S_1S_2)/\Omega(\mathbb{C}^*) \simeq H^1_{S_1S_2}(\mathbb{C}^*, \Omega_{\mathbb{C}^*})$$

in $H^1_{\underline{S}_1\underline{S}_2}(\mathbb{C}^*, \Omega_{\mathbb{C}^*})$ is the same as the image of $[\bar\partial\underline{\omega}_1 \star \bar\partial\underline{\omega}_2]$ by the canonical morphism

$$H^1(\Gamma_{\underline{S}_1\underline{S}_2}(\mathbb{C}^*, \mathcal{C}^{(1,\bullet)}_{\infty,\mathbb{C}^*})) \to H^1_{\underline{S}_1\underline{S}_2}(\mathbb{C}^*, \Omega_{\mathbb{C}^*}).$$

Let us note $p_1, p_2\colon\ \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^*$, the two canonical projections and consider the commutative diagram

$$\begin{tikzcd}
\mathbb{C}^* \times \mathbb{C}^* \arrow[rr, shift left, "\phi"] \arrow[dr, "\mu"'] & & \mathbb{C}^* \times \mathbb{C}^* \arrow[ll, shift left, "\psi"] \arrow[dl, "p_2"] \\
& \mathbb{C}^*, &
\end{tikzcd}$$

where $\phi(z_1, z_2) = (z_1, z_1z_2)$ and $\psi(\zeta, z) = (\zeta, z/\zeta)$. Since $\phi \circ \psi = \mathrm{id} = \psi \circ \phi$, we have

$$\int_\mu = \int_{p_2} \circ \int_\phi = \int_{p_2} \circ\, \psi^*.$$

Therefore,

$$\begin{aligned}
\bar\partial\underline{\omega}_1 \star \bar\partial\underline{\omega}_2 &= \int_\mu (\bar\partial\underline{\omega}_1 \boxtimes \bar\partial\underline{\omega}_2) \\
&= \int_{p_2} (\psi^*(p_1^*\bar\partial\underline{\omega}_1 \wedge p_2^*\bar\partial\underline{\omega}_2)) \\
&= \int_{p_2} (p_1^*\bar\partial\underline{\omega}_1 \wedge h^*\bar\partial\underline{\omega}_2),
\end{aligned}$$

where $h(\zeta, z) = z/\zeta$. Since

$$\bar\partial\underline{\omega}_1 = \frac{\partial \underline{f}_1}{\partial \bar z}(z) d\bar z \wedge dz \quad \text{and} \quad \bar\partial\underline{\omega}_2 = \frac{\partial \underline{f}_2}{\partial \bar z}(z) d\bar z \wedge dz,$$

we have

$$\begin{aligned}
h^*\bar\partial\underline{\omega}_2 &= \frac{\partial \underline{f}_2}{\partial \bar z}\Big(\frac{z}{\zeta}\Big) d\Big(\frac{\bar z}{\bar\zeta}\Big) \wedge d\Big(\frac{z}{\zeta}\Big) \\
&= \frac{\partial \underline{f}_2}{\partial \bar z}\Big(\frac{z}{\zeta}\Big) \frac{\bar\zeta d\bar z - \bar z\, d\bar\zeta}{\bar\zeta^2} \wedge \frac{\zeta\, dz - z\, d\zeta}{\zeta^2}
\end{aligned}$$

and

$$p_1^*\bar{\partial}\underline{\omega}_1 \wedge h^*\bar{\partial}\underline{\omega}_2 = \frac{\partial \underline{f}_1}{\partial \bar{z}}(\zeta)\frac{\partial \underline{f}_2}{\partial \bar{z}}\Big(\frac{z}{\zeta}\Big)\frac{d\bar{\zeta}}{\bar{\zeta}} \wedge \frac{d\zeta}{\zeta} \wedge d\bar{z} \wedge dz.$$

Therefore,

$$\bar{\partial}\underline{\omega}_1 \star \bar{\partial}\underline{\omega}_2 = \Big(\int_{\mathbb{C}^*} \frac{\partial \underline{f}_1}{\partial \bar{z}}(\zeta)\frac{\partial \underline{f}_2}{\partial \bar{z}}\Big(\frac{z}{\zeta}\Big)\frac{d\bar{\zeta}}{\bar{\zeta}} \wedge \frac{d\zeta}{\zeta}\Big)d\bar{z} \wedge dz.$$

Since $\underline{f}_1$ coincides with f_1 on $\mathbb{C}^* \setminus \underline{S}_1$, one has

$$\operatorname{supp}\Big(\zeta \mapsto \frac{\partial \underline{f}_1}{\partial \bar{z}}(\zeta)\Big) \subset \underline{S}_1.$$

Similarly, one has

$$\operatorname{supp}\Big(\zeta \mapsto \frac{\partial \underline{f}_2}{\partial \bar{z}}\Big(\frac{z}{\zeta}\Big)\Big) \subset z\underline{S}_2^{-1}.$$

Hence,

$$\zeta \mapsto \frac{\partial \underline{f}_1}{\partial \bar{z}}(\zeta)\frac{\partial \underline{f}_2}{\partial \bar{z}}\Big(\frac{z}{\zeta}\Big)$$

is an infinitely differentiable function on $\mathbb{C}^*$ supported by $\underline{S}_1 \cap z\underline{S}_2^{-1}$ which is a compact subset of $\mathbb{C}^*$.

Since U is a relatively compact open subset of $\mathbb{C}^*$ and $\underline{S}_1$ and $\underline{S}_2$ are convolvable closed subsets of $\mathbb{C}^*$,

$$K = \underline{S}_1 \cap \bar{U}\underline{S}_2^{-1}$$

is a compact subset of $\mathbb{C}^*$. Let c be a singular infinitely differentiable 2-chain of $\mathbb{C}^*$ such that

$$[c] \in H_2(\mathbb{C}^*, \mathbb{C}^* \setminus K)$$

is the relative orientation class $\alpha_{\mathbb{C}^*,K}$. Then, on U, one has

$$\bar{\partial}\underline{\omega}_1 \star \bar{\partial}\underline{\omega}_2 = \Big(\int_c \frac{\partial \underline{f}_1}{\partial \bar{z}}(\zeta)\frac{\partial \underline{f}_2}{\partial \bar{z}}\Big(\frac{z}{\zeta}\Big)\frac{d\bar{\zeta}}{\bar{\zeta}} \wedge \frac{d\zeta}{\zeta}\Big)d\bar{z} \wedge dz,$$

since the integrated form is supported by $\underline{S}_1 \cap z\underline{S}_2^{-1} \subset K$ for any $z \in U$. Moreover, the function $\underline{f}_2$ is infinitely differentiable on $\mathbb{C}^*$ and the chain c is supported by a compact subset of $\mathbb{C}^*$. Thus, the function

$$f\colon z \mapsto \int_c \frac{\partial \underline{f}_1}{\partial \bar{z}}(\zeta)\underline{f}_2\Big(\frac{z}{\zeta}\Big)\, d\bar{\zeta} \wedge \frac{d\zeta}{\zeta}$$

is infinitely differentiable on $\mathbb{C}^*$ and

$$\frac{\partial f}{\partial \bar{z}}(z) = \int_c \frac{\partial \underline{f}_1}{\partial \bar{z}}(\zeta)\frac{\partial \underline{f}_2}{\partial \bar{z}}\Big(\frac{z}{\zeta}\Big)\frac{d\bar{\zeta}}{\bar{\zeta}} \wedge \frac{d\zeta}{\zeta}.$$

Therefore, on U, one has

$$\bar{\partial}\underline{\omega}_1 \star \bar{\partial}\underline{\omega}_2 = \bar{\partial}\omega$$

where $\omega = f(z)\,dz$. Since $\operatorname{supp}(\bar{\partial}\underline{\omega}_1 \star \bar{\partial}\underline{\omega}_2) \subset \underline{S}_1\underline{S}_2$, the function f is holomorphic on $U \setminus \underline{S}_1\underline{S}_2$ and it follows from what precedes that

$$([\omega_1] \star [\omega_2])|_U = [\omega|_U]$$

in

$$\Omega(U\setminus\underline{S}_1\underline{S}_2)/\Omega(U) \simeq H^1_{(\underline{S}_1\underline{S}_2)\cap U}(U, \Omega_{\mathbb{C}^*}).$$

Let us now show how to compute $[\omega|_U]$ in $\Omega(U \setminus \overline{V})/\Omega(U)$ by means of f_1 and f_2 alone. Since V is an open neighborhood of $\underline{S}_1\underline{S}_2$,

$$\underline{S}_1 \cap (\overline{U} \setminus V)\underline{S}_2^{-1} = \emptyset.$$

Therefore,

$$\mathbb{C}^* = (\mathbb{C}^* \setminus \underline{S}_1) \cup \big(\mathbb{C}^* \setminus ((\overline{U} \setminus V)\underline{S}_2^{-1})\big)$$

and, replacing c if necessary by a barycentric subdivision, we may assume that $c = c_1 + c_2$, where

$$\operatorname{supp} c_1 \subset \mathbb{C}^* \setminus \underline{S}_1 \quad \text{and} \quad \operatorname{supp} c_2 \subset \mathbb{C}^* \setminus ((\overline{U} \setminus V)\underline{S}_2^{-1}).$$

Since $\operatorname{supp} \frac{\partial f_1}{\partial \bar{z}} \subset \underline{S}_1$, it is then clear that

$$f(z) = \int_{c_2} \frac{\partial \underline{f}_1}{\partial \bar{z}}(\zeta)\underline{f}_2\Big(\frac{z}{\zeta}\Big)\, d\bar{\zeta} \wedge \frac{d\zeta}{\zeta}.$$

Moreover, for any $z \in \overline{U} \setminus V$ one has

$$\mathbb{C}^* \setminus z\underline{S}_2^{-1} \supset \mathbb{C}^* \setminus ((\overline{U} \setminus V)\underline{S}_2^{-1}) \supset \operatorname{supp} c_2$$

and since the function $\zeta \mapsto \underline{f}_2(z/\zeta)$ is holomorphic on $\mathbb{C}^* \setminus z\underline{S}_2^{-1}$, it follows that

$$\begin{aligned} f(z) &= \int_{c_2} \frac{\partial}{\partial \bar{\zeta}}\Big(\underline{f}_1(\zeta)\underline{f}_2\Big(\frac{z}{\zeta}\Big)\frac{1}{\zeta}\Big)\, d\bar{\zeta} \wedge d\zeta \\ &= \int_{\partial c_2} \underline{f}_1(\zeta)\underline{f}_2\Big(\frac{z}{\zeta}\Big)\frac{d\zeta}{\zeta}. \end{aligned}$$

By construction,

$$\operatorname{supp}(\partial c) \subset \mathbb{C}^* \setminus K = (\mathbb{C}^* \setminus \underline{S}_1) \cup (\mathbb{C}^* \setminus \overline{U}\underline{S}_2^{-1}).$$

Replacing c, if necessary, by one of its barycentric subdivisions, we may thus assume that $\partial c = c_1' + c_2'$ where $\operatorname{supp} c_1' \subset \mathbb{C}^* \setminus \underline{S}_1$ and $\operatorname{supp} c_2' \subset \mathbb{C}^* \setminus \overline{U}\underline{S}_2^{-1}$. Since

$$\partial c_1 + \partial c_2 = \partial c = c_1' + c_2',$$

there is a chain c_3 such that

$$\partial c_2 - c_2' = c_3 = c_1' - \partial c_1.$$

Since $\operatorname{supp} c_2' \subset \mathbb{C}^* \setminus \overline{U}\underline{S}_2^{-1}$, the function

$$z \mapsto \int_{c_2'} \underline{f}_1(\zeta)\underline{f}_2\Big(\frac{z}{\zeta}\Big)\,\frac{d\zeta}{\zeta}$$

is clearly holomorphic on U. Hence, the image of $[\omega|_U]$ in $\Omega(U\setminus\overline{V})/\Omega(U)$ is $[g(z)\,dz]$, where g is the holomorphic function on $U \setminus \overline{V}$ defined by setting

$$g(z) = \int_{c_3} \underline{f}_1(\zeta)\underline{f}_2\Big(\frac{z}{\zeta}\Big)\,\frac{d\zeta}{\zeta}.$$

Since

$$\operatorname{supp}(\partial c_2 - c_2') \subset \big(\mathbb{C}^* \setminus ((\overline{U} \setminus V)\underline{S}_2^{-1})\big) \cup (\mathbb{C}^* \setminus \overline{U}\underline{S}_2^{-1}) = \mathbb{C}^* \setminus ((\overline{U} \setminus V)\underline{S}_2^{-1})$$

and

$$\operatorname{supp}(c_1' - \partial c_1) \subset (\mathbb{C}^* \setminus \underline{S}_1) \cup (\mathbb{C}^* \setminus \underline{S}_1) = \mathbb{C}^* \setminus \underline{S}_1,$$

it is clear that

$$\operatorname{supp} c_3 \subset (\mathbb{C}^* \setminus \underline{S}_1) \cap \big(\mathbb{C}^* \setminus ((\overline{U} \setminus V)\underline{S}_2^{-1})\big).$$

Therefore, we have in fact

$$g(z) = \int_{c_3} f_1(\zeta)f_2\Big(\frac{z}{\zeta}\Big)\,\frac{d\zeta}{\zeta}$$

for any $z \in U \setminus \overline{V}$. Moreover, since $\partial c_3 = \partial c_1' = -\partial c_2'$, it is clear that

$$\operatorname{supp} \partial c_3 \subset (\mathbb{C}^* \setminus \underline{S}_1) \cap (\mathbb{C}^* \setminus \overline{U}\underline{S}_2^{-1}).$$

So,

$$c_3 \in Z_1\big((\mathbb{C}^* \setminus \underline{S}_1) \cap \big(\mathbb{C}^* \setminus ((\overline{U} \setminus V)\underline{S}_2^{-1})\big), (\mathbb{C}^* \setminus \underline{S}_1) \cap (\mathbb{C}^* \setminus \overline{U}\underline{S}_2^{-1})\big)$$

and it follows by construction that it is a relative Hadamard cycle for $\underline{S}_1$ with respect to $\overline{U}\underline{S}_2^{-1}$ in $\mathbb{C}^* \setminus (\overline{U} \setminus V)\underline{S}_2^{-1}$ (apply Remark 5.4 with $F = \underline{S}_1$, $G = \overline{U}\underline{S}_2^{-1}$ and $W = \mathbb{C}^* \setminus ((\overline{U} \setminus V)\underline{S}_2^{-1})$). Thus, c_3 is also a relative Hadamard cycle for S_1 with respect to $\overline{U}S_2^{-1}$ in $\mathbb{C}^* \setminus (\overline{U} \setminus V)S_2^{-1}$.

To conclude, it remains to show that if c_3' is another relative Hadamard cycle for S_1 with respect to $\overline{U}S_2^{-1}$ in $\mathbb{C}^* \setminus (\overline{U} \setminus V)S_2^{-1}$ and if

$$\check{g}(z) = \int_{c_3'} f_1(\zeta)f_2\Big(\frac{z}{\zeta}\Big)\,\frac{d\zeta}{\zeta}$$

for any $z \in U \setminus \overline{V}$, then $[g(z)\,dz] = [\check{g}(z)\,dz]$ in $\Omega(U\setminus\overline{V})/\Omega(U)$. For such a c_3', we have $[c_3] = [c_3']$ in

$$H_1\big((\mathbb{C}^* \setminus \underline{S}_1) \cap (\mathbb{C}^* \setminus (\overline{U} \setminus V)\underline{S}_2^{-1}), (\mathbb{C}^* \setminus \underline{S}_1) \cap (\mathbb{C}^* \setminus \overline{U}\underline{S}_2^{-1})\big).$$

Therefore, $c_3' = c_3 + c_4 + \partial c_5$ where c_4 is a 1-chain of $(\mathbb{C}^* \setminus \underline{S}_1) \cap (\mathbb{C}^* \setminus \overline{U}\underline{S}_2^{-1})$ and c_5 is a 2-chain of $(\mathbb{C}^* \setminus \underline{S}_1) \cap (\mathbb{C}^* \setminus (\overline{U} \setminus V)\underline{S}_2^{-1})$. It follows that the function

$$\check{g}\colon z \mapsto \int_{c_3'} f_1(\zeta) f_2\Big(\frac{z}{\zeta}\Big)\,\frac{d\zeta}{\zeta}$$

is a holomorphic function on $U \setminus \overline{V}$ and that

$$\check{g}(z) = g(z) + \int_{c_4} f_1(\zeta) f_2\Big(\frac{z}{\zeta}\Big)\,\frac{d\zeta}{\zeta}$$

on $U \setminus \overline{V}$. Since

$$z \mapsto \int_{c_4} f_1(\zeta) f_2\Big(\frac{z}{\zeta}\Big)\,\frac{d\zeta}{\zeta}$$

is clearly holomorphic on U, we have $[g(z)\,dz] = [\check{g}(z)\,dz]$ in $\Omega(U\setminus\overline{V})/\Omega(U)$ as expected. □

§6. The case of strongly convolvable sets

It is natural to ask whether one can compute the holomorphic cohomological multiplicative convolution on $\mathbb{C}^*$ thanks to a global formula, by adding extra conditions on S_1 and S_2. Recalling Definition 2.3, we are led to introduce the following one.

Definition 6.1. Let S_1 and S_2 be two convolvable proper closed subsets of $\mathbb{C}^*$ such that $S_1S_2 \neq \mathbb{C}^*$. These two closed sets are said to be *strongly convolvable* if, furthermore, $\overline{S}_1$ and $\overline{S}_2$ are star-eligible, that is to say, if $\overline{S}_1 \times \overline{S}_2 \subset M$. (Here $\overline{(.)}$ denotes the closure in $\mathbb{P}$).

Remark 6.2. One can find convolvable proper closed subsets of $\mathbb{C}^*$ which are not strongly convolvable. For example, consider

$$S_1 = \big\{(2m)! : m \in \mathbb{N}\big\} \quad \text{and} \quad S_2 = \big\{\tfrac{1}{(2n+1)!} : n \in \mathbb{N}\big\}.$$

We shall now highlight the link with the generalized Hadamard product. Recall Definitions 2.4 and 2.5.

Proposition 6.3. *Let S_1 and S_2 be two strongly convolvable proper closed subsets of $\mathbb{C}^*$. Assume that $\omega_1 = f_1\,dz$ and $\omega_2 = f_2\,dz$ with $f_1 \in \mathcal{O}(\mathbb{C}^* \setminus S_1)$ and $f_2 \in$*

$\mathcal{O}(\mathbb{C}^* \setminus S_2)$. For all $z \in \mathbb{C}^* \setminus S_1S_2$, let c_z be a generalized Hadamard cycle for $\bar{S}_1$ in $\mathbb{P} \setminus (z\bar{S}_2^{-1} \cup (\{0,\infty\} \setminus \bar{S}_1))$. Then

$$[\omega_1] \star [\omega_2] = [f\,dz] \in \Omega(\mathbb{C}^* \setminus S_1S_2)/\Omega(\mathbb{C}^*),$$

where

$$f(z) = -\int_{c_z} f_1(\zeta) f_2\Big(\frac{z}{\zeta}\Big)\,\frac{d\zeta}{\zeta}$$

for all $z \in \mathbb{C}^* \setminus S_1S_2$.

Proof. Let U be a relatively compact open subset of $\mathbb{C}^*$ and V an open neighborhood of S_1S_2 in $\mathbb{C}^*$. Let c be a relative Hadamard cycle for S_1 with respect to $\bar{U}S_2^{-1}$ in $\mathbb{C}^* \setminus (\bar{U} \setminus V)S_2^{-1}$. Then, by a similar argument to the proof of Lemma 2.6, it is clear that the image of $[c_z]$ by the sequence of canonical maps

$$\begin{array}{c}
H_1(\mathbb{P} \setminus (\bar{S}_1 \cup z\bar{S}_2^{-1} \cup \{0\} \cup \{\infty\})) = H_1(\mathbb{C}^* \setminus (S_1 \cup zS_2^{-1})) \\
\downarrow \\
{}^{\mathrm{BM}}H_1(\mathbb{C}^* \setminus (S_1 \cup zS_2^{-1})) \\
\downarrow \\
{}^{\mathrm{BM}}H_1\big((\mathbb{C}^* \setminus S_1) \cap (\mathbb{C}^* \setminus (\bar{U} \setminus V)S_2^{-1})\big) \\
\downarrow \\
H_1\big((\mathbb{C}^* \setminus S_1) \cap (\mathbb{C}^* \setminus (\bar{U} \setminus V)S_2^{-1}), (\mathbb{C}^* \setminus S_1) \cap (\mathbb{C}^* \setminus \bar{U}S_2^{-1})\big)
\end{array}$$

is $[-c]$ for all $z \in \bar{U} \setminus V$. Hence

$$\int_c f_1(\zeta) f_2\Big(\frac{z}{\zeta}\Big)\,\frac{d\zeta}{\zeta} = -\int_{c_z} f_1(\zeta) f_2\Big(\frac{z}{\zeta}\Big)\,\frac{d\zeta}{\zeta} \quad \forall\, z \in U \setminus \bar{V}.$$

Since this argument is valid for all U and all V, the conclusion follows from Theorem 5.5. □

In this context, we set $(f_1 \star f_2)(z) = \frac{1}{2i\pi} \int_{c_z} f_1(\zeta) f_2(\frac{z}{\zeta})\,\frac{d\zeta}{\zeta}$. If $f_1 \in \mathcal{O}(\mathbb{P} \setminus \bar{S}_1)$ and $f_2 \in \mathcal{O}(\mathbb{P} \setminus \bar{S}_2)$, this really coincides with the generalized Hadamard product.

Remark 6.4. Let S_1 and S_2 be two strongly convolvable proper closed subsets of $\mathbb{C}^*$. Let us make an identification $f\,dz \leftrightarrow -2i\pi f$ between holomorphic 1-forms and holomorphic functions. Then, by the previous proposition, the holomorphic cohomological convolution morphism

$$H^1_{S_1}(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \otimes H^1_{S_2}(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \to H^1_{S_1S_2}(\mathbb{C}^*, \Omega_{\mathbb{C}^*})$$

can be seen as a bilinear map

$$\mathcal{O}(\mathbb{C}^* \setminus S_1)/\mathcal{O}(\mathbb{C}^*) \times \mathcal{O}(\mathbb{C}^* \setminus S_2)/\mathcal{O}(\mathbb{C}^*) \to \mathcal{O}(\mathbb{C}^* \setminus S_1S_2)/\mathcal{O}(\mathbb{C}^*),$$

which can be computed by

$$[f_1] \star [f_2] = [f_1 \star f_2].$$

For the following example, we use the notation $D(0, R) = \{z \in \mathbb{C} : |z| < R\}$ with $R > 0$.

Example 6.5. Let $S = \mathbb{C}^* \setminus D(0, s)$ and $T = \mathbb{C}^* \setminus D(0, t)$ with $s > 0$, $t > 0$ and let

$$f \in \mathcal{O}(\mathbb{C}^* \setminus S) = \mathcal{O}(D(0, s)\setminus\{0\}) \quad \text{and} \quad g \in \mathcal{O}(\mathbb{C}^* \setminus T) = \mathcal{O}(D(0, t)\setminus\{0\})$$

be two holomorphic functions. Then S and T are strongly convolvable proper closed subsets of $\mathbb{C}^*$ and we can write $f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n$, $g(z) = \sum_{n=-\infty}^{+\infty} b_n z^n$. Since the polar part of f (resp. g) is holomorphic on $\mathbb{C}^*$, we have $[f] = [\sum_{n=0}^{+\infty} a_n z^n]$ in $\mathcal{O}(D(0, s)\setminus\{0\})/\mathcal{O}(\mathbb{C}^*)$ and $[g] = [\sum_{n=0}^{+\infty} b_n z^n]$ in $\mathcal{O}(D(0, t)\setminus\{0\})/\mathcal{O}(\mathbb{C}^*)$. Using the preceding remark, we see that the holomorphic cohomological convolution $[f] \star [g]$ is given by

$$[f \star g] = \left[\sum_{n=0}^{+\infty} a_n b_n z^n\right],$$

since the generalized Hadamard product coincides with the usual one in this case.

Let us now state a trivial proposition.

Proposition 6.6. *Let S_1 and S_2 be two convolvable closed subsets of $\mathbb{C}^*$ and $S_1' \subset S_1$, $S_2' \subset S_2$ two closed subsets. Then S_1' and S_2' are convolvable and the diagram*

$$\begin{tikzcd}
H^1_{S_1}(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \otimes H^1_{S_2}(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \arrow[r] & H^1_{S_1S_2}(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \\
H^1_{S_1'}(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \otimes H^1_{S_2'}(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \arrow[r] \arrow[u] & H^1_{S_1'S_2'}(\mathbb{C}^*, \Omega_{\mathbb{C}^*}), \arrow[u]
\end{tikzcd}$$

where the horizontal arrows are given by the holomorphic cohomological convolution morphisms, is commutative.

Example 6.5 combined with Proposition 6.6 allows us to compute several other examples.

Example 6.7. Let $S_1 = S_2 = (-\infty, -1]$. The principal determination of the function $z \mapsto \ln(1+z)$ is holomorphic on $\mathbb{C}^* \setminus S_1$. Moreover, S_1 and S_2 are strongly convolvable and thus, there is $g \in \mathcal{O}(\mathbb{C}^* \setminus [1, +\infty))$ such that

$$[\ln(1+z)] \star [\ln(1+z)] = [g].$$

Using the previous results, one has

$$\begin{aligned}
([\ln(1+z)] \star [\ln(1+z)])|_{D(0,1)} &= [\ln(1+z)|_{D(0,1)}] \star [\ln(1+z)|_{D(0,1)}] \\
&= \left[\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} z^n\right] \star \left[\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} z^n\right] \\
&= \left[\sum_{n=1}^{\infty} \frac{z^n}{n^2}\right] \\
&= [\mathrm{Li}_2(z)]|_{D(0,1)},
\end{aligned}$$

where Li_2 is the principal dilogarithm function, holomorphic on $\mathcal{O}(\mathbb{C}^* \setminus [1, +\infty))$. Hence, there is $h \in \mathcal{O}(\mathbb{C}^*)$ such that

$$g|_{D(0,1)} - \mathrm{Li}_2|_{D(0,1)} = h.$$

By the uniqueness of the analytic continuation, one deduces that $g - \mathrm{Li}_2 = h$ on $\mathbb{C}^* \setminus [1, +\infty)$ and, thus, that

$$[\ln(1+z)] \star [\ln(1+z)] = [\mathrm{Li}_2(z)]$$

in $\mathcal{O}(\mathbb{C}^* \setminus S_1 S_2) / \mathcal{O}(\mathbb{C}^*)$.

§7. Further applications

As we saw, the holomorphic cohomological convolution is well fitted to study the Hadamard product in the noncompact setting. It should therefore be a good tool to study Hadamard convolution operators associated with convolvable closed subsets of $\mathbb{C}^*$ (see e.g. [11] for the compact setting). Actually, this point of view has already been fruitful in the additive version of the holomorphic cohomological convolution. In [5], we defined a natural notion of convolution between analytic functionals with noncompact convex carrier (generalizing the work of Méril in [13]) and showed compatibility with the additive holomorphic cohomological convolution, modulo some growth conditions. We also explained that this convolution is transformed into a product by the enhanced Laplace transform studied in [6]. Hence, the cohomological framework offers additional clarity concerning these contour-integration transformations.

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