LIÈGE université
Sciences Appliquées

Structural & Stochastic Dynamics
Urban & Environmental Engineering

Perturbation analysis of a Multi-Degree-Of-Freedom system equipped with only one tuned mass damper

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Dynamical Systems :Theory and Applications
December 2-5, 2019, Łódź, POLAND

UEE
Urban & Environmental Engineering

Context

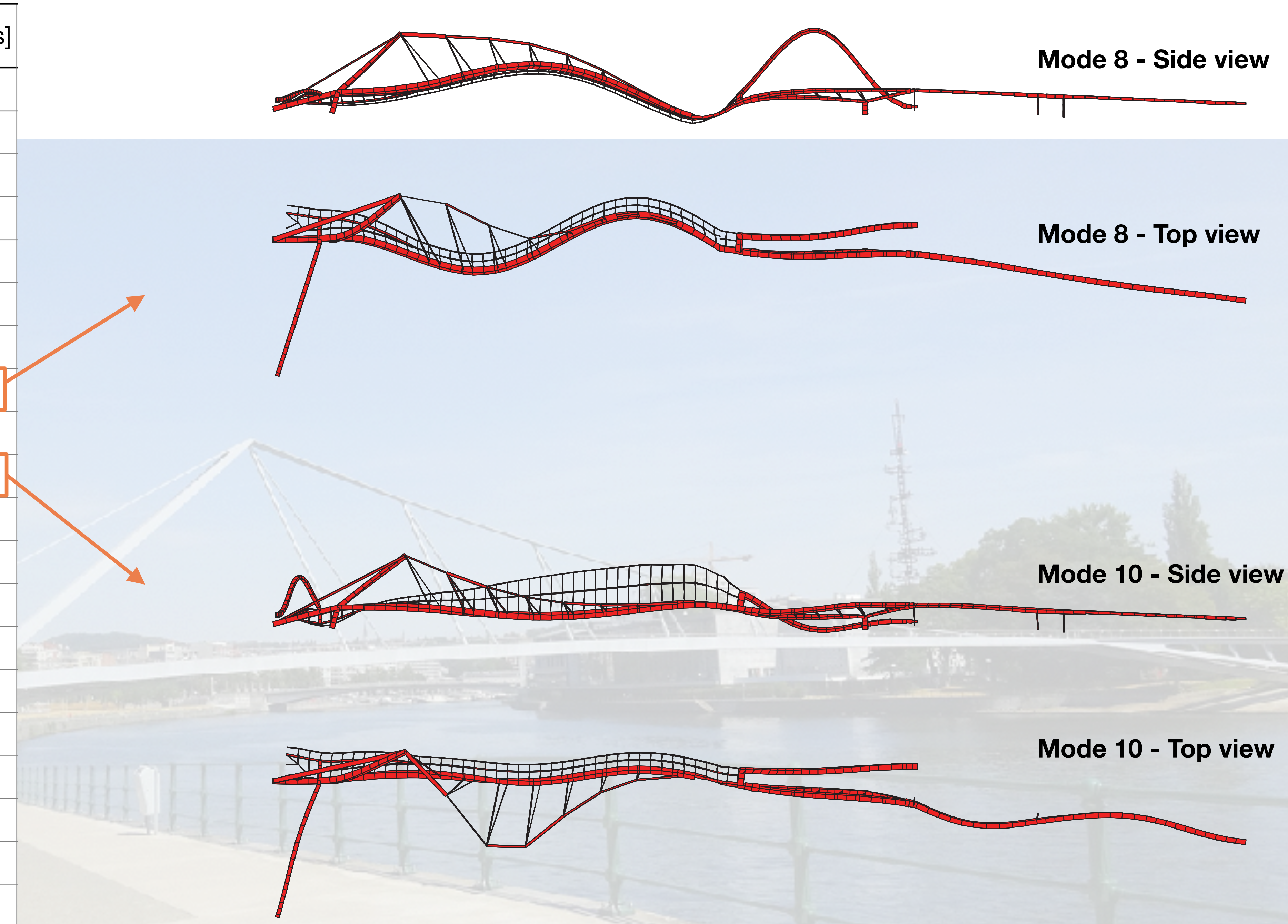
Structures with clustered natural frequencies & similar mode shapes

La belle liégeoise, Liège, BELGIUM
Design: Greisch, Construction: Poncin
Photo: Wikipedia

Context

Structures with clustered natural frequencies & similar mode shapes

Mode	Frequency [Hz]	Mass [tons]
1	0,390	114
2	0,661	97
3	0,826	162
4	1,055	75
5	1,103	57
6	1,338	21
7	1,343	95
8	1,411	79
9	1,555	33
10	1,644	39
11	1,656	19
12	1,663	15
13	1,673	35
14	1,896	91
15	2,138	15
16	2,324	31
17	2,394	73
18	2,453	44
19	2,572	49
20	2,728	31
21	2,875	955
22	2,938	55



Context

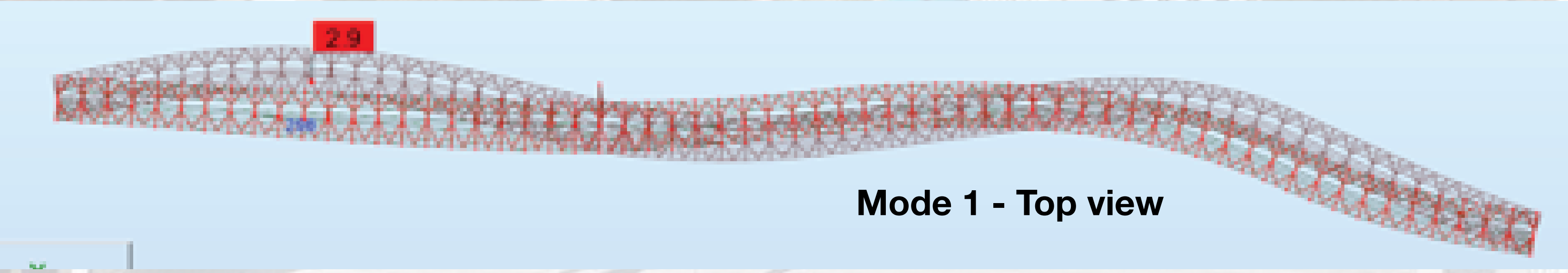
Structures with clustered natural frequencies & similar mode shapes

Passerelle de Mantes-la-Jolie-Limay, FRANCE
Design: Terrell Group, Construction: Viry
Photo: M. Fieschi , le Parisien

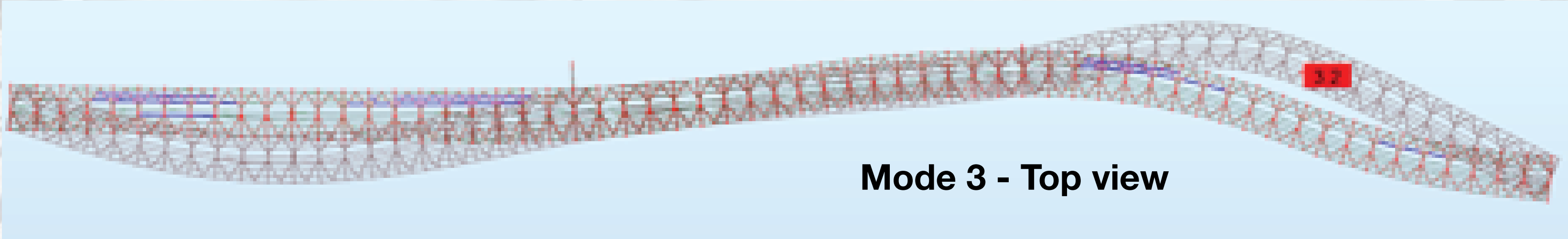
Context

Structures with clustered natural frequencies & similar mode shapes

Mode	Frequency [Hz]	Mass [ton]
1	1,12	86
2	1,20	83
3	1,26	77
4	1,55	65
5	2,11	49
6	2,26	40
7	2,61	60



Mode 1 - Top view



Mode 3 - Top view

Illustrative problem

Structures with clustered natural frequencies & similar mode shapes

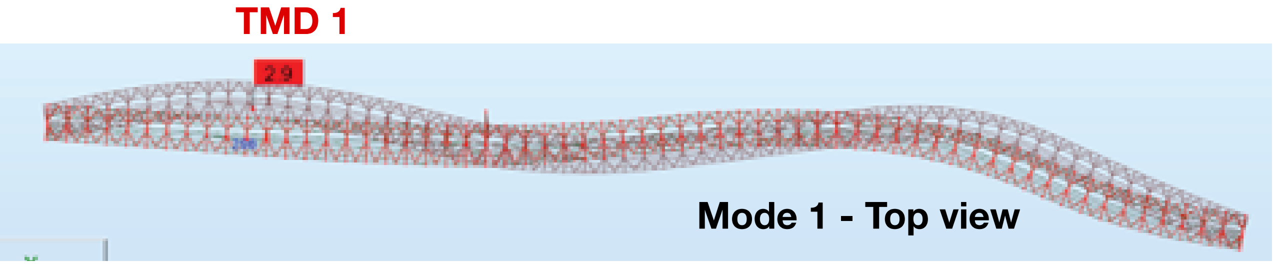
Equipped with several Tuned Mass Dampers (TMD)

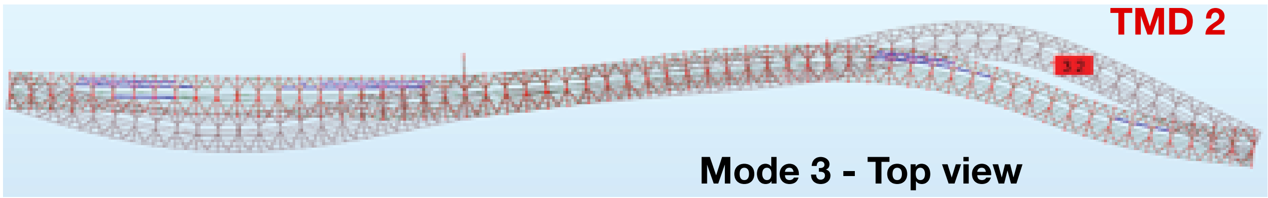
7 TMDs

3 TMDs

$$\left(\begin{array}{c} \text{1 Vertical mode} \\ \boxed{\text{2 Horizontal modes}} \end{array} \right)$$

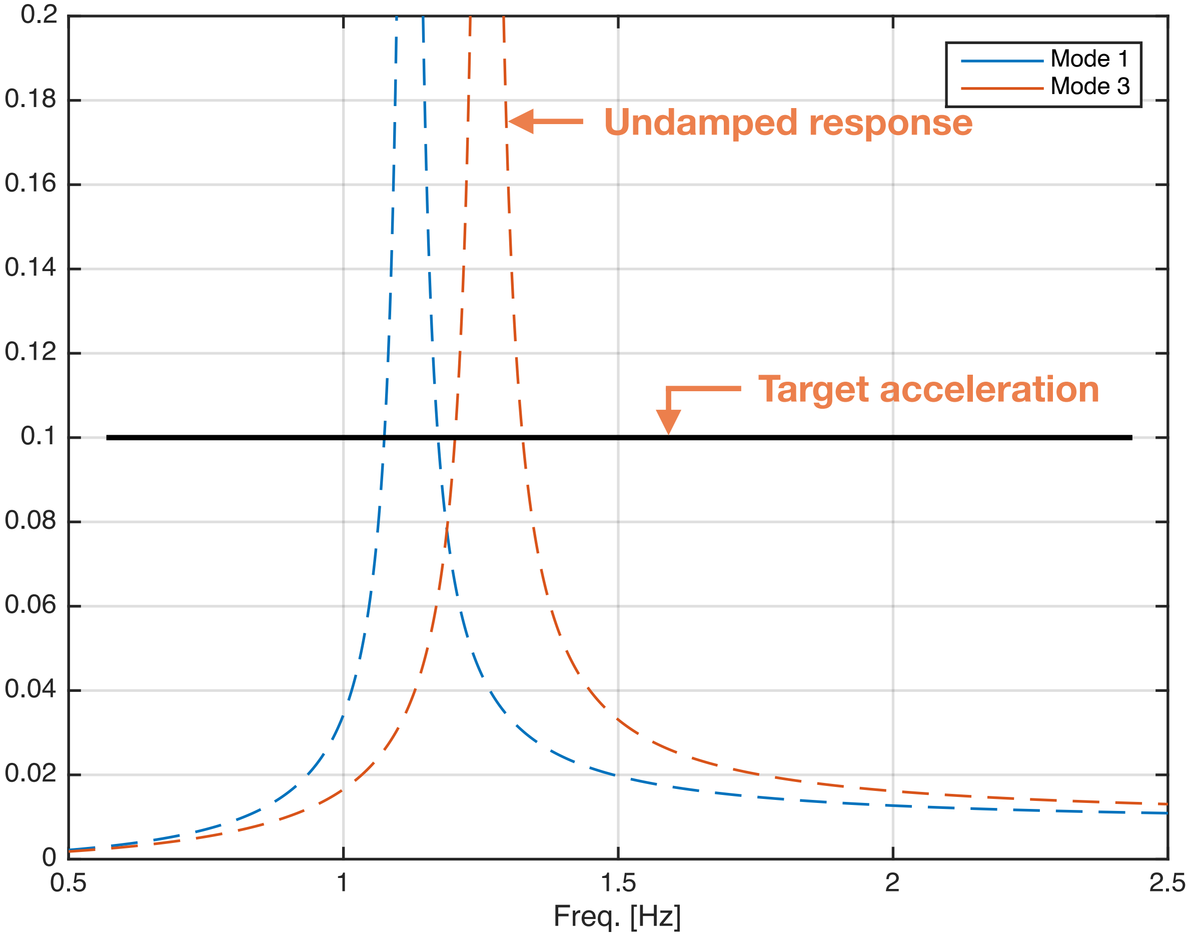
Illustrative problem





Mode	Frequency [Hz]	Mass [tons]	TMD 1	TMD 2
1	1,12	86	1 (antinode)	0,65
3	1,26	77	-0,9	1 (antinode)

Structural (inherent) damping: 0.4% (in each mode)
Generalized load: 750 N (in each mode)
TMD1: 1400 kg, TMD2: 1400 kg

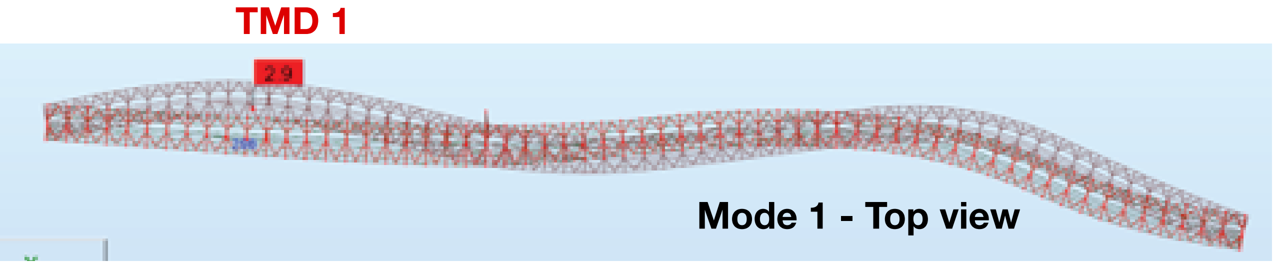


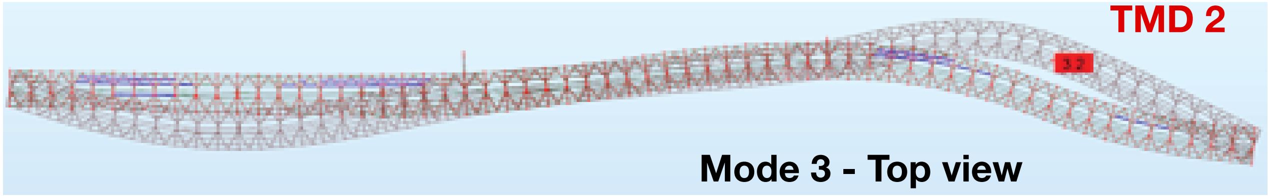
Addition of 2 TMD

Use 1-DOF design formula
(e.g. den Hartog, Warburton)

i.e. treat each mode separately

Illustrative problem





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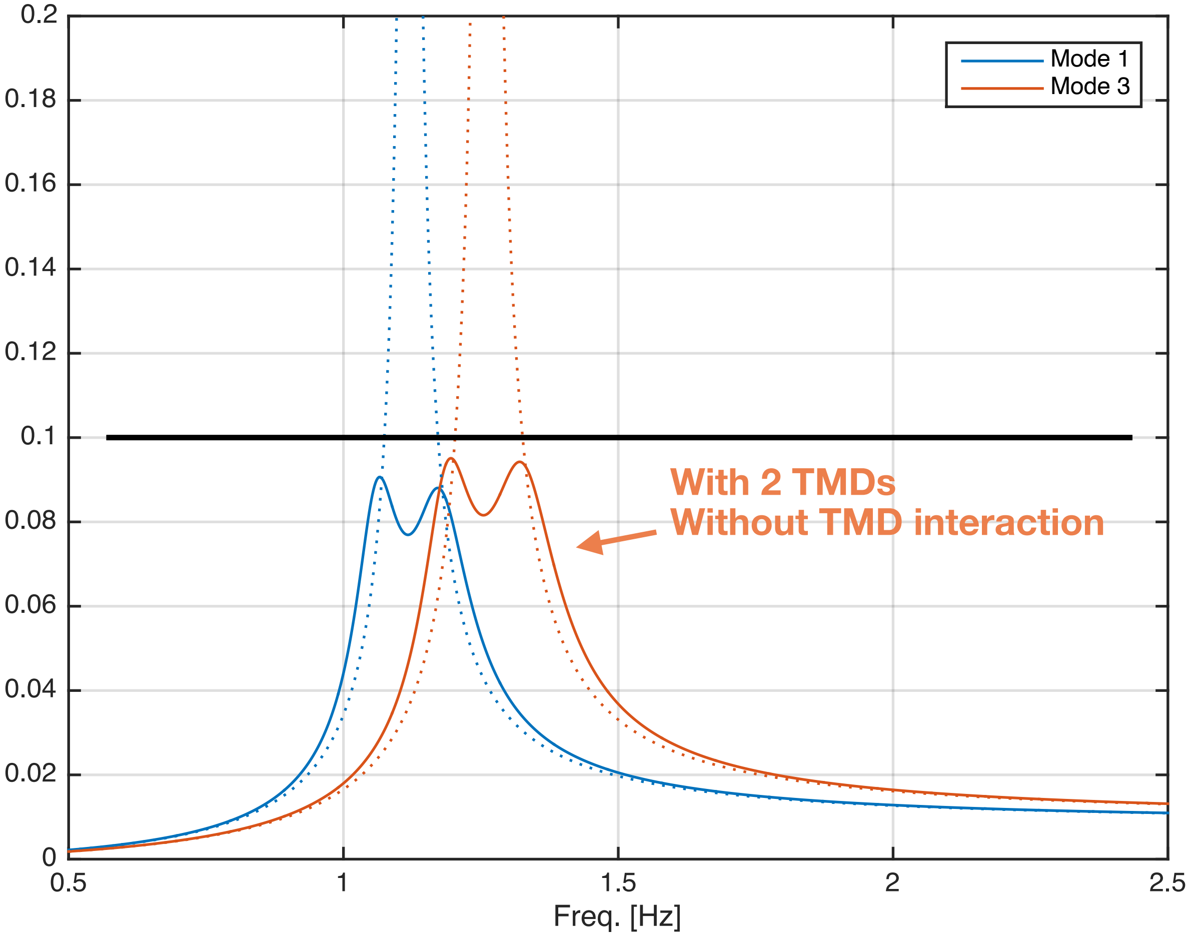


Addition of 2 TMD

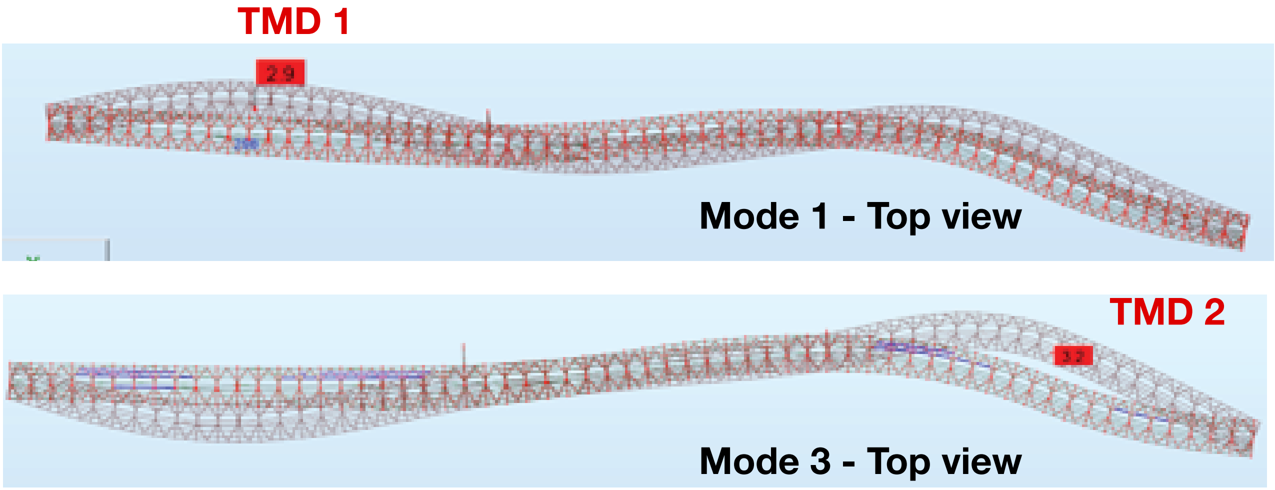
Use 1-DOF design formula
(e.g. den Hartog, Warburton)

i.e. treat each mode separately

This is how you believe the structure will behave …

But in fact, there is coupling …

Illustrative problem



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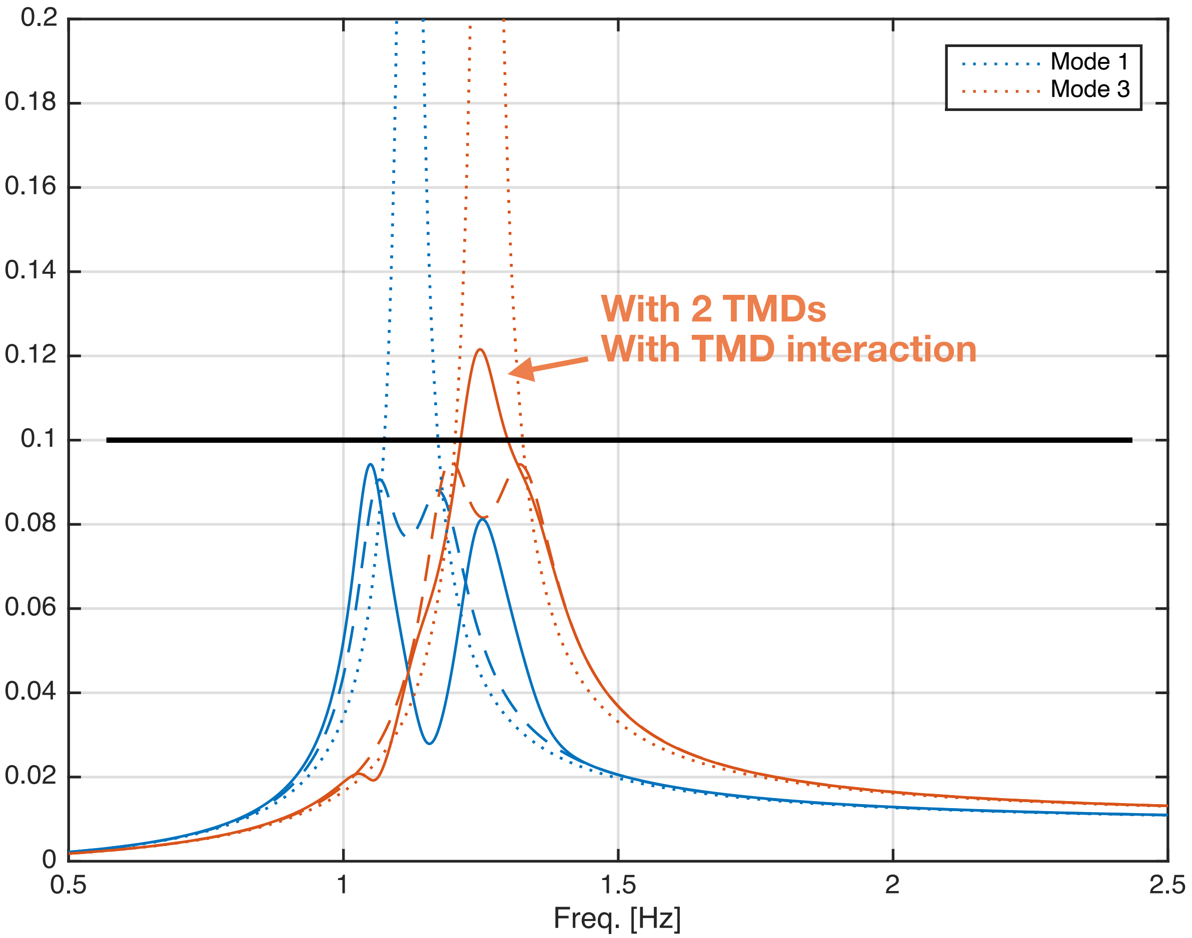


Addition of 2 TMD

Use 1-DOF design formula
(e.g. den Hartog, Warburton)

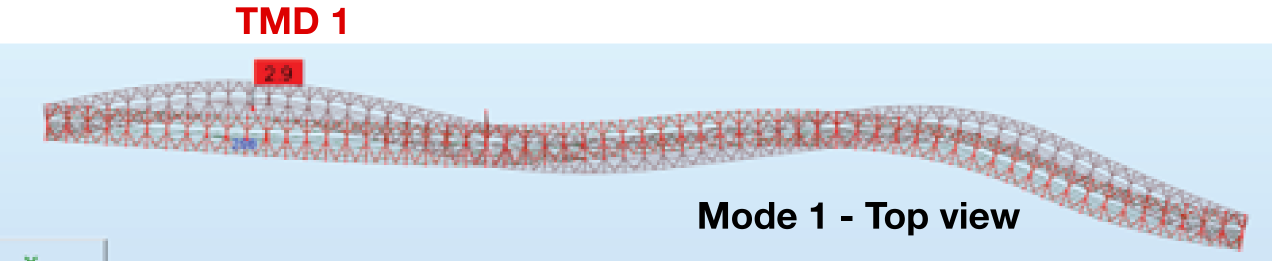
i.e. treat each mode separately

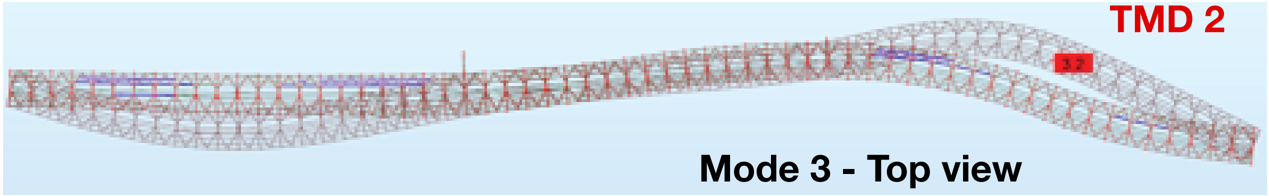
This is how you believe the structure will behave …

But in fact, there is coupling …

Loss of accuracy of 25% !

Illustrative problem





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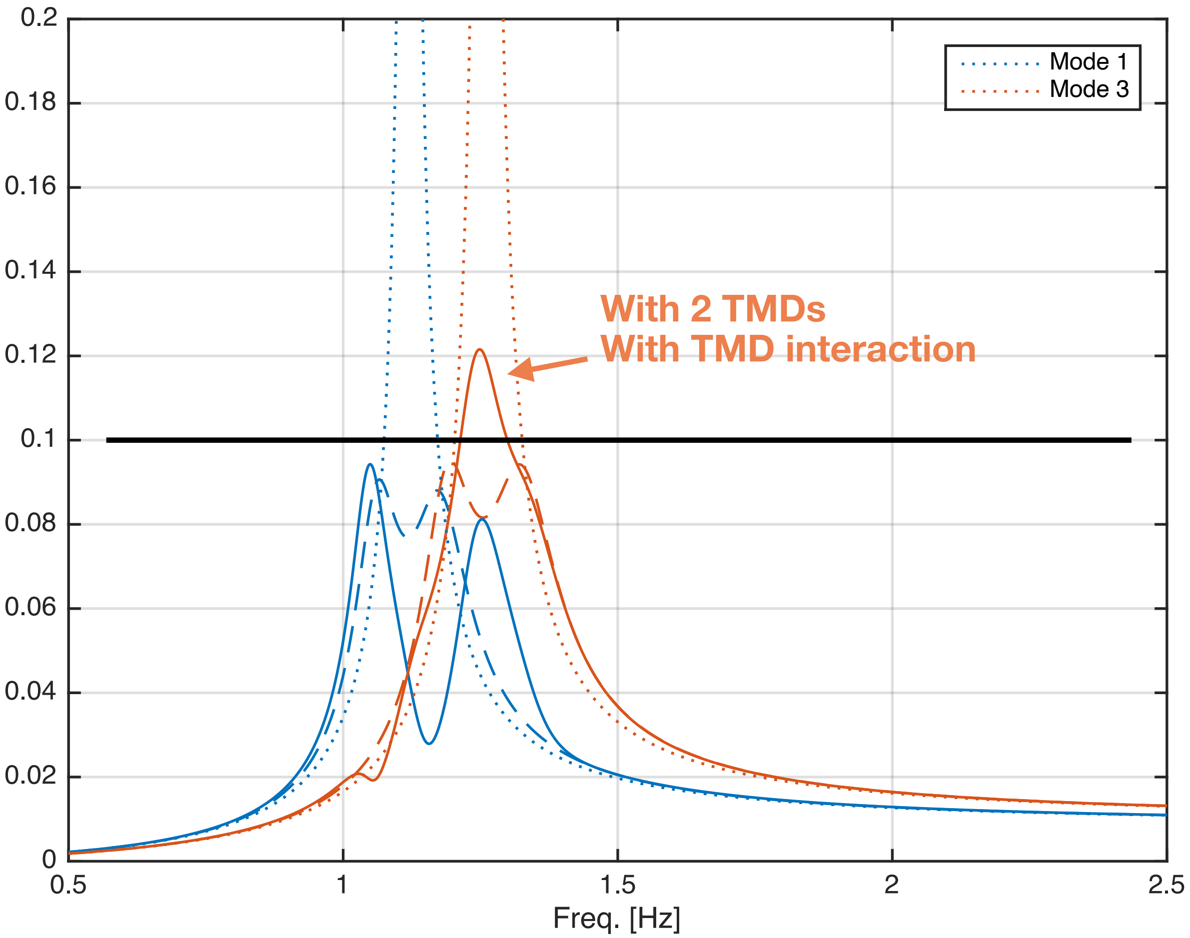
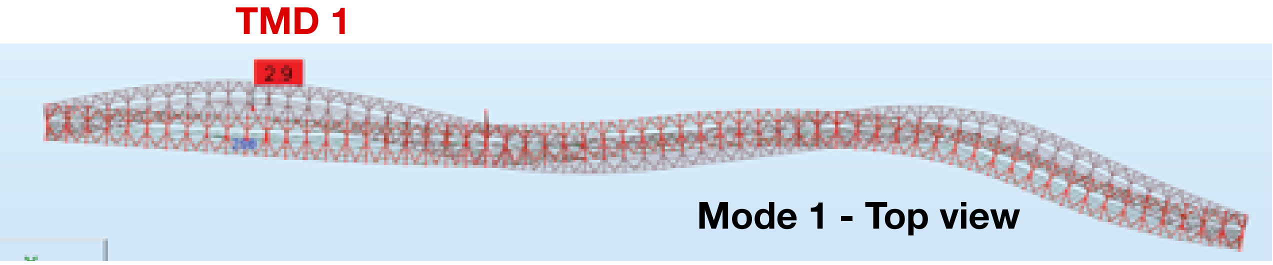


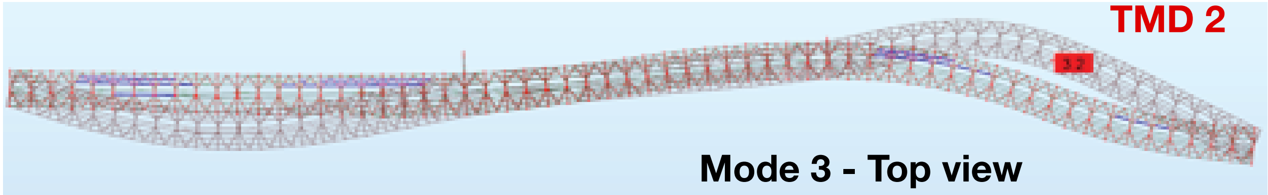
To **use classical 1-DOF tuning formula** for close natural frequencies is **inaccurate**.

Use 2 TMD for 2 modes

How to design them ?

Illustrative problem





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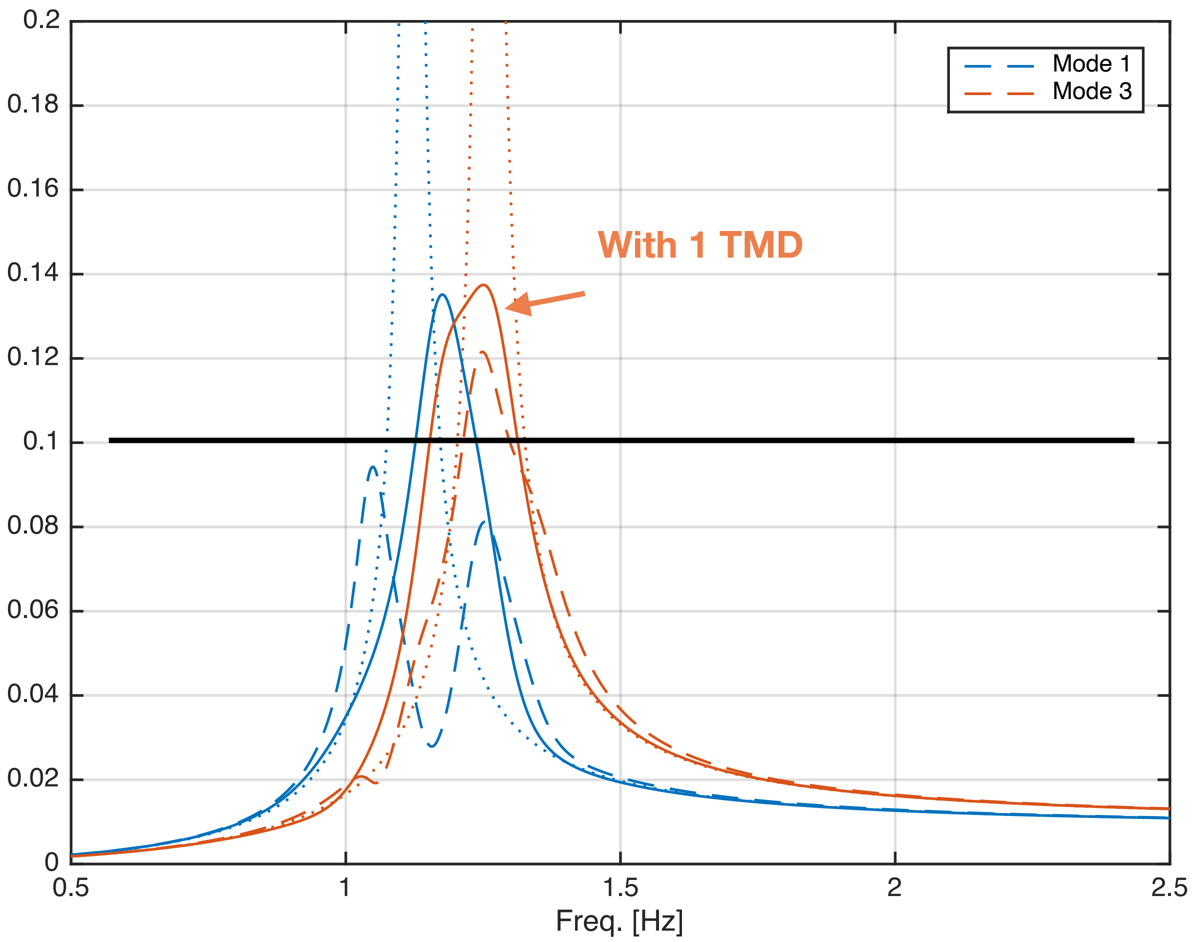
TMD1: 1400 kg, TMD2: 1400 kg

(Use just 1 TMD instead with 4000 kg)



Use 1 TMD for 2 modes

How to design it ?

Considered Problem

Given a structure with known **mode shapes** and **natural frequencies**,

Design a damping system with less devices (TMD, TID) than the number of modes

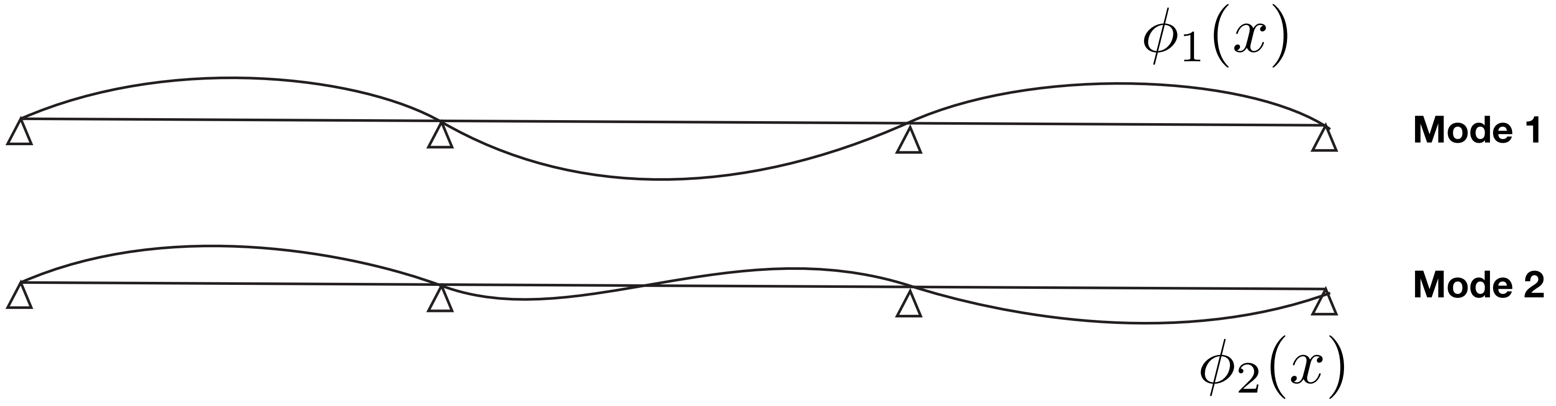
Modal matrices

$$\mathbf{M} = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \\ & \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \\ & \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \\ & \end{bmatrix}$$

Mode shapes



Considered Problem

Given a structure with known **mode shapes** and **natural frequencies**,

Design a damping system with less devices (TMD, TID) than the number of modes

Add damping device & find optimal parameters

Modal matrices

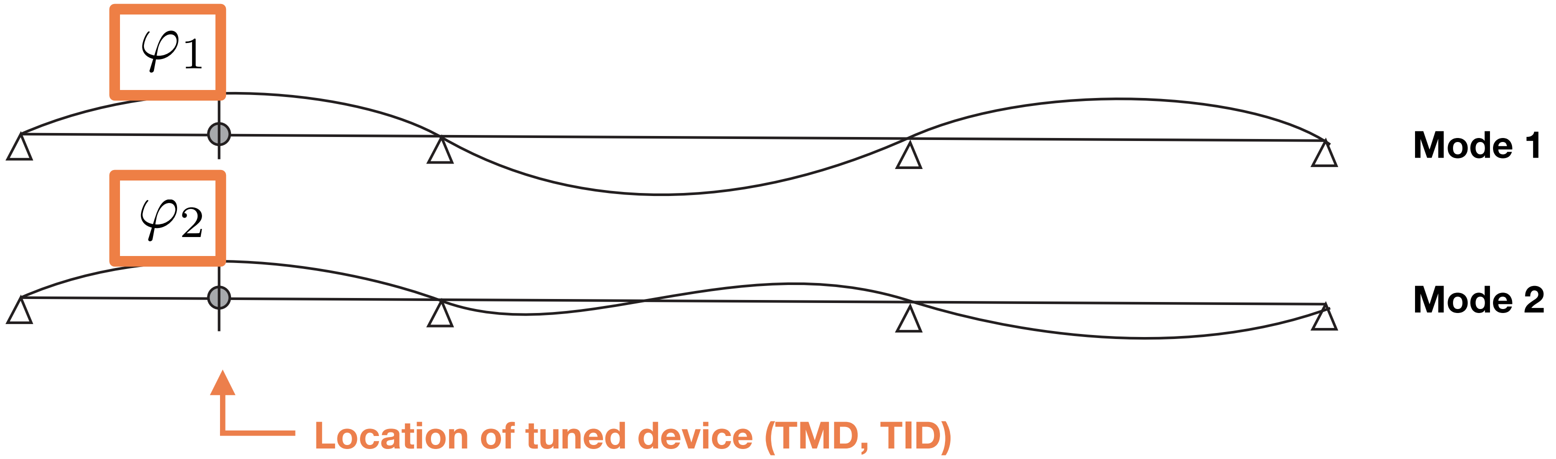
$$\mathbf{M} = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \boxed{m} \end{bmatrix}$$

$$\mathbf{y} = \begin{pmatrix} q_1 \\ q_2 \\ x_{\text{TMD}} \end{pmatrix}$$

$$\mathbf{K} = \begin{bmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \boxed{k} \begin{bmatrix} \varphi_1^2 & \varphi_1\varphi_2 & -\varphi_1 \\ \varphi_1\varphi_2 & \varphi_2^2 & -\varphi_2 \\ -\varphi_1 & -\varphi_2 & 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \boxed{c} \begin{bmatrix} \varphi_1^2 & \varphi_1\varphi_2 & -\varphi_1 \\ \varphi_1\varphi_2 & \varphi_2^2 & -\varphi_2 \\ -\varphi_1 & -\varphi_2 & 1 \end{bmatrix}$$

Coupled equations



Considered Problem

Given the augmented structural matrices, find TMD location and parameters such that the Frequency Response Function

$$\mathbf{H}(\omega) = \left(-\mathbf{M}\omega^2 + i\omega\mathbf{C} + \mathbf{K}\right)^{-1}$$

is smaller than a target value, in a certain norm.

Option 1 : numerics (loose understanding of problem)

Option 2 : analytics (heavy expressions, useless for design, hard to extend)

Option 3 : approximate analytical solution (perturbation method)

1. Compute new modal basis $(\mathbf{K} - \lambda^2\mathbf{M})\boldsymbol{\phi} = 0$
2. Compute new modal matrices $\mathbf{M}^* = \boldsymbol{\phi}^T\mathbf{M}\boldsymbol{\phi} \quad ; \quad \mathbf{C}^* = \boldsymbol{\phi}^T\mathbf{C}\boldsymbol{\phi} \quad ; \quad \mathbf{K}^* = \boldsymbol{\phi}^T\mathbf{K}\boldsymbol{\phi}$
3. Notice that $\mathbf{C}^*$ is not diagonal (non proportional damping)
4. Solve problem in new modal basis (coupled through damping only)
5. Return to original variables

Solution Methodology — Perturbation Method

Observe existence of small numbers :

- mass ratio : $\mu = \dfrac{m}{M_1} = \varepsilon^2\,\mu_1$
- mistuning : $\nu = \dfrac{\omega_{\mathrm{TMD}} - \omega_1}{\omega_1} = \varepsilon\nu_1$
- Struct./TMD displacement : $\sim \varepsilon$
- close nat. frequencies : $\beta = \dfrac{\omega_2 - \omega_1}{\omega_1} = \varepsilon\,\beta_1$
- damping ratios : $\xi_s = \dfrac{C_1}{2M_1\omega 1} = \varepsilon\,\xi_{s1}$

$$\xi_{\mathrm{TMD}} = \frac{c}{2m\omega_{\mathrm{TMD}}} = \varepsilon\,\xi_{t1}$$

Without TMD

Modifications due to TMD

Re-write structural matrices :

$$\begin{aligned} \mathbf{M} &= \boxed{\mathbf{M}_0} \\ \mathbf{C} &= \boxed{\mathbf{C}_0} + \mathbf{C}_1\varepsilon + \mathrm{ord}\left(\varepsilon^2\right) \\ \mathbf{K} &= \boxed{\mathbf{K}_0} + \mathbf{K}_1\varepsilon + \mathrm{ord}\left(\varepsilon^2\right) \end{aligned}$$

Perturbed Modal Basis

1. Compute new modal basis $(\mathbf{K}-\lambda\mathbf{M})\,\boldsymbol{\phi}=0$

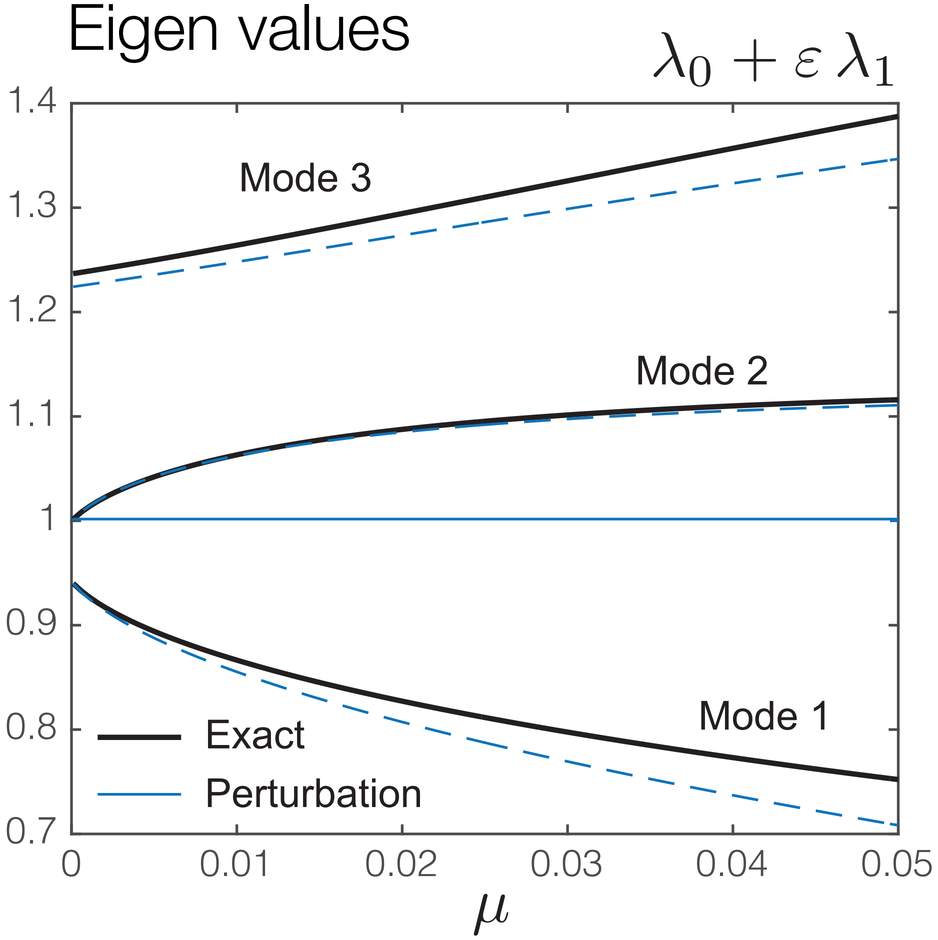
$$\left[\mathbf{K}_0+\varepsilon\mathbf{K}_1-(\lambda_0+\varepsilon\lambda_1)\,\mathbf{M}_0\right]\boldsymbol{\phi}=\mathcal{O}(\varepsilon^2)$$

Assume anstaz : $\lambda=\lambda_0+\varepsilon\lambda_1+\mathrm{ord}(\varepsilon^2)$ $\boldsymbol{\phi}=\boldsymbol{\phi}_0+\varepsilon\boldsymbol{\phi}_1+\mathrm{ord}(\varepsilon^2)$

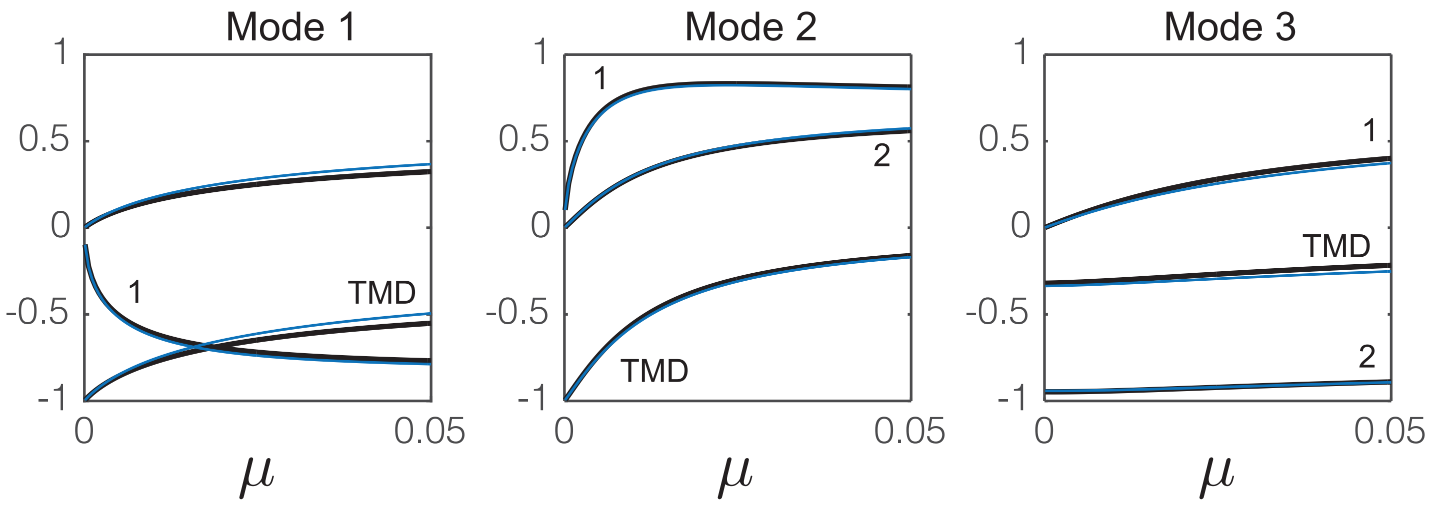
$\mathrm{ord}(\varepsilon^0):(K_0-\lambda_0\mathbf{M}_0)\boldsymbol{\phi}_0=0,$ (as if without TMD)

$\mathrm{ord}(\varepsilon^1):(K_1-\lambda_1\mathbf{M}_0)\boldsymbol{\phi}_0=0.$ (correction, simple since $\mathbf{M}_0$ is diagonal)

Eigen values



Eigen modes



$$\varphi_1=1;\ \varphi_2=-0.8;\ \xi_s=0.4\%$$

$$\beta=1.112;\ \nu\text{ and }\xi_{\mathrm{TMD}}=\text{variable}$$

(den Hartog on mode 1)

Perturbed Modal Basis

2. Compute new modal matrices $\mathbf{M}^* = \boldsymbol{\phi}^T \mathbf{M} \boldsymbol{\phi} \quad ; \quad \mathbf{C}^* = \boldsymbol{\phi}^T \mathbf{C} \boldsymbol{\phi} \quad ; \quad \mathbf{K}^* = \boldsymbol{\phi}^T \mathbf{K} \boldsymbol{\phi}$

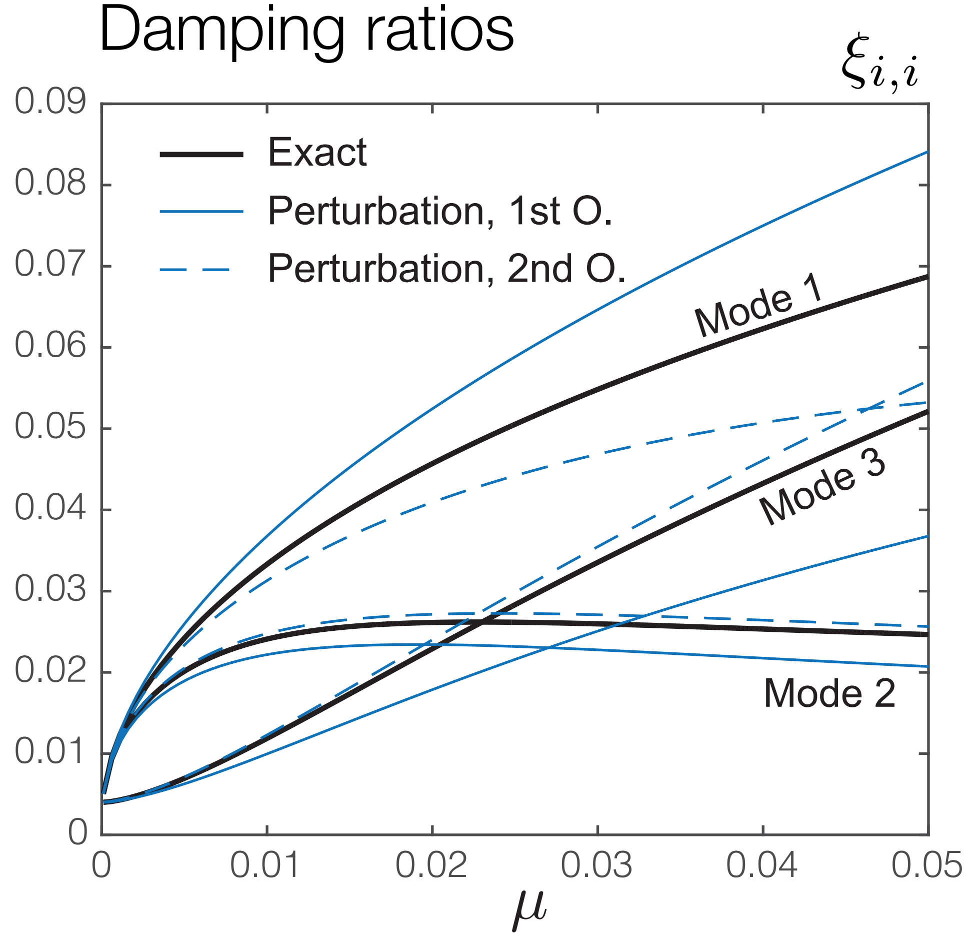
at leading order : $\mathbf{M}_0^* = \boldsymbol{\phi}_0^T \mathbf{M}_0 \boldsymbol{\phi}_0$, $\mathbf{K}_0^* = \boldsymbol{\phi}_0^T \mathbf{K}_0 \boldsymbol{\phi}_0$, …

at second order : $\mathbf{K}_1^* = \boldsymbol{\phi}_0^T \mathbf{K}_1 \boldsymbol{\phi}_0$, …

Determine, for instance, damping ratios (at leading order): $\xi_i = \dfrac{C^*_{i,i}}{2\sqrt{M^*_{i,i} K^*_{i,j}}}$

$$\xi_{i,i} = \frac{\Gamma\,\xi_s + \xi_{\mathrm{TMD}}}{\Gamma + 1} \qquad \text{with} \quad \Gamma = \mu_1 \left(\frac{\varphi_1}{\lambda_1^i}\right)^2 + \mu_1 \frac{M_1}{M_2} \left(\frac{\varphi_2}{\lambda_1^i - 2\beta_1}\right)^2$$

Damping ratios



- Accurate for moderate values of mass ratio
- Captures the right trend, even at leading order

$$\varphi_1 = 1;\ \varphi_2 = -0.8;\ \xi_s = 0.4\%$$

$$\beta = 1.112;\ \nu \text{ and } \xi_{\mathrm{TMD}} = \text{variable}$$

(den Hartog on mode 1)

Frequency Response Function

3. Notice that $\mathbf{C}^*$ is not diagonal (non proportional damping)
4. Solve problem in new modal basis (coupled through damping only)

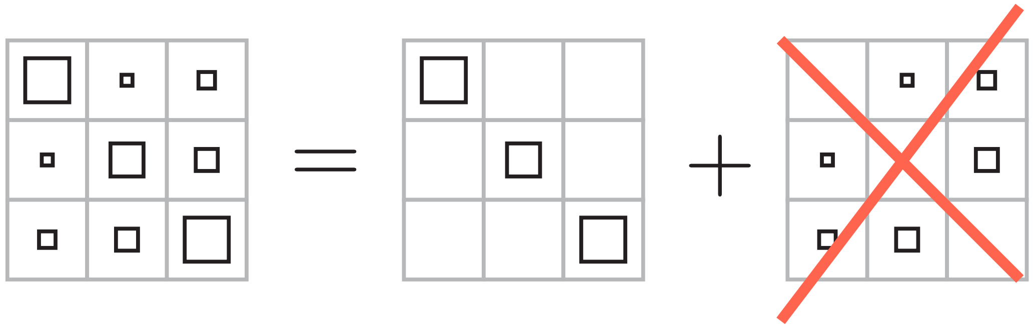
$$\mathbf{H}^*(\omega) = \left(-\mathbf{M}^*\omega^2 + i\omega\mathbf{C}^* + \mathbf{K}^*\right)^{-1}$$

does not take a simple analytical solution

$$\mathbf{H}^*(\omega) = \left[-\underset{\text{Diagonal}}{\underset{\uparrow}{\mathbf{M}_0^*}}\omega^2 + i\omega\left(\underset{\text{Full}}{\underset{\uparrow}{\mathbf{C}_0^*}} + \varepsilon \underset{\text{Full}}{\underset{\uparrow}{\mathbf{C}_1^*}}\right) + \left(\underset{\text{Diagonal}}{\underset{\uparrow}{\mathbf{K}_0^*}} + \varepsilon \underset{\text{Diagonal}}{\underset{\uparrow}{\mathbf{K}_1^*}}\right)\right]^{-1}$$

… even with series expansion

Split $\mathbf{C}^*$ into (large) diagonal elements and (small) out-of-diagonal elements



$$\mathbf{C}^* = \mathbf{C}^*_{\mathrm{d}} + \mathbf{C}^*_{\mathrm{o}}$$

Neglect out-of-diagonal elements : $\mathbf{H}^*_{\mathrm{diag}}(\omega) = \left[-\underset{\text{Diagonal}}{\underset{\uparrow}{\mathbf{M}_0^*}}\omega^2 + i\omega\left(\underset{\text{Diagonal}}{\underset{\uparrow}{\mathbf{C}^*_{0,\mathrm{d}}}} + \varepsilon \underset{\text{Diagonal}}{\underset{\uparrow}{\mathbf{C}^*_{1,\mathrm{d}}}}\right) + \left(\underset{\text{Diagonal}}{\underset{\uparrow}{\mathbf{K}_0^*}} + \varepsilon \underset{\text{Diagonal}}{\underset{\uparrow}{\mathbf{K}_1^*}}\right)\right]^{-1}$

Partly consider the coupling: $\mathbf{H}^*_{\mathrm{corr}}(\omega) \simeq \left(\mathbf{I} - i\omega\mathbf{H}^*_{\mathrm{diag}}(\omega)\left(\mathbf{C}^*_{0,\mathrm{o}} + \varepsilon\mathbf{C}^*_{1,\mathrm{o}}\right)\right)\mathbf{H}^*_{\mathrm{diag}}(\omega)$

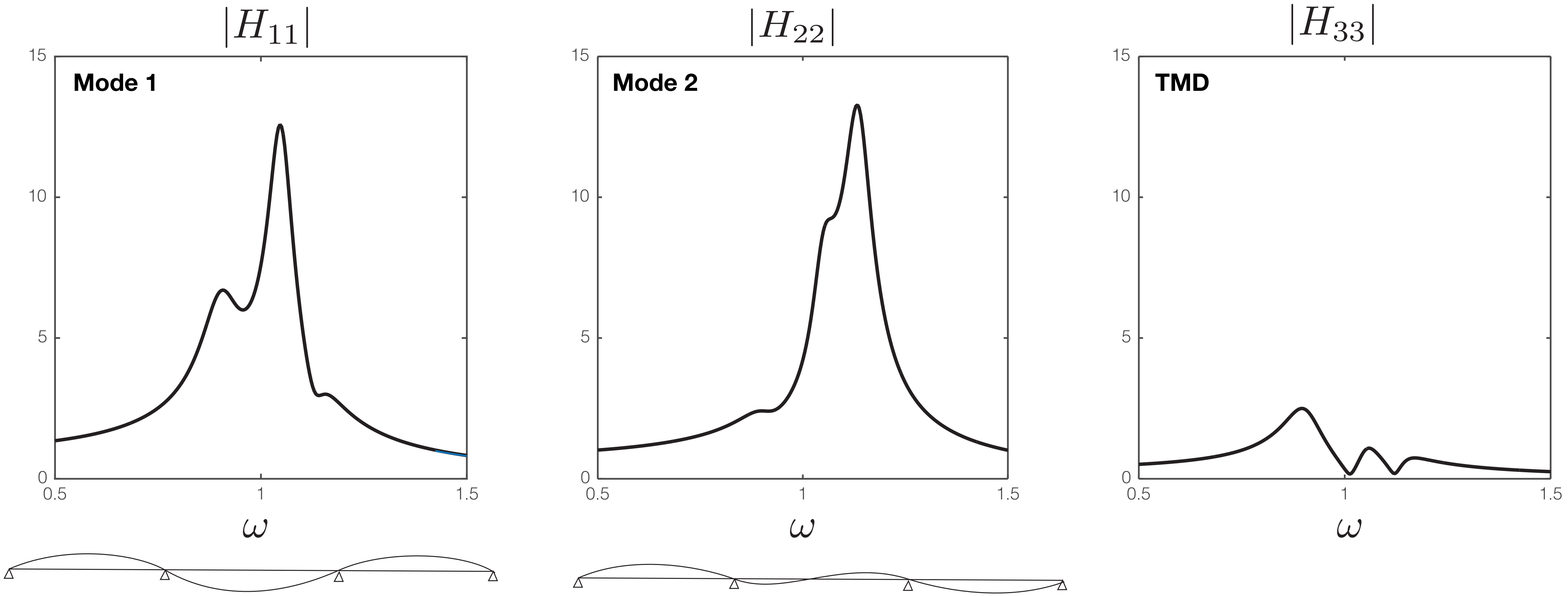
No matrix inversion

Denoël, V., & Degée, H. (2009). Asymptotic expansion of slightly coupled modal dynamic transfer functions. *Journal of Sound and Vibration*, *328*(1-2), 1-8.

Frequency Response Function

5. Return to original variables

$$\beta = 1.112;\ \mu = 3\%; \nu = 0.971;\ \xi_{\mathrm{TMD}} = 10.4\%$$
$$\varphi_1 = 1;\ \varphi_2 = -0.8;\ \xi_s = 0.4\%$$



$$\mathbf{H}(\omega) = \left(-\mathbf{M}\omega^2 + i\omega\mathbf{C} + \mathbf{K}\right)^{-1}$$

Frequency Response Function

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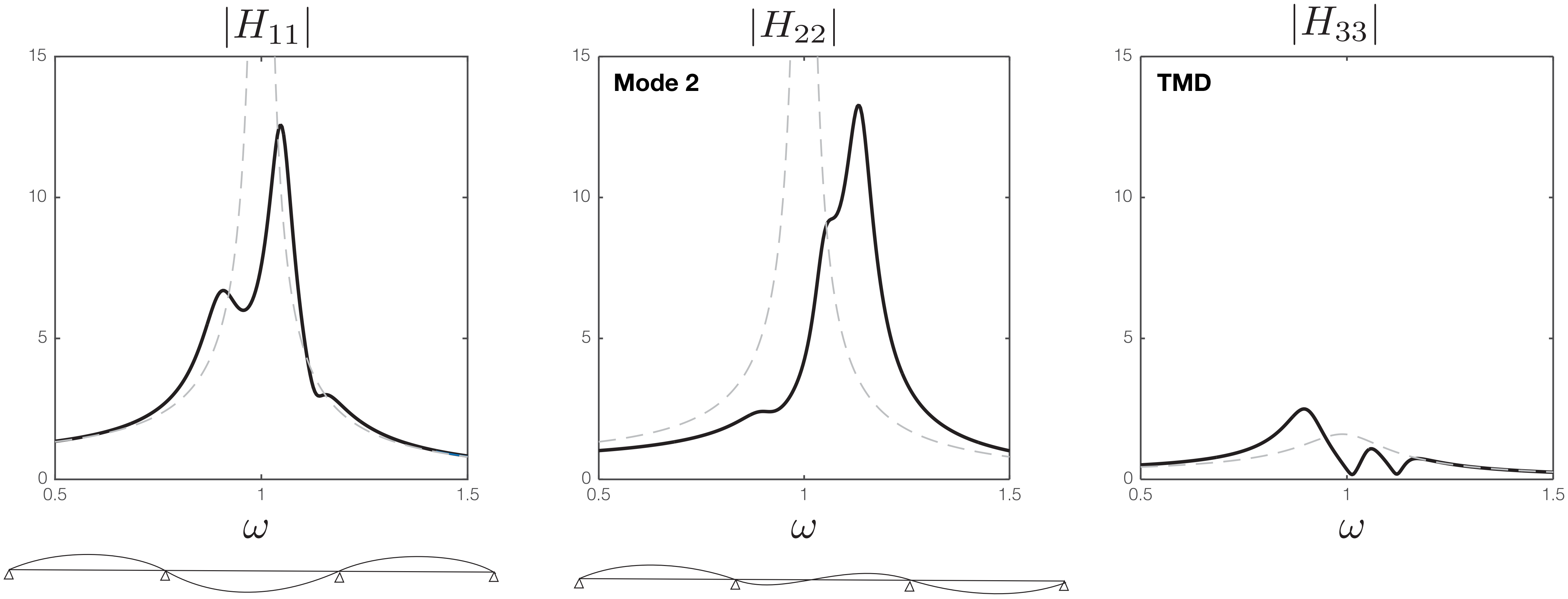

— $\mathbf{H}(\omega) = \left(-\mathbf{M}\omega^2 + i\omega\mathbf{C} + \mathbf{K}\right)^{-1}$ **(Exact)**

– – $\mathbf{H}(\omega) = \left(-\mathbf{M}_0\omega^2 + i\omega\mathbf{C}_0 + \mathbf{K}_0\right)^{-1}$

Frequency Response Function

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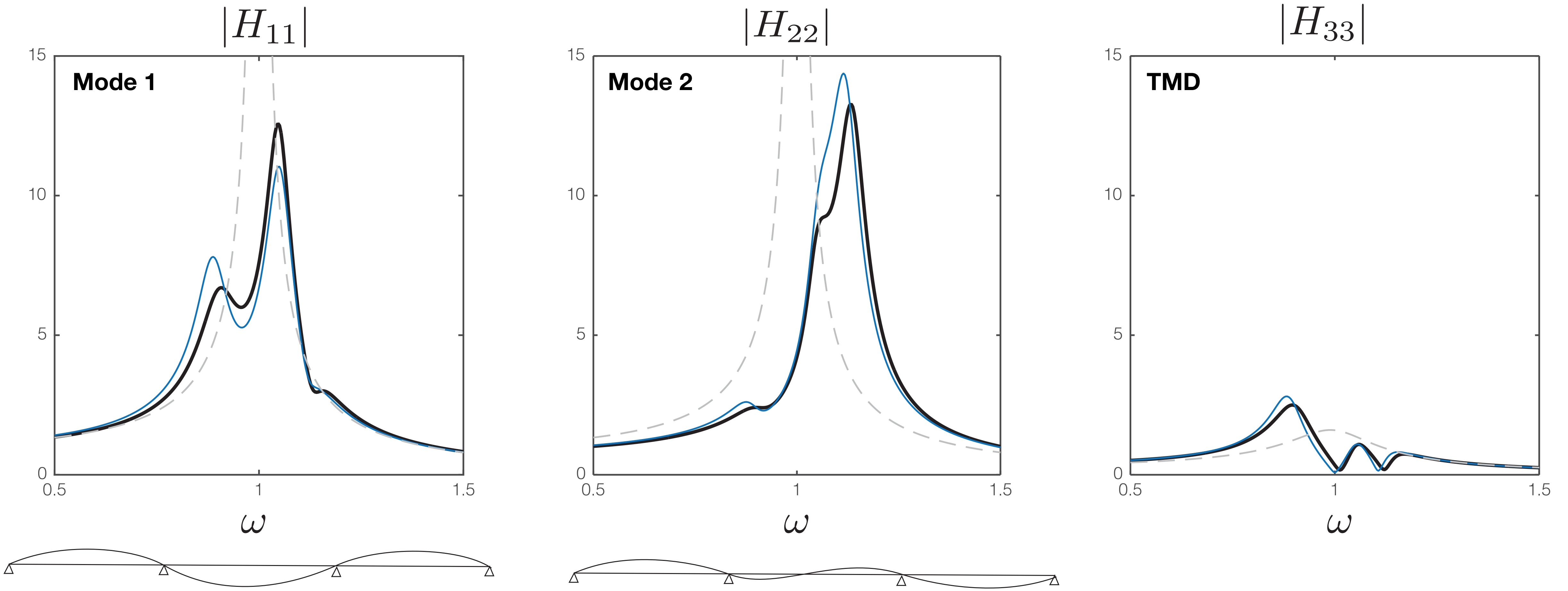

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- - $\mathbf{H}(\omega) = \left(-\mathbf{M}_0\omega^2 + i\omega\mathbf{C}_0 + \mathbf{K}_0\right)^{-1}$

— $\mathbf{H}(\omega) = \left[-\mathbf{M}_0\omega^2 + i\omega\left(\mathbf{C}_0 + \varepsilon\mathbf{C}_1\right) + \left(\mathbf{K}_0 + \varepsilon\mathbf{K}_1\right)\right]^{-1}$

Frequency Response Function

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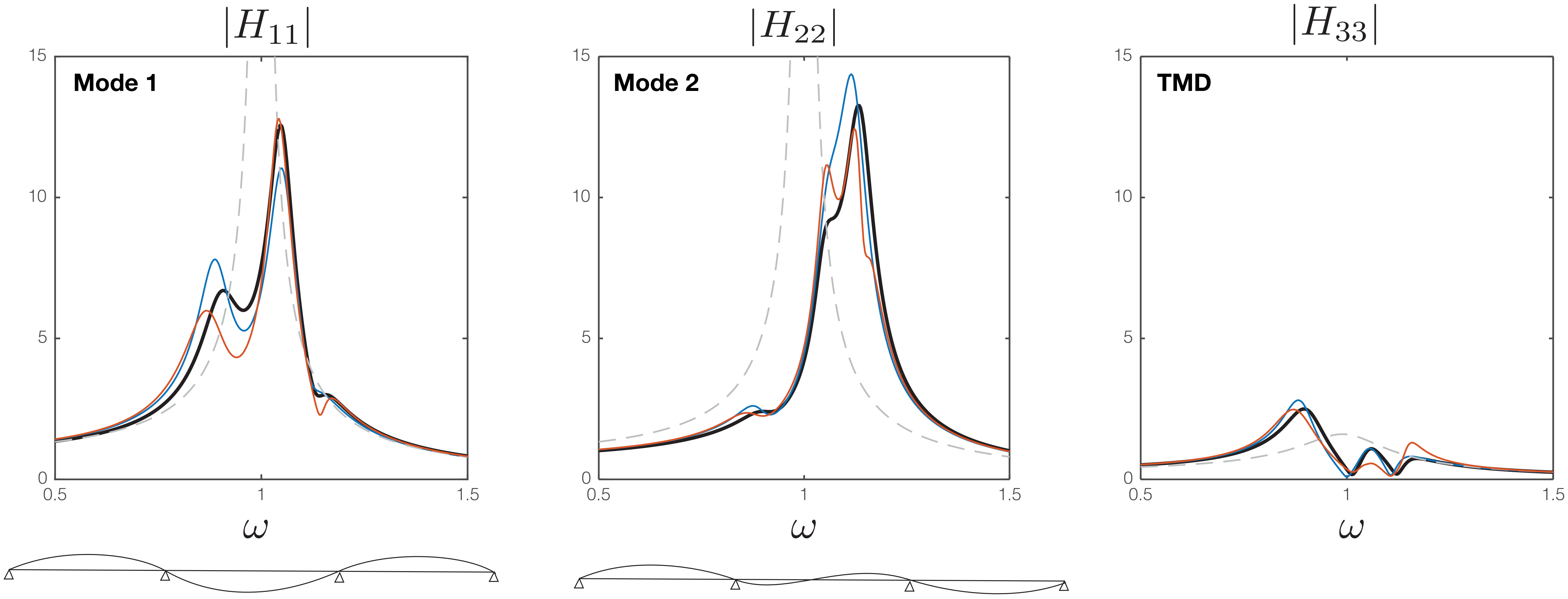


— $\mathbf{H}(\omega) = \left(-\mathbf{M}\omega^2 + i\omega\mathbf{C} + \mathbf{K}\right)^{-1}$ **(Exact)**

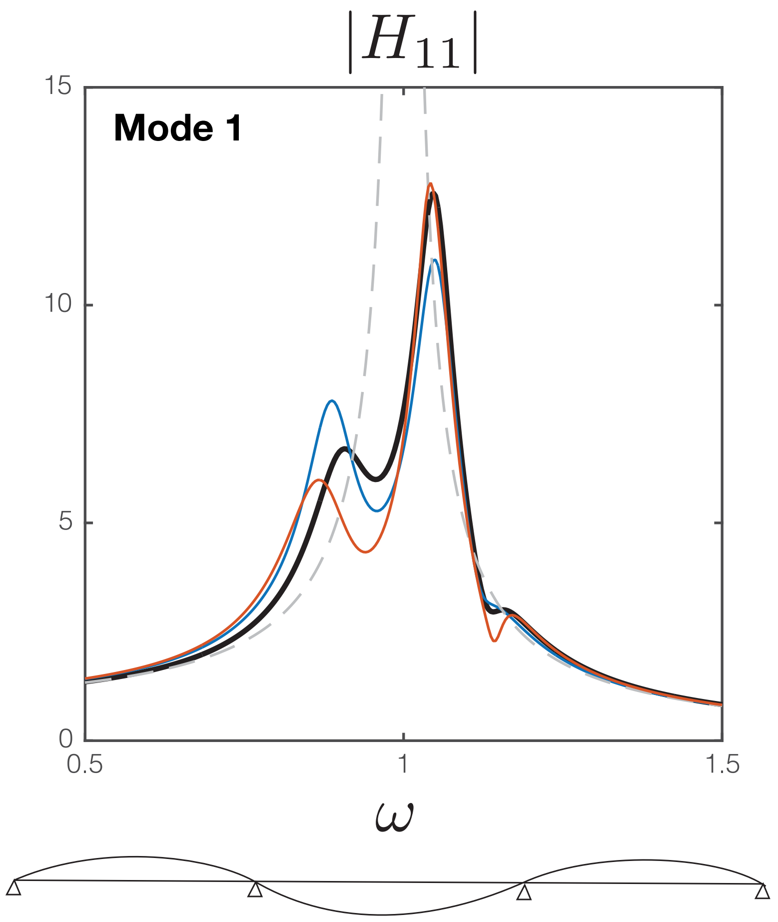
- - $\mathbf{H}(\omega) = \left(-\mathbf{M}_0\omega^2 + i\omega\mathbf{C}_0 + \mathbf{K}_0\right)^{-1}$

— $\mathbf{H}(\omega) = \left[-\mathbf{M}_0\omega^2 + i\omega\left(\mathbf{C}_0 + \varepsilon\mathbf{C}_1\right) + \left(\mathbf{K}_0 + \varepsilon\mathbf{K}_1\right)\right]^{-1}$

— $\mathbf{H}(\omega) = \boldsymbol{\phi}_0\left(\mathbf{I} - \mathrm{i}\omega\mathbf{H}^*_{\mathrm{diag}}(\omega)\left(\mathbf{C}^*_{0,\mathrm{o}} + \varepsilon\mathbf{C}^*_{1,\mathrm{o}}\right)\right)\mathbf{H}^*_{\mathrm{diag}}(\omega)\boldsymbol{\phi}_0^T$ **(Proposed)**

Results

Response to broad-band excitation



$$\boldsymbol{\Sigma}_q = \int_{-\infty}^{+\infty} \mathbf{H}(\omega)\mathbf{S}_p(\omega)\bar{\mathbf{H}}^{\mathrm{T}}(\omega)\mathrm{d}\omega$$

Variance of the acceleration in structural mode shapes :

$$\sigma^2_{\ddot{q}_i} = \frac{\boxed{S_{p^*_{i,i}}(\omega_i)}}{\boxed{M_i^{*2}}} \frac{\pi\boxed{\omega_i}}{2\xi_{i,i}}$$

$$\xi_{i,i} = \frac{\Gamma\,\xi_s + \xi_{\mathrm{TMD}}}{\Gamma + 1}$$

— $\mathbf{H}(\omega) = \left(-\mathbf{M}\omega^2 + i\omega\mathbf{C} + \mathbf{K}\right)^{-1}$ **(Exact)**

- - - $\mathbf{H}(\omega) = \left(-\mathbf{M}_0\omega^2 + i\omega\mathbf{C}_0 + \mathbf{K}_0\right)^{-1}$

— $\mathbf{H}(\omega) = \left[-\mathbf{M}_0\omega^2 + i\omega\left(\mathbf{C}_0 + \varepsilon\mathbf{C}_1\right) + \left(\mathbf{K}_0 + \varepsilon\mathbf{K}_1\right)\right]^{-1}$

— $\mathbf{H}(\omega) = \boldsymbol{\phi}_0 \boxed{\left(\mathbf{I} - \mathrm{i}\omega\mathbf{H}^*_{\mathrm{diag}}(\omega)\left(\mathbf{C}^*_{0,\mathrm{o}} + \varepsilon\mathbf{C}^*_{1,\mathrm{o}}\right)\right)}\boxed{\mathbf{H}^*_{\mathrm{diag}}(\omega)}\boldsymbol{\phi}_0^T$ **(Proposed)**

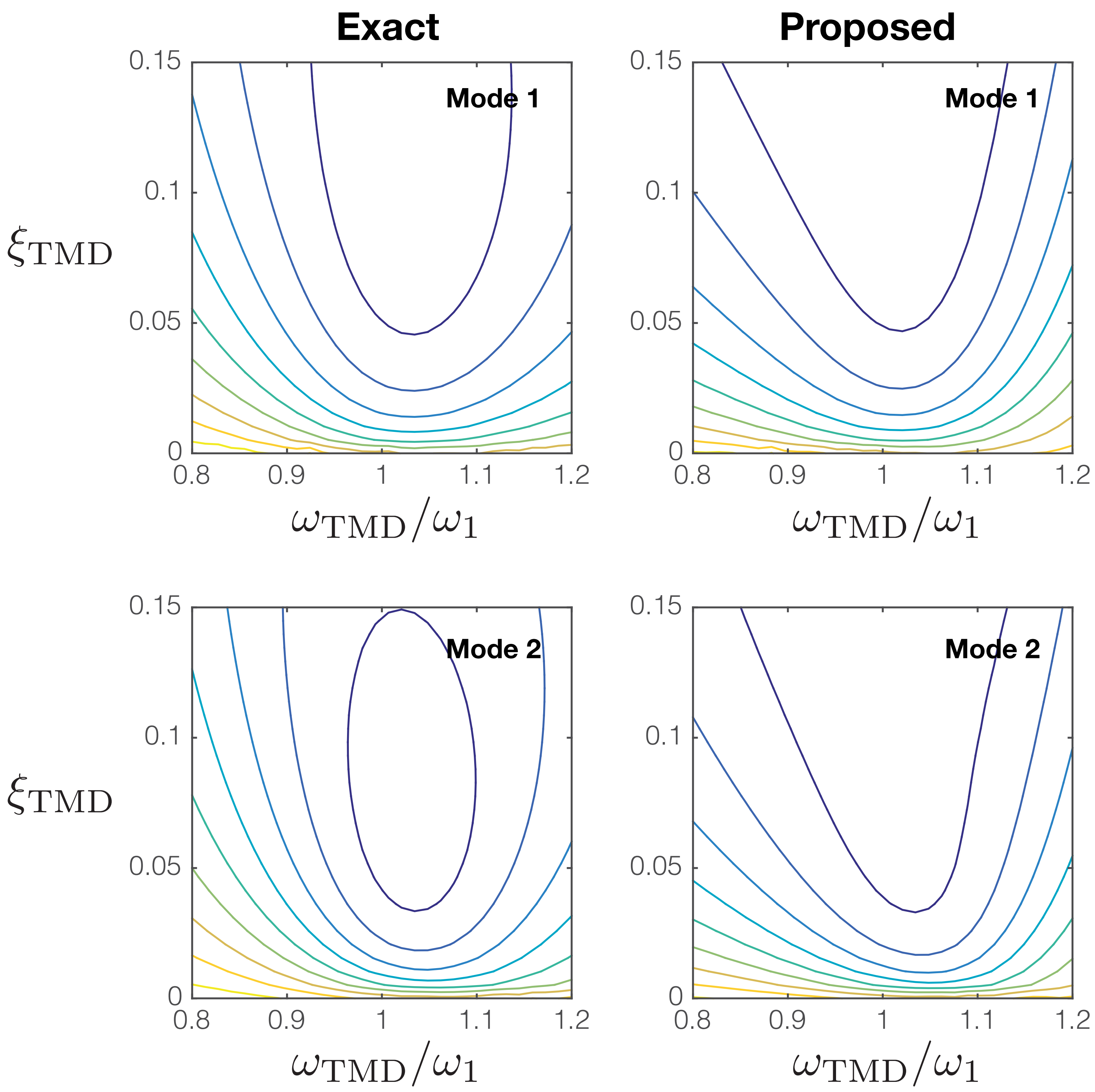
Results

Response to broad-band excitation

Optimality plot (find optimum TMD parameters)

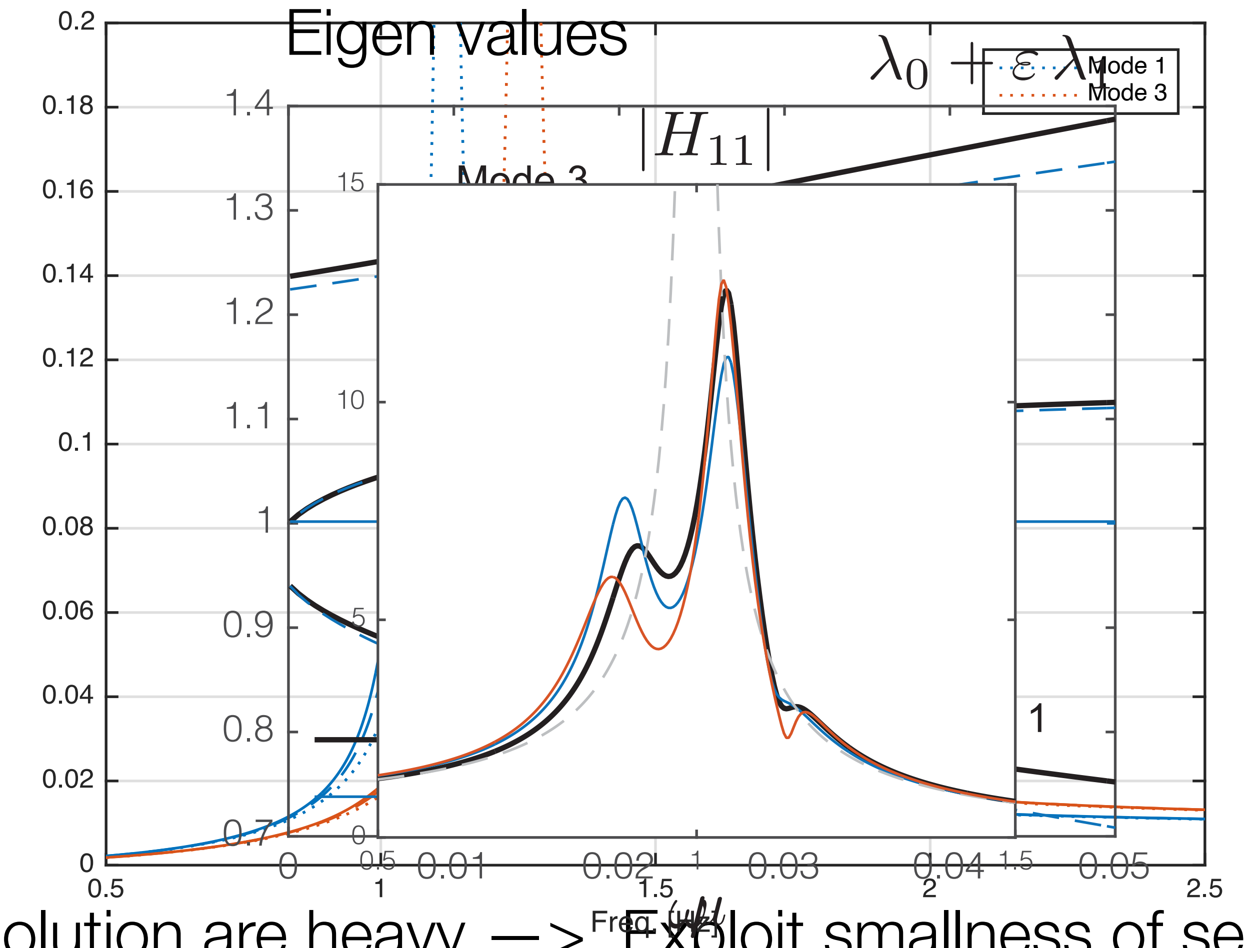
$$\beta = 1.112;\ \mu = 3\%$$

$$\varphi_1 = 1;\ \varphi_2 = -0.8;\ \xi_s = 0.4\%$$



- Proposed method is asymptotically accurate as mistuning and damping tend to zero
- Proposed method is based on a simple analytical formula

Summary



- Exact analytical solution are heavy —> Exploit smallness of several small numbers
- [illegible] modes (not presented here)
- Equivalent uncoupled system

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Thank you !
Questions ? Comments ?

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December 2-5, 2019, Łódź, POLAND

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