Perturbation analysis of a Multi-Degree-Of-Freedom system equipped with only one tuned mass damper

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Abstract: This paper develops a simple mathematical model for the analysis of a 2-dof modal model equipped with a tuned mass damper. The final aim of this formulation is to develop a simple design procedure to assess the influence and possibility of using a single tuned mass damper to mitigate the vibrations in two structural modes having close natural frequencies and mode shapes. The method is based on the coupled analysis of modal responses under stochastic loading. Random processes are employed to represent the loading and the acceleration in the structural mode shapes. After a proper rescaling offering an appropriate distinguished limit, a perturbation method of the eigenvalue problem is developed and the variances of these responses are expressed at first order by accurate and simple analytical expressions.

1. Introduction

The discomfort occasioned by dynamic loads such as crowds or turbulent winds constitutes a topic of growing concern in the conception of modern footbridges. The discomfort is generally expressed in terms of acceleration. Guidelines such as that edited by the Sétra [10] or the final report of the HIVoSS project [6] determine acceleration thresholds to not exceed. When thresholds are exceeded it is usual to mitigate the vibrations with tuned mass dampers (TMDs). Their design therefore targets the minimization of the response until it meets the comfort criteria. In the case of SDOF structures, several formulations for the optimum damper design exist [4], among which the famous Den Hartog's formula [5], which provides the optimal parameters for one TMD to damp vibrations in one structural mode. For MDOF structures, a classical solution is to dedicate a damper for each mode which leads to the repeated use of Den Hartog's formula. This solution does not include the coupling between the modal responses; each mode is indeed treated separately, i.e. considered like a SDOF system. This could be detrimental to the optimal functioning of the dampers. Also, in some cases of mode shapes with close natural frequencies and mode shapes, it happens that a single TMD could be used to mitigate vibrations in both modes at the same time. This work contributes to the development of simple design methods to account for the existing coupling between several modes. More specifically, we develop the design formulae to assess the efficiency of a single TMD to mitigate vibrations in two structural modes. This conceptual solution has been made feasible thanks to the concept of Tuned Inerter Damper [11] which allows to virtually reach high mass ratios [8].

2. The considered problem

Let us consider a structure represented by its modal mass matrix $\mathbf{M}_s$, modal damping matrix $\mathbf{C}_s$ and modal stiffness matrix $\mathbf{K}_s$. It is assumed that the inherent damping is of classical type [1], so that all three $M \times M$ matrices (with M the number of modes) are diagonal. These matrices are associated with the $M \times 1$ vector $\mathbf{q}(t)$ of modal coordinates of the structure without TMD.

In this paper we study the influence of a single tuned-mass damper TMD on the structural response. The TMD is modeled by a mass m, a spring k, and a viscous damping c. It is located at a position of the structure where the modal amplitudes are $\boldsymbol{\varphi} = [\varphi_1, \cdots, \varphi_M]^T$. Besides, the TMD brings a new degree of freedom to the system and the global system {Structure + Damper} has (M+1) DOF. It is characterized by the following structural matrices

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_s & 0 \\ 0 & m \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} \mathbf{C}_s & 0 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} \boldsymbol{\varphi}\boldsymbol{\varphi}^T & -\boldsymbol{\varphi} \\ -\boldsymbol{\varphi}^T & 1 \end{bmatrix}; \tag{1}$$

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_s & 0 \\ 0 & 0 \end{bmatrix} + k \begin{bmatrix} \boldsymbol{\varphi}\boldsymbol{\varphi}^T & -\boldsymbol{\varphi} \\ -\boldsymbol{\varphi}^T & 1 \end{bmatrix}, \tag{2}$$

and an augmented response vector $\mathbf{y}(t) = \left[\mathbf{q}^T, q_{TMD}\right]^T$ where q_{TMD} corresponds to the amplitude of the TMD. The new stiffness matrix $\mathbf{K}$ being fully populated, it might be advantageous to compute the modal basis of the global system. It is obtained by solving $\left(\mathbf{K} - \mathbf{M}\omega^2\right)\Phi = \mathbf{0}$. In the new basis defined by the mode shapes Φ, the structural matrices $\mathbf{M}^* = \Phi^T\mathbf{M}\Phi$ and $\mathbf{K}^* = \Phi^T\mathbf{K}\Phi$ are diagonal matrices, by construction, while it is certainly not the case for $\mathbf{C}^* = \Phi^T\mathbf{C}\Phi$. In this new basis, the equation of motion reads

$$\mathbf{M}^*\ddot{\mathbf{z}} + \mathbf{C}^*\dot{\mathbf{z}} + \mathbf{K}^*\mathbf{z} = \mathbf{p}^* \tag{3}$$

where $\mathbf{z}(t)$, defined as $\mathbf{y}(t) = \boldsymbol{\Phi}\mathbf{z}(t)$, is the vector of generalized coordinates of the augmented problem and $\mathbf{p}^*(t) = \boldsymbol{\Phi}^T\mathbf{p}(t)$ represents the modal (generalized) loads. It is obtained as a function of the loads on the augmented problem, which read $\mathbf{p}(t) = \left[\mathbf{p}_s^T(t), 0\right]^T$ where $\mathbf{p}_s$ represents the modal loads on the structure (with the modal basis of the original structure). The upper dot denotes derivatives with respect to time t.

The use of the stationary stochastic loading considered in the following developments suggests the frequency domain formulation of (3),

$$\mathbf{Z}(\omega) = \mathbf{H}^{*}(\omega)\mathbf{P}^{*}(\omega) \tag{4}$$

where

$$\mathbf{H}^{*}(\omega) = \left(-\mathbf{M}^{*}\omega^{2} + i\omega\mathbf{C}^{*} + \mathbf{K}^{*}\right)^{-1}. \tag{5}$$

The loading model can be described using one of these approaches : deterministic or stochastic. With a deterministic approach, a precise description is provided for the loading. Not all the phenomena can be described with such a model. For instance, wind and crowd loads cannot be described with such certainty. As soon as environmental loads or loads resulting from human activity are concerned, which is customary in civil engineering applications, the treatment of such phenomena is done with the attribution of a certain randomness. Accordingly, the loading is modeled as a random process characterized by statistical characteristics. It is assumed that the loading is Gaussian and stationary, so that it is represented by its Power Spectral Density (PSD). In order to provide a simple loading model, we have chosen band-limited white noises for the loading, i.e a constant power spectral density S_0 in the range $[-\omega_{\max}, \omega_{\max}]$. This is not a limitation; any other more realistic power spectral density could be considered.

3. Solution methodologies

Three solution methodologies are presented in the sequel. They are termed *exact, uncoupled* and *corrected* (referring to the first correction of an asymptotic series). These three solutions will be compared; they are established sequentially.

3.1. Solution 1 : Exact solution

The *exact* solution is the formal stochastic analysis (REF) and does not formulate any assumption about the generalized damping matrix $\mathbf{C}^{*}$, the transfer function $\mathbf{H}^{*}(\omega)$ being therefore obtained through a full matrix inversion as in (5).The PSD matrix of the response (the displacement) is given by [1]

$$\mathbf{S}_z = \mathbf{H}^{*}\mathbf{S}_{p^*}\overline{\mathbf{H}}^{*T}. \tag{6}$$

The integral of the PSD matrix over the frequency space yields the covariance matrix $\mathbf{\Sigma}_z$. The diagonal components of $\mathbf{\Sigma}_z$ represent the variances of the augmented vector $\mathbf{z}(t)$ while the off-diagonal components incarnate the interaction between the responses in the different modes. Simple expressions of the variance and covariance of the modal accelerations are derived by [9], based on a multiple timescale approach[2]. A white noise approximation is employed, that consists in the replacement of the PSD of the modal forces by a white noise, such that the intensity is considered at the level of the resonance frequency. All in all, the variance and covariance of the modal responses read

$$\sigma^2_{\ddot{z}_m} = \frac{S_{p^*_{m,m}}}{M^{*2}_m}(2\omega_{\max}+\frac{\pi\omega_m}{2\xi});\ \sigma_{\ddot{z}_{m,n}} = \frac{S_{p^*_{m,n}}}{K^*_m K^*_n}\Big(2\omega_{\max}\omega^2_m\omega^2_n+\frac{1}{4}\Big(\frac{\omega_m+\omega_n}{2}\Big)^4\frac{\pi(\overline{\xi}-i\varepsilon)}{2\varepsilon(\varepsilon^2+\xi^2)}\Big) \tag{7}$$

with ξ the damping ratio in the considered mode, $\overline{\xi}$ is the average damping ratio in the two considered modes (please notice that a refined version with a longer expression is available in [3]) and ε a parameter related to the relative distance between natural frequencies.

Following the definition $\mathbf{y}(t) = \mathbf{\Phi z}(t)$, the variances of the structural response in the original mode shapes $\sigma^2_{\ddot{q}}$ are then computed using the modal decomposition. They are expressed as a function of the quantities given in (7) :

$$\sigma^2_{\ddot{q}} = \sum_{m=1}^{M+1}\phi^2_m\sigma^2_{\ddot{z}_m} + 2\sum_{m=1}^{M+1}\sum_{n\neq m}^{M+1}\phi_m\ \phi_n\sigma_{\ddot{z}_{m;n}}. \tag{8}$$

It turns out that these analytical expressions involve parameters that are not defined explicitly, such as the natural frequencies of the augmented system (ω_m, ω_n), or the corresponding modal damping ratio ξ. The purpose of sections 4.2 and 4.3 is to derive analytical formulations of these modal properties, in order to obtain the variance by means of (8).

The *exact* solution is accurate but requires heavy numerical developments and prohibits therefore any clear understanding of the solution.

3.2. Solution 2 : Uncoupled/Diagonal solution

At the opposite, the *uncoupled* solution is obtained by eliminating the off-diagonal components in the generalized damping matrix $\mathbf{C}^*_d$. Assuming a diagonal damping matrix $\mathbf{C}^*_d$, this results in a diagonal transfer matrix $\mathbf{H}^*_d$, such that

$$\mathbf{H}^*_d(\omega) = (-\mathbf{M}^*\omega^2 + i\omega\mathbf{C}^*_d + \mathbf{K}^*)^{-1}. \tag{9}$$

It presents the advantage to be interpretable and simple to compute. It is, nevertheless, not accurate as the information contained in the off-diagonal components is lost.

3.3. Solution 3 : Corrected solution

According to the method developed by [12], the damping matrix $\mathbf{C}^*$ can be seen as the sum of an exclusively diagonal matrix $\mathbf{C}^*_d$ and an off-diagonal matrix $\mathbf{C}^*_0$. Using the same notations as in [12], the transfer function can be written

$$\mathbf{H}^*(\omega) = (\mathbf{I} + \mathbf{J}_d^{-1}\mathbf{J}_0)^{-1}\mathbf{J}_d^{-1}\mathbf{M}^{*-1} \tag{10}$$

where $\mathbf{J}_d = -\mathbf{I}\omega^2 + \mathrm{i}\omega\mathbf{M}^{*-1}\mathbf{C}_d^* + \mathbf{M}^{*-1}\mathbf{K}^*$ and $\mathbf{J}_0 = \mathrm{i}\omega\mathbf{M}^{*-1}\mathbf{C}_0^*$. Whenever $\mathbf{J}_d^{-1}\mathbf{J}_0$ is one order of magnitude less than $\mathbf{I}$, it is permitted to write the following expansion [12]

$$(\mathbf{I} + \mathbf{J}_d^{-1}\mathbf{J}_0)^{-1} \simeq (\mathbf{I} - \mathbf{J}_d^{-1}\mathbf{J}_0) = (\mathbf{I} - \mathbf{H}_d^*\mathbf{M}^*\mathbf{J}_0) = \mathbf{I} - \mathrm{i}\omega\mathbf{H}_d^*\mathbf{C}_0^*. \tag{11}$$

An approximate expression of $\mathbf{H}^*$ is therefore obtained, which avoids the full matrix inversion. It is, as stated before, a sort of an intermediate approach. Indeed, its expression contains the contribution of diagonal components through $\mathbf{H}_d^*(\omega)$, and off-diagonal components through $\mathbf{C}_0^*$. This leads to the *corrected* transfer function defined as

$$\mathbf{H}_c^*(\omega) = (\mathbf{I} - \mathrm{i}\omega\mathbf{H}_d^*(\omega)\mathbf{C}_0^*)\,\mathbf{H}_d^*(\omega). \tag{12}$$

In a deterministic approach, the computation of the response $\mathbf{Z}_c(\omega)$ is obtained by replacing $\mathbf{H}^*(\omega)$ by $\mathbf{H}_c^*(\omega)$ in (4); this yields

$$\mathbf{Z}_c = \mathbf{H}_c^*\mathbf{P}^* = \mathbf{Z}_d - \mathrm{i}\omega\mathbf{H}_d^*(\omega)\mathbf{C}_0^*\mathbf{Z}_d \tag{13}$$

where $\mathbf{Z}_d = \mathbf{H}_d^*\mathbf{P}^*$ is the response of the uncoupled problem, see Solution 2. The response $\mathbf{Z}_d$ computed through the uncoupled approach, is completed by a residual term $\Delta\mathbf{Z}_c := -\mathrm{i}\omega\mathbf{H}_d^*(\omega)\mathbf{C}_0^*\mathbf{Z}_d$, that is a function of $\mathbf{Z}_d$. The correction of the uncoupled approach provides the *corrected* solution $\mathbf{Z}_c$. Consequently, the reintroduction of $\mathbf{Z}_c$ as an input permits the computation of a corrected and more accurate response. In an iterative manner, this approach offers a recurrence scheme approaching the "exact" result.

4. Analysis of a 3-DOF model

Given a 2-DOF structure, the implementation of one TMD leads to a global system of 3 DOFs. The matrices of mass, damping and stiffness are thus of size 3x3.

$$\mathbf{M} = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & m \end{bmatrix}; \mathbf{K} = \begin{bmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + k \begin{bmatrix} \varphi_1^2 & \varphi_1\varphi_2 & -\varphi_1 \\ \varphi_1\varphi_2 & \varphi_2^2 & -\varphi_2 \\ -\varphi_1 & -\varphi_2 & 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} \varphi_1^2 & \varphi_1\varphi_2 & -\varphi_1 \\ \varphi_1\varphi_2 & \varphi_2^2 & -\varphi_2 \\ -\varphi_1 & -\varphi_2 & 1 \end{bmatrix}$$

4.1. Perturbation method

A perturbation method is employed for the mathematical developments. This method consists in comparing the contributions of the problem parameters. The parameters that have small effect are identified and repelled to a higher order, keeping the essence of the information but with less complexity. In this way, a distinguished limit is constructed [7].

The method is built around a dimensionless parameter $0 < \varepsilon \ll 1$ which symbolizes a sort of a scale of the problem. Starting from assumptions about the orders of magnitude of the physical parameters, the next step is to express each of them as a series expansion of ε.

The purpose of the method in the current context is to approximate the modal properties such as the natural frequencies and the damping ratios of the damped system. Before that, it is important to particularize the definition of the properties of the TMD ($\omega_{\mathrm{TMD}} = \sqrt{k/m}$), and make assumptions about their order of magnitude. A distinguished limit is obtained by defining

$$m = \mu\ M = (\varepsilon^2\mu_1)M_1\ ;\ k = m\omega_{\mathrm{TMD}}^2 = \varepsilon^2\mu_1(1+2\varepsilon\nu_2)K_1\ ;\ \omega_{\mathrm{TMD}} = (1+\varepsilon\nu_2)\Omega_1. \quad (14)$$

where μ_1 and ν_2 are of order 1. Two other dimensionless parameters introduced in order to express the relative magnitudes of the modals masses and frequencies in the structural modes : $\overline{M}_2 = M_2/M_1$ and $\beta = \Omega_2/\Omega_1 = \sqrt{K_2/M_2}/\sqrt{K_1/M_1}$. The latter parameter measures the distance between the natural frequencies and can be written as $\beta = 1 + b\varepsilon$, with $b = \mathrm{ord}(1)$. The range of study of β is limited to values close to the unity. The targeted range of β is $[1; 1.2]$, because for higher values the effect of the modal coupling is limited.

The quantities above are established as a function of the properties of the first mode (M_1, K_1, Ω_1). This is an arbitrary choice; it could as well have been possible to perform the developments with respect to the second mode properties. For the sole purpose to lighten the equations, M_1 and K_1 are taken equal to 1; this could be formalized by introducing a dimensionless time. The resulting matrices are

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \overline{M_2} & 0 \\ 0 & 0 & \varepsilon^2\mu_1 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \beta^2\overline{M_2} & 0 \\ 0 & 0 & \varepsilon^2\mu_1 \end{bmatrix} + \varepsilon^2\mu_1(1+2\varepsilon\nu_2)\begin{bmatrix} \varphi_1^2 & \varphi_1\varphi_2 & -\varphi_1 \\ \varphi_1\varphi_2 & \varphi_2^2 & -\varphi_2 \\ -\varphi_1 & -\varphi_2 & 1 \end{bmatrix} \quad (15)$$

The mode matrix $\mathbf{\Phi}$ is a 3×3 matrix gathering the three mode shapes. Each of them takes a form similar to $\boldsymbol{\phi} = \begin{bmatrix} \phi_{s,1} & \phi_{s,2} & \phi_{\mathrm{TMD}} \end{bmatrix}^T$ where $\phi_{s,j}$ is the amplitude of the mode

($j = 1, 2$) of the original structure, whereas ϕ_{TMD} is the amplitude of the response of the TMD. The latter is one order of magnitude larger than the structural amplitudes. In order to maintain the consistency of the dimensionless development, the re-scaled amplitude of the TMD must be of the same order as the structural amplitude. For that purpose, a re-scaled mode shape $\tilde{\boldsymbol{\phi}}$ is defined as

$$\tilde{\boldsymbol{\phi}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\varepsilon \end{bmatrix} \boldsymbol{\phi} := \mathbf{A}(\varepsilon)\boldsymbol{\phi}. \tag{16}$$

This scaling leads to the establishment of the matrices $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{K}}$ as power series expansion of ε,

$$\tilde{\mathbf{M}} = \mathbf{A}^T\mathbf{M}\mathbf{A} = \mathbf{M}_0 \qquad \text{and} \qquad \tilde{\mathbf{K}} = \mathbf{A}^T\mathbf{K}\mathbf{A} = \mathbf{K}_0 + \mathbf{K}_1\varepsilon + \mathrm{ord}\left(\varepsilon^2\right) \tag{17}$$

where

$$\mathbf{M}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \overline{M_2} & 0 \\ 0 & 0 & \mu_1 \end{bmatrix}; \quad \mathbf{K}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \overline{M_2} & 0 \\ 0 & 0 & \mu_1 \end{bmatrix}; \quad \mathbf{K_1} = \begin{bmatrix} 0 & 0 & -\mu_1\varphi_1 \\ 0 & 2b\overline{M_2} & -\mu_1\varphi_2 \\ -\mu_1\varphi_1 & -\mu_1\varphi_2 & 2\nu_2 \end{bmatrix}.$$

Similarly the damping matrices associated with the structural damping on one hand, and with the damping of the TMD, on the other hand can be written

$$\tilde{\mathbf{C}}_s = \mathbf{A}^T\mathbf{C}_s\mathbf{A} = \mathbf{C}_{s,0} + \mathbf{C}_{s,1}\varepsilon + \mathrm{ord}\left(\varepsilon^2\right) \qquad \text{and} \qquad \tilde{\mathbf{C}}_d = \mathbf{A}^T\mathbf{C}_d\mathbf{A} = \mathbf{C}_{d,0} + \mathbf{C}_{d,1}\varepsilon + \mathrm{ord}\left(\varepsilon^2\right) \tag{18}$$

where

$$\mathbf{C}_{s,0} = \begin{bmatrix} 2\xi_s & 0 & 0 \\ 0 & 2\overline{M_2}\xi_s & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad \mathbf{C}_{s,1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2b\overline{M_2}\xi_s & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

$$\mathbf{C}_{d,0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2\mu_1\xi_{TMD} \end{bmatrix}; \quad \mathbf{C}_{d,1} = 2\mu_1\xi_{TMD} \begin{bmatrix} 0 & 0 & -\varphi_1 \\ 0 & 0 & -\varphi_2 \\ -\varphi_1 & -\varphi_2 & \nu_2 \end{bmatrix}.$$

In order to simplify the following notations, the tilde symbol is dropped.

4.2. Determination of the natural frequencies

The eigenvalues are the solutions of $(\mathbf{K} - \lambda^2\mathbf{M})\boldsymbol{\phi} = 0$ with $\lambda = \omega^2/\Omega_1^2$. The eigen vectors and eigen values are sought in the form of the following ansatz $\lambda = \lambda_0 + \varepsilon\lambda_1 + \mathrm{ord}(\varepsilon^2)$

and $\boldsymbol{\phi} = \boldsymbol{\phi}_0 + \varepsilon\boldsymbol{\phi}_1 + \mathrm{ord}(\varepsilon^2)$. Application of standard perturbation methods yields, after balancing the likewise powers of ε,

$$\begin{aligned} &\mathrm{ord}(\varepsilon^0): \quad (K_0 - \lambda_0 \mathbf{M}_0)\boldsymbol{\phi}_0 = 0, \\ &\mathrm{ord}(\varepsilon^1): \quad (K_1 - \lambda_1 \mathbf{M}_0)\boldsymbol{\phi}_0 = 0. \end{aligned} \tag{19}$$

The solution of the problem at first order is readily obtained as $\lambda_0 = (1,1,1)^T$, while the mode shapes $\boldsymbol{\phi_0}$ cannot be computed as the matrix $(K_0 - \lambda_0 \mathbf{M}_0)$ is nil. At second order, the value of λ_1 is seen to satisfy the following 3rd degree polynomial equation

$$\lambda_1^3 - 2\,(b+\nu_2)\,\lambda_1^2 + \left(4b\nu_2 - \mu_1\left(\varphi_1^2 + \frac{\varphi_2^2}{\overline{M}_2}\right)\right)\lambda_1 + 2\mu_1\varphi_1^2 b = 0. \tag{20}$$

The solution of this equation provides the three roots associated with the three different natural frequencies of the coupled problem.

4.3. Determination of the equivalent damping ratios

Stepping back to the eigenvalue problem at order ε^1. The eigenvectors $\boldsymbol{\phi}_0$ are expressed as a function of λ_1 which is known from leading order, i.e. as one of the roots of (20). In order to simplify the following notations, symbol λ_1 is kept instead of its complete analytical expression. With this, the i^{th} eigen mode ($i = 1, ...3$), at leading order, is given by

$$\boldsymbol{\phi}_0^i = \left(-\frac{\mu_1\varphi_1}{\lambda_1^i}, -\frac{\mu_1\varphi_2}{(\lambda_1^i - 2b)\overline{M}_2}, 1\right)^T. \tag{21}$$

This first order approximation of the mode shapes is used to determine the leading order approximation of modal matrices,

$$\begin{aligned} \mathrm{M}^*_{i,j} &= \boldsymbol{\phi}_0^{iT}\,\mathbf{M}_0\,\boldsymbol{\phi}_0^j = \frac{\mu_1^2\varphi_1^2}{\lambda_1^i\lambda_1^j} + \frac{1}{\overline{M}_2}\frac{\mu_1^2\varphi_2^2}{(\lambda_1^i - 2b)(\lambda_1^j - 2b)} + \mu_1 \\ \mathrm{K}^*_{i,j} &= \boldsymbol{\phi}_0^{iT}\,\mathbf{K}_0\,\boldsymbol{\phi}_0^j = \frac{\mu_1^2\varphi_1^2}{\lambda_1^i\lambda_1^j} + \frac{\beta^2}{\overline{M}_2}\frac{\mu_1^2\varphi_2^2}{(\lambda_1^i - 2b)(\lambda_1^j - 2b)} + \mu_1. \end{aligned}$$

Similarly, the generalized damping matrices are given by

$$\begin{aligned} \mathrm{C}^*_{s,i,j} &= \boldsymbol{\phi}_0^{iT}\,\mathbf{C}_{s,0}\,\boldsymbol{\phi}_0^j = 2\Big(\frac{\mu_1^2\varphi_1^2}{\lambda_1^i\lambda_1^j} + \frac{1}{\overline{M}_2}\frac{\mu_1^2\varphi_2^2}{(\lambda_1^i - 2b)(\lambda_1^j - 2b)}\Big)\xi_s \\ \mathrm{C}^*_{d,i,j} &= \boldsymbol{\phi}_0^{iT}\,\mathbf{C}_{d,0}\,\boldsymbol{\phi}_0^j = 2\mu_1\xi_{\mathrm{TMD}}. \end{aligned}$$

A dimensionless viscosity quantifying the modal coupling (defined as an extension of the damping ratio) is approximated, at first order, by

$$\xi_{i,j} = \frac{\mathrm{C}^*_{s,i,j} + \mathrm{C}^*_{d,i,j}}{2\sqrt{\mathrm{M}^*_{i,j}\mathrm{K}^*_{i,j}}} = \frac{\mu_1\big(\frac{\varphi_1^2}{\lambda_1^i\lambda_1^j} + \frac{1}{\overline{M}_2}\frac{\varphi_2^2}{(\lambda_1^i - 2b)(\lambda_1^j - 2b)}\big)\xi_s + \xi_{TMD}}{\mu_1\big(\frac{\varphi_1^2}{\lambda_1^i\lambda_1^j} + \frac{1}{\overline{M}_2}\frac{\varphi_2^2}{(\lambda_1^i - 2b)(\lambda_1^j - 2b)}\big) + 1}. \tag{22}$$

As a particular case, when $i = j$, one recovers the modal damping ratios. After some simplifications, they take the simple form

$$\xi_{i,i} = \frac{\Gamma\,\xi_s + \xi_{\mathrm{TMD}}}{\Gamma + 1}, \tag{23}$$

i.e. a weighted average of ξ_s and ξ_{TMD}, where $\Gamma = \Gamma_1 + \Gamma_2$ is the sum of two positive quantities

$$\Gamma_1 = \mu_1 \left(\frac{\varphi_1}{\lambda_1^i}\right)^2 \qquad \text{and} \qquad \Gamma_2 = \frac{\mu_1}{\overline{M}_2}\left(\frac{\varphi_2}{\lambda_1^i - 2b}\right)^2. \tag{24}$$

These dimensionless groups play a major role in the understanding of this problem. Indeed, two different structures equipped with possibly different tuned mass dampers will exhibit the same dynamical behavior (at leading order) as soon as they are characterized by these two same dimensionless groups.

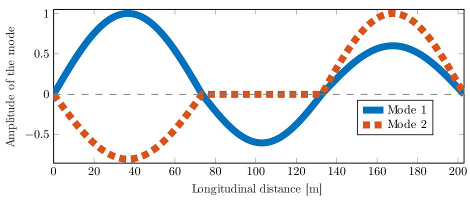
As a first limit case, the modal damping ratio tends towards ξ_s, when $\lambda_1 \to 0$ or $\lambda_1 \to 2b$. In the first configuration, Γ_1 is the dominant term, while it is Γ_2 in the second. As a second limit case, the damping ratio converges towards ξ_{TMD} when $\lambda_1 \to \infty$; in this case both Γ_1 and Γ_2 are small quantities. The intermediate case appears therefore as the most interesting one. It corresponds to the case where $\Gamma_1 \sim \Gamma_2$. This can drive the designer to the determination of the optimal placement of a TMD, i.e. at a location that satisfies

$$\frac{\varphi_1}{\varphi_2} = (1 + \frac{2b}{\lambda_1^i - 2b})\frac{1}{\sqrt{\overline{M}_2}}. \tag{25}$$

Equations (24) can provide a much wide outlook over this problem. For instance, they show that, when the damping is governed by Γ_2, the modal damping ratio is expressed as a function of $\mu_1\varphi_2^2/\overline{M}_2$. This indicates that the optimal configuration should be such that φ_2 is multiplied by $\sqrt{\theta}$ when $\overline{M_2}$ is multiplied by θ. Same observations apply for changes in μ_1.

5. Validation

The method is inspired by the example of the footbridge of Mantes Limay in Paris (France). This structure composed by 3 spans, is modeled by a 2-DOF structural system. The modal characteristics are summarized in the following table. The structure is subjected to crowd excitation, inducing lateral accelerations. The modal forms are idealized, they are represented in the Figure 1 where the position of the TMD is indicated by a star.



	Mode 1	Mode 2
Frequency (Hz)	1	1.12
Modal mass (t)	150	150
Damping ratio (%)	0.4	0.4

Figure 1. Modal characteristics of the 2DOF model

The PSD of the acceleration is displayed in the following figure. The corrected plot converges better towards the exact result. The shape and the amplitude is well approximated, comparing to the "diagonal" plot refering to the uncoupled approach.

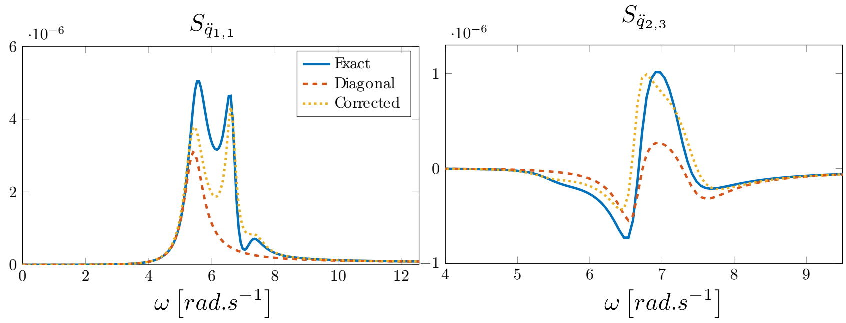


Figure 2. PSD - Different approaches

The natural frequencies and the modal damping ratios are well approximated with the analytical expressions. The table below evaluates the relative error of the analytical expressions with respect to the numerical computation, for a mass ratio of 5%. The errors are smaller for smaller mass ratios.

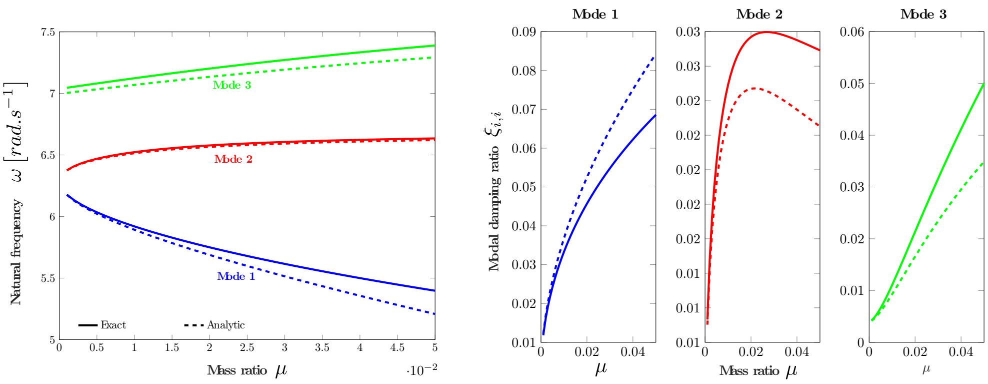


Figure 3. Approximation of the modal properties

Relative error	Mode 1	Mode 2	Mode 3
Natural frequency ω_i	1.3%	0.2%	3.5%
Damping ratio $\xi_{i,i}$	23%	16%	30%

Table 1. Relative error with respect to the numerical results

The loading being applied on the first DOF, the following figure displays the nodal variance of the acceleration (equation 8) in the $(\alpha, \xi_{\mathrm{TMD}})$ space. The minimum variance is identified by the cercle.

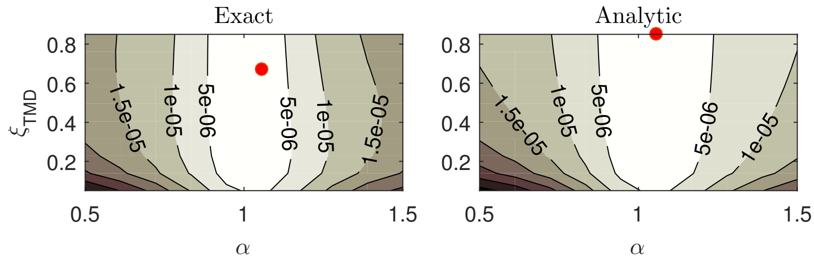


Figure 4. Variance as a function of the parameters of the TMD

The brighter region containing the minimal values of the variance, is well reproduced by the "Analytic" map. It matches the criteria on the optimum tuning α as well, but not the criteria on the optimum ξ_{TMD}.

The method reproduces in a favorable way the dynamic behavior of the MDOF structure.

equations that are lighter and more explicit, favorable for direct interpretations of the different phenomena involved.

References

[1] Ray W Clough, Joseph Penzien, and DS Griffin. Dynamics of structures, 1977.

[2] Vincent Denoël. Estimation of modal correlation coefficients from background and resonant responses. *Structural Engineering and Mechanics*, 32(6):725–740, 2009.

[3] Vincent Denoël. Multiple timescale spectral analysis. *Probabilistic Engineering Mechanics*, 39:69–86, 2015.

[4] Yozo Fujino and Masato Abé. Design formulas for tuned mass dampers based on a

perturbation technique. *Earthquake engineering & structural dynamics*, 22(10):833–854, 1993.

[5] J. P. Den Hartog. *Mechanical Vibrations*. McGraw-Hill, New York, 4th. edition, 1956.

[6] Stephen Hicks, Markus Feldmann, Christoph Heinemeyer, Mladen Lukic, Elsa Caetano, Alvaro Cunha, Arndt Goldack, Andreas Keil, Mike Schlaich, Andy Smith, Oliver Hechler, Renata Obiala, Flavio Galanti, and Paul Waarts. *Human-induced vibration of steel structures (Hivoss)*. European Research Area - European Commission, 2010.

[7] EJ Hinch. Perturbation methods. 1991.

[8] IF Lazar, SA Neild, and DJ Wagg. Using an inerter-based device for structural vibration suppression. *Earthquake Engineering & Structural Dynamics*, 43(8):1129–1147, 2014.

[9] Anass Mayou. Modal coupling in the damped structures, effect of the coupling on the vibration behavior of footbridges. 2019.

[10] Service d'Études Techniques des Routes et Autoroutes (Sétra). *Technical guide - Footbridges: Assessment of vibrational behaviour of footbridges under pedestrian loading*. Ministère des Transports, de l'Equipement, du Tourisme, et de la Mer (France), 2006.

[11] Malcolm C Smith. Synthesis of mechanical networks: the inerter. *IEEE Transactions on automatic control*, 47(10):1648–1662, 2002.

[12] Vincent Denoël and Hervé Degée. Asymptotic expansion of slightly coupled modal dynamic transfer functions. *Journal of Sound and Vibration*, 328(1-8), 2009.