A way to extend Pascal's triangle to words

Joint work with Julien Leroy and Michel Rigo (ULiège)

Manon Stipulanti (Hofstra University) BAEF Fellow

Hofstra Math. Seminar
Long Island (NY, USA)
November 6, 2019

A little about me

Manon	Stipulanti	Born in
		

- Kingdom of Belgium
  (we have a King!)
- Three official languages:
  Dutch, French and German
  (and many dialects)
- Capital: Brussels
  Capital of the European Union

What is Belgium famous for?

Chocolate

Liège waffle

Brussels waffle

Beers

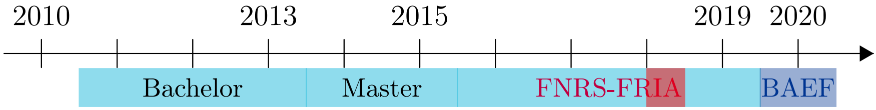
~~French~~ fries

Comics

THE UNIVERSITY OF WINNIPEG

LIÈGE université Mathematics

HOFSTRA UNIVERSITY



Scientific interests:

- combinatorics on words
- numeration systems
- formal language theory
- automata theory

fnrs FREEDOM TO RESEARCH

BAEF

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Pascal's triangle

$\mathrm{P}\colon (m,k) \in \mathbb{N}\times\mathbb{N} \mapsto \binom{m}{k} \in \mathbb{N}$

$\binom{m}{k}$		k								
		0	1	2	3	4	5	6	7	$\cdots$
	0	1	0	0	0	0	0	0	0	
	1	1	1	0	0	0	0	0	0	
	2	1	2	1	0	0	0	0	0	
m	3	1	3	3	1	0	0	0	0	
	4	1	4	6	4	1	0	0	0	
	5	1	5	**10**	**10**	5	1	0	0	
	6	1	6	15	**20**	15	6	1	0	
	7	1	7	21	35	35	21	7	1	
	$\vdots$									$\ddots$

Binomial coefficients:

$$\binom{m}{k} = \frac{m!}{(m-k)!\,k!}$$

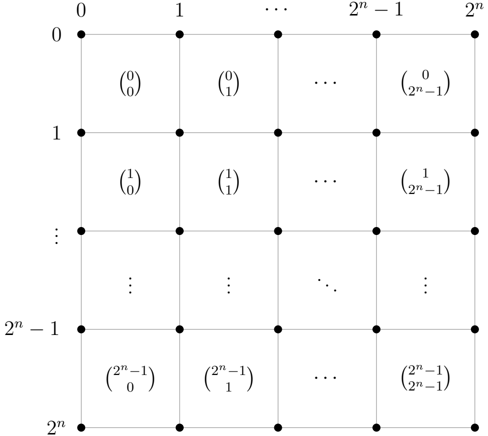
Pascal's rule:

$$\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}$$

A specific construction

- Grid: first 2^n rows and columns of the Pascal's triangle

$$\left(\binom{m}{k}\right)_{0\le m,k<2^n}$$



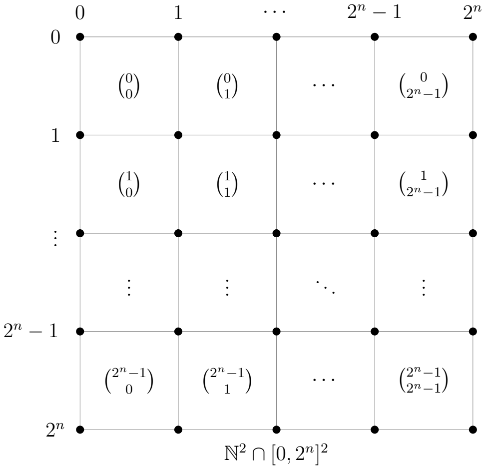
$\mathbb{N}^2 \cap [0, 2^n]^2$

A specific construction

- Grid: first 2^n rows and columns of the Pascal's triangle

$$\left(\binom{m}{k}\right)_{0\leq m,k<2^n}$$

- Color each square in
  - white if $\binom{m}{k} \equiv 0 \bmod 2$
  - black if $\binom{m}{k} \equiv 1 \bmod 2$



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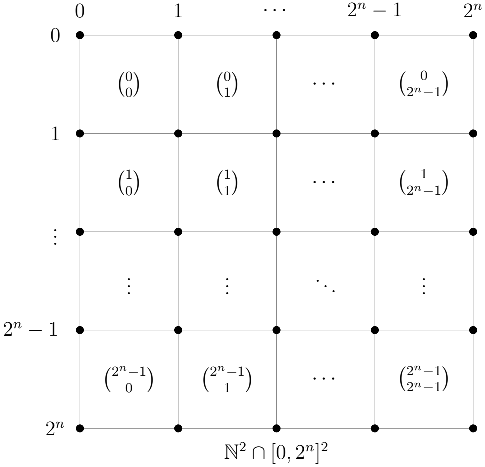
- Normalize by a homothety of ratio $1/2^n$ (bring into $[0,1]^2$)



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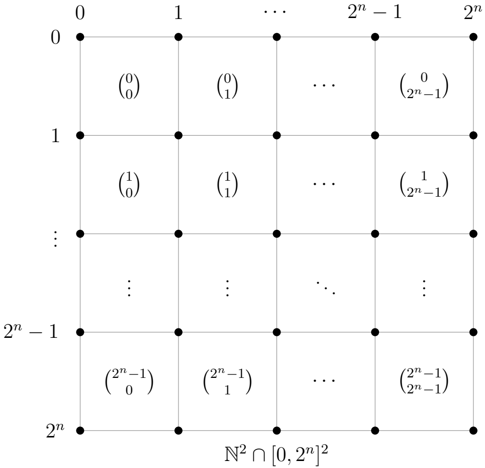
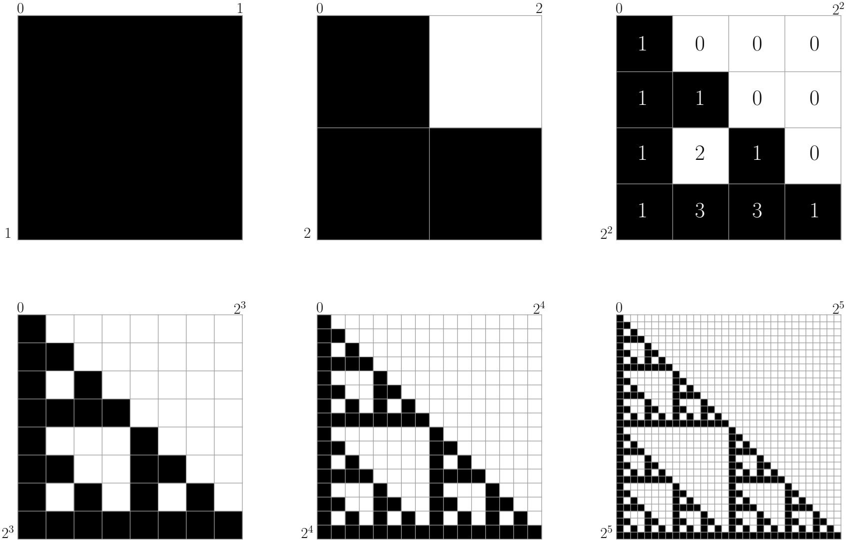
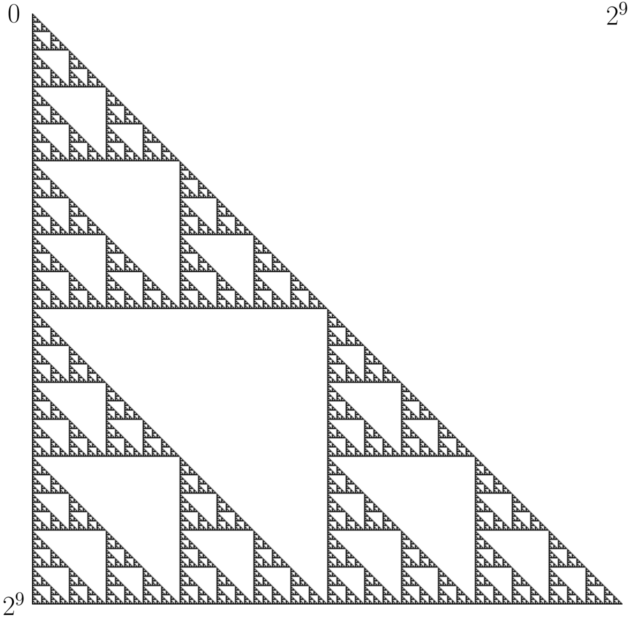
- Normalize by a homothety of ratio $1/2^n$ (bring into $[0,1]^2$) $\rightsquigarrow$ sequence of compact sets belonging to $[0,1]^2$



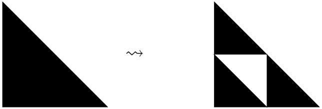
The first six elements of the sequence



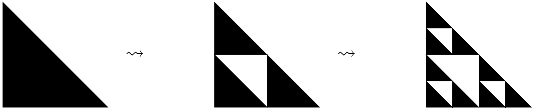
The tenth element of the sequence



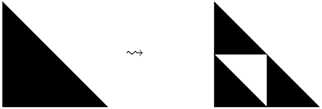
Sierpiński's triangle

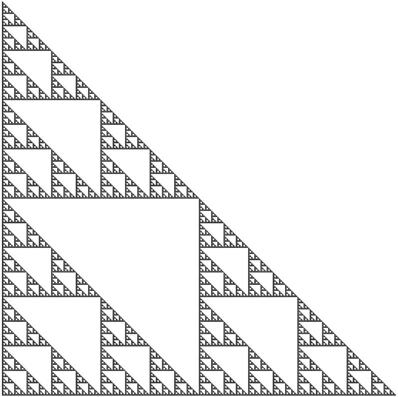


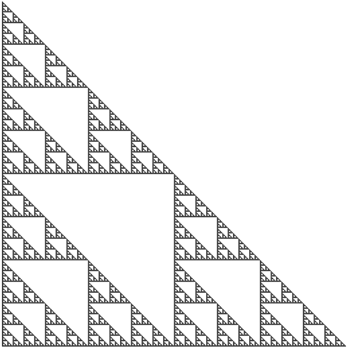
Sierpiński's triangle



Sierpiński's triangle







Folklore fact

The latter sequence of compact sets converges to Sierpiński's triangle (w.r.t. the Hausdorff distance).

Definitions:

- ϵ-*fattening* of a subset $S \subset \mathbb{R}^2$

$$[S]_\epsilon = \bigcup_{x \in S} B(x, \epsilon)$$

- $(\mathcal{H}(\mathbb{R}^2), d_h)$ complete space of the non-empty compact subsets of $\mathbb{R}^2$ equipped with the *Hausdorff distance* d_h

$$d_h(S, S') = \inf\{\epsilon \in \mathbb{R}_{>0} \mid S \subset [S']_\epsilon \quad \text{and} \quad S' \subset [S]_\epsilon\}$$

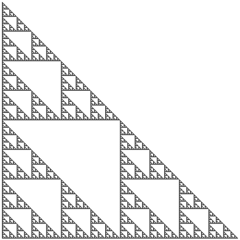
Theorem (von Haeseler, Peitgen, and Skordev, 1992)

Let p be a prime and $s > 0$.
The sequence of compact sets corresponding to

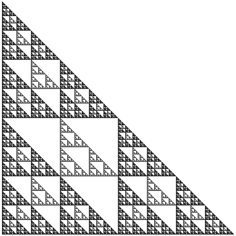
$$\left(\binom{m}{k} \bmod p^s\right)_{0 \le m,k < p^n}$$

converges when n tends to infinity (w.r.t. the Hausdorff distance).

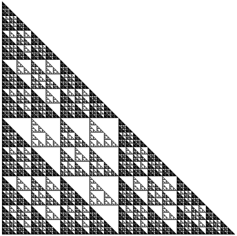
$p = 2,\ s = 1$



$p = 2,\ s = 2$



$p = 2,\ s = 3$

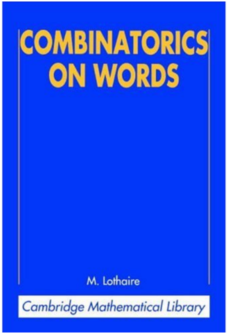


New idea: CoW

Replace integers by finite words.

Combinatorics on words (CoW)

- new area of discrete mathematics (± 1900)
- study sequences of symbols (called letters)
- topics include:
    - regularities and patterns in words
    - important types of words (e.g. automatic, regular, de Bruijn, Lyndon, Sturmian)
    - coding of structures (e.g. paths, trees or curves in the plane)



M. Lothaire, 1983.

Binomial coefficient of finite words

Definition: A *finite word* is a finite sequence of letters belonging to a finite set called the *alphabet*.

Example: $101, 101001 \in \{0,1\}^*$

Binomial coefficient of words

Let u, v be two finite words.
The *binomial coefficient* $\binom{u}{v}$ of u and v is the number of times v occurs as a subsequence of u (meaning as a "scattered" subword).

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<u>Example</u>: $u = 101001$ $\qquad v = 101$

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Binomial coefficient of words

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The *binomial coefficient* $\binom{u}{v}$ of u and v is the number of times v occurs as a subsequence of u (meaning as a "scattered" subword).

Example: $u = \mathbf{101}001$ $\qquad$ $v = 101$ $\qquad$ 1 occurrence

Binomial coefficient of finite words

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Example: $101, 101001 \in \{0,1\}^*$

Binomial coefficient of words

Let u, v be two finite words.
The *binomial coefficient* $\binom{u}{v}$ of u and v is the number of times v occurs as a subsequence of u (meaning as a "scattered" subword).

Example: $u = \mathbf{10}100\mathbf{1}$ $\qquad v = 101$ $\qquad$ 2 occurrences

Binomial coefficient of finite words

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Example: $101, 101001 \in \{0,1\}^*$

Binomial coefficient of words

Let u, v be two finite words.
The *binomial coefficient* $\binom{u}{v}$ of u and v is the number of times v occurs as a subsequence of u (meaning as a "scattered" subword).

Example: $u = 101001$ $v = 101$ 3 occurrences

Binomial coefficient of finite words

Definition: A *finite word* is a finite sequence of letters belonging to a finite set called the *alphabet*.

Example: $101, 101001 \in \{0,1\}^*$

Binomial coefficient of words

Let u, v be two finite words.
The *binomial coefficient* $\binom{u}{v}$ of u and v is the number of times v occurs as a subsequence of u (meaning as a "scattered" subword).

Example: $u = 101001$ $\quad v = 101 \quad$ 4 occurrences

Binomial coefficient of finite words

Definition: A *finite word* is a finite sequence of letters belonging to a finite set called the *alphabet*.

Example: $101, 101001 \in \{0,1\}^*$

Binomial coefficient of words

Let u, v be two finite words.
The *binomial coefficient* $\binom{u}{v}$ of u and v is the number of times v occurs as a subsequence of u (meaning as a "scattered" subword).

Example: $u = 101001$ $v = 101$ 5 occurrences

Binomial coefficient of finite words

Definition: A *finite word* is a finite sequence of letters belonging to a finite set called the *alphabet*.

Example: $101, 101001 \in \{0,1\}^*$

Binomial coefficient of words

Let u, v be two finite words.
The *binomial coefficient* $\binom{u}{v}$ of u and v is the number of times v occurs as a subsequence of u (meaning as a "scattered" subword).

Example: $u = 101001$ $\quad v = 101$ $\quad$ 6 occurrences

Definition: A *finite word* is a finite sequence of letters belonging to a finite set called the *alphabet*.

Example: $101, 101001 \in \{0,1\}^*$

Binomial coefficient of words

Let u, v be two finite words.
The *binomial coefficient* $\binom{u}{v}$ of u and v is the number of times v occurs as a subsequence of u (meaning as a "scattered" subword).

Example: $u = 101001$ $\qquad v = 101$

$$\Rightarrow \binom{101001}{101} = 6$$

Remark:

Natural generalization of binomial coefficients of integers

If a is a letter

$$\binom{a^m}{a^k} = \binom{\overbrace{a \cdots a}^{m \text{ times}}}{\underbrace{a \cdots a}_{k \text{ times}}} = \binom{m}{k} \quad \forall\, m, k \in \mathbb{N}$$

Let $(A,<)$ be a totally ordered alphabet.
Let $L \subset A^*$ be an infinite language (set of words) over A.
The words in L are genealogically ordered

$$w_0 <_{\mathrm{gen}} w_1 <_{\mathrm{gen}} w_2 <_{\mathrm{gen}} \cdots .$$

The *generalized Pascal's triangle* P_L associated with L is defined by

$$\mathrm{P}_L \colon (m,k) \in \mathbb{N} \times \mathbb{N} \mapsto \binom{w_m}{w_k} \in \mathbb{N} .$$

Questions:

- With a similar construction, can we expect the convergence to an analogue of Sierpiński's triangle?
- In particular, where should we cut to normalize a given generalized Pascal's triangle?
- Could we describe this limit object?

Base 2

Definitions:

- $\text{rep}_2(n)$ greedy base-2 representation of $n \in \mathbb{N}_{>0}$ starting with 1
- $\text{rep}_2(0) = \varepsilon$ where ε is the empty word

n	$n = \sum_{i=0}^{\ell} c_i 2^i$ with $c_i \in \{0,1\}$	$\text{rep}_2(n) = c_\ell \cdots c_0$
0		ε
1	1×2^0	1
2	$1 \times 2^1 + 0 \times 2^0$	10
3	$1 \times 2^1 + 1 \times 2^0$	11
4	$1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0$	100
5	$1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$	101
6	$1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$	110
$\vdots$	$\vdots$	$\vdots$
		$L_2 = 1\{0,1\}^* \cup \{\varepsilon\}$

Generalized Pascal's triangle P_2 in base 2

$\binom{\text{rep}_2(m)}{\text{rep}_2(k)}$		rep$_2(k)$								
		ε	1	10	11	100	101	110	111	$\cdots$
	ε	1	0	0	0	0	0	0	0	
	1	1	1	0	0	0	0	0	0	
	10	1	1	1	0	0	0	0	0	
	11	1	**2**	0	**1**	0	0	0	0	
rep$_2(m)$	100	1	1	2	0	1	0	0	0	
	101	1	2	1	1	0	1	0	0	
	110	1	2	2	1	0	0	1	0	
	111	1	3	0	**3**	0	0	0	1	
	$\vdots$									$\ddots$

Rule (not local):

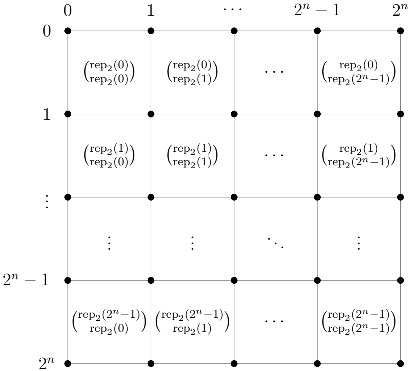
$$\binom{ua}{vb} = \binom{u}{vb} + \delta_{a,b}\binom{u}{v}$$

$\binom{\mathrm{rep}_2(m)}{\mathrm{rep}_2(k)}$		$\mathrm{rep}_2(k)$								
		ε	**1**	10	**11**	100	101	110	**111**	$\cdots$
	ε	**1**	0	0	0	0	0	0	0	
	1	**1**	**1**	0	0	0	0	0	0	
	10	1	1	1	0	0	0	0	0	
	11	**1**	**2**	0	**1**	0	0	0	0	
$\mathrm{rep}_2(m)$	100	1	1	2	0	1	0	0	0	
	101	1	2	1	1	0	1	0	0	
	110	1	2	2	1	0	0	1	0	
	111	**1**	**3**	0	**3**	0	0	0	**1**	
	$\vdots$									$\ddots$

The usual Pascal's triangle

Same construction

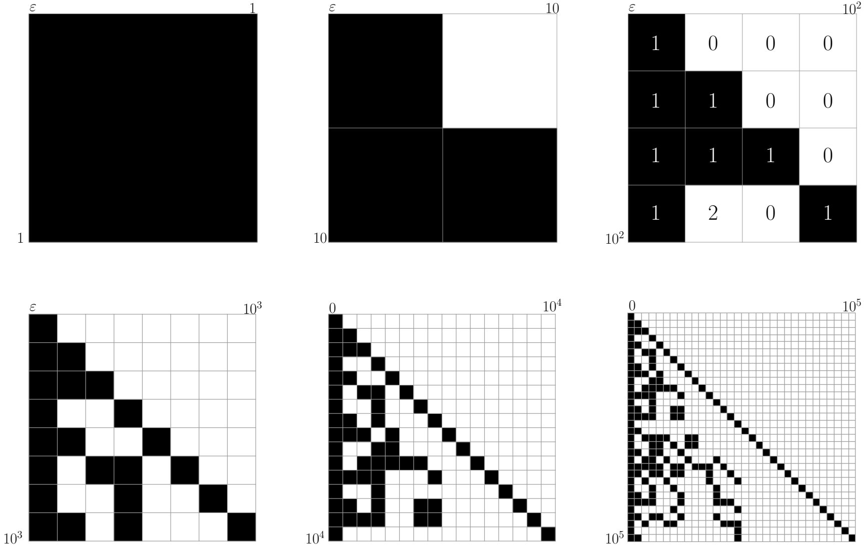
- Grid: first 2^n rows and columns of P_2
- Color each square in
  - white if $\binom{\text{rep}_2(m)}{\text{rep}_2(k)} \equiv 0 \bmod 2$
  - black if $\binom{\text{rep}_2(m)}{\text{rep}_2(k)} \equiv 1 \bmod 2$
- Normalize by a homothety of ratio $1/2^n$ (bring into $[0,1]^2$) $\rightsquigarrow$ sequence of compact sets belonging to $[0,1]^2$



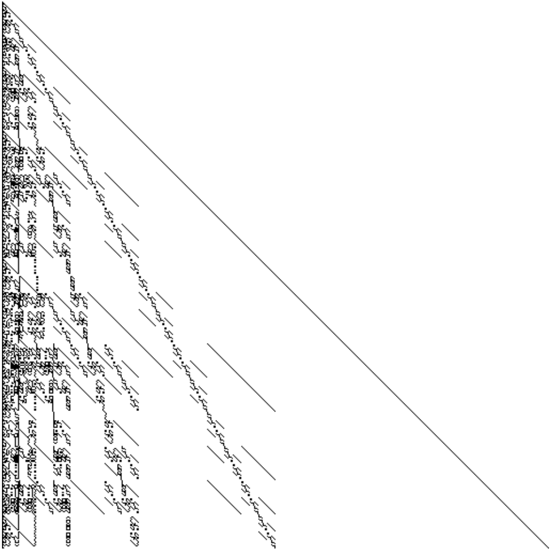
$\mathbb{N}^2 \cap [0, 2^n]^2$

$$\mathcal{U}_n = \frac{1}{2^n} \bigcup_{\substack{u,v \in L_2,\ |u|,|v| \le n \\ \binom{u}{v} \equiv 1 \bmod 2}} \text{val}_2(v,u) + [0,1]^2$$

The elements $\mathcal{U}_0, \ldots, \mathcal{U}_5$



The element $\mathcal{U}_9$



Lines of different slopes: $1, 2, 4, 8, 16, \ldots$

The $(\star)$ condition

$(\star)$

(u,v) satisfies $(\star)$ iff $\begin{cases} u, v \neq \varepsilon \\ \binom{u}{v} \equiv 1 \bmod 2 \\ \binom{u}{v0} = 0 = \binom{u}{v1} \end{cases}$

Example: $(u,v) = (101, 11)$ satisfies $(\star)$

$$\binom{101}{11} = 1 \qquad \binom{101}{110} = 0 \qquad \binom{101}{111} = 0$$

Lemma: Completion

(u,v) satisfies $(\star)$ $\Rightarrow$ $(u0,v0),(u1,v1)$ satisfy $(\star)$

Proof: Since (u,v) satisfies $(\star)$

$$\binom{u}{v} \equiv 1 \bmod 2, \quad \binom{u}{v0} = 0 = \binom{u}{v1}$$

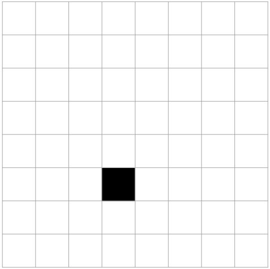
Proof for $(u0,v0)$:

$$\binom{u0}{v0} = \underbrace{\binom{u}{v0}}_{=0 \text{ since } (\star)} + \underbrace{\binom{u}{v}}_{\equiv 1 \bmod 2} \equiv 1 \bmod 2$$

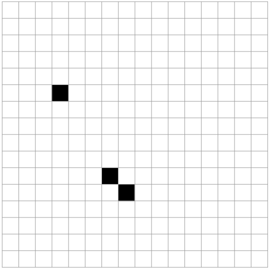
If $\binom{u0}{v00} > 0$ or $\binom{u0}{v01} > 0$, then $v0$ is a subsequence of u.
This contradicts $(\star)$.

Same proof for $(u1,v1)$. $\square$

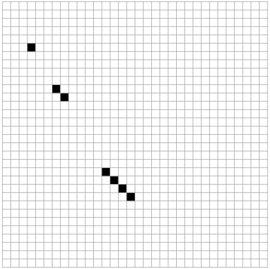
Example: $(u,v) = (101, 11)$ satisfies $(\star) \Rightarrow \binom{u}{v} \equiv 1 \bmod 2$



$\mathcal{U}_3$



$\mathcal{U}_4$

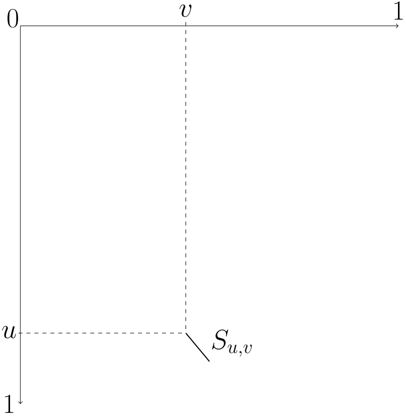


$\mathcal{U}_5$

$\rightsquigarrow$ Creation of a segment of slope 1
Endpoint $(3/8, 5/8) = (\mathrm{val}_2(11)/2^3, \mathrm{val}_2(101)/2^3)$
Length $\sqrt{2} \cdot 2^{-3}$

Segments of slope 1

The $(\star)$ condition describes lines of slope 1 in $[0,1]^2$.



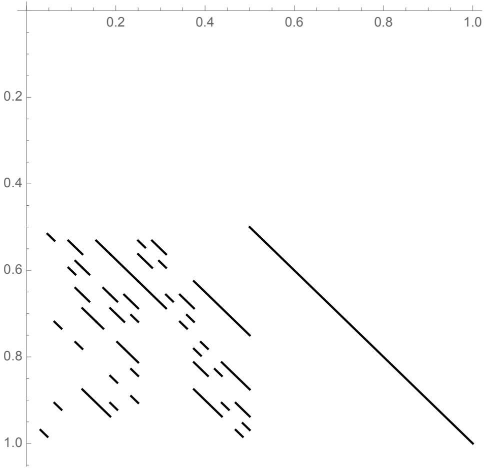
$(u,v) \in L_2 \times L_2$ satisfying $(\star)$
$\rightsquigarrow$ closed segment $S_{u,v}$

- slope 1
- length $\sqrt{2} \cdot 2^{-|u|}$
- origin

$$A_{u,v} = \mathrm{val}_2(v,u)/2^{|u|}$$
$$= (0.0^{|u|-|v|}v, 0.u)$$

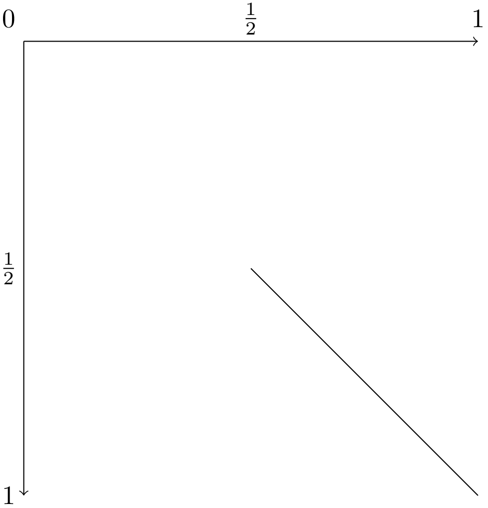
Definition: New compact set containing those lines

$$\mathcal{A}_0 = \overline{\bigcup_{\substack{(u,v)\\ \text{satisfying } (\star)}} S_{u,v}} \subset [0,1]^2$$



Modifying the slope

Two maps $c\colon (x,y) \mapsto (x/2, y/2)$ and $h\colon (x,y) \mapsto (x, 2y)$



Example: $(1,1)$ satisfies $(\star)$

Segment $S_{1,1}$
endpoint $(1/2, 1/2)$
length $\sqrt{2} \cdot 2^{-1}$

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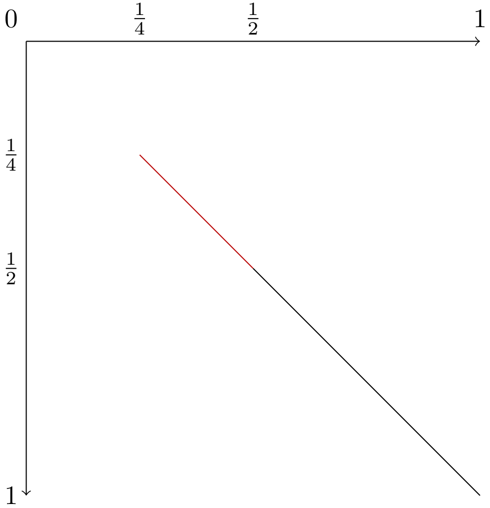

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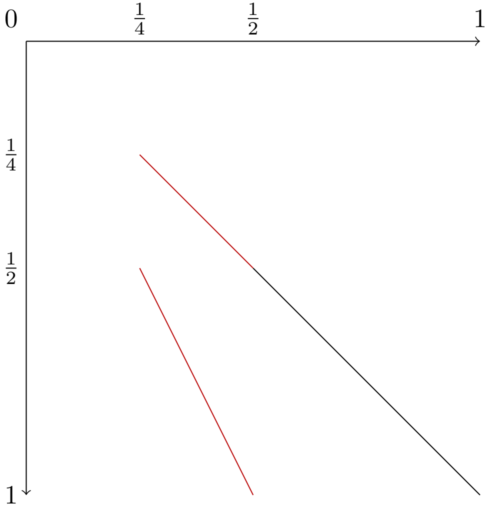

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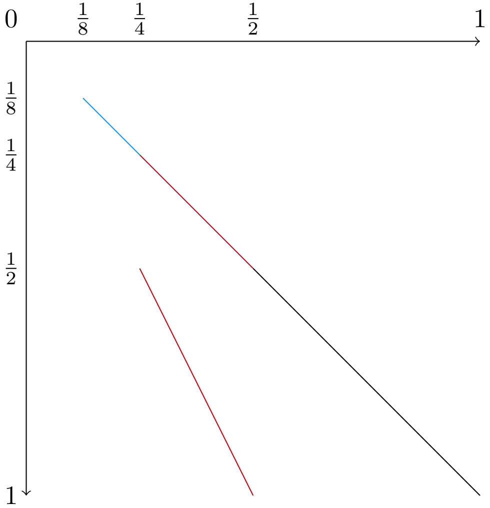


Example: $(1,1)$ satisfies $(\star)$

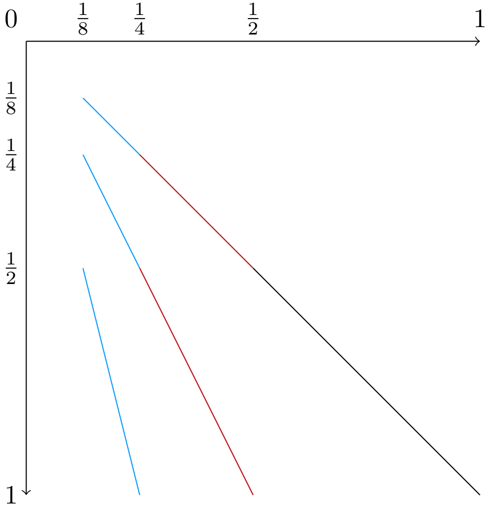
Segment $S_{1,1}$
endpoint $(1/2, 1/2)$
length $\sqrt{2} \cdot 2^{-1}$

Modifying the slope

Two maps $c\colon (x,y) \mapsto (x/2, y/2)$ and $h\colon (x,y) \mapsto (x, 2y)$

Example: $(1,1)$ satisfies $(\star)$

Segment $S_{1,1}$
endpoint $(1/2, 1/2)$
length $\sqrt{2} \cdot 2^{-1}$

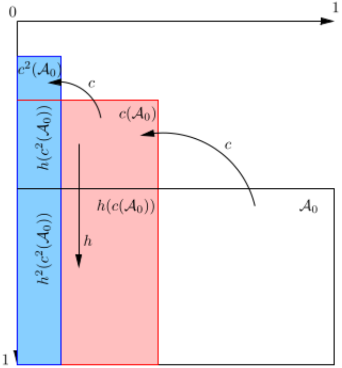


Definition: New compact set containing lines of slopes 1, 2, 2^2, ..., 2^n

$c\colon (x,y) \mapsto (x/2, y/2)$

$h\colon (x,y) \mapsto (x, 2y)$

$$\mathcal{A}_n = \bigcup_{0\leq j\leq i\leq n} h^j(c^i(\mathcal{A}_0))$$



The compact sets $(\mathcal{A}_n)_{n\geq 0}$ are increasingly nested and their union is bounded.

Thus $(\mathcal{A}_n)_{n\geq 0}$ converges to

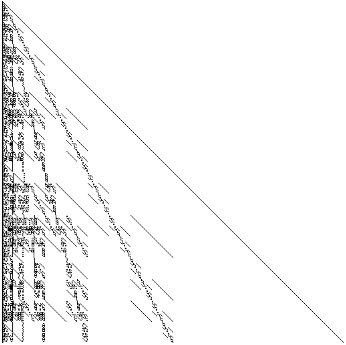
$$\mathcal{L} = \overline{\bigcup_{n\geq 0} \mathcal{A}_n}$$

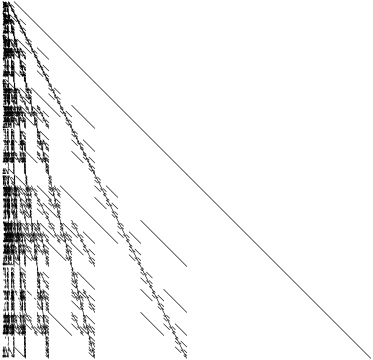
(w.r.t. the Hausdorff distance).

A key result

Theorem (Leroy, Rigo, S., 2016)

The sequence $(\mathcal{U}_n)_{n\geq 0}$ of compact sets converges to the compact set $\mathcal{L}$ when n tends to infinity (w.r.t. the Hausdorff distance).





"Simple" characterization of $\mathcal{L}$: $(\star)$ condition

Extension modulo p

Previous result: even and odd coefficients

Theorem (Lucas, 1878)

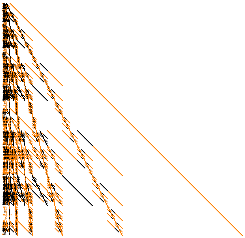
Let p be a prime number.
If $m = m_k p^k + \cdots + m_1 p + m_0$ and $n = n_k p^k + \cdots + n_1 p + n_0$ then

$$\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{m_i}{n_i} \bmod p.$$

Theorem (Leroy, Rigo, S., 2016)

Let p be a prime and $0 < r < p$.
When considering binomial coefficients congruent to $r \pmod p$, the sequence $(\mathcal{U}_{n,p,r})_{n\geq 0}$ converges to a well-defined compact set $\mathcal{L}_{p,r}$ (w.r.t. the Hausdorff distance).

Example: $\mathcal{L}_{3,1} \cup \mathcal{L}_{3,2}$

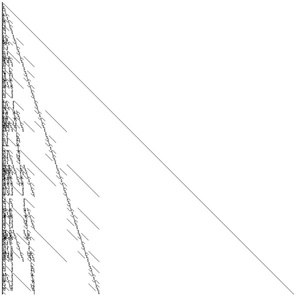


Extension to any integer base

Everything still holds for binomial coefficients $\equiv r \bmod p$ with

- integer base $b \geq 2$
- language L_b of greedy base-b representations of integers
- p a prime
- $r \in \{1, \ldots, p-1\}$

Example: base 3, $\equiv 1 \bmod 2$



Fibonacci numeration system

Definitions:

- Fibonacci numbers $(F(n))_{n\geq 0}$
  $F(0)=1,\ F(1)=2,\ F(n+2)=F(n+1)+F(n)\quad \forall\, n\geq 0$
  $1,2,3,5,8,13,21,34,55,89,144,233,377,610,987,1597,\ldots$
- $\mathrm{rep}_F(n)$ greedy Fibonacci representation of $n\in\mathbb{N}_{>0}$ starting with 1
- $\mathrm{rep}_F(0)=\varepsilon$ where ε is the empty word

n	$n=\sum_{i=0}^{\ell} c_i F(i)$ with $c_i\in\{0,1\}$	$\mathrm{rep}_F(n)=c_\ell\cdots c_0$
0		ε
1	$1\times F(0)$	1
2	$1\times F(1)+0\times F(0)$	10
3	$1\times F(2)+0\times F(1)+0\times F(0)$	100
4	$1\times F(2)+0\times F(1)+1\times F(0)$	101
5	$1\times F(3)+0\times F(2)+0\times F(1)+0\times F(0)$	1000
6	$1\times F(3)+0\times F(2)+0\times F(1)+1\times F(0)$	1001
$\vdots$	$\vdots$	$\vdots$
		$L_F=1\{0,01\}^*\cup\{\varepsilon\}$

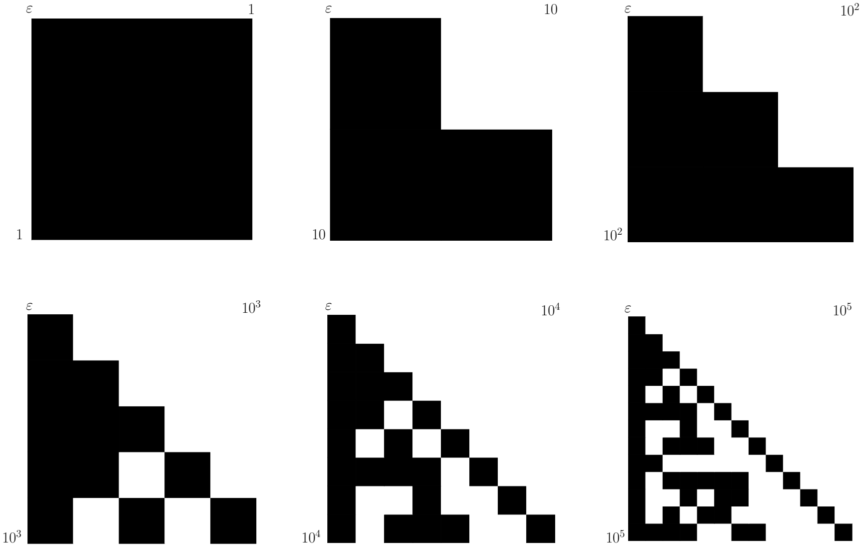
Generalized Pascal's triangle P_F in Fibonacci base

	$\binom{\text{rep}_F(m)}{\text{rep}_F(k)}$	$\text{rep}_F(k)$: ε	1	10	100	101	1000	1001	1010	…
$\text{rep}_F(m)$	ε	1	0	0	0	0	0	0	0	
	1	1	1	0	0	0	0	0	0	
	10	1	1	1	0	0	0	0	0	
	100	1	1	2	1	0	0	0	0	
	101	1	2	1	0	1	0	0	0	
	1000	1	1	3	3	0	1	0	0	
	1001	1	2	2	1	2	0	1	0	
	1010	1	2	3	1	1	0	0	1	
	⋮									⋱

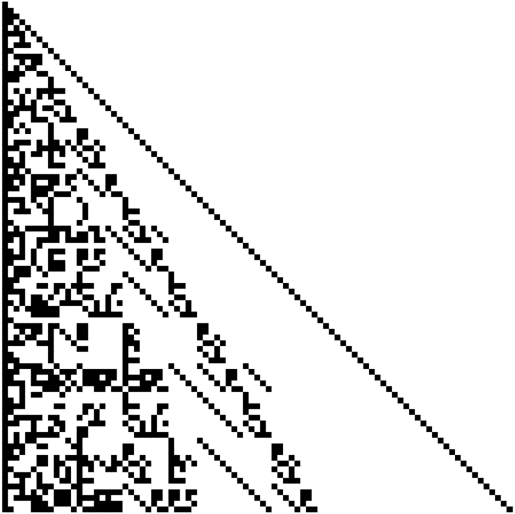
Rule (not local):

$$\binom{ua}{vb} = \binom{u}{vb} + \delta_{a,b}\binom{u}{v}$$

The first six elements of the sequence $(\mathcal{U}'_n)_{n\geq 0}$

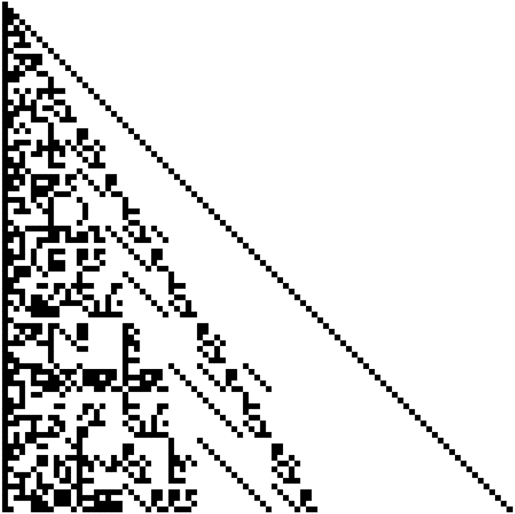


The tenth element



Lines of different slopes:

The tenth element



Lines of different slopes: φ^n, $n \geq 0$, with $\varphi = \frac{1+\sqrt{5}}{2}$ (Golden Ratio)

The $(\star')$ condition

Recall

- (u,v) satisfies $(\star)$ iff $\binom{u}{v} \equiv 1 \bmod 2$, and $\binom{u}{v0} = 0 = \binom{u}{v1}$.
- Completion: (u,v) satisfies $(\star) \Rightarrow (u0,v0),(u1,v1)$ satisfy $(\star)$

Problem: we cannot always add a letter 1 as a suffix in L_F.

Solution: add a 0 if necessary

$$p(u,v) = \begin{cases} 1 & \text{if } u \text{ or } v \text{ ends with } 1, \\ 0 & \text{otherwise.} \end{cases}$$

So $u0^{p(u,v)}w, v0^{p(u,v)}w \in L_F$ for all $w \in 0^*L_F$.

$(\star')$

(u,v) satisfies $(\star')$ iff $u = v = \varepsilon$ or $\begin{cases} u,v \neq \varepsilon \\ \binom{u0^{p(u,v)}}{v0^{p(u,v)}} \equiv 1 \pmod 2 \\ \binom{u0^{p(u,v)}}{v0^{p(u,v)}a} = 0 \quad \forall\, a \in \{0,1\}. \end{cases}$

Similar technique

- Completion lemma with the $(\star')$ condition
- Creation of segments of slope 1
- New compact set $\mathcal{A}'_0$ containing those lines

$$\mathcal{A}'_0 = \overline{\bigcup_{\substack{(u,v) \\ \text{satisfying } (\star')}} S_{u,v}} \subset [0,1]^2$$

- Modification of the slopes with $c\colon (x,y) \mapsto (x/\varphi, y/\varphi)$ and $h\colon (x,y) \mapsto (x, \varphi y)$
- New compact set $\mathcal{A}'_n$ containing lines of slopes $1, \varphi, \varphi^2, \ldots, \varphi^n$

$$\mathcal{A}'_n = \bigcup_{0 \le j \le i \le n} h^j(c^i(\mathcal{A}'_0))$$

- $(\mathcal{A}'_n)_{n \ge 0}$ converges to

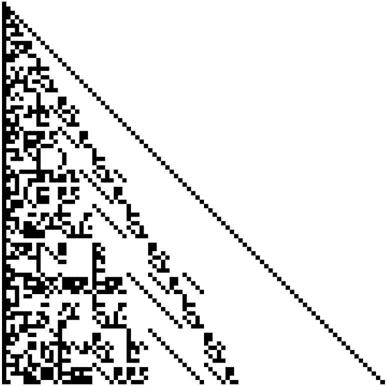
$$\mathcal{L}' = \overline{\bigcup_{n \ge 0} \mathcal{A}'_n}$$

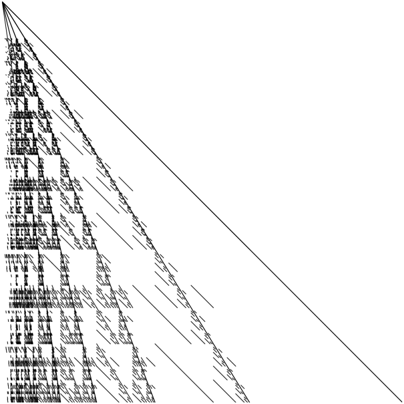
(w.r.t. the Hausdorff distance).

Similar convergence result

Theorem (S., 2018)

The sequence $(\mathcal{U}'_n)_{n\geq 0}$ of compact sets converges to the compact set $\mathcal{L}'$ when n tends to infinity (w.r.t. the Hausdorff distance).





"Simple" characterization of $\mathcal{L}'$: $(\star')$ condition

Linear numeration system

Definition

A *numeration system* is a sequence $U = (U(n))_{n\geq 0}$ of integers s.t.

- U increasing
- $U(0) = 1$
- $\sup_{n\geq 0} \frac{U(n+1)}{U(n)}$ bounded by a constant $\rightsquigarrow$ finite alphabet.

A numeration system U is *linear* if $\exists k \geq 1, \exists a_0, \ldots, a_{k-1} \in \mathbb{Z}$ s.t.

$$U(n+k) = a_{k-1}\, U(n+k-1) + \cdots + a_0\, U(n) \quad \forall\, n \geq 0.$$

Greedy representation in $(U(n))_{n\geq 0}$:

$$n = \sum_{i=0}^{\ell} c_i U(i) \qquad \text{with} \qquad \sum_{i=0}^{j-1} c_i U(i) < U(j)$$

$$\text{rep}_U(n) = c_\ell \cdots c_0 \in \underbrace{L_U = \text{rep}_U(\mathbb{N})}_{\text{numeration language}}$$

Example: integer base $(b^n)_{n\geq 0}$ with $b \in \mathbb{N}_{>1}$

Fibonacci numeration system $(F(n))_{n\geq 0}$

Parry numbers

$\beta \in \mathbb{R}_{>1}$ $\qquad A_\beta = \{0, 1, \ldots, \lceil \beta \rceil - 1\}$

$$x \in [0,1] \rightsquigarrow x = \sum_{j=1}^{+\infty} c_j \beta^{-j}, \quad c_j \in A_\beta$$

Greedy way: $c_j \beta^{-j} + c_{j+1}\beta^{-j-1} + \cdots < \beta^{-(j-1)}$
β-expansion of x: $d_\beta(x) = c_1 c_2 c_3 \cdots$

Definition

$\beta \in \mathbb{R}_{>1}$ is a *Parry number* if $d_\beta(1)$ is ultimately periodic.

Example: $b \in \mathbb{N}_{>1}$: $d_b(1) = (b-1)^\omega$
Golden ratio φ: $d_\varphi(1) = 110^\omega$

Parry numeration system

Parry number $\beta \in \mathbb{R}_{>1} \rightsquigarrow$ linear numeration system $(U_\beta(n))_{n\geq 0}$

- $d_\beta(1) = t_1 \cdots t_m 0^\omega$

$$\begin{array}{rcll} U_\beta(0) & = & 1 & \\ U_\beta(i) & = & t_1 U_\beta(i-1) + \cdots + t_i U_\beta(0) + 1 & \forall\, 1 \leq i \leq m-1 \\ U_\beta(n) & = & t_1 U_\beta(n-1) + \cdots + t_m U_\beta(n-m) & \forall\, n \geq m \end{array}$$

- $d_\beta(1) = t_1 \cdots t_m (t_{m+1} \cdots t_{m+k})^\omega$

$$\begin{array}{rcll} U_\beta(0) & = & 1 & \\ U_\beta(i) & = & t_1 U_\beta(i-1) + \cdots + t_i U_\beta(0) + 1 & \forall\, 1 \leq i \leq m+k-1 \\ U_\beta(n) & = & t_1 U_\beta(n-1) + \cdots + t_{m+k} U_\beta(n-m-k) & \forall\, n \geq m+k \\ & & +U_\beta(n-k) & \\ & & -t_1 U_\beta(n-k-1) - \cdots - t_m U_\beta(n-m-k) & \end{array}$$

Examples:

$b \in \mathbb{N}_{>1} \rightsquigarrow (b^n)_{n\geq 0}$ base b

Golden ratio $\varphi \rightsquigarrow (F(n))_{n\geq 0}$ Fibonacci numeration system

Extension to Parry numeration systems

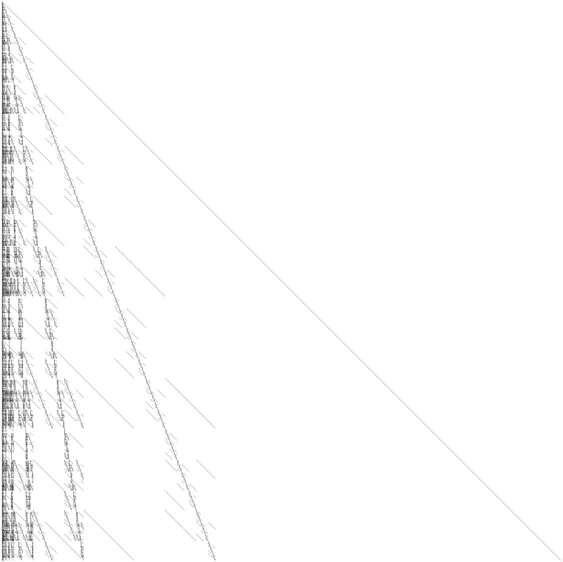
- Parry number $\beta \in \mathbb{R}_{>1}$
- Parry numeration system $(U_\beta(n))_{n\geq 0}$
- Numeration language L_{U_β}
- Generalized Pascal's triangle P_β in $(U_\beta(n))_{n\geq 0}$ indexed by words of L_{U_β}
- Sequence of compact sets extracted from P_β (first $U_\beta(n)$ rows and columns of P_β)
- Convergence to a limit object (same technique)
  - Lines of different slopes: β^n, $n \geq 0$
  - $(\star')$ condition and description of segments of slope 1
  - Two maps $c\colon (x, y) \mapsto (x/\beta, y/\beta)$ and $h\colon (x, y) \mapsto (x, \beta y)$
  - Sequence of sets $\mathcal{A}_n^\beta$ containing lines of slopes 1, β, β^2, …, β^n
  - $\mathcal{A}_n^\beta$ converges to

$$\mathcal{L}^\beta = \overline{\bigcup_{n\geq 0} \mathcal{A}_n^\beta}$$

- Works modulo any prime number

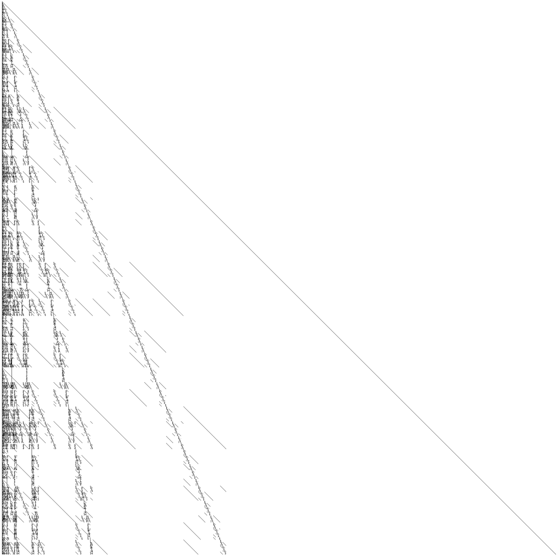
Example 1

φ^2



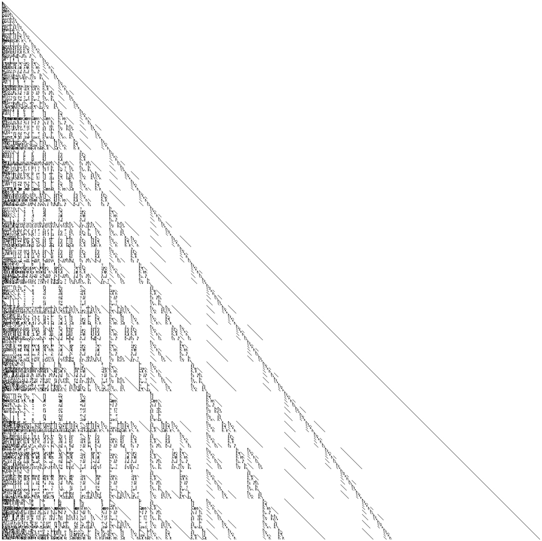
Example 2

$\beta_1 \approx 2.47098$ dominant root of $P(X) = X^4 - 2X^3 - X^2 - 1$



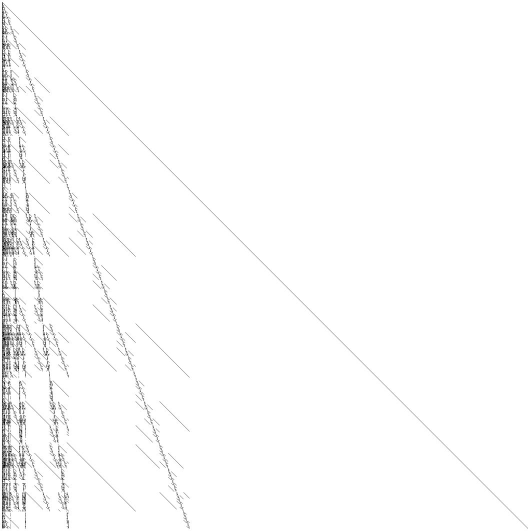
Example 3

$\beta_2 \approx 1.38028$ dominant root of $P(X) = X^4 - X^3 - 1$



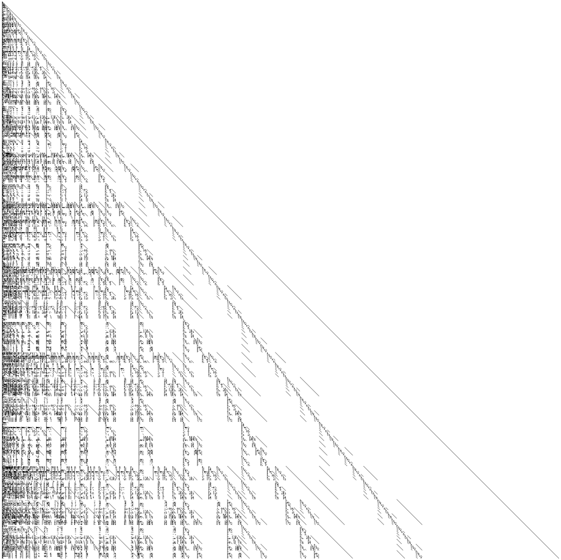
Example 4

$\beta_3 \approx 2.80399$ dominant root of $P(X) = X^4 - 2X^3 - 2X^2 - 2$



Example 5

$\beta_4 \approx 1.32472$ dominant root of polynomial $P(X) = X^5 - X^4 - 1$



In this talk:

Numeration system	Generalized Pascal's triangle	Convergence modulo a prime
Base 2	✓	✓
Integer base	✓	✓
Fibonacci	✓	✓
Parry	✓	✓

Other related work

- Regularity of the sequence counting subword occurrences: result for any integer base b and the Fibonacci numeration system
- Behavior of the summatory function: result for any integer base b (exact behavior) and the Fibonacci numeration system (asymptotics)

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