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**BASIC EQUATIONS OF GEOPHYSICAL FLUID DYNAMICS**  
and their application to ocean-atmosphere weather and climate models

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#### Introduction.

There is a growing interest for interactive models of the ocean and the atmosphere, at all scales, from local air-sea interactions to hurricane thermodynamics and climate forecasting.

Although the basic equations describing the atmosphere and the ocean are the same Geophysical Fluid Dynamics equations, interactive ocean-atmosphere models frequently come up against a — rather unexpected — difficulty when trying to set up a common frame of study.

Because scales of motions have different orders of magnitude in the atmosphere and the ocean and effects which are dominant in the former at a given scale may not be so in the latter, equations fundamentally similar, have often been treated differently, submitted to specific approximations, calibrated with data sampled on totally different networks and finite-differenced with incompatible grids.

Systems of equations taken from standard textbooks of meteorology and oceanography may show little more than a formal similitude — if any at all — and tracing all the hypotheses which underlie any particular set may prove to be much more difficult than one would expect.

The macroscopic equations of a geofluid which, like the atmosphere is a mixture of, say, dry air, water vapor, water droplets, ..., for instance, are well-known to include terms proportional to the squares of the so-called "diffusion velocities" (i.e. the velocities of the individual components in a frame of reference moving with the mixture's center of mass). These terms are absent from the classical Geophysical Fluid Dynamics equations and it is not clear whether they have been neglected (and might have to be reintroduced in some cases) or whether they have been taken into account in the definitions and parameterizations of the constitutive equations.



Changes of phase are essential mechanisms in the atmosphere and quite irrelevant in the ocean. Is there a common description of both the atmosphere and the ocean where they are properly represented?

Ocean Hydrodynamics is very often described in the scope of the so-called Boussinesq approximation which regards the dynamical state of the geofluid as a first order perturbation of some reference state. How much is this applicable to the atmosphere and can it be extended to include specific thermodynamic features like changes of phase? What is the appropriate state of reference for the atmosphere and how close can it be to a state of equilibrium?

The derivation of macroscopic equations for a mixture taking into account chemical reactions and (or) changes of phases is a fairly classical problem which has been abundantly discussed in reputed textbooks and monographs (e.g. Landau and Lifshitz, 1959; Meixner and Reik, 1959; Truesdell and Noll, 1965; Hayday, 1966; Truesdell, 1969; Gyarmati, 1970; Glansdorff and Prigogine, 1971; Woods, 1975).

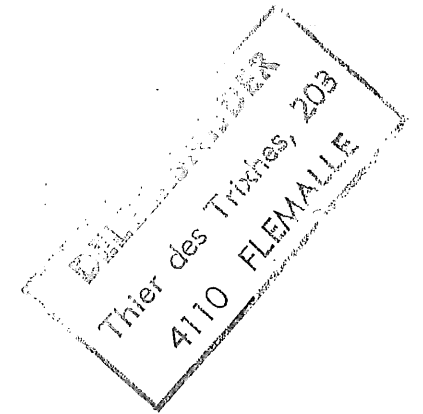
However, in the words of Becker (1972), "Differing formulations can be found in the extensive literature. The field abounds with ad hoc assumptions. No consensus has yet been reached and no formulation of the basic laws is completely satisfying."

Furtermore, although the Boussinesq approximation is commonly used in some atmospheric studies, the subjacent assumptions and subsequent limitations are not easy to identify and the separate consideration, in the meteorology tradition, of dry air, dry air with water vapour but no condensation, clouds, falling rain, etc., may be rather confusing.

The development of ambitious and expensive interactive ocean-atmosphere models justifies that, today, at the risk of being accused of repetition or reordering, one takes the time to reflect on the basic mathematical model and on what one might call the fundamental "geohydrodynamic" equations applicable to such compound physical system.

The present course is an attempt to provide the elements of such a reflection with, hopefully, enough novelty in the arguments or their presentation and interpretation to foster a constructive discussion.

Professor Jacques C.J. Nihoul.



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### The equations of mass conservation

The atmosphere and the ocean are composed of innumerable constituents, among which some - like water in the sea - are found in largely dominant concentrations, some - like water in the atmosphere - are only minor components, and some - like trace metals in the ocean - are in very small amounts.

In principle, one can attach, to a given constituent "i", a specific mass  $\rho^i$  (mass of the constituent per unit volume of the mixture, in  $\text{kg m}^{-3}$ ) and a velocity  $\mathbf{v}^i$  and write an equation of mass conservation in the form

$$(1) \quad \frac{\partial \rho^i}{\partial t} + \nabla \cdot (\rho^i \mathbf{v}^i) = Q^i$$

where  $Q^i$  is the rate of production (or destruction) of constituent "i" and  $\nabla$  the nabla operator  $\nabla = \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{\partial}{\partial x_3}$

Bulk variables  $\rho, \mathbf{v}$ , can then be defined by

$$(2) \quad \rho = \sum \rho^i$$

$$(3) \quad \rho \mathbf{v} = \sum \rho^i \mathbf{v}^i$$

where the sum is over all constituents.

Summing over all constituents and taking into account that  $\sum Q^i = 0$ , one gets from eq. (1), the well-known "continuity equation"

$$(4) \quad \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0$$

This equation contains only the bulk variables. Unfortunately, other basic equations and, in particular, the energy equation which is essential in climate modelling, cannot be as easily written in terms of the bulk variables only.

Yet, one cannot contemplate a model taking into account, individually, all the constituents of the geofluids.

With essentially hydrodynamic and thermodynamic phenomena in mind — the situation would be different if pollution was at stake — one can consider the atmosphere and the ocean as "mixtures of conventional components": for the atmosphere, dry air, water vapor, condensed water (liquid or solid) in clouds, condensed water (rain, snow, ...) in precipitation; for the ocean, pure water, dissolved salts and — to complete the analogy with the atmosphere although they play a comparatively negligible part — particles in suspension and sedimenting flocs\*.

The specific masses of these constituents will be denoted  $\rho^a$  ( $a = 1, 2, 3, 4$ ). In particular cases, for a more speaking presentation, a letter will be used for superscript (in the atmosphere, "a" for dry air, "v" for water vapor, "c" for clouds, "p" for precipitation; in the ocean, "w" for water, "s" for salts, "t" for suspension, "f" for sedimenting particles).

Neglecting small contributions, one can write

$$(5) \quad \rho = \sum \rho^a \quad a = 1, 2, 3, 4$$

Let

$$(6) \quad c^a = \frac{\rho^a}{\rho}$$

Eq. (5) can be written

$$(5') \quad (5'') \quad \sum c^a = 1, \quad \sum dc^a = 0$$

\* The description can be refined and, for instance, a further distinction between water, snow and ice can be introduced easily.

The general formalism which follows is completely general and no modification is required if the number of conventional constituents is increased.

The density of water vapor in the atmosphere,  $c^v$  ( $\equiv c^2$ ) is the "humidity"\*.

Mass conservation equations for the  $\rho^a$ 's can be obtained from eq. (1) by summing over the corresponding class of constituents (e.g. over all dissolved substances in the sea). One gets

$$(7) \quad \frac{\partial \rho^a}{\partial t} + \nabla \cdot (\rho^a \mathbf{v}^a) = Q^a$$

where

$$(8) \quad \rho^a \mathbf{v}^a = \sum_{\text{class } a} \rho^i \mathbf{v}^i$$

$$(9) \quad Q^a = \sum_{\text{class } a} Q^i$$

$Q^a$  refers now to interactions which transfer a constituent from one compartment to another (condensation, evaporation, coalescence and formation of rain drops, dissolving, flocculation, ...).

Introducing the bulk velocity  $\mathbf{v}$ , eq. (7) can be written

$$(10) \quad \frac{\partial \rho^a}{\partial t} + \nabla \cdot (\rho^a \mathbf{v}) = Q^a - \nabla \cdot \mathbf{q}^a$$

where

$$(11) \quad (12) \quad \mathbf{q}^a = \rho^a \tilde{\mathbf{v}}^a; \quad \tilde{\mathbf{v}}^a = \mathbf{v}^a - \mathbf{v}$$

\* The  $c^a$ 's are "specific" values. Related variables, sometimes used in the literature, are obtained by dividing  $\rho^a$  by the specific mass of the major constituent instead of that of the mixture or by its saturation value  $\rho_s^a$ . One finds, for instance, the "absolute humidity"  $\rho^v/\rho^a$  and the "relative humidity"  $\rho^v/\rho_s^v$ ;  $\rho^a$  ( $\equiv \rho^1$ ) and  $\rho^v$  ( $\equiv \rho^2$ ) denoting respectively the specific masses of dry air and water vapor.



It is readily seen that one must have

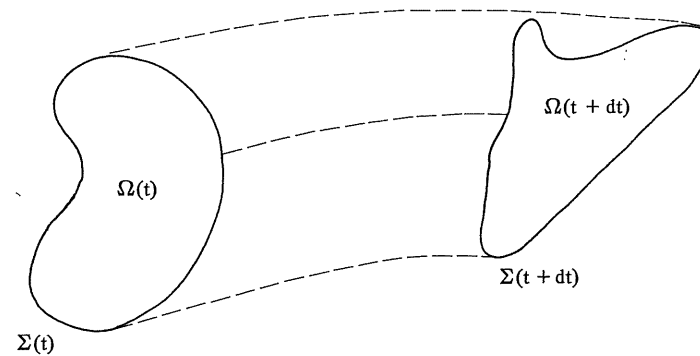
$$(13) \quad \Sigma \mathbf{q}^a = 0 \quad ; \quad \Sigma Q^a = 0$$

Eq. (10) can also be written

$$(15) \quad \rho \frac{dc^a}{dt} \equiv \rho \left( \frac{\partial c^a}{\partial t} + \mathbf{v} \cdot \nabla c^a \right) = Q^a - \nabla \cdot \mathbf{q}^a$$

with the condition

$$(5'') \quad \Sigma dc^a = 0$$



If one considers an arbitrary volume  $\Omega$  (enclosed by a surface  $\Sigma$ ) transported by the geofluid (with the bulk velocity  $\mathbf{v}$ ) and the total mass

$$\int_{\Omega} \rho^a d\Omega$$

of the constituent "a" contained in the volume  $\Omega$ , one can write

$$(16) \quad \frac{d}{dt} \int_{\Omega} \rho^a d\Omega = \int_{\Omega} Q^a d\Omega - \int_{\Sigma} \rho^a \mathbf{v}^a \cdot \mathbf{n} d\Sigma$$

Eq. (16) is a common sense expression of the conservation of mass : the increase in a time  $dt$  of the total mass of "a" contained in  $\Omega$  is equal to the quantity produced inside of  $\Omega$  minus what has leaked out through the surface  $\Sigma$  with the normal velocity  $\mathbf{v}^a \cdot \mathbf{n}$  ( $\mathbf{n}$  is the unit vector along the normal to  $\Sigma$ , pointing outwards).

Inverting the orders of derivation with respect to time and integration over the moving volume  $\Omega$ , and taking into account that, for any field variable  $y$ , one has (e.g. Nihoul, 1977)

$$(17) \quad \frac{d}{dt} \int_{\Omega} \rho y d\Omega = \int_{\Omega} \left[ \frac{d(\rho y)}{dt} + \nabla \cdot (\rho y \mathbf{v}) \right] d\Omega = \int_{\Omega} \rho \frac{dy}{dt} d\Omega$$

one obtains, writing  $\rho^a = \rho c^a$  and transforming the surface integral into a volume integral by Gauss' theorem,

$$(18) \quad \int_{\Omega} \rho \frac{dc^a}{dt} d\Omega = \int_{\Omega} Q^a d\Omega - \int_{\Omega} \nabla \cdot \mathbf{q}^a d\Omega$$

The volume  $\Omega$  being arbitrary, eq. (15) follows.

The demonstration above is a classical one (e.g. Nihoul, 1977). It is reproduced here, in brief, because, on the one hand, the same reasoning will be applied throughout this paper to establish the basic equations and because, on the other hand, it provides a simple and clear illustration of the signification of the fluxes  $\mathbf{q}^a$ .

Each flux  $\mathbf{q}^a$  can be further separated into a "molecular flux"  $\mathbf{q}_m^a$  and a "migration flux"  $\mathbf{q}_M^a$ .

Molecular diffusion is the common lot of all substances in a mixture. Molecular fluxes can be parameterized in terms of concentration gradients with the help of appropriate diffusivities  $\lambda^a$ , viz, in the simplest case

$$(19) \quad \mathbf{q}_m^a = - \lambda^a \nabla \rho^a$$

Migration, a relative motion at *macroscopic* scale (as opposed to diffusion at molecular scale) is the attribute of falling rain drops, sedimenting particles or flocs, rising bubbles ... The determination of the migration fluxes requires the knowledge or the parameterization of the migration velocities  $\mathbf{v}_M^a$ .

#### The momentum equations.

Considering again an arbitrary volume  $\Omega$  moving with the mixture and the momentum  $\int \rho^a \mathbf{v}^a d\Omega$  of the constituent "a" contained in  $\Omega$ , one has, by Newton's law (e.g. Truesdell and Noll, 1965, Truesdell, 1969)

$$(20) \quad \frac{d}{dt} \int_{\Omega} \rho^a \mathbf{v}^a d\Omega = \int_{\Omega} \rho^a \mathbf{F}^a d\Omega + \int_{\Omega} \mathbf{f}^a d\Omega + \int_{\Sigma} \mathbf{t}^a d\Sigma + \int_{\Omega} \mathbf{M}^a d\Omega - \int_{\Sigma} (\rho^a \mathbf{v}^a) \hat{\mathbf{v}}^a \cdot \mathbf{n} d\Sigma$$

In this equation

(i)  $\mathbf{F}^a$  is the external force (per unit mass), i.e., in axes rotating with the Earth

$$(21) \quad \mathbf{F}^a = -2 \boldsymbol{\Omega} \wedge \mathbf{v}^a + \mathbf{g}$$

$\boldsymbol{\Omega}$  is the Earth's rotation vector and  $\mathbf{g}$  the acceleration of gravity\*.

\* Electromagnetic forces which play a role only in the ionosphere have been neglected.

Tide-generating and centrifugal forces may be included in  $\mathbf{g}$  by redefining it as the gradient of an appropriate (gravitational + tidal + centrifugal) potential.

This is however relatively academic. In ocean-troposphere models, it is generally sufficient to take for  $\mathbf{g}$  the local acceleration of gravity.

(ii)  $\mathbf{f}^a$  is the internal force (per unit volume) associated with mutual interactions between different components ;

(iii)  $\mathbf{t}^a$  is the force (per unit area) which acts on the surface  $\Sigma$  enclosing  $\Omega$  i.e. (e.g. Truesdell, 1969; Nihoul, 1977)

$$(22) \quad \mathbf{t}^a = \mathbf{T}^a \cdot \mathbf{n}$$

$\mathbf{T}^a$  is the partial stress tensor and can be written

$$(23) \quad \mathbf{T}^a = -p^a \mathbf{I} + \mathbf{T}_v^a$$

to display the "partial pressure"  $p^a$

( $\mathbf{I} = \mathbf{e}_1 \mathbf{e}_1 + \mathbf{e}_2 \mathbf{e}_2 + \mathbf{e}_3 \mathbf{e}_3$  is the "identity" tensor);

(iv)  $\mathbf{M}^a$  is the rate of change of the momentum  $\rho^a \mathbf{v}^a$  of the constituent  $a$  which results from changes of phase.

A single equation for the bulk velocity  $\mathbf{v}$  can be obtained by summing eqs (20) over all constituents.

One obviously has

$$(24), (25) \quad \Sigma \mathbf{f}^a = 0, \quad \Sigma \mathbf{M}^a = 0$$

Setting

$$(26) \quad \rho \mathbf{F} = \Sigma \rho^a \mathbf{F}^a = -2 \boldsymbol{\Omega} \wedge \rho \mathbf{v} + \rho \mathbf{g}$$

$$(27) \quad p = \Sigma p^a$$

$$(28) \quad \mathbf{T}_v = \Sigma (\mathbf{T}_v^a - \rho^a \hat{\mathbf{v}}^a \hat{\mathbf{v}}^a)$$



one obtains

$$(29) \quad \frac{d}{dt} \int_{\Omega} \rho v \, d\Omega = \int_{\Omega} \rho F \, d\Omega + \int_{\Sigma} (-pI + T_v) \cdot n \, d\Sigma$$

Using eq. (17) to compute the time derivative and Gauss' theorem to transform the surface integral, one gets, taking into account that the volume  $\Omega$  is arbitrary

$$(30) \quad \rho \frac{dv}{dt} = \rho F - \nabla p + \nabla \cdot T_v$$

#### The energy equations.

Applying the same reasoning as above to the energy  $\rho^a (\epsilon^a + \frac{1}{2} v^{a2})$  of the constituent  $a$  ( $\epsilon^a$  is the internal energy), one gets

$$(31) \quad \begin{aligned} \frac{d}{dt} \int_{\Omega} \rho^a (\epsilon^a + \frac{1}{2} v^{a2}) \, d\Omega &= \int_{\Omega} \rho^a F^a \cdot v^a \, d\Omega + \int_{\Omega} R^a \, d\Omega \\ &+ \int_{\Sigma} S^a \, d\Omega + \int_{\Sigma} \left\{ t^a \cdot v^a - q_h^a \cdot n - \rho^a (\epsilon^a + \frac{1}{2} v^{a2}) \hat{v}^a \cdot n \right\} d\Sigma \end{aligned}$$

where

- (i)  $\rho^a F^a \cdot v^a$  is the power developed by the external forces acting on "a".
- (ii)  $R^a$  is the rate of change of energy due to radiation.
- (iii)  $S^a$  is the sum of the rates of change of energy due to the work done by internal forces, the release of heat by internal friction and changes of phase, the flow work done by the pressures to in-

roduce the created phases into their new systems and the changes in internal and kinetic energy resulting from changes of phase.

- (iv)  $t^a \cdot v^a$  is the power developed by the surface forces.
- (v)  $q_h^a$  is the heat flux associated with the constituent  $a$ .

One has (e.g. Woods, 1975)

$$(32) \quad \Sigma S^a = 0$$

Thus setting

$$(33) \quad R = \Sigma R^a$$

$$(34) \quad \epsilon = \Sigma c^a (\epsilon^a + \frac{1}{2} v^{a2})$$

Using eqs (23) and (28) and taking into account that the volume  $\Omega$  is arbitrary, one obtains

$$(35) \quad \begin{aligned} \rho \frac{d\epsilon}{dt} + \rho \frac{d}{dt} (\frac{1}{2} v^2) &= \Sigma \rho^a F^a \cdot v^a + R + \nabla \cdot (-pv + T_v \cdot v) \\ &+ \nabla \cdot \left\{ \Sigma [-p^a \hat{v}^a + T_v^a \cdot \hat{v}^a - q_h^a - \rho^a \hat{v}^a (\epsilon^a + \frac{1}{2} v^{a2})] \right\} \end{aligned}$$

The scalar product of eq. (29) by  $v$  yields

$$(36) \quad \rho \frac{d}{dt} (\frac{1}{2} v^2) = \rho F \cdot v + \nabla \cdot (-pv + T_v \cdot v) + p \nabla \cdot v - T_v : \nabla v$$

Subtracting eq. (36) from eq. (35), one gets

$$(37) \quad \rho \frac{d\epsilon}{dt} = \Sigma \rho^a F^a \cdot \hat{v}^a + R + T_v : \nabla v - p \nabla \cdot v - \nabla \cdot q_e$$

where

$$(38) \quad \mathbf{q}_\epsilon = \Sigma [\mathbf{q}_h^a + \rho^a \hat{\mathbf{v}}^a (\epsilon^a + \frac{1}{2} \hat{\mathbf{v}}^{a2}) + \hat{\mathbf{v}}^a \cdot (p^a \mathbf{I} - \mathbf{T}_v^a)]$$

and where

$$(39) \quad \begin{aligned} \Sigma \rho^a \mathbf{F}^a \cdot \hat{\mathbf{v}}^a &= \Sigma \rho^a [- (2 \Omega \wedge \mathbf{v}^a) \cdot \hat{\mathbf{v}}^a + \mathbf{g} \cdot \hat{\mathbf{v}}^a] \\ &= (-2 \Omega \wedge \mathbf{v} + \mathbf{g}) \cdot \Sigma \mathbf{q}^a - \Sigma \rho^a (2 \Omega \wedge \hat{\mathbf{v}}^a) \cdot \hat{\mathbf{v}}^a \end{aligned}$$

is zero in the particular case considered here but is retained in the subsequent analysis to maintain the general level of the exposé.

The question arises whether  $\mathbf{T}_v$ , defined by eq. (28),  $\epsilon$ , defined by eq. (34) and  $\mathbf{q}_\epsilon$ , defined by eq. (38), are the equivalent of the viscous stress tensor, the internal energy and the heat flux introduced in the theory of a single fluid.

Some authors consider that

$$\mathbf{T}_v' = \Sigma \mathbf{T}_v^a \quad ; \quad \epsilon' = \Sigma c^a \epsilon^a \quad ; \quad \mathbf{q}_\epsilon' = \Sigma \mathbf{q}_h^a$$

are the appropriate analogues and that using  $\mathbf{T}_v$ ,  $\epsilon$  and  $\mathbf{q}_\epsilon$  is tantamount to assuming that the relative velocities  $\hat{\mathbf{v}}^a$  are small and that their square products can be neglected.

Others argue that  $\mathbf{T}_v$ ,  $\epsilon$  and  $\mathbf{q}_\epsilon$  are the correct expressions; the additional terms containing the relative velocities being necessary corrections to account accurately for the dispersion of individual particle velocities around the *bulk velocity*  $\mathbf{v}$ .

If one approaches the problem along the lines of gaz kinetics and, generally speaking, statistical physics, one must admit that the second opinion does have some ground.

Let  $f^a(t, \mathbf{x}, \mathbf{u})$  denote the distribution function of the particles (molecules ...) of the constituent  $a$ . If  $m^a$  is the mass of one of these particles, the number  $n^a$  of particles per unit volume, the

specific mass  $\rho^a$  and the velocity  $\mathbf{v}^a$  can be defined by

$$n^a = \int f^a \, d\mathbf{u} \quad ; \quad \rho^a = n^a m^a \quad ; \quad n^a \mathbf{v}^a = \int \mathbf{u} f^a \, d\mathbf{u}$$

where  $\mathbf{u}$  denotes the velocity of an individual particle.

(i) In the macroscopic momentum equation, both the advection term and the stress tensor arise from integrals of the form

$$m^a \int \mathbf{u} \mathbf{u} f^a \, d\mathbf{u}$$

which can be written

$$\begin{aligned} m^a \int \mathbf{u} \mathbf{u} f^a \, d\mathbf{u} &= m^a \int (\mathbf{u} - \mathbf{v}^a) (\mathbf{u} - \mathbf{v}^a) f^a \, d\mathbf{u} + m^a n^a \mathbf{v}^a \mathbf{v}^a \\ &= p^a \mathbf{I} - \mathbf{T}_v^a + m^a n^a \mathbf{v}^a \mathbf{v}^a \\ &= p^a \mathbf{I} - \mathbf{T}_v^a + m^a n^a (\mathbf{v} \mathbf{v} + \mathbf{v} \hat{\mathbf{v}}^a + \hat{\mathbf{v}}^a \mathbf{v} + \hat{\mathbf{v}}^a \hat{\mathbf{v}}^a) \end{aligned}$$

Hence

$$\Sigma m^a \int \mathbf{u} \mathbf{u} f^a \, d\mathbf{u} = p \mathbf{I} - \Sigma (\mathbf{T}_v^a - \rho^a \hat{\mathbf{v}}^a \hat{\mathbf{v}}^a) + \rho \mathbf{v} \mathbf{v}$$

On the other hand, one can also write

$$\begin{aligned} \Sigma m^a \int \mathbf{u} \mathbf{u} f^a \, d\mathbf{u} &= \Sigma m^a \int (\mathbf{u} - \mathbf{v}) (\mathbf{u} - \mathbf{v}) f^a \, d\mathbf{u} + \rho \mathbf{v} \mathbf{v} \\ &= p \mathbf{I} - \mathbf{T}_v + \rho \mathbf{v} \mathbf{v} \end{aligned}$$

It seems thus reasonable to let

$$\mathbf{T}_v = \Sigma (\mathbf{T}_v^a - \rho^a \hat{\mathbf{v}}^a \hat{\mathbf{v}}^a)$$

(ii) If one takes

$$\rho^a \epsilon^a = \frac{1}{2} m^a \int (\mathbf{u} - \mathbf{v}^a) \cdot (\mathbf{u} - \mathbf{v}^a) f^a \, d\mathbf{u}$$

$$\rho \epsilon = \Sigma \frac{1}{2} m^a \int (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) f^a \, d\mathbf{u}$$

the same calculation as above (scalar products replacing dyadic products) yields

$$\epsilon = \Sigma c^a (\epsilon^a + \frac{1}{2} \hat{\mathbf{v}}^{a2})$$

(iii) Setting

$$\mathbf{q}_h^a = \frac{1}{2} m^a \int (\mathbf{u} - \mathbf{v}^a) \cdot (\mathbf{u} - \mathbf{v}^a) (\mathbf{u} - \mathbf{v}^a) f^a d\mathbf{u}$$

$$\mathbf{q}_h = \Sigma \frac{1}{2} m^a \int (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) (\mathbf{u} - \mathbf{v}) f^a d\mathbf{u}$$

One can write

$$\Sigma [q_h^a + \rho^a \hat{v}^a (\epsilon^a + \frac{1}{2} \hat{v}^{a2}) + \hat{v}^a \cdot (\rho^a \mathbf{I} - \mathbf{T}_v^a)]$$

$$= \Sigma \frac{1}{2} m^a \int (\mathbf{u} - \mathbf{v}^a) \cdot (\mathbf{u} - \mathbf{v}^a) (\mathbf{u} - \mathbf{v}^a) f^a d\mathbf{u}$$

$$+ \Sigma \frac{1}{2} \rho^a \hat{v}^a \hat{v}^{a2}$$

$$+ \Sigma \frac{1}{2} m^a \hat{v}^a \int (\mathbf{u} - \mathbf{v}^a) \cdot (\mathbf{u} - \mathbf{v}^a) f^a d\mathbf{u}$$

$$+ \Sigma m^a \hat{v}^a \int (\mathbf{u} - \mathbf{v}^a) (\mathbf{u} - \mathbf{v}^a) f^a d\mathbf{u}$$

$$= \Sigma \frac{1}{2} m^a \int (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) (\mathbf{u} - \mathbf{v}) f^a d\mathbf{u}$$

$$= \mathbf{q}_h \equiv \mathbf{q}_e$$

In the following,  $\mathbf{T}_v$ ,  $\epsilon$  and  $\mathbf{q}_e$  as defined by eqs (28), (34) and (38) will be taken as the viscous stress tensor, the internal energy and the heat flux, equivalent, for the mixture, of their analogues for a single fluid.

**The entropy equation.**

Assuming local equilibrium, one can write (e.g. Woods, 1975)

$$\rho T d\eta = \rho T \left[ \left( \frac{\partial \eta}{\partial T} \right)_{p,c^a} dT + \left( \frac{\partial \eta}{\partial p} \right)_{T,c^a} dp + \Sigma \left( \frac{\partial \eta}{\partial c^a} \right)_{T,p,c^b} dc^a \right]$$

$$(40) \quad = \rho c_p dT - \beta T dp + T \rho \Sigma \tilde{\eta}_a dc^a$$

$$= \rho d\epsilon - \frac{p}{\rho} d\rho - \rho \Sigma \tilde{\mu}^a dc^a$$

where

$T$  is the absolute temperature (K)  
 $\eta$  is the entropy ( $m^2 s^{-2} K^{-1}$ )

$$(41) \quad c_p = T \left( \frac{\partial \eta}{\partial T} \right)_{p,c^a}$$

is the specific heat at constant pressure ( $m^2 s^{-2} K^{-1}$ )

$$(42) \quad \beta = - \frac{1}{\rho} \left( \frac{\partial \rho}{\partial T} \right)_{p,c^a} = - \rho \left( \frac{\partial \eta}{\partial p} \right)_{T,c^a}$$

is the thermal expansion coefficient ( $K^{-1}$ )

$\mu^a$  is the "chemical potential" of the constituent  $a$   
 $\eta_a$  stands for  $\left( \frac{\partial \eta}{\partial c^a} \right)_{p,T,c^b}$

In eq. (40), the constraint  $\Sigma dc^a = 0$  has been taken into account and one of the  $c^a$ 's -  $c^n$ , say - has been eliminated.

A tilde  $\sim$  over a symbol such as  $\eta_a$  or  $\mu^a$  indicates a difference i.e.  $\tilde{\eta}_a = \eta_a - \eta_n$ ,  $\tilde{\mu}^a = \mu^a - \mu^n$  ...

By convention, when the summation sign  $\Sigma$  is followed by a tilde, the sum is made on all the constituents from  $a = 1$  to  $a = n-1$ . When the summation sign is not followed by a tilde, the sum goes from  $a = 1$  to  $a = n$ .

If  $\chi$  denotes the enthalpy, one has, by Gibbs' relation

$$(43) \quad d\chi = T d\eta + \frac{dp}{\rho} + \Sigma \mu^a dc^a$$

Hence \*

$$(44) \quad \tilde{\chi}_a = \tilde{\mu}^a + T \tilde{\eta}_a$$

\* The partial derivative

$$\chi_a \equiv \left( \frac{\partial \chi}{\partial c^a} \right)_{p,T,c^b} \dots$$



Using eqs. (37) and (39), one can write eq. (40) in the form

$$\begin{aligned}
 \rho \frac{d\eta}{dt} &= \frac{\rho}{T} \frac{d\varepsilon}{dt} - \frac{p}{\rho T} \frac{d\rho}{dt} - \sum \frac{\tilde{\mu}^a}{T} \rho \frac{dc^a}{dt} \\
 &= \frac{1}{T} (R + \mathbf{T}_v : \nabla \mathbf{v}) - \frac{p}{T} (\nabla \cdot \mathbf{v} + \frac{1}{\rho} \frac{d\rho}{dt}) \\
 &\quad - \frac{1}{T} \nabla \cdot \mathbf{q}_\varepsilon - \sum \frac{\tilde{\mu}^a}{T} (\dot{Q}^a - \nabla \cdot \mathbf{q}^a) \\
 &= \frac{R}{T} - \nabla \cdot \mathbf{q}_\eta - \sum \dot{Q}^a \frac{\tilde{\mu}^a}{T} + \mathbf{T}_v : \frac{\nabla \mathbf{v}}{T} \\
 &\quad - \sum \mathbf{q}^a \cdot \frac{\nabla \tilde{\mu}^a}{T} - \mathbf{q}_\eta \cdot \frac{\nabla T}{T}
 \end{aligned}
 \tag{45}$$

../'..

is equal to the enthalpy  $\chi^a$  of the constituent  $a$  if the  $\chi^a$ 's are functions of  $T$  and  $p$  only.

In the atmosphere, for instance (Van Mieghem and Dufour, 1975)

$$\chi^a \sim \chi_0^a + c_p^a (T - T_0) \quad ; \quad \chi^v \sim \chi_0^v + c_p^v (T - T_0) \quad ;$$

$$\chi^c \sim \chi_0^c + c_p^c (T - T_0) + \frac{p}{\rho_*} \sim \chi^p$$

where  $\rho_*^c$  is the mass of condensed water per unit volume of condensed water (not per unit volume of the atmospheric mixture ;  $\rho_*^c \neq \rho^c$ )  
One has also (Van Mieghem, 1975)

$$\begin{aligned}
 \eta^a &\sim \eta_0^a + c_p^a \ln \frac{T}{T_0} - R_a \ln \frac{p^a}{p_0^a} \quad ; \quad p^a \sim c^a R_a \rho T \\
 \eta^v &\sim \eta_0^v + c_p^v \ln \frac{T}{T_0} - R_v \ln \frac{p^v}{p_0^v} \quad ; \quad p^v \sim c^v R_v \rho T \\
 \eta^c &\sim \eta_0^c + c_p^c \ln \frac{T}{T_0} \sim \eta^p \quad ; \quad R_a = c_p^a - c_v^a
 \end{aligned}$$

( $p^a$  and  $p^v$  are the partial pressures).  
Thus ( $p, T, c^b$  constant)

$$\begin{aligned}
 \eta_a &= \frac{\partial \eta}{\partial c^a} = \frac{\partial}{\partial c^a} (\sum c^\gamma \eta^\gamma) = \eta^a + \sum c^\gamma \frac{\partial \eta^\gamma}{\partial c^a} \\
 &\sim \eta^a - \frac{c^a R_a}{p^a} \frac{\partial p^a}{\partial c^a} - \frac{c^v R_v}{p^v} \frac{\partial p^v}{\partial c^a} \\
 &\sim \eta^a - \frac{1}{\rho T} \frac{\partial (p^a + p^v)}{\partial c^a} \\
 &\sim \eta^a - \frac{1}{\rho T} \frac{\partial p}{\partial c^a} \sim \eta^a
 \end{aligned}$$

where

$$\mathbf{q}_\eta = \mathbf{T}^{-1} (\mathbf{q}_\varepsilon - \sum \mathbf{q}^a \tilde{\mu}^a)$$

is the flux of entropy.

With the assumption that the viscous stress tensor is symmetric (Woods, 1975), the entropy production rate  $\sigma$  is given by

$$\mathbf{T} \sigma = \mathbf{T}_v : \mathbf{D} - \mathbf{q}_\eta \cdot \nabla T - \sum \mathbf{q}^a \cdot \nabla \tilde{\mu}^a - \sum \dot{Q}^a \tilde{\mu}^a$$

where  $\mathbf{D}$  denotes the symmetrical part of  $\nabla \mathbf{v}$ , i.e.

$$D_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

The entropy production rate  $\sigma$  has the general form

$$\sigma = \sum_i J_i X_i$$

where the  $X_i$ 's, assumed to be independently assignable, are the "affinities" while the conjugate scalars  $J_i$ , assumed to be dependent on the affinities, are the "processes".

One generally assumes linear relationships between processes and affinities, viz

$$J_i = \sum_j L_{ij} X_j \quad i = 1, 2, \dots$$

Eqs. (50) are the "constitutive equations" providing the required parameterization of the fluxes  $\mathbf{q}^a$ ,  $\mathbf{q}_\eta$ , the viscous stress tensor ...

The adoption of linear constitutive relations would seem to be strictly valid only for small deviations from equilibrium and might appear to somewhat restrict the range of validity of process thermodynamics. However, well-known linear laws such as those of heat and electrical conduction, mass diffusion ... have been found experimentally to apply to systems well removed from equilibrium.

In practice, comparatively simple constitutive equations are used, inspired by semi-empirical relationships like the proportionality of the viscous stress tensor to the deformation tensor, the heat flux to the gradient of temperature (Fourier's law) and the molecular mass fluxes to the gradients of concentrations (Fick's law)\* ...

In most geophysical problems, molecular diffusion processes, are negligible as compared with that part of the advection which is the deed of macroscopic microscale turbulent motions creating a dispersion, similar to molecular diffusion but several orders of magnitude more efficient.

If one keeps in mind that there is a molecular sink and that microscale turbulence is organized to transfer variance to it and must be parameterized accordingly, one may neglect molecular diffusion. So doing, one does not ignore molecular effects but simply relegates them to infinite wave number.

\* The parameterization of the migration flux associated, for instance, with falling rain in the atmosphere is another special case of eq. (50). The affinities

$$T^{-1} \cdot \nabla \tilde{\mu}^p,$$

with  $\tilde{\mu}^p = \chi_p - T \eta_p - \mu^v \sim \phi^p(T) - R_v T \ln p^v + \frac{p}{\rho_c^c}$

give contributions proportional to

$$\left( -\frac{R_v T}{p^v} \frac{\partial p^v}{\partial x_3} + \frac{1}{\rho_c^c} \frac{\partial p}{\partial x_3} \right) \mathbf{e}_3 \sim \frac{g(\rho_c^c - \rho)}{\rho_c^c} \mathbf{e}_3$$

The vertical migration flux, proportional to  $g(\rho_c^c - \rho)$  may be identified as the product of a vertical migration velocity — the terminal velocity of a representative falling rain drop (Batchelor, 1967) — by the specific mass of condensed water.

### Radiation.

Photons as the  $n+1$  constituent.

Let  $n^{\nu\delta} dv dV d\gamma$  denote the number of photons in the frequency interval  $dv$  centered around  $\nu$ , in the volume  $dV$  and with directions of propagation within the solid angle  $d\gamma$  about the direction  $\delta$ .

Conservation of photons can be expressed by an equation similar to eq. (1), viz.

$$(51) \quad \frac{\partial n^{\nu\delta}}{\partial t} + \nabla \cdot (c \delta n^{\nu\delta}) = Q^{\nu\delta}$$

where  $c$  is the speed of light ( $c \sim c_0 = 3 \cdot 10^8 \text{ ms}^{-1}$  in the atmosphere,  $c \sim 2.3 \cdot 10^8 \text{ ms}^{-1}$  in the ocean),  $\delta$  the unit vector in the direction of propagation and  $Q^{\nu\delta}$  the local rate of photon production.

$Q^{\nu\delta}$  can be written as the sum of four contributions i.e.

$$(52) \quad Q^{\nu\delta} = E^{\nu\delta} - A^{\nu\delta} + S_i^{\nu\delta} - S_0^{\nu\delta}$$

where

$E^{\nu\delta}$  and  $S_i^{\nu\delta}$  represent the gain of photons per unit time respectively by spontaneous emission and by scattering in the direction  $\delta$  from the directions  $\delta'$ ;

$A^{\nu\delta}$  and  $S_0^{\nu\delta}$  represent the loss of photons per unit time by absorption (induced emission deducted) and by scattering in other directions, respectively.

Eq. (51) can be transformed into an equation for the radiation energy by multiplying all terms by  $h\nu$ , i.e. the energy of a photon of frequency  $\nu$  ( $h$  is the Planck constant).

Neglecting the time derivative as compared to the divergence of the flux which is proportional to the speed of light and setting

$$(53) \quad i^{\nu\delta} = h\nu n^{\nu\delta} c \delta$$



$$(54) \quad r^{\nu\delta} = h\nu (A^{\nu\delta} - E^{\nu\delta})$$

$$(55) \quad s^{\nu\delta} = h\nu (S_1^{\nu\delta} - S_0^{\nu\delta})$$

one gets

$$(56) \quad \nabla \cdot i^{\nu\delta} = s^{\nu\delta} - r^{\nu\delta}$$

The vector  $i^{\nu\delta}$  is known in optics as the (spectral) radiance. Its magnitude or intensity  $i^{\nu\delta}$  specifies the radiant energy flux per unit spectral band-width and steradian of the beam  $\delta$  at any given point.

Introducing the scattering coefficient  $d^{\nu}$  and the absorption coefficient  $a^{\nu}$ \*, one can write  $s^{\nu\delta}$  and  $r^{\nu\delta}$  in the form

$$(57) \quad s^{\nu\delta} = \frac{d^{\nu}}{4\pi} \int_{4\pi} [i^{\nu\delta'} \phi^{\nu}(\delta', \delta) - i^{\nu\delta} \phi^{\nu}(\delta, \delta')] d\gamma'$$

$$(58) \quad r^{\nu\delta} = a^{\nu} (i^{\nu\delta} - i^{\nu b})$$

The scattering (or "phase") function  $\phi$  specifies the fraction of energy scattered from a monochromatic beam of direction  $\delta$ , per unit travel distance, into another particular direction  $\delta'$ . The scattering function is normalized so that

$$(59) \quad \frac{1}{4\pi} \int_{4\pi} \phi^{\nu}(\delta, \delta') d\gamma' = 1$$

\* The absorption coefficient is independent of the direction of the incident radiation. The scattering coefficient is independent of the direction of the incident radiation if the scattered particles are distributed at random.

$i^{\nu b}$  denotes the black body radiance. It has been assumed that spontaneous emission obeys Kirchoff's law.

The sum  $\kappa_e^{\nu} = a^{\nu} + d^{\nu}$  is called the "extinction" coefficient and

$$(60) \quad \alpha^{\nu} = \frac{d^{\nu}}{a^{\nu} + d^{\nu}}$$

Using eqs. (57), (58), (59) and (60) and introducing the "directional derivative"  $\delta \cdot \nabla$ , eq. (56) may be written

$$(61) \quad \delta \cdot \nabla i^{\nu\delta} = \kappa_e^{\nu} (j^{\nu\delta} - i^{\nu\delta})$$

where

$$(62) \quad j^{\nu\delta} = (1 - \alpha^{\nu}) i^{\nu b} + \frac{\alpha^{\nu}}{4\pi} \int_{4\pi} i^{\nu\delta'} \phi^{\nu}(\delta, \delta') d\gamma'$$

The energy lost by the photons is gained by the fluid.

Thus

$$(63) \quad R = \int_0^{\infty} d\nu \int_{4\pi} r^{\nu\delta} d\gamma$$

Integrating eq. (57), one has

$$(64) \quad \int_0^{\infty} d\nu \int_{4\pi} s^{\nu\delta} d\gamma = 0$$

Hence, introducing the radiation flux

$$(65) \quad \mathbf{q}_r = \int_0^{\infty} d\nu \int_{4\pi} i^{\nu\delta} d\gamma$$

and integrating eq. (56) over all solid angles and frequencies, one gets

$$(66) \quad R = \int_0^{\infty} dv \int_{4\pi} r^{v\delta} d\gamma = - \nabla \cdot \mathbf{q}_r$$

#### The temperature equation and the equation of state.

Combining eqs. (37), (40) and (44), one gets

$$(67) \quad \rho c_p \frac{dT}{dt} - \beta T \frac{dp}{dT} + \rho \sum \tilde{\chi}_a \frac{dc^a}{dt} = R + \mathbf{T}_v : \nabla \mathbf{v} - \nabla \cdot \mathbf{q}_e$$

The specific mass  $\rho$  must be regarded as a function of  $T, p$  and the  $c^a$ 's - or, alternatively,  $\eta, p$  and the  $c^a$ 's -, the exact form of which is specified by the equation of state.

Thus, using eqs. (40), (41) and (42)

$$(68) \quad \begin{aligned} d\rho &= -\beta\rho dT + \frac{d\rho}{a^2} + \sum \tilde{\rho}_a dc^a \\ &= -\frac{\beta\rho T}{c_p} d\eta + \frac{d\rho}{a^{*2}} + \sum \tilde{\rho}_a^* dc^a \end{aligned}$$

where  $a$  and  $a^*$  denote respectively the speeds of sound at constant temperature and at constant entropy, with

$$(69) \quad \frac{1}{a^{*2}} = \frac{1}{a^2} - \frac{\beta^2 T}{c_p}$$

and where  $\rho_a$  and  $\rho_a^*$  are the partial derivatives of  $\rho$  with respect to  $c^a$ , respectively at constant temperature and at constant entropy.

Introducing the Gibbs function

$$(70) \quad \zeta = \epsilon + \frac{p}{\rho} - T\eta$$

one can write, using eq. (40),

$$(71) \quad d\zeta = -\eta dT + \frac{dp}{\rho} + \sum \tilde{\mu}^a dc^a$$

and

$$(72) \quad \tilde{\rho}_a = -\rho^2 \frac{\partial^2 \zeta}{\partial p \partial c^a} = -\rho^2 \left( \frac{\partial \tilde{\mu}^a}{\partial p} \right)_{T, c^a}$$

$$(73) \quad \tilde{\eta}_a = -\frac{\partial^2 \zeta}{\partial T \partial c^a} = -\left( \frac{\partial \tilde{\mu}^a}{\partial T} \right)_{p, c^a}$$

Thus, by eqs. (40), (41) and (42)

$$(74) \quad \tilde{\rho}_a^* = \tilde{\rho}_a + \frac{\beta\rho T}{c_p} \tilde{\eta}_a = -\rho^2 \left[ \left( \frac{\partial \tilde{\mu}^a}{\partial p} \right)_{T, c^a} + \frac{\beta T}{\rho c_p} \left( \frac{\partial \tilde{\mu}^a}{\partial T} \right)_{p, c^a} \right]$$

#### The reference state.

It is convenient to define a (hypothetical) state of reference which displays the main features of most geohydrodynamic situations and to which actual situations can be referred as (small) perturbations.

The state of reference, indicated by the subscript "0", has the following characteristics

- (i) horizontal homogeneity
- (ii) constant composition\* and constant entropy, i.e.,

\* The condition (75) does not require an ocean of pure water or a "standard" atmosphere of dry air (e.g. Iribarne and Godson, 1973). It allows constant salinity and humidity as close as possible to natural values [within the limits of eq. (79) for the humidity].



$$(75), (76) \quad \frac{dc_0^a}{dx_3} = 0 \quad ; \quad \frac{dn_0}{dx_3} = 0$$

(iii) macroscopic equilibrium

$$(77), (78), (79) \quad v_0 = 0 \quad ; \quad \frac{dp_0}{dx_3} = -\rho_0 g \quad ; \quad Q^a = 0$$

Eq. (45) can be written in the alternative form

$$(80) \quad \rho T \frac{dn}{dt} = R - \nabla \cdot \mathbf{q}_e - \sum \tilde{\mu}^a \rho \frac{dc^a}{dt} + T_v : \nabla v$$

which reduces, in the present case, to

$$(81) \quad R_0 - \nabla \cdot \mathbf{q}_{e,0} = 0$$

It is assumed that eq. (81) is satisfied by equilibrium of the radiation field ( $R_0 = 0$ ) and constant heat flux ( $\nabla \cdot \mathbf{q}_{e,0} = 0$ ) i.e. constant temperature gradient.

Taking eqs. (75) and (76) into account, one gets, from eq. (40)

$$(82) \quad \rho_0 c_p \frac{dT_0}{dx_3} = \beta_0 T_0 \frac{dp_0}{dx_3}$$

hence, using eq. (78)

$$(83) \quad \frac{dT_0}{dx_3} = -\gamma \equiv -\frac{\beta_0 g T_0}{c_p}$$

In the atmosphere

$$\beta T \sim 1 \quad , \quad \gamma \sim \frac{g}{c_p} \sim 0(10^{-2})$$

In the ocean

$$\gamma \sim 0(10^{-4})$$

Neglecting second order corrections, one may regard  $\gamma$  — and thus the temperature gradient — in the reference state as a constant.

In the ocean, the temperature variations in the reference state (less than 1 K from surface to bottom) are so small that one can regard the reference temperature as uniform. The situation is different in the atmosphere (1 K per 100 m). In the atmosphere,  $\gamma$  is called the "dry adiabatic temperature gradient".

In the reference state, eq. (74) can be written, using eqs. (75), (78), (79) and (83)

$$(84) \quad \begin{aligned} * \tilde{\rho}_{a,0} &= \tilde{\rho}_{a,0} + \frac{\beta_0 \rho_0}{c_p} T_0 \tilde{\eta}_{a,0} \\ &= -\rho^2 \left[ \left( \frac{\partial \tilde{\mu}^a}{\partial p} \right)_{T,c^a} + \frac{\beta T}{\rho c_p} \left( \frac{\partial \tilde{\mu}^a}{\partial T} \right)_{p,c^a} \right]_0 = \frac{\rho_0}{g} \tilde{\gamma}^a \end{aligned}$$

where  $\tilde{\gamma}^a$  stands in brief for  $\frac{d\tilde{\mu}_0^a}{dx_3}$

Combining eqs. (68) and (78), one gets

$$(85) \quad \frac{1}{\rho_0} \frac{d\rho_0}{dx_3} = -\frac{g}{*a_0^2} \sim \begin{cases} 0(10^{-4}) & \text{in the atmosphere} \\ 0(10^{-5}) & \text{in the ocean} \end{cases}$$

The variation of the specific mass  $\rho$  in the reference state can be neglected in the ocean (a length scale of 100 km is much larger than ocean depths).

Eq. (68) can be linearized around the reference state, i.e., using eq. (84)

$$(86) \quad \frac{d\rho}{dt} = -\frac{\beta_0 T_0}{c_p} \rho_0 \frac{d\eta}{dt} + \frac{1}{*a_0^2} \frac{dp}{dt} + \Sigma * \tilde{\rho}_{a,0} \frac{dc^a}{dt}$$

Setting

$$(87), (88) \quad \frac{d\rho_0}{dt} = v_3 \frac{d\rho_0}{dx_3} ; \quad \frac{dp_0}{dt} = v_3 \frac{dp_0}{dx_3}$$

one can write

$$(89) \quad \frac{d\rho_0}{dt} = \frac{1}{*a_0^2} \frac{dp_0}{dt}$$

and, subtracting eq. (89), from eq. (86)

$$(90) \quad \frac{d}{dt} (\rho - \rho_0) - \frac{1}{*a_0^2} \frac{d}{dt} (p - p_0) = -\frac{\beta_0 T_0 \rho_0}{c_p} \frac{d\eta}{dt} + \Sigma * \tilde{\rho}_{a,0} \frac{dc^a}{dt}$$

Typical geofluid accelerations are much smaller than the acceleration of gravity. The gravity force is then essentially balanced by the vertical pressure gradient and a situation close to hydrostatic equilibrium prevails in the vertical direction, independently of the geofluid motion. Thus, to first order,

$$(91) \quad -\frac{1}{*a_0^2} \frac{d}{dt} (p - p_0) \sim -\frac{v_3}{*a_0^2} \frac{d}{dx_3} (p - p_0) \sim \frac{v_3}{*a_0^2} (\rho - \rho_0) g$$

$$\sim -\frac{\rho - \rho_0}{\rho_0} v_3 \frac{d\rho_0}{dx_3} \sim -\frac{\rho - \rho_0}{\rho_0} \frac{d\rho_0}{dt}$$

Combining eqs. (90) and (91) and setting

$$(92) \quad b = -\frac{\rho - \rho_0}{\rho_0} g$$

one obtains, the geohydrodynamic "equation of state"

$$(93) \quad \frac{db}{dt} = \gamma \frac{d\eta}{dt} - \Sigma \tilde{\gamma}^a \frac{dc^a}{dt}$$

The vertical momentum equation contains the difference of two very large terms  $-\frac{\partial p}{\partial x_3} - \rho g$ . It is convenient to rewrite them as follows

$$(94) \quad -\frac{\partial p}{\partial x_3} - \rho g = -\frac{\partial (p - p_0)}{\partial x_3} - (\rho - \rho_0) g$$

i.e.,  $p_0$  being a function of  $x_3$  only,

$$(95) \quad -\nabla p + \rho g = -\nabla (p - p_0) + \rho_0 b$$

where

$$(96) \quad \mathbf{b} = b \mathbf{e}_3$$

is the "buoyancy".

**The geohydrodynamic approximation.**

In the right-hand side of eq. (94), the small difference  $\rho - \rho_0$  is multiplied by  $g$  and cannot be neglected.

In all other terms of the basic equations,  $\rho$  can be replaced by  $\rho_0$  with a very good approximation.



Eqs. (4) and (30) can then be written, using eq. (95)

$$(97) \quad \nabla \cdot (\rho_0 \mathbf{v}) = 0$$

$$(98) \quad \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + 2 \boldsymbol{\Omega} \wedge \mathbf{v} = - \frac{1}{\rho_0} \nabla (p - p_0) + \mathbf{b} + 2\nu \nabla \cdot \mathbf{D}$$

where\*

$$(99) \quad \nu = \frac{\mu}{\rho_0}$$

is the "kinematic viscosity" ( $\mu$  is the "dynamic viscosity").

In many geophysical situations, these equations can be further simplified. Going back to eq. (85), one can see that the typical length scale of variation of  $\rho_0$ ,  $\Lambda \sim \frac{g}{\omega^2 a_0}$ , is of the order of 100 km in the ocean and 10 km in the atmosphere. Thus, if one excepts very large scale atmospheric processes, one may reasonably neglect the variations of  $\rho_0$ .

Eqs. (97) and (98) can then be written

$$(100) \quad \nabla \cdot \mathbf{v} = 0$$

$$(101) \quad \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + 2 \boldsymbol{\Omega} \wedge \mathbf{v} = - \nabla \pi + \mathbf{b} + \nu \nabla^2 \mathbf{v}$$

where

$$(102) \quad \pi = \frac{p - p_0}{\rho_0}$$

In large scale atmospheric motions, it may be necessary to take the variations of  $\rho_0$  into account. However such motions are characterized by quasi-hydrostatic equilibrium in the vertical with vertical

\* The viscous stress tensor is assumed proportional to the deformation tensor, viz

$$\mathbf{T}_v = 2\mu \mathbf{D} \quad ; \quad \nabla \cdot \mathbf{T}_v = 2\mu \nabla \cdot \mathbf{D}$$

accelerations much smaller than  $g$  and vertical velocities much smaller than horizontal velocities. Then, if one sets

$$(103) \quad 2 \boldsymbol{\Omega} = m \mathbf{e}_2 + f \mathbf{e}_3$$

one can write eqs. (97) and (98) in the form

$$(104) \quad \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{1}{\rho_0} \frac{\partial (\rho_0 v_3)}{\partial x_3} = 0$$

$$(105) \quad \frac{\partial v_1}{\partial t} + \mathbf{v} \cdot \nabla v_1 - f v_2 + \underline{m v_3} = - \frac{1}{\rho_0} \frac{\partial (p - p_0)}{\partial x_1} + \underline{\nu \nabla^2 v_1} + \underline{\nu \frac{\partial}{\partial x_1} (\nabla \cdot \mathbf{v})}$$

$$(106) \quad \frac{\partial v_2}{\partial t} + \mathbf{v} \cdot \nabla v_2 + f v_1 = - \frac{1}{\rho_0} \frac{\partial (p - p_0)}{\partial x_2} + \underline{\nu \nabla^2 v_2} + \underline{\nu \frac{\partial}{\partial x_2} (\nabla \cdot \mathbf{v})}$$

$$(107) \quad \frac{\partial v_3}{\partial t} + \mathbf{v} \cdot \nabla v_3 - m v_1 = - \frac{1}{\rho_0} \frac{\partial (p - p_0)}{\partial x_3} + b + \underline{\nu \nabla^2 v_3} + \underline{\nu \frac{\partial}{\partial x_3} (\nabla \cdot \mathbf{v})}$$

where terms representing negligible contributions have been underlined.

Let

$$(108) \quad z = \int_0^{x_3} r \, dx_3 \quad ; \quad dz = r \, dx_3$$

$$(109) \quad w = r v_3 \quad ; \quad v_3 \frac{\partial}{\partial x_3} = w \frac{\partial}{\partial z}$$

with

$$r = \frac{\rho_0}{\rho_0}$$

where  $\rho_0$  is an appropriate constant reference value of  $\rho_0$ .



Changing variables in eqs. (104), (105), (106) and (107) and taking  $r \sim 1$  in the underlined terms which are, in any case, negligible corrections, one gets

$$(110) \quad \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial w}{\partial z} = 0$$

$$(111) \quad \frac{\partial v_1}{\partial t} + \mathbf{v} \cdot \nabla v_1 - f v_2 + m w = - \frac{\partial \pi}{\partial x_1} + \nu \nabla^2 v_1$$

$$(112) \quad \frac{\partial v_2}{\partial t} + \mathbf{v} \cdot \nabla v_2 + f v_1 = - \frac{\partial \pi}{\partial x_2} + \nu \nabla^2 v_2$$

$$(113) \quad \frac{\partial w}{\partial t} + \mathbf{v} \cdot \nabla w - m v_1 = - \frac{\partial \pi}{\partial z} + b + \nu \nabla^2 w$$

where

$$(114) \quad \mathbf{v} \cdot \nabla = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + v_3 \frac{\partial}{\partial x_3} = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + w \frac{\partial}{\partial z}$$

and

$$(115) \quad \pi = \frac{P - P_0}{\rho_0}$$

[The definition of  $\pi$  is the same as in eq. (103) since, in eq. (103),  $\rho_0$  is regarded as a constant, i.e.  $\rho_0 \sim \rho_0$ ].

Eqs. (100) and (101) are thus universally applicable if  $\mathbf{v}$  and  $\nabla$  are redefined by

$$(116) \quad \mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + w \mathbf{e}_3$$

$$(117) \quad \nabla = \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{\partial}{\partial z}$$

$w$  and  $z$  reducing respectively to  $v_3$  and  $x_3$  in most cases.

Combining eqs. (84) and (93), one obtains

$$(118) \quad \frac{db}{dt} = \gamma \left( \frac{d\eta}{dt} - \sum \tilde{\eta}_{a,0} \frac{dc^a}{dt} \right) - \sum \frac{\tilde{\rho}_{a,0} g}{\rho_0} \frac{dc^a}{dt}$$

The contributions due to changes in the concentration  $c^a$  are small except, in the atmosphere, in the case of a change of phase. In this case, the third term in the right-hand side of eq. (118) is negligible as compared with the second term which includes a contribution proportional to the rate of evaporation and condensation with the corresponding latent heat as a factor.

Neglecting the last term in the right-hand side of eq. (118) and using eq. (15) and (80), one gets

$$(119) \quad \frac{db}{dt} = \frac{\beta_0 g}{\rho_0 c_p} (R^T - \nabla \cdot \mathbf{q}_e^T)$$

where

$$(120) \quad R^T = R + \mathbf{T}_v : \nabla \mathbf{v} - \sum Q^a \tilde{\chi}_0^a$$

and

$$(121) \quad \mathbf{q}_e^T = \mathbf{q}_e - \sum \tilde{\chi}_0^a \mathbf{q}^a$$

may be regarded as "total heat source" and "total heat flux" respectively\*.

Using eq. (40), eqs. (118) and (119) can also be written

$$(122) \quad \frac{db}{dt} = \beta_0 g \left( \frac{dT}{dt} - \frac{\beta_0 T_0}{\rho_0 c_p} \frac{dp}{dt} \right) = \frac{\beta_0 g}{\rho_0 c_p} (R^T - \nabla \cdot \mathbf{q}_e^T)$$

Because of the conditions of near-hydrostatic equilibrium, one may write, with a very good approximation

$$(123) \quad - \frac{\beta_0 T_0}{\rho_0 c_p} \frac{dp}{dt} \sim \frac{\beta_0 T_0 g}{c_p} v_3 \sim \gamma v_3$$

where, as pointed out before,  $\gamma$  may be regarded as a constant. Similarly the variations of  $\tilde{\chi}_0^a/c_p$  may be neglected.

\* The minus sign before the sums in eqs. (120) and (121) should not be misleading. For condensation, for instance

$$- Q^c (\chi_0^c - \chi_0^c) = - Q^c (\chi_0^c - \chi_0^c) > 0$$

Introducing then the "equivalent temperature"  $\theta$  (Matveyev, 1964)\*

$$(124) \quad \theta = T + \gamma v_3 + \sum \frac{\tilde{\chi}_0^a}{c_p} c^a$$

one can write eq. (122) in the form

$$(125) \quad \rho_0 \frac{d\theta}{dt} = Q^\theta - \nabla \cdot \mathbf{q}^\theta$$

where \*\*

$$(126) \quad Q^\theta = \frac{R + \mathbf{T}_v : \nabla \mathbf{v}}{c_p} \sim \frac{R}{c_p}$$

$$(127) \quad \mathbf{q}^\theta = \frac{\mathbf{q}_e}{c_p}$$

The second term in the right-hand side of eq. (119) represents the effect of a molecular flux of buoyancy. Taking into account that molecular fluxes are small and neglecting small corrections, one may write

$$(128) \quad - \frac{\beta_0 g}{\rho_0 c_p} \nabla \cdot \mathbf{q}_e^T \sim - \nabla \cdot \frac{\beta_0 g}{\rho_0 c_p} \mathbf{q}_e^T \sim \lambda \nabla^2 b$$

where  $\lambda$  is the appropriate "diffusivity coefficient."

Introducing then the "buoyancy production rate"

$$(129) \quad B = \frac{\beta_0 g}{\rho_0 c_p} R^T$$

one may write eq. (122) in the simple form

$$(130) \quad \frac{\partial b}{\partial t} + \mathbf{v} \cdot \nabla b = B + \lambda \nabla^2 b$$

Eqs. (100), (101) and (130) constitute, with the definitions (96) and (103) the *basic geohydrodynamic equations*.

\* In the ocean, where there is no change of phase and where  $\gamma$  is very small, one may usually take

$$\theta \sim T \quad (\text{in the ocean}).$$

\*\* The heat produced by viscous dissipation may usually be neglected.

If  $B$  is negligible or can be expressed as a function of  $b$  only, they constitute a closed system of five partial differential equations for  $\mathbf{v}$ ,  $\pi$  and  $b$ . These equations are known as the "*Boussinesq Equations*".

In the other cases and in particular when the distinction must be made between  $\theta$  and  $T$ , changes of phase must be taken into account, etc., the Boussinesq equations must be complemented by the mass conservation equations (15) and the thermodynamic law of phase changes.

#### Variability of the atmosphere and the ocean.

The basic geohydrodynamic equations established in the previous sections correspond to a definite *selection of the state variables* which are used to describe the geofluids in mathematical terms.

As the equations stand, however, in general form, they are applicable to all kinds of geohydrodynamic phenomena from molecular scales to climatic scales. The model must, in practice, further be tuned to the signals one wishes to investigate, i.e., after selecting the *nature* of the state variables, one must specify their *resolution*, both in physical space and in Fourier space.

Although the variability of the atmosphere and the ocean is universally recognized (e.g. Monin, 1972; Monin et al., 1974; Nihoul, 1980), nothing like an unanimous spectral description of geohydrodynamic phenomena appears to be available in the literature which often offers conflicting views based on bits and parts of energy spectra plotted in irreconcilable units.

Yet, to define the "*nature*" and "*resolution*" of appropriate state variables for ocean-atmosphere models, one must identify the "*spectral windows*" of the geofluids.

A simple — and, in first approximation, probably appropriate — spectral scenario takes form if one recognizes that the basic equations of Geophysical Fluid Dynamics contain three characteristic frequencies :

(i) the Brunt-Väisälä frequency  $n$  is a measure of the stratification ( $n^2$  is defined as the vertical gradient of buoyancy). The maximum Brunt-Väisälä frequency is of the order of  $10^{-2} \text{ s}^{-1}$  in the ocean and a similar value may presumably be used for the atmosphere;



(ii) the Coriolis frequency  $f$  is a measure of the effect of the Earth's rotation ( $f$  is defined as twice the vertical component of the Earth's rotation vector). In mid-latitudes,  $f \sim 10^{-4} \text{ s}^{-1}$ ;

(iii) the Kibel frequency  $j$  is a measure of the effect of the Earth's curvature. If  $\beta$  denotes the gradient of  $f$  and  $\epsilon$  the energy dissipation rate, the Kibel frequency is of the order  $j \sim \epsilon^{1/5} \beta^{2/5}$  i.e., for  $\beta \sim 10^{-11} \text{ m}^{-1} \text{ s}^{-1}$ ,  $\epsilon \sim 10^{-9} \text{ m}^2 \text{ s}^{-3}$  in the ocean at corresponding scales,  $\epsilon \sim 5 \cdot 10^{-4} \text{ m}^2 \text{ s}^{-3}$  in the atmosphere

$$\text{in the ocean} \quad j \sim 0.6 \cdot 10^{-6} \text{ s}^{-1} \quad (t_\beta \sim 18 \text{ days})$$

$$\text{in the atmosphere} \quad j \sim 10^{-5} \text{ s}^{-1} \quad (t_\beta \sim 1 \text{ day})$$

The characteristic wave-number associated with the Kibel frequency is (Nihoul, 1980)  $\kappa_\beta \sim \epsilon^{-1/5} \beta^{3/5}$ , i.e.

$$\text{in the ocean} \quad \kappa_\beta \sim 1.6 \cdot 10^{-5} \quad (\lambda_\beta \sim 60 \text{ km})$$

$$\text{in the atmosphere} \quad \kappa_\beta \sim 10^{-6} \quad (\lambda_\beta \sim 1000 \text{ km})$$

Schematically, one can distinguish between

(i) *microscale* motions, for frequencies larger than  $n \sim 10^{-2} \text{ s}^{-1}$ , unaffected by the Earth's curvature, the Earth's rotation and the stratification and constituted mainly by three-dimensional turbulence;

(ii) *mesoscale* motions, in the range of frequencies between the Coriolis and Brunt-Väisälä frequencies, affected by the Earth's rotation and by the stratification and constituted by inertial oscillations, tides, diurnal variations, internal waves and *bliny-turbulence\**;

(iii) *synoptic* motions, for frequencies smaller than the Coriolis frequency ( $f \sim 10^{-4} \text{ s}^{-1}$ ) and larger, say, than the frequency of annual variations ( $\sim 10^{-7} \text{ s}^{-1}$ ), constituted mainly of the so-called "synoptic eddies" of pseudo-dimensional rosbby-turbulence\*\*, with a peak of energy about the Kibel frequency;

(iv) *climatic* motions, for frequencies in the range  $10^{-7}-10^{-8} \text{ s}^{-1}$ , corresponding to periods of 1 to 20 years, characteristics of man's capabilities of exploiting a climate forecast;

(v) *"paleo-climatic"* motions for still smaller frequencies.

\* A "bliny" (from the Russian *blini*) is a pancake-shaped eddy, contributing to an energy cascade to smaller scales via epidermic instabilities and internal waves.

\*\* A "rosby" (from the scientist Rosby) is a pseudo two-dimensional eddy of scale of the order of the Rosby radius of deformation.

To study geohydrodynamic phenomena in a particular range of scales, one must smooth out smaller scale fluctuations by appropriate time averaging and find a suitable representation of larger scales forcing.

From the basic equations, one thus derives

(i) *primitive equations* by filtering microscale "eddy" turbulence;

(ii) *synoptic (weather) equations* by filtering both mesoscale waves, *bliny-turbulence* and microscale turbulence;

(iii) *climatic equations* by filtering out rosbby turbulence, *bliny turbulence* and eddy turbulence.

#### Weathers and climates of the atmosphere and the ocean.

In the words of Wiin-Nielsen, Secretary General of W.M.O.,

Normally, we consider "weather" as the state of the atmosphere at a particular place and at a particular time. It is possible to talk about the weather during any individual day, week, month, season or year.

"Climate", on the other hand is often referred to as "average weather", or to use the definition of the World Climate Programme: "Climate is the synthesis of weather over the whole of a period essentially long enough to establish its statistical ensemble properties (mean values, variances, probability of extreme events ...) and is largely independent of any instantaneous state". In brief, we may say that climate is the probability of occurrence of different types of weather at a given location for a nominated calendar period, week, month or season.

These definitions are not, of course, universally accepted. During the development of GARP, a different concept emerged. The distinction between "weather" and "climate" became associated with the limits of predictability of atmospheric states.

The definition of "climate" in this statement may seem to be different from the definition given in the preceding section and based on the concept of spectral windows and "climatic" time scales and length scales.

Climate models are often questioned in this respect and often suspected of dealing with "long range weather forecasts" rather than true climatic predictions.

Long range weather prediction, however, consists really in forecasting the day to day variations of the weather and carrying the forecast for a long time (ten years, say) while climate modelling (even considered for the same decade) is concerned with the evolution of mean - monthly or seasonal - values described by averaged equations derived from the primitive

equations with the necessary parameterization of non-linear interaction residues.

The parameterization, in fact, implies a knowledge of the variability of smaller time scale processes which is nothing else than what meteorologists call the statistical ensemble properties of weather.

Hence, the scale definition of climate is as much "statistical" as any other and any small difference there may be is largely ascribable to the priorities of the different approaches.

#### Climatic averages.

If one defines the climatic state variables as averages over a suitable period of time (a month, a season, a year, ...), then mathematical models of climates must aim at forecasting the evolution of these climatic variables over one or two decades, i.e., a span of time compatible with human planning.

The time of averaging, however, cannot be chosen arbitrarily. To preserve the statistical significance of the mean, it must correspond to a valley in the spectral distribution of energy.

For the atmosphere, a time of averaging of the order of a month ( $2.5 \cdot 10^6$  s) seems to be appropriate for climate models. It is more than ten times larger than the characteristic time of the rosbies ("synoptic eddies").

In the ocean, however, the rosbies have a much longer life time and a much lower mobility than their atmospheric analogues (travelling cyclones and anticyclones in mid latitudes). A meaningful statistical procedure which involves a succession of these oceanic features locally would require an averaging period of at least two or three years. It therefore would not resolve the annual cycle of various air-sea interaction processes which are essential for our understanding of the climate system as a whole.

As far as length scales are concerned, the situation is just the opposite: the characteristic horizontal area of the oceanic rosbies tends to be more than a hundred times smaller than that of their atmospheric counterparts. One therefore can get a reduced representation of the large

scale distribution of climate variables in the ocean by appropriate *horizontal* averaging.

Such an average is always included in the numerical modelling process as the selection of a numerical grid implies the filtering out of sub-grid scale processes. A spatial resolution of a few degrees of the great circle ( $\sim 300-600$  km) smoothes out most of the perturbations introduced by travelling ocean rosbies and appears to be adequate.

Thus the combination of time and space averaging, eliminating most of the variance within time intervals of about one month and within distances of about 300-600 km, provides the appropriate low pass filter for the definition of the climatic state variables and the formulation of ocean-atmosphere climate models.

#### Examples of application of the geohydrodynamic equations to ocean-atmosphere climate models.

Among the models which have been developed up to now in order to simulate the climates, one of the most recent and realistic ones is the three-dimensional general circulation model elaborated at the National Center for Atmospheric Research (USA) by Washington et al, (1980).

This model links separate existing models of the atmosphere, the ocean and the sea ice, and covers the entire globe. The oceanic part of the model is based on the primitive equations and uses the hydrostatic and Boussinesq approximations with a rigid lid. Unfortunately, this does not allow net mass fluxes across boundaries; thus, the water flux at the surface due to precipitation and evaporation is represented by an equivalent boundary salt flux in the salinity equation. The runoff from the land into the oceans is not included. This explains especially why the computed salinity values are too large.

The model of the mixed layer of the ocean (Fig. 1) is taken from an earlier model developed by Bryan et al (1975) in which the wind mixing is parameterized and heat is redistributed downwards through successive layers so that heat is conserved and the potential energy gained during the mixing is equal to the energy supplied by the wind.



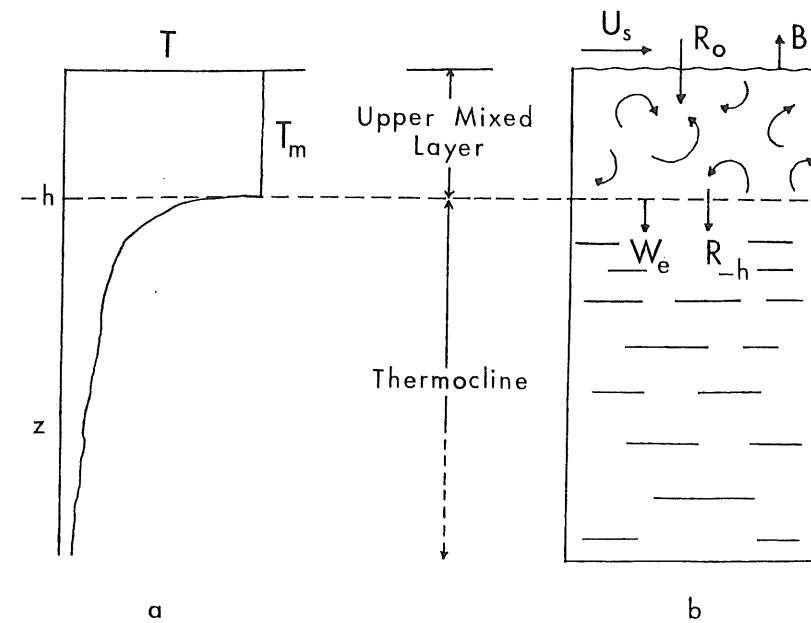


Fig. 1.

- a. Vertical temperature profile of a typical upper oceanic mixed layer of temperature  $T_m$  and depth  $h$ .
- b. Typical mixed layer showing turbulent motions with quiescent stratified water below, the insolation at the sea surface and mixed layer bottom,  $R_o$  and  $R_{-h}$  respectively, the back flux  $B$ , surface wind  $U_s$ , and the entrainment rate  $w_e$ .

When the model becomes gravitationally unstable, a convective adjustment is performed in order to return adjacent layers to a statistically stable state.

The ocean is subdivided into four layers with thicknesses, from top to bottom, of 50, 450, 1500 and 2000 m (Kim, 1979; Fig. 2). Heat is exchanged by diffusion between adjacent layers in proportion to the temperature gradient between them (thermal diffusivity  $\sim 10^{-4} \text{ m}^2 \text{ s}^{-1}$ ).

Bottom topography is included on a scale consistent with the resolution ( $5^\circ$  horizontal mesh).

The sea ice model is a simple thermodynamic model using a simplified calculation of heat flux through sea ice.

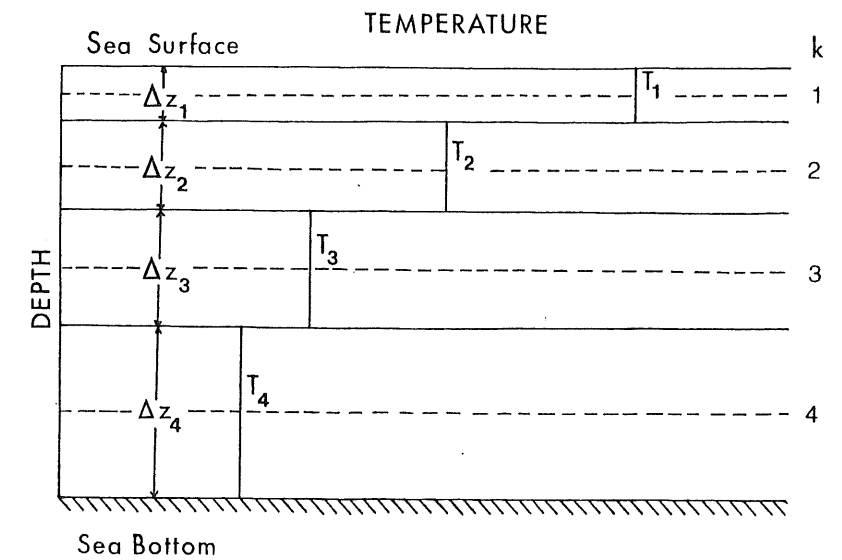


Fig. 2.

Representation of the layer temperatures  $T_k$  and corresponding layer thicknesses  $\Delta z_k$  of the four layer OGCM (Kim, 1979), where  $k$  may have the integer values 1,2,3,4 corresponding to the four layers.

The growth and decay of the latter are predicted on the basis of precipitation and (or) flux budgets and the heat flux within the ice is assumed to be independent of depth.

The vertical eddy diffusivity for heat and momentum is taken equal to  $10^{-4} \text{ m}^2 \text{ s}^{-1}$ , while horizontal eddy diffusivities for momentum and sensible heat of  $10^6 \text{ m}^2 \text{ s}^{-1}$  and  $2 \cdot 10^4 \text{ m}^2 \text{ s}^{-1}$ , respectively, are chosen. The results show that these values overestimate the effects of the oceanic eddies.

In order to compute the sensible and latent heat fluxes over the ocean, monthly averages of the speed of the surface wind  $U$ , temperature difference  $\Delta T$  and humidity difference  $\Delta q$  are used. The coefficient of momentum exchange  $C_D$  is taken to be a function of the bulk Richardson number of stability and the same drag coefficient is used for energy fluxes. It is worth noting that the model works with time-averaged momentum flux rather than stresses computed from averaged winds.



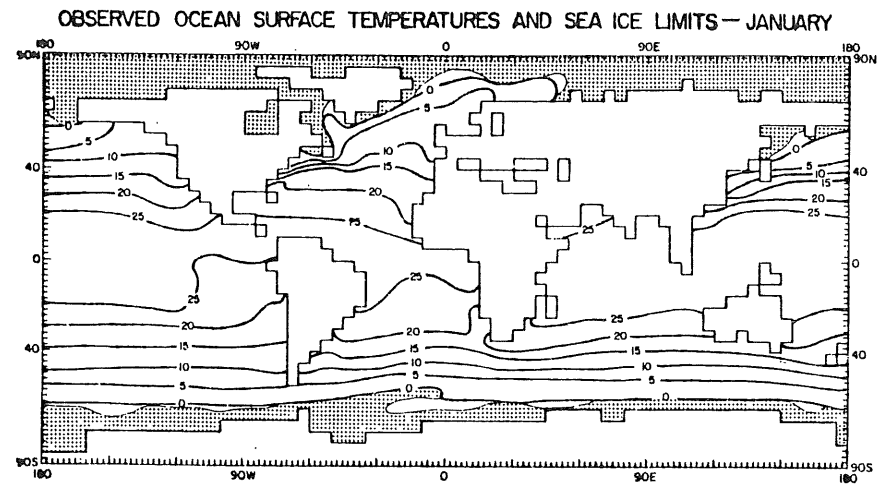
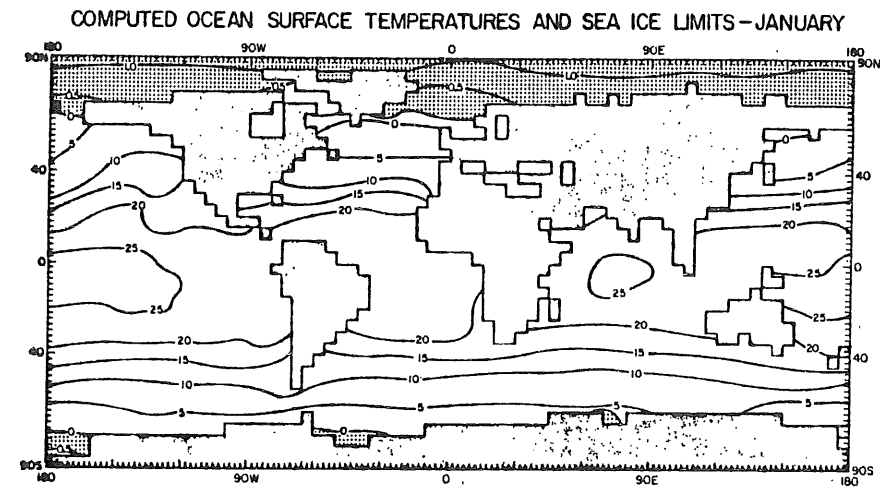


Fig. 3a.

Computed and observed ocean surface temperature ( $^{\circ}\text{C}$ ) and sea ice limits for January. Observed values from Alexander and Mobley (1976). Large stippled pattern indicates extent of sea ice. Sea ice thickness contours on computed maps are in meters.

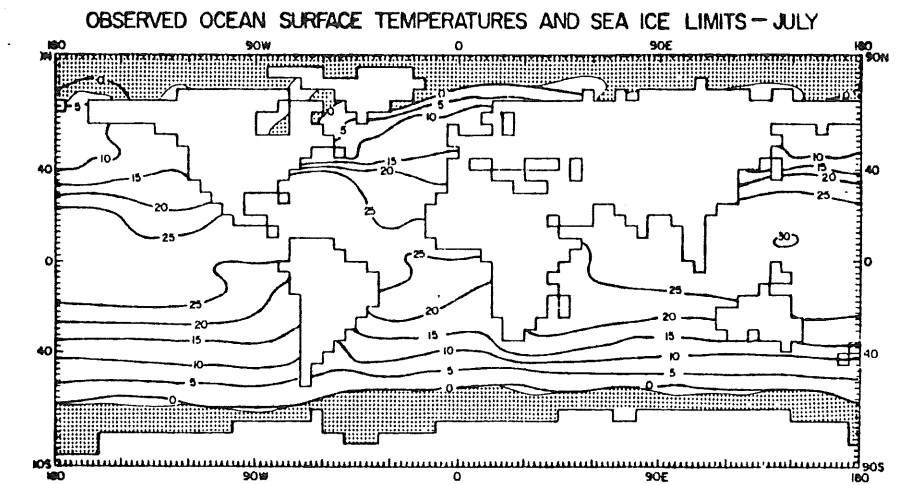
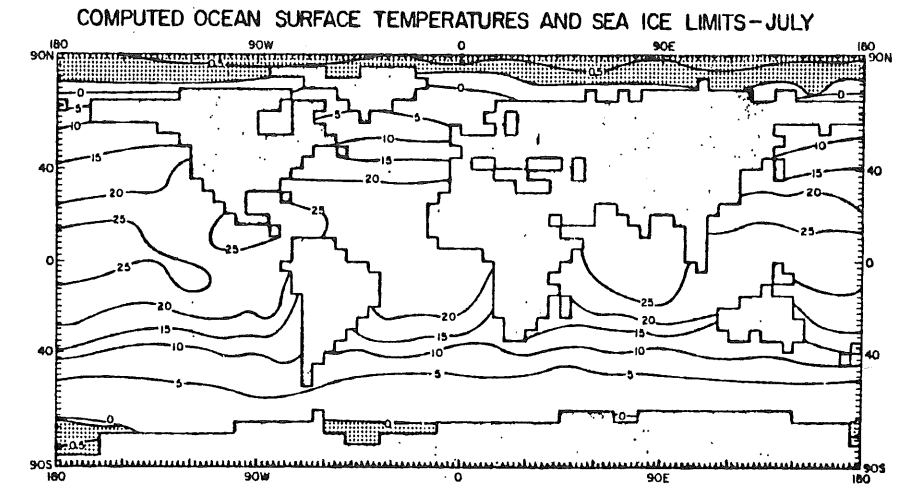


Fig. 3b.

Computed and observed ocean surface temperature ( $^{\circ}\text{C}$ ) and sea ice limits for July. Observed values from Alexander and Mobley (1976). Large stippled pattern indicates extent of sea ice. Sea ice thickness contours on computed maps are in meters.

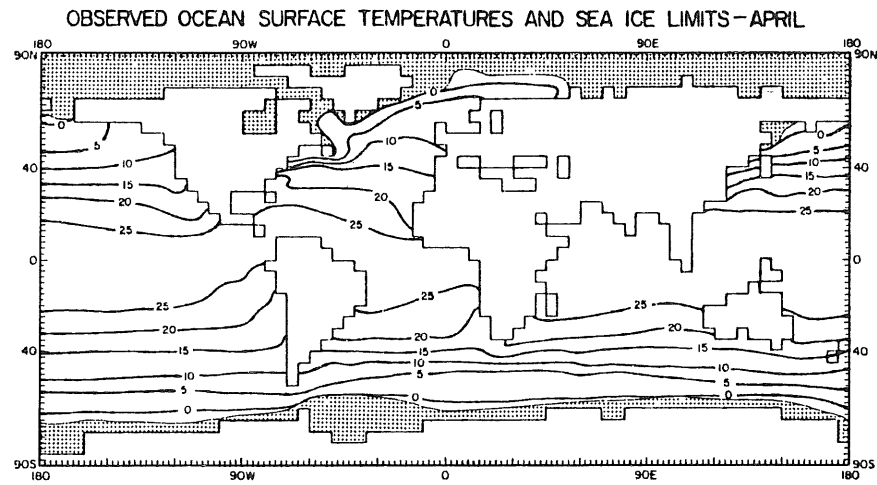
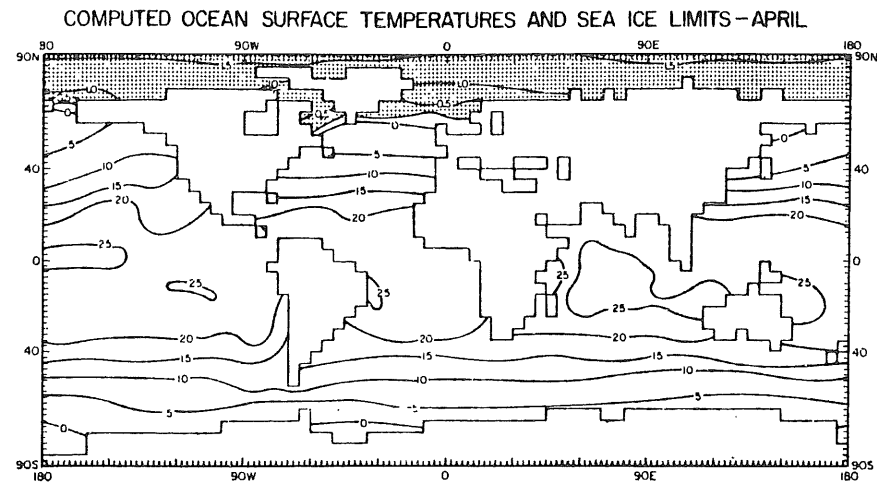


Fig. 4a.

Computed and observed ocean surface temperature ( $^{\circ}\text{C}$ ) and sea ice limits for April. Observed values from Alexander and Mobley (1976). Large stippled pattern indicates extent of sea ice. Sea ice thickness contours on computed maps are in meters.

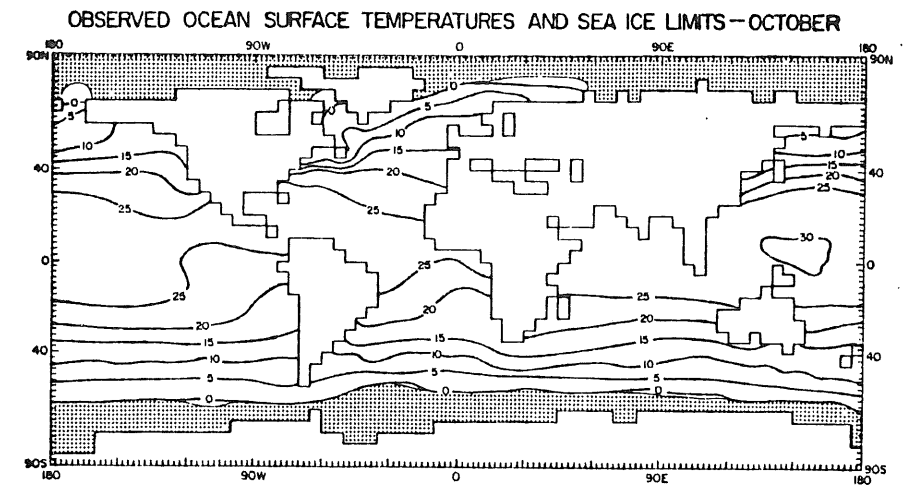
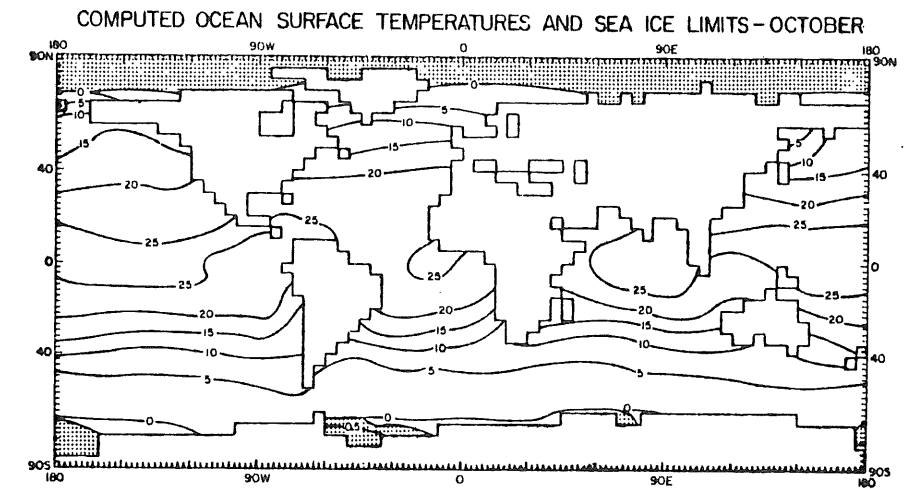


Fig. 4b.

Computed and observed ocean surface temperature ( $^{\circ}\text{C}$ ) and sea ice limits for October. Observed values from Alexander and Mobley (1976). Large stippled pattern indicates extent of sea ice. Sea ice thickness contours on computed maps are in meters.



This global circulation model simulates reasonably well the ocean surface temperature pattern. Fig. 3 and 4 compare computed and observed values of SST and sea ice limits for January, April, July and October. The observed and computed values of the amplitude of the seasonal cycle of SST agree in phase, but not in magnitude. The calculated ocean temperatures are as much as 3°C too cold locally in the tropics and up to 4°C too warm in high latitudes. It is well known (Washington and Chervin, 1980) that small changes in ocean surface temperature ( $\leq 2^\circ\text{C}$ ) result in significant changes in atmospheric structure. A better treatment of the oceanic subsystem is thus needed.

Only the seasonal shift of isotherms towards the summer pole is quite well reproduced. The error on the temperatures affects the distribution of sea ice : the Antarctic region (which is more influenced by warmer oceans than the Arctic region) is not well simulated. The thickness and geographical extent of sea ice in the Antarctic are much less than observed.

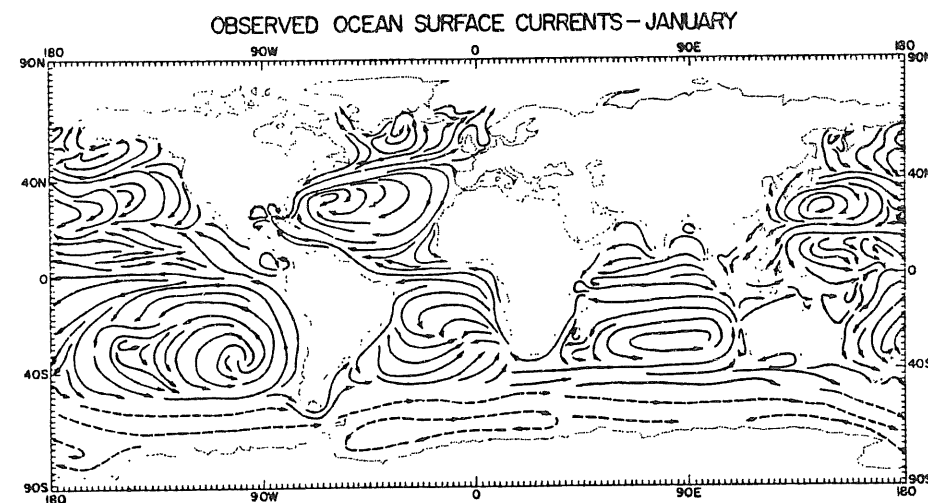


Fig. 5a.

Observed ocean currents for January (from Meehl, 1980)

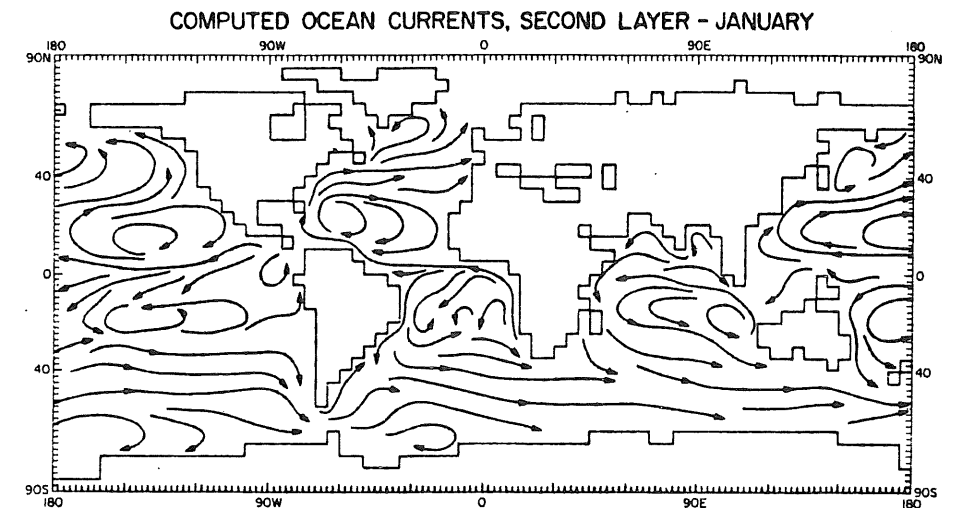
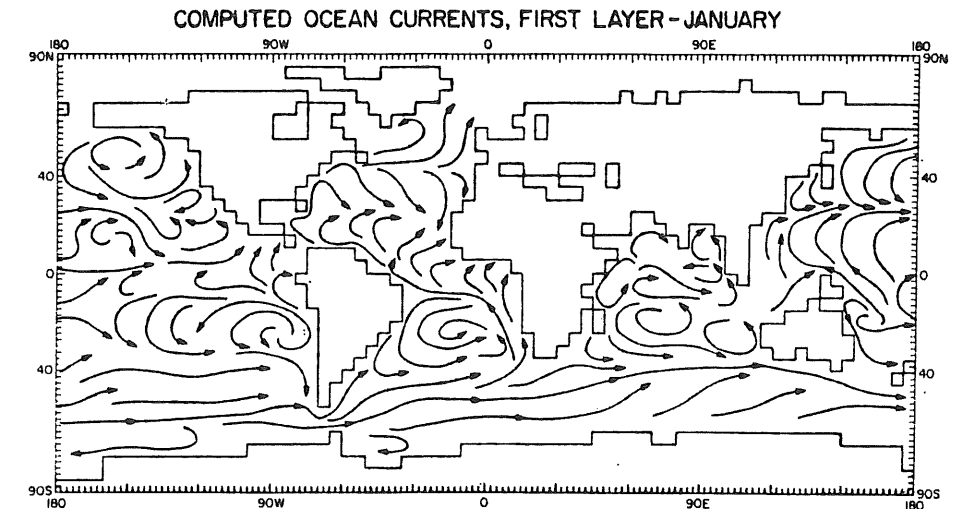


Fig. 5b.

Computed ocean currents for January

Fig. 5 and 6 (Meehl, 1980) show the January and July ocean currents streamlines for computed surface currents, second-layer currents and observed surface currents. (Note that the latter represents the long-term mean flow). The major ocean current systems are better reproduced in the second layer. This layer dominates the long-term heat storage while the other three layers contribute much less.

Fig. 7 shows computed and observed annual values of wind stresses. Some of the large-scale features such as trade wind and westerly wind stresses are well reproduced, but the computed values are smaller than the observed ones by at least a factor 2, and the computed directions of the Northern Hemisphere westerly wind stress have a larger southward component than observed.

Fig. 8 shows annual mean vertical velocity at the bottom of the upper mixed layer (50 m). Observed and computed values agree, although the model underestimates strong upwellings along coastal regions.

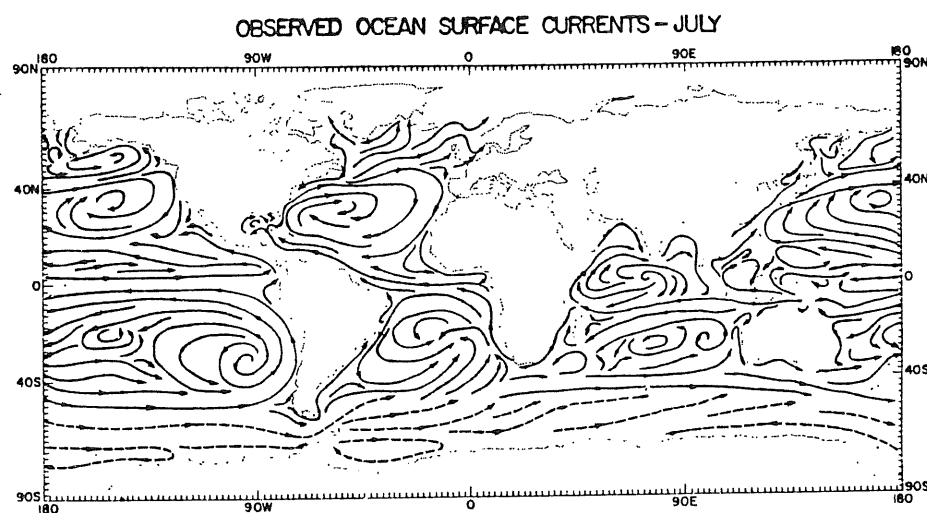


Fig. 6a.

Observed ocean currents for July (from Meehl, 1980)

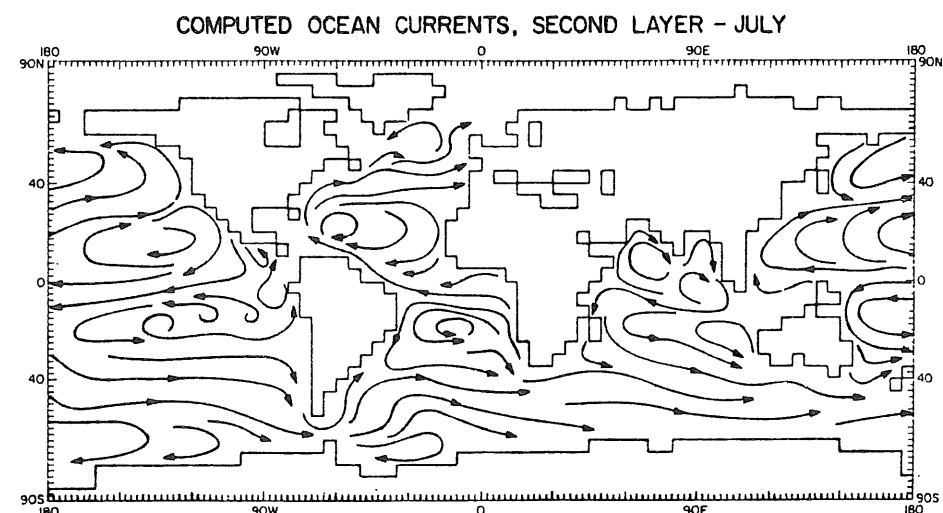
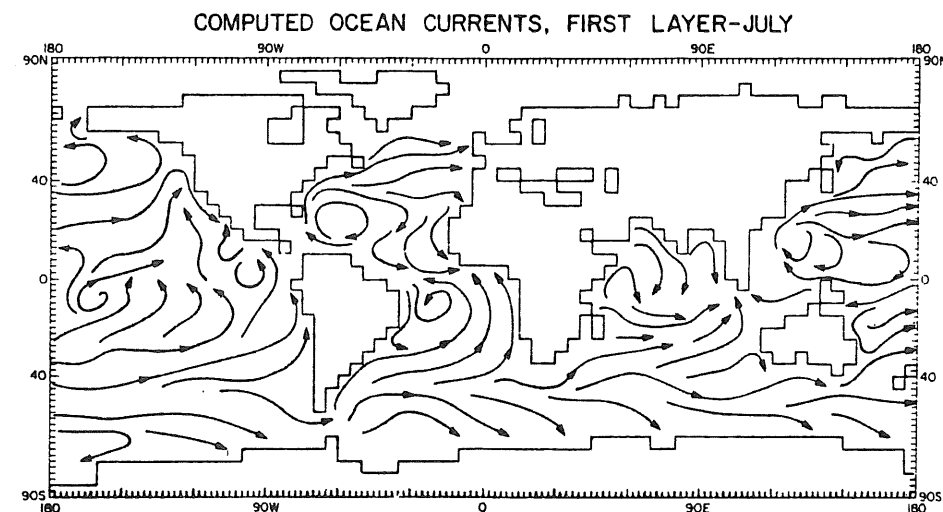


Fig. 6b.

Computed ocean currents for July



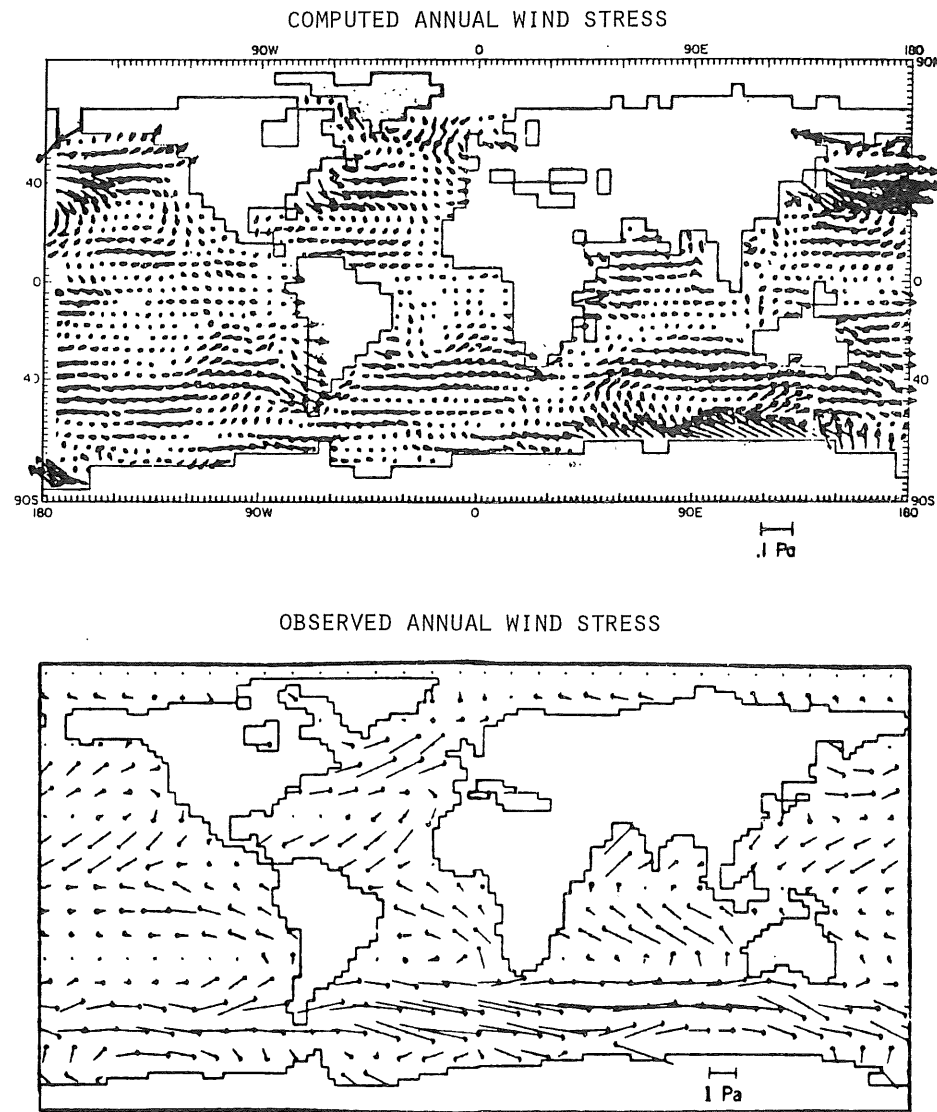


Fig. 7.

Computed and observed annually averaged wind stress. Observed values are Hellerman's (1967) from Bryan and Lewis (1979).

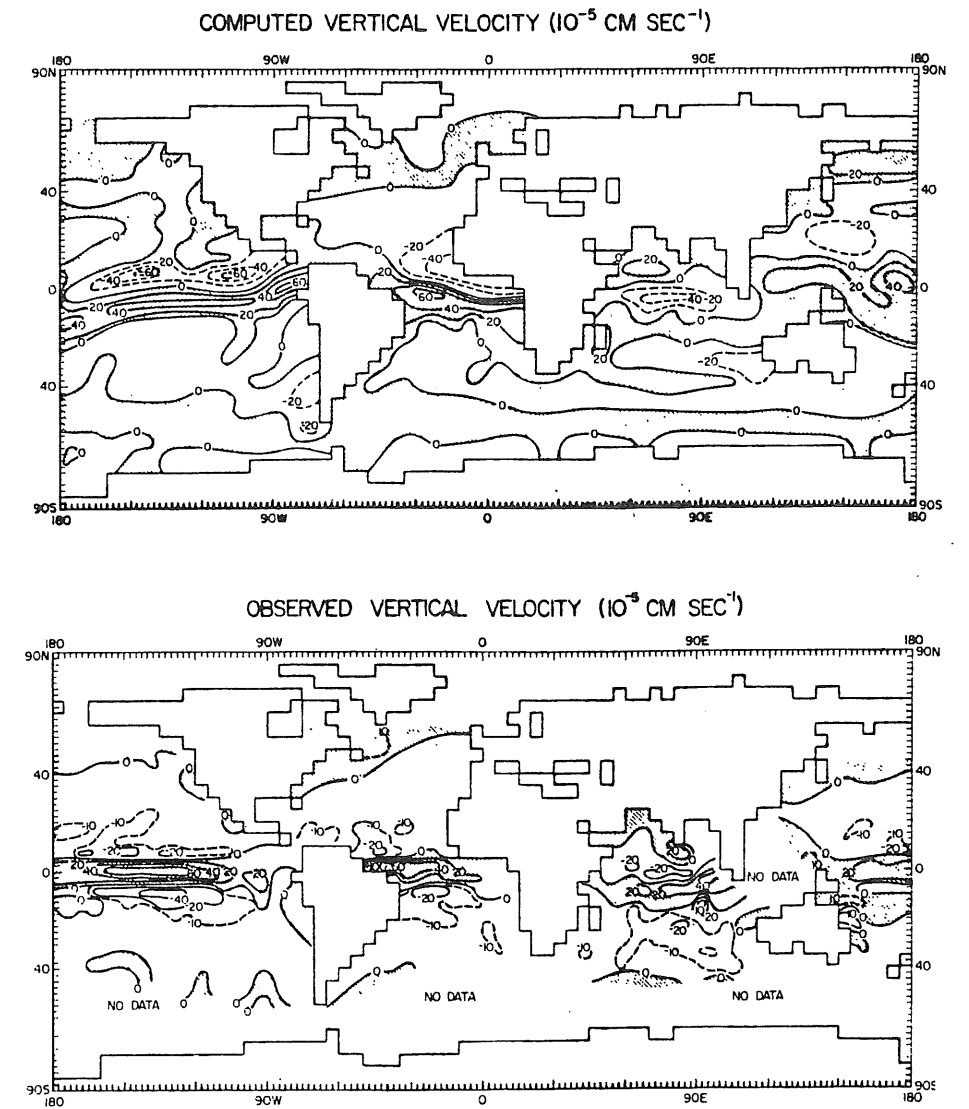


Fig. 8.

Computed January and observed annual mean vertical velocity at bottom of mixed layer. Observed values are from Stommel (1964) [bottom]. Cross-hatched areas indicate positive (upward) vertical motion.

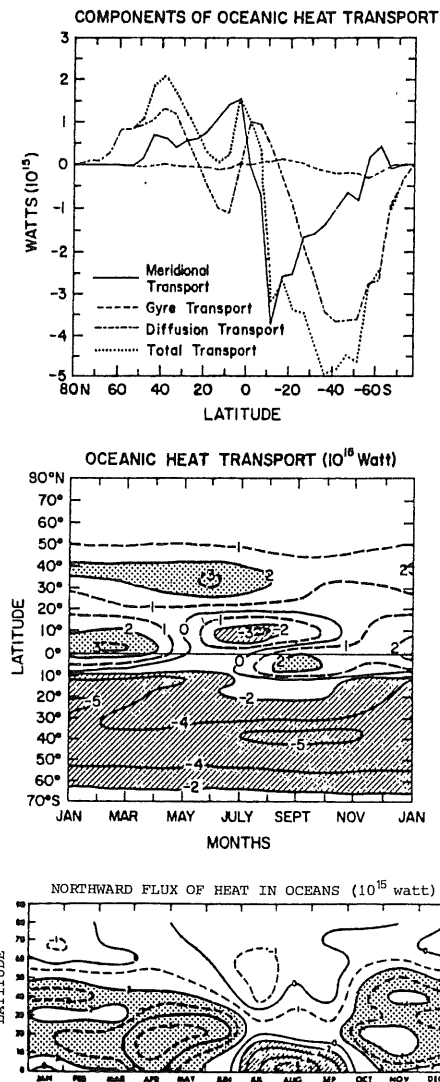


Fig. 9.

Top : computed oceanic heat transport as a function of latitude separated into components; middle : computed horizontal oceanic heat as function of time; bottom : observed horizontal oceanic heat transport as function of time (Oort and Vonder Haar, 1976). On middle and bottom figures, cross hatching indicates southward transport  $> 2 \times 10^{15}$  W; stippling indicates northward transport  $> 2 \times 10^{15}$  W.

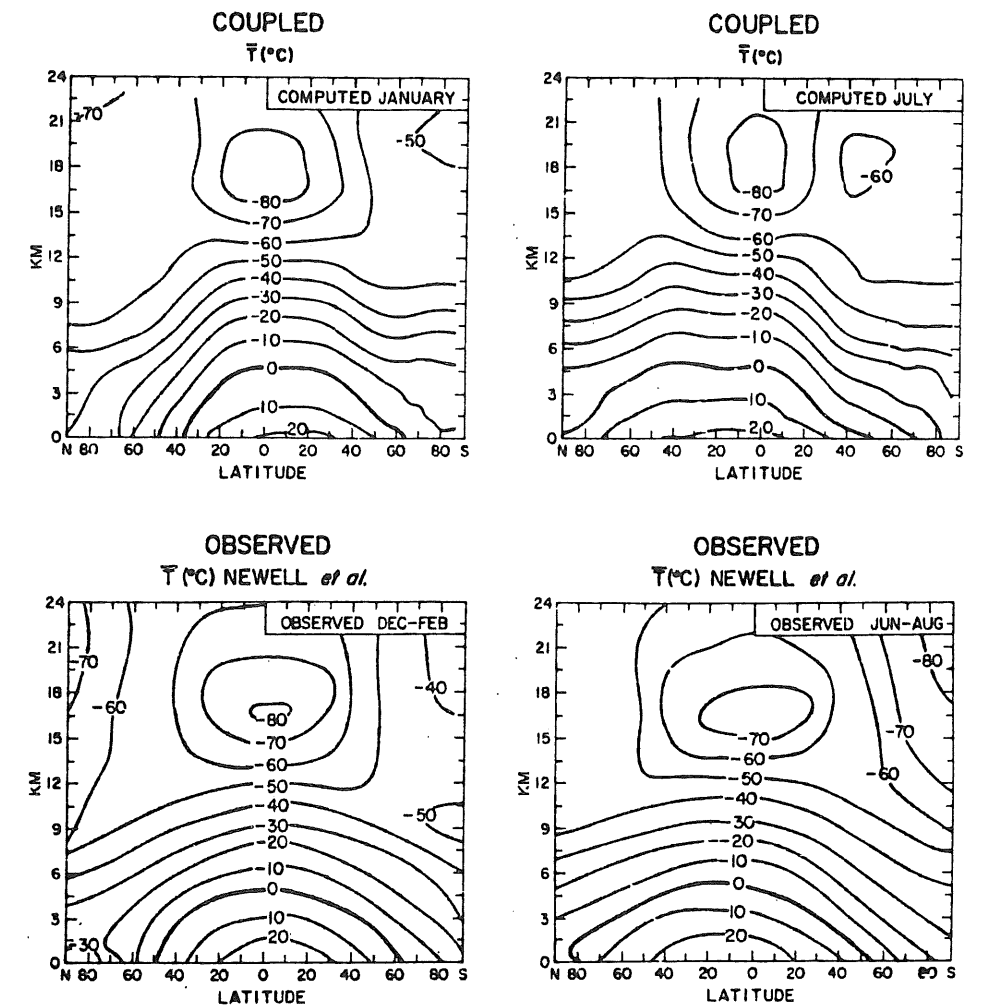


Fig. 10.

Computed and observed zonally averaged atmospheric temperature  $\bar{T}$  distribution for January and July. Top : with coupled model; bottom : observed.



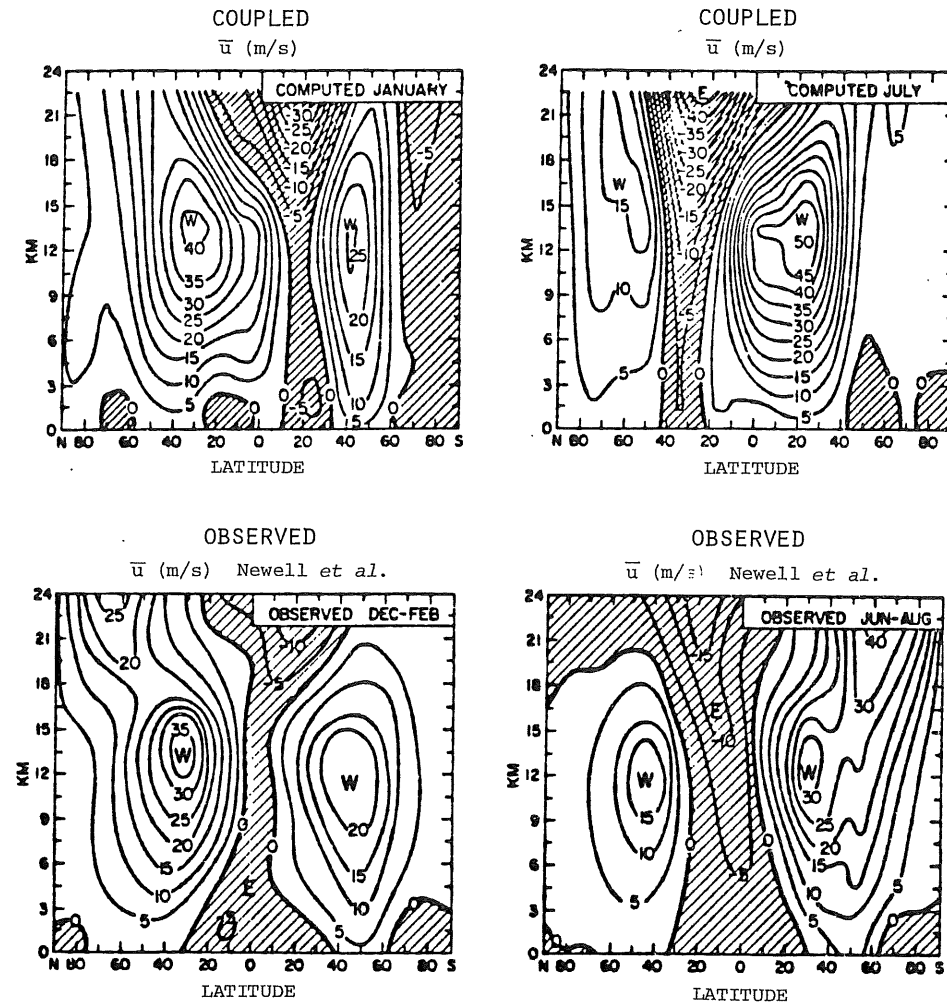


Fig. 11.

Computed and observed zonally averaged wind component  $u$  for January and July. Top : with coupled model; bottom : observed.

If one compares the computed horizontal ocean heat transport (middle of Fig. 9) with that calculated by Oort and Vonder Haar (1976) as a residual term in the earth's heat balance (bottom of Fig. 9) by using available data for the Northern Hemisphere, one can see that these values are similar, but the computed heat transport maxima are smaller in magnitude, occur two months before and are centered farther south. The total transport (top of Fig. 9) is in good agreement with results found by Bryden and Hall (1979) in their recent estimates of annually averaged observed oceanic heat transport across the North Atlantic at  $25^\circ$  N.

To determine which mechanisms in the ocean contribute to the transport, the authors divide the total heat transport (top of Fig. 9) into three components : meridional, gyre and diffusive transports. Fig. 9 gives the latitudinal annual time average of each transport. The meridional term is dominant in the tropics and subtropics, the diffusion term dominates in the higher latitudes, while the gyre term contributes little to the total heat transport (due especially to the coarse resolution used :  $5^\circ$  of horizontal resolution).

Fig. 10 and 11 show computed and observed zonally averaged atmospheric temperature  $T$  distribution and wind component  $u$  for two months : January and July. The agreement is quite good, however the computed temperatures are too small and the tropical belt of easterlies is displaced in the summer hemisphere.

#### Conclusions.

Faced with dramatic food, energy and environmental problem, man is today required to provide better and better forecasts of the "weathers" and climates of the atmosphere and the ocean.

This requires sophisticated *interactive* ocean-atmosphere models corresponding to distinct spectral windows of geophysical processes and, in particular, climate models able to predict, over one or two decades, the evolution of, say, monthly mean values.



With the recent expansion of computer facilities and the progressive constitution of extensive data banks — incorporating in particular the data from large scale oceanographic experiments and from remote sensing — such ambitious models are coming within reach of the new generations of geophysical fluid dynamicists and modellers.

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