State Complexity of the Multiples of the Thue-Morse Set

Adeline Massuir
Joint work with Émilie Charlier and Célia Cisternino

September 2nd, 2019

What do we want to do ?

Definition

Let $b \in \mathbb{N}_{\geq 2}$. A subset X of $\mathbb{N}$ is *b-recognizable* if $\text{rep}_b(X)$ is regular.

What do we want to do ?

Definition

Let $b \in \mathbb{N}_{\geq 2}$. A subset X of $\mathbb{N}$ is *b-recognizable* if $\text{rep}_b(X)$ is regular.

It is equivalent to work with $0^* \text{rep}_b(X)$.

What do we want to do ?

Definition

Let $b \in \mathbb{N}_{\geq 2}$. A subset X of $\mathbb{N}$ is *b-recognizable* if $\mathrm{rep}_b(X)$ is regular.

It is equivalent to work with $0^* \mathrm{rep}_b(X)$.

Theorem

Let $b \in \mathbb{N}_{\geq 2}$ and $m \in \mathbb{N}$. If $X \subseteq \mathbb{N}$ is b-recognizable, so is mX.

Theorem [Alexeev, 2004]

The state complexity of the language $0^* \operatorname{rep}_b(m\mathbb{N})$ is

$$\min_{N\geq 0}\left\{\frac{m}{\gcd(m,b^N)}+\sum_{n=0}^{N-1}\frac{b^n}{\gcd(b^n,m)}\right\}$$

Thue-Morse

0

Thue-Morse

0

1

Thue-Morse

01

1

Thue-Morse

01

10

Thue-Morse

0110

10

Thue-Morse

0110

1001

Thue-Morse

01101001

1001

Thue-Morse

01101001

10010110

Thue-Morse

0110100110010110

10010110

Thue-Morse

$$0110100110010110\dots$$

Thue-Morse

$0110100110010110\ldots$

Definition

The Thue-Morse set is the set

$$\mathscr{T} = \{n \in \mathbb{N} : |\operatorname{rep}_2(n)|_1 \in 2\,\mathbb{N}\}.$$

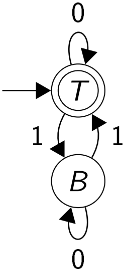
Thue-Morse

0110100110010110...

Definition

The Thue-Morse set is the set

$$\mathscr{T} = \{n \in \mathbb{N} : |\operatorname{rep}_2(n)|_1 \in 2\,\mathbb{N}\}\,.$$

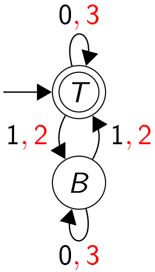


$$0110100110010110\ldots$$

Definition

The Thue-Morse set is the set

$$\mathcal{T} = \{n \in \mathbb{N} : |\operatorname{rep}_2(n)|_1 \in 2\mathbb{N}\}.$$



Definition

Let $p, q \in \mathbb{N}_{\geq 2}$. We say that p and q are *multiplicatively independent* if

$$p^a = q^b \Rightarrow a = b = 0.$$

They are said *multiplicatively dependent* otherwise.

Definition

Let $p, q \in \mathbb{N}_{\geq 2}$. We say that p and q are *multiplicatively independent* if

$$p^a = q^b \Rightarrow a = b = 0.$$

They are said *multiplicatively dependent* otherwise.

Theorem [Cobham, 1969]

- Let b, b' two multiplicatively independent bases. A subset of $\mathbb{N}$ is both b-recognizable and b'-recognizable iff it is a finite union of arithmetic progressions.
- Let b, b' two multiplicatively dependent bases. A subset of $\mathbb{N}$ is b-recognizable iff it is b'-recognizable.

The result

$$\mathscr{T} = \{n \in \mathbb{N} : |\operatorname{rep}_2(n)|_1 \in 2\,\mathbb{N}\}$$

Theorem

Let $m \in \mathbb{N}$ and $p \in \mathbb{N}_{\geq 1}$.
Then the state complexity of the language $0^* \operatorname{rep}_{2^p}(m\mathscr{T})$ is

$$2k + \left\lceil \frac{z}{p} \right\rceil$$

if $m = k2^z$ with k odd.

The method

Automaton	Language accepted
$\mathscr{A}_{\mathscr{T},2^p}$	$(0,0)^* \operatorname{rep}_{2^p}(\mathscr{T} \times \mathbb{N})$

The method

Automaton	Language accepted
$\mathscr{A}_{\mathscr{T},2^p}$	$(0,0)^* \operatorname{rep}_{2^p}(\mathscr{T} \times \mathbb{N})$
$\mathscr{A}_{m,2^p}$	$(0,0)^* \operatorname{rep}_{2^p}(\{(n, mn) : n \in \mathbb{N}\})$

The method

Automaton	Language accepted
$\mathscr{A}_{\mathscr{T},2^p}$	$(0,0)^* \operatorname{rep}_{2^p}(\mathscr{T} \times \mathbb{N})$
$\mathscr{A}_{m,2^p}$	$(0,0)^* \operatorname{rep}_{2^p}(\{(n, mn) : n \in \mathbb{N}\})$
$\mathscr{A}_{\mathscr{T},2^p} \times \mathscr{A}_{m,2^p}$	$(0,0)^* \operatorname{rep}_{2^p}(\{(t, mt) : t \in \mathscr{T}\})$

The method

Automaton	Language accepted
$\mathscr{A}_{\mathscr{T},2^p}$	$(0,0)^* \operatorname{rep}_{2^p}(\mathscr{T} \times \mathbb{N})$
$\mathscr{A}_{m,2^p}$	$(0,0)^* \operatorname{rep}_{2^p}(\{(n, mn) : n \in \mathbb{N}\})$
$\mathscr{A}_{\mathscr{T},2^p} \times \mathscr{A}_{m,2^p}$	$(0,0)^* \operatorname{rep}_{2^p}(\{(t, mt) : t \in \mathscr{T}\})$
$\pi\left(\mathscr{A}_{\mathscr{T},2^p} \times \mathscr{A}_{m,2^p}\right)$	$0^* \operatorname{rep}_{2^p}(m\mathscr{T})$

The automaton $\mathscr{A}_{\mathscr{T},2^p}$

$(0,0)^* \{\mathrm{rep}_{2^p}(t,n) : t \in \mathscr{T}, n \in \mathbb{N}\}$

The automaton $\mathscr{A}_{\mathscr{T},2^p}$

$$(0,0)^* \{\mathrm{rep}_{2^p}(t,n) : t \in \mathscr{T}, n \in \mathbb{N}\}$$

States	T, B
Initial state	T
Final states	T
Alphabet	$\{0, \ldots, 2^p - 1\}^2$
Transitions	$\delta_{\mathscr{T},2^p}(X,(a,b)) = \begin{cases} X & \text{if } a \in \mathscr{T} \\ \overline{X} & \text{else.} \end{cases}$

The automaton $\mathscr{A}_{\mathscr{T},2^p}$

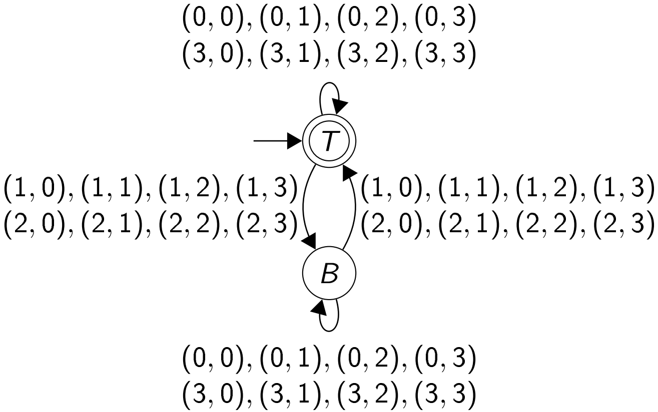
$(0,0)^* \{\text{rep}_{2^p}(t,n) : t \in \mathscr{T}, n \in \mathbb{N}\}$

States	T, B
Initial state	T
Final states	T
Alphabet	$\{0,\dots,2^p-1\}^2$
Transitions	$\delta_{\mathscr{T},2^p}(X,(a,b)) = \begin{cases} X & \text{if } a \in \mathscr{T} \\ \overline{X} & \text{else.} \end{cases}$

For all $u, v \in \{0,\dots,2^p-1\}^*$,

$$\delta_{\mathscr{T},2^p}(X,(u,v)) = \begin{cases} X & \text{if } \text{val}_{2^p}(u) \in \mathscr{T} \\ \overline{X} & \text{else.} \end{cases}$$

The automaton $\mathscr{A}_{\mathscr{T},4}$



The automaton $\mathscr{A}_{m,b}$

$(0,0)^* \{\mathrm{rep}_b(n, mn) : n \in \mathbb{N}\}$

The automaton $\mathscr{A}_{m,b}$

$(0,0)^* \{\text{rep}_b(n, mn) : n \in \mathbb{N}\}$

States	$0, \dots, m-1$
Initial state	0
Final states	0
Alphabet	$\{0, \dots, b-1\}^2$
Transitions	$\delta_{m,b}(i,(d,e)) = j \Leftrightarrow bi + e = md + j$

The automaton $\mathscr{A}_{m,b}$

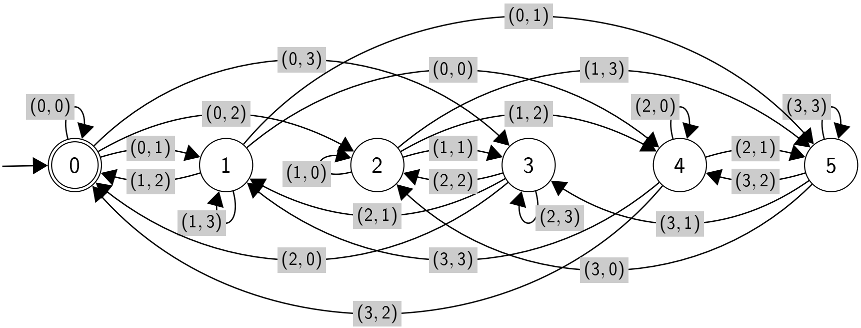
$(0,0)^* \{\text{rep}_b(n, mn) : n \in \mathbb{N}\}$

States	$0, \ldots, m-1$
Initial state	0
Final states	0
Alphabet	$\{0, \ldots, b-1\}^2$
Transitions	$\delta_{m,b}(i,(d,e)) = j \Leftrightarrow bi + e = md + j$

For all $i, j \in \{0, \ldots, m-1\}$, for all $u, v \in \{0, \ldots, b-1\}^*$,

$$\delta_{m,b}(i,(u,v)) = j \Leftrightarrow b^{|(u,v)|} i + \text{val}_b(v) = m\,\text{val}_b(u) + j.$$

The automaton $\mathscr{A}_{6,4}$



The product automaton $\mathscr{A}_{m,2^p} \times \mathscr{A}_{\mathscr{T},2^p}$

$(0,0)^* \{\mathrm{rep}_{2^p}(t, mt) : t \in \mathscr{T}\}$

The product automaton $\mathscr{A}_{m,2^p} \times \mathscr{A}_{\mathscr{T},2^p}$

$(0,0)^* \{\mathrm{rep}_{2^p}(t, mt) : t \in \mathscr{T}\}$

States	$(0,T), \ldots, (m-1,T), (0,B), \ldots, (m-1,B)$
Initial state	$(0,T)$
Final states	$(0,T)$
Alphabet	$\{0, \ldots, 2^p - 1\}^2$
Transitions	$\delta_{\mathscr{T},2^p}((i,X),(u,v)) = (j,Y)$ $\Leftrightarrow 2^{p\lvert(u,v)\rvert} i + \mathrm{val}_{2^p}(v) = m\,\mathrm{val}_{2^p}(u) + j$ and $Y = \begin{cases} X \text{ if } \mathrm{val}_{2^p}(u) \in \mathscr{T} \\ \overline{X} \text{ else.} \end{cases}$

The product automaton $\mathscr{A}_{m,2^p} \times \mathscr{A}_{\mathscr{T},2^p}$

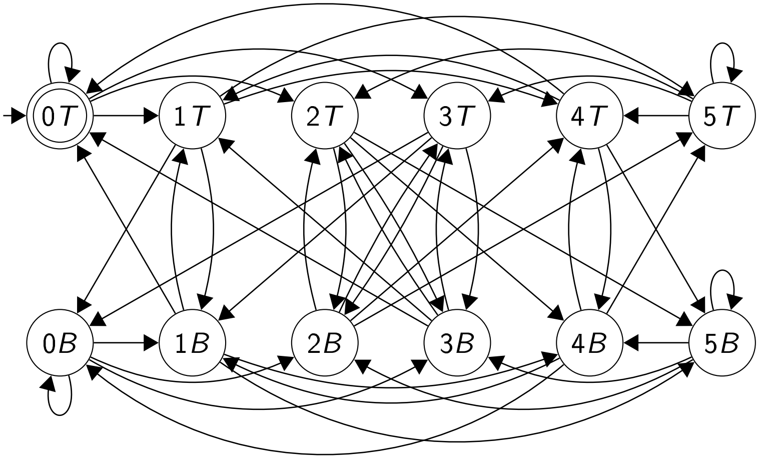
$(0,0)^* \{\mathrm{rep}_{2^p}(t, mt) : t \in \mathscr{T}\}$

States	$(0,T),\dots,(m-1,T),(0,B),\dots,(m-1,B)$
Initial state	$(0,T)$
Final states	$(0,T)$
Alphabet	$\{0,\dots,2^p-1\}^2$
Transitions	$\delta_{\mathscr{T},2^p}((i,X),(u,v)) = (j,Y)$ $\Leftrightarrow 2^{p\lvert(u,v)\rvert} i + \mathrm{val}_{2^p}(v) = m\,\mathrm{val}_{2^p}(u) + j$ and $Y = \begin{cases} X & \text{if } \mathrm{val}_{2^p}(u) \in \mathscr{T} \\ \overline{X} & \text{else.} \end{cases}$

Remark

If i, X, v are fixed, there exist unique j, Y, u such that we have a transition labeled by (u,v) from (i,X) to (j,Y).

The automaton $\mathscr{A}_{6,4} \times \mathscr{A}_{\mathscr{T},4}$

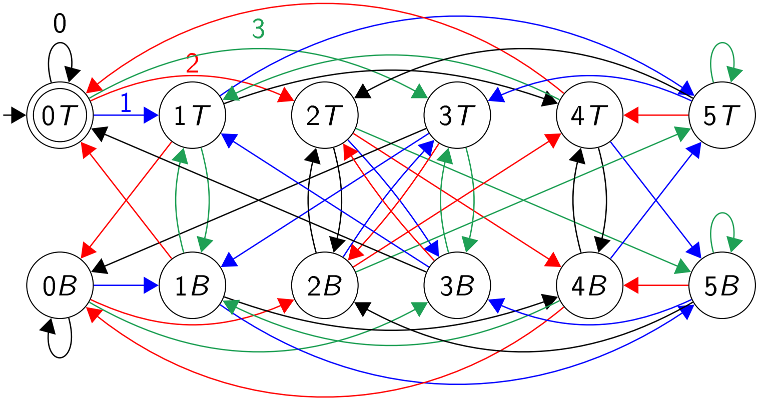


The projected automaton $\pi\left(\mathscr{A}_{m,2^p} \times \mathscr{A}_{\mathscr{T},2^p}\right)$

$$0^* \operatorname{rep}_{2^p}(m\mathscr{T}) = 0^* \{\operatorname{rep}_{2^p}(mt) : t \in \mathscr{T}\}$$

The projected automaton $\pi\left(\mathscr{A}_{m,2^p} \times \mathscr{A}_{\mathscr{T},2^p}\right)$

$$0^* \operatorname{rep}_{2^p}(m\mathscr{T}) = 0^* \{\operatorname{rep}_{2^p}(mt) : t \in \mathscr{T}\}$$



Proposition

The automaton $\pi\left(\mathscr{A}_{m,2^p} \times \mathscr{A}_{\mathscr{T},2^p}\right)$ is

- deterministic,
- accessible,
- coaccessible.

Proposition

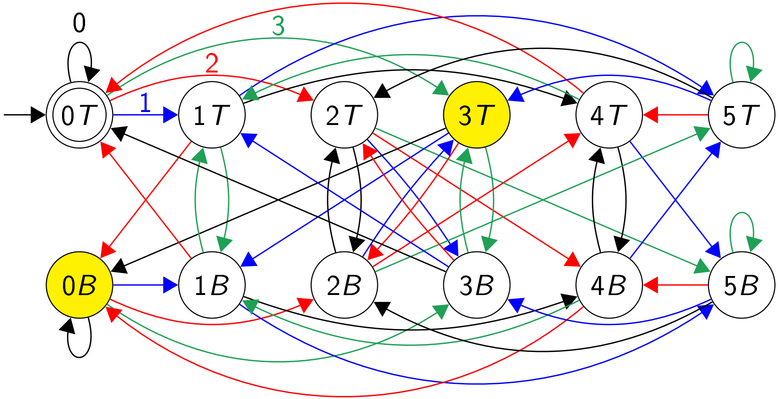
The automaton $\pi\left(\mathscr{A}_{m,2^p} \times \mathscr{A}_{\mathscr{T},2^p}\right)$ is

- deterministic,
- accessible,
- coaccessible.

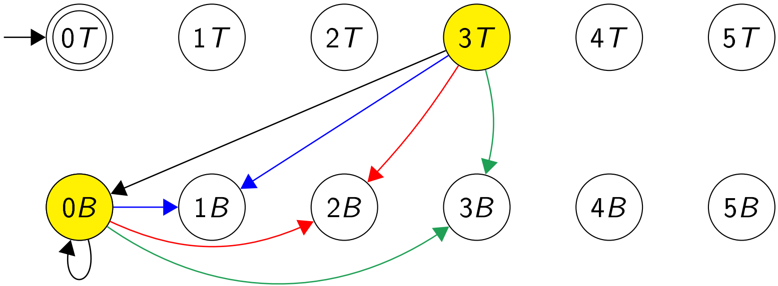
Proposition

In the automaton $\pi\left(\mathscr{A}_{m,2^p} \times \mathscr{A}_{\mathscr{T},2^p}\right)$, the states (i, T) and (i, B) are disjoined for all $i \in \{0, \ldots, m-1\}$.

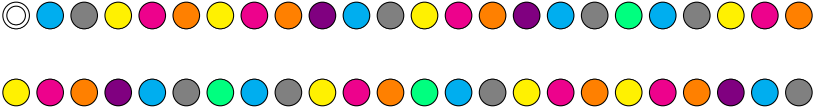
The automaton $\pi(\mathscr{A}_{6,4} \times \mathscr{A}_{\mathscr{T},4})$



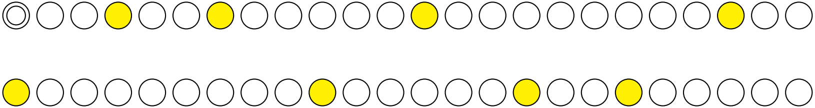
The automaton $\pi(\mathscr{A}_{6,4} \times \mathscr{A}_{\mathscr{T},4})$



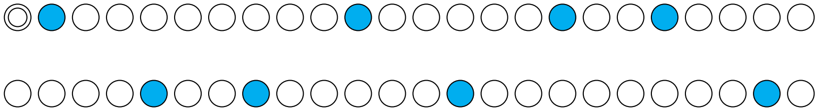
The automaton $\pi(\mathscr{A}_{24,4} \times \mathscr{A}_{\mathscr{T},4})$



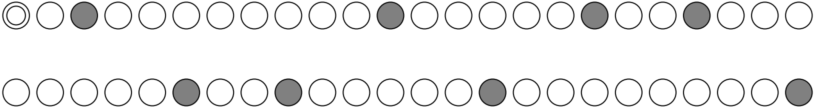
The automaton $\pi(\mathscr{A}_{24,4} \times \mathscr{A}_{\mathscr{T},4})$



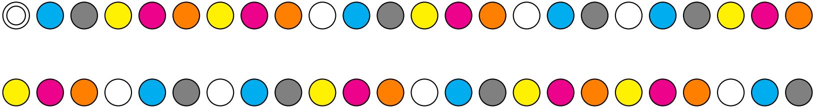
The automaton $\pi(\mathscr{A}_{24,4} \times \mathscr{A}_{\mathscr{T},4})$



The automaton $\pi(\mathscr{A}_{24,4} \times \mathscr{A}_{\mathscr{T},4})$



The automaton $\pi(\mathscr{A}_{24,4} \times \mathscr{A}_{\mathscr{T},4})$



Recall

For $m \in \mathbb{N}, p \in \mathbb{N}_{\geq 1}$, we write $m = k2^z$ with z odd.

Recall

For $m \in \mathbb{N}, p \in \mathbb{N}_{\geq 1}$, we write $m = k2^z$ with z odd.

For all $n \in \mathbb{N}$, we set

$$T_n := \begin{cases} T \text{ if } n \in \mathscr{T} \\ B \text{ else.} \end{cases}$$

Recall

For $m \in \mathbb{N}, p \in \mathbb{N}_{\geq 1}$, we write $m = k2^z$ with z odd.

For all $n \in \mathbb{N}$, we set

$$T_n := \begin{cases} T \text{ if } n \in \mathscr{T} \\ B \text{ else.} \end{cases}$$

Definition

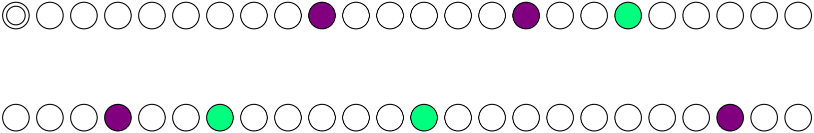
For all $j \in \{1, \dots, k-1\}$, we set

$$[(j, T)] := \{(j + k\ell, T_\ell) : 0 \leq \ell \leq 2^z - 1\}$$
$$[(j, B)] := \left\{\left(j + k\ell, \overline{T_\ell}\right) : 0 \leq \ell \leq 2^z - 1\right\}.$$

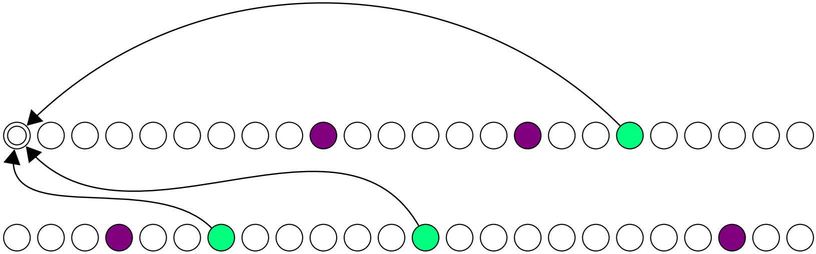
We also set

$$[(0, T)] := \{(0, T)\} \text{ and } [(0, B)] := \left\{\left(k\ell, \overline{T_\ell} : 0 \leq \ell \leq 2^z - 1\right)\right\}.$$

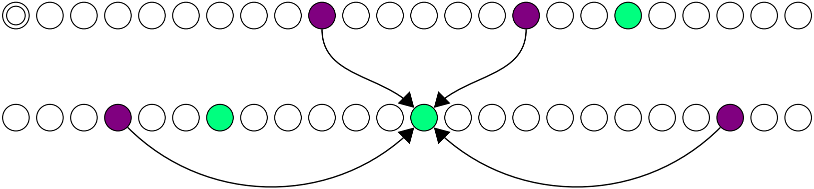
The automaton $\pi(\mathscr{A}_{24,4} \times \mathscr{A}_{\mathscr{T},4})$



The automaton $\pi(\mathscr{A}_{24,4} \times \mathscr{A}_{\mathscr{T},4})$



The automaton $\pi(\mathscr{A}_{24,4} \times \mathscr{A}_{\mathscr{T},4})$



Definition

For all $\alpha \in \{0, \ldots, z-1\}$, we set

$$C_\alpha := \left\{ \left(k2^{z-\alpha-1} + k2^{z-\alpha}\ell, \overline{T_\ell}\right) : 0 \leq \ell \leq 2^\alpha - 1 \right\}.$$

For all $\beta \in \left\{0, \ldots, \left\lceil \frac{z}{p} \right\rceil - 2\right\}$, we set

$$\Gamma_\beta := \bigcup_{\alpha \in \{\beta p, \ldots, (\beta+1)p-1\}} C_\alpha.$$

We also set

$$\Gamma_{\left\lceil \frac{z}{p} \right\rceil - 1} := \bigcup_{\alpha \in \left\{\left(\left\lceil \frac{z}{p} \right\rceil - 1\right)p, \ldots, z-1\right\}} C_\alpha.$$

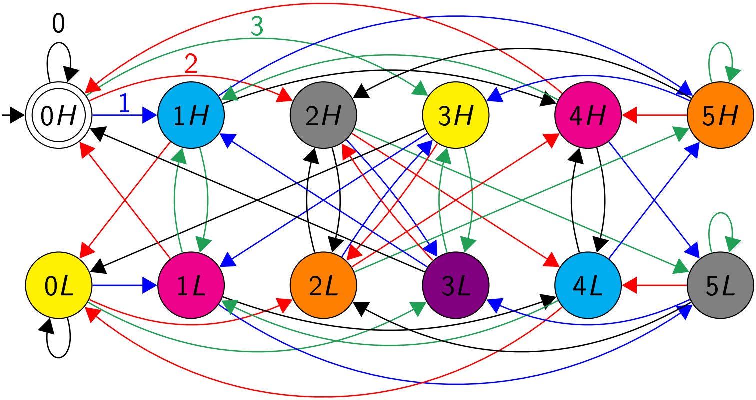
We can build a new automaton

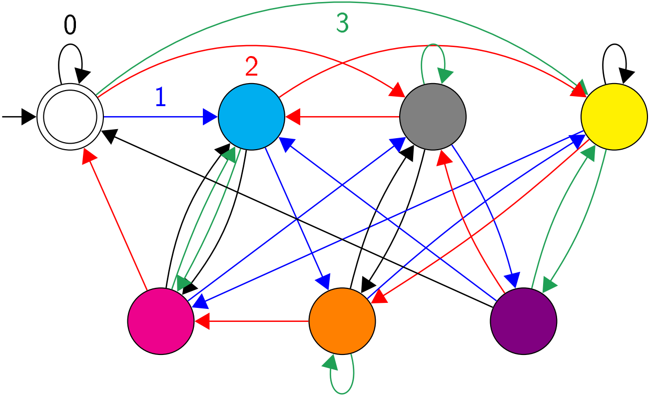
We can build a new automaton which is

- accessible

We can build a new automaton which is

- accessible
- reduced.


0
3
2
1
0H
1H
2H
3H
4H
5H
0L
1L
2L
3L
4L
5L


0
3
2
1

Theorem

Let $m \in \mathbb{N}$ and $p \in \mathbb{N}_{\geq 1}$. Then the state complexity of the language $0^* \operatorname{rep}_{2^p}(m\mathscr{T})$ is equal to

$$2k + \left\lceil \frac{z}{p} \right\rceil$$

if $m = k2^z$ with k odd.

Theorem

Let $m \in \mathbb{N}$ and $p \in \mathbb{N}_{\geq 1}$. Then the state complexity of the language $0^* \operatorname{rep}_{2^p}(m\mathscr{T})$ is equal to

$$2k + \left\lceil \frac{z}{p} \right\rceil$$

if $m = k2^z$ with k odd.

$2 \times 3 + \left\lceil \frac{1}{2} \right\rceil = 7$

Corollary

Given any 2^p-recognizable set Y (via a finite automaton $\mathscr{A}$ recognizing it), it is decidable whether $Y = m\mathscr{T}$ for some $m \in \mathbb{N}$. The decision procedure can be run in time $O(N^2)$ where N is the number of states of the given automaton $\mathscr{A}$.

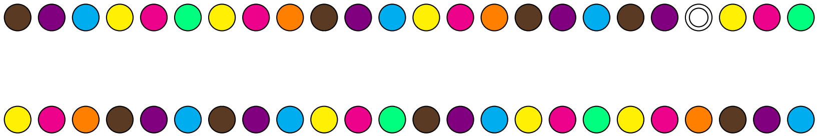
To go further

What about the language

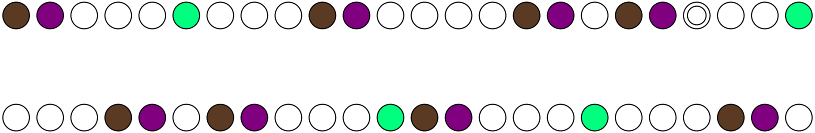
$$0^* \operatorname{rep}_{2^p}(m\mathscr{T} + r)$$

where $r \in \{0, \ldots, m-1\}$?

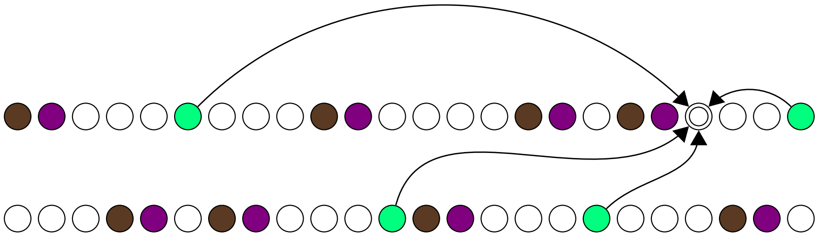
The automaton $\pi\left(\mathscr{A}_{24,4}^{20} \times \mathscr{A}_{\mathscr{T},4}\right)$



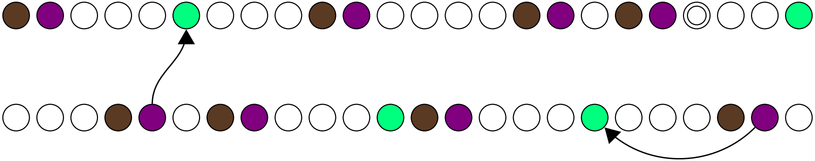
The automaton $\pi\left(\mathscr{A}_{24,4}^{20} \times \mathscr{A}_{\mathscr{T},4}\right)$



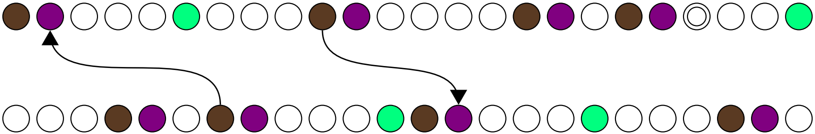
The automaton $\pi\left(\mathscr{A}^{20}_{24,4} \times \mathscr{A}_{\mathscr{T},4}\right)$



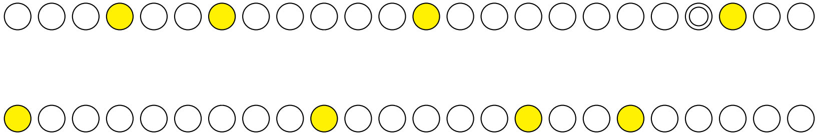
The automaton $\pi\left(\mathscr{A}_{24,4}^{20} \times \mathscr{A}_{\mathscr{T},4}\right)$



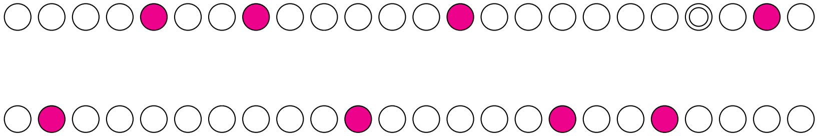
The automaton $\pi\left(\mathscr{A}_{24,4}^{20} \times \mathscr{A}_{\mathscr{T},4}\right)$



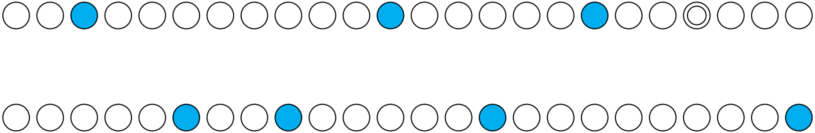
The automaton $\pi\left(\mathscr{A}_{24,4}^{20} \times \mathscr{A}_{\mathscr{T},4}\right)$



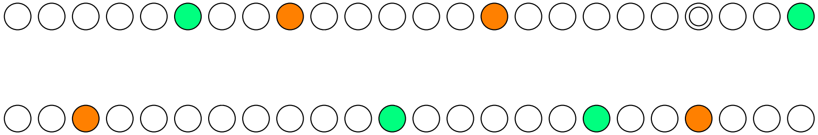
The automaton $\pi\left(\mathscr{A}_{24,4}^{20} \times \mathscr{A}_{\mathscr{T},4}\right)$



The automaton $\pi\left(\mathscr{A}_{24,4}^{20} \times \mathscr{A}_{\mathscr{T},4}\right)$



The automaton $\pi\left(\mathscr{A}_{24,4}^{20} \times \mathscr{A}_{\mathscr{T},4}\right)$



Definition

For all $0 \leq \alpha \leq \max\left\{\left\lceil \frac{z}{p} \right\rceil, |\operatorname{rep}_{2^p}(r)|\right\} =: L$,

$$R'_\alpha = \begin{cases} \left\{\left(\left\lfloor \frac{r}{2^{\alpha p}} \right\rfloor + \ell k 2^{z-\alpha p}, X_\ell\right) : 0 \leq \ell \leq 2^{\alpha p} - 1\right\} & \text{if } \alpha \leq \left\lfloor \frac{z}{p} \right\rfloor \\ \left\{\left(\left\lfloor \frac{r}{2^{\alpha p}} \right\rfloor + \ell k, X_\ell\right) : 0 \leq \ell \leq 2^{z} - 1\right\} & \text{else.} \end{cases}$$

and $R_\alpha = R'_\alpha \setminus \bigcup_{i=0}^{\alpha-1} R'_i$.

Definition

For all $0 \leq \alpha \leq \max\left\{\left\lceil \frac{z}{p} \right\rceil, |\operatorname{rep}_{2^p}(r)|\right\} =: L$,

$$R'_\alpha = \begin{cases} \left\{\left(\left\lfloor \frac{r}{2^{\alpha p}} \right\rfloor + \ell k 2^{z-\alpha p}, X_\ell\right) : 0 \leq \ell \leq 2^{\alpha p} - 1\right\} & \text{if } \alpha \leq \left\lfloor \frac{z}{p} \right\rfloor \\ \left\{\left(\left\lfloor \frac{r}{2^{\alpha p}} \right\rfloor + \ell k, X_\ell\right) : 0 \leq \ell \leq 2^{z} - 1\right\} & \text{else.} \end{cases}$$

and $R_\alpha = R'_\alpha \setminus \bigcup_{i=0}^{\alpha-1} R'_i$.

For all $1 \leq j \leq k-1$ and $Y \in \{T, B\}$ and for $j = 0$ and $Y = \overline{X}$,

$$S_j^Y = [(j, Y)] \setminus \bigcup_{\alpha=0}^{L} R_\alpha.$$

Theorem

Let $m \in \mathbb{N}, r \in \{0, \ldots, m-1\}$ and $p \in \mathbb{N}_{\geq 1}$. Let $X = \mathscr{T}$ or $\mathbb{N} \setminus \mathscr{T}$. Then, the state complexity of the language $0^* \operatorname{rep}_{2^p}(mX + r)$ is equal to

$$2k + \left\lceil \frac{z}{p} \right\rceil$$

if $m = k2^z$ with k odd.