Computing the k-binomial complexity of the Thue–Morse word

August 09, 2019

Marie Lejeune (FNRS grantee)
Joint work with Julien Leroy and Michel Rigo

Let's look at the Thue–Morse word

$$\mathbf{t} = 01101001100101101001011001101001\cdots$$

and more precisely at its factors of a given length:

k-binomial complexity of Thue–Morse

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$$\mathsf{Fac}_{\mathbf{t}}(1) = \{0, \ \}.$$

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The factor complexity

$$p_{\mathbf{t}} : n \mapsto \#\mathrm{Fac}_{\mathbf{t}}(n)$$

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is not bounded by a constant while k-binomial complexity

$$\mathbf{b}_{\mathbf{t}}^{(k)} : n \mapsto \#(\mathrm{Fac}_{\mathbf{t}}(n)/\sim_k)$$

is bounded.

k-binomial complexity of Thue–Morse

Factors and subwords

Let $u = u_1 u_2 \cdots u_m$ be a finite or infinite word.

Definition

A **(scattered) subword** of u is a finite subsequence of the sequence $(u_j)_{j=1}^{m}$. A **factor** of u is a contiguous subword.

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Factor complexity

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where $u \sim_{=} v \Leftrightarrow u = v$.

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We can replace $\sim_{=}$ with other equivalence relations.

Other equivalence relations

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We will deal with the last one.

k-binomial equivalence

Definition (Reminder)

Let u and v be two finite words. They are **k-binomially equivalent** if

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$$\binom{u}{a} = 2 = \binom{v}{a}, \binom{u}{b} = 4 = \binom{v}{b}, \binom{u}{aa} = 1 = \binom{v}{aa},$$
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The words $u = bb\color{red}{ab}\color{black}{ab}b$ and $v = b\color{red}{ab}\color{black}{bab}$ are 2-binomially equivalent. Indeed,

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1. For all words u, v and for every nonnegative integer k,

$$u \sim_{k+1} v \Rightarrow u \sim_k v.$$

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   Indeed, the words u and v are 1-abelian equivalent if

$$\binom{u}{a} = |u|_a = |v|_a = \binom{v}{a} \ \forall a \in A.$$

k-binomial complexity

Definition

If $\mathbf{w}$ is an infinite word, we can define the function

$$\mathbf{b}_{\mathbf{w}}^{(k)} : \mathbb{N} \to \mathbb{N} : n \mapsto \#(\mathrm{Fac}_{\mathbf{w}}(n)/\sim_k),$$

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We have an order relation between the different complexity functions:

$$\rho_{\mathbf{w}}^{ab}(n) \leq \mathbf{b}_{\mathbf{w}}^{(k)}(n) \leq \mathbf{b}_{\mathbf{w}}^{(k+1)}(n) \leq p_{\mathbf{w}}(n) \quad \forall n \in \mathbb{N}, k \in \mathbb{N}^+$$

where $\rho_{\mathbf{w}}^{ab}$ is the abelian complexity function of the word $\mathbf{w}$.

A famous word...

Let us define the **Thue–Morse morphism**

$$\varphi : \{0,1\}^* \to \{0,1\}^* : \begin{cases} 0 \mapsto 01; \\ 1 \mapsto 10. \end{cases}$$

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$$\begin{aligned} \varphi(0) &= 01, \\ \varphi^2(0) &= 0110, \\ \varphi^3(0) &= 01101001, \\ &\dots \end{aligned}$$

A famous word...

Let us define the **Thue–Morse morphism**

$$\varphi : \{0,1\}^* \to \{0,1\}^* : \begin{cases} 0 \mapsto 01; \\ 1 \mapsto 10. \end{cases}$$

We have

$$\begin{aligned} \varphi(0) &= 01, \\ \varphi^2(0) &= 0110, \\ \varphi^3(0) &= 01101001, \\ &\dots \end{aligned}$$

We can thus define the **Thue–Morse word** as one of the fixed points of the morphism φ :

$$\mathbf{t} := \varphi^\omega(0) = 0110100110010110\cdots$$

k-binomial complexity of Thue–Morse

About Morse–Hedlund theorem

A lot of properties about factor complexity are known.

Theorem (Morse–Hedlund)

Let $\mathbf{w}$ be an infinite word on an ℓ-letter alphabet. The three following assertions are equivalent.

1. The word $\mathbf{w}$ is ultimately periodic: there exist finite words u and v such that $\mathbf{w} = u \cdot v^\omega$.
2. There exists $n \in \mathbb{N}$ such that $p_{\mathbf{w}}(n) < n + \ell - 1$.
3. The function $p_{\mathbf{w}}$ is bounded by a constant.

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3. The function $p_{\mathbf{w}}$ is bounded by a constant.

Aperiodic words with minimal complexity

A **Sturmian word** is an infinite word having, as factor complexity, $p(n) = n + 1$ for all $n \in \mathbb{N}$.

Let $\mathbf{w}$ be a Sturmian word. We have, for every $n \geq 2$,

$$n < p_{\mathbf{w}}(n) < p_{\mathbf{t}}(n).$$

However, results are quite different when regarding the k-binomial complexity function.

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Theorem (M. Rigo, P. Salimov, 2015)
Let $\mathbf{w}$ be a Sturmian word. We have $\mathbf{b}_{\mathbf{w}}^{(k)}(n) = p_{\mathbf{w}}(n) = n + 1$ for all $k \geq 2$.

Sturmian words vs. Thue–Morse word

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Theorem (M. Rigo, P. Salimov, 2015)
For every $k \geq 1$, there exists a constant $C_k > 0$ such that, for every $n \in \mathbb{N}$,

$$\mathbf{b}_{\mathbf{t}}^{(k)}(n) \leq C_k.$$

The exact value of $\mathbf{b_t}^{(k)}$

Theorem (M. L., J. Leroy, M. Rigo, 2018)
Let k be a positive integer. For every $n \leq 2^k - 1$, we have

$$\mathbf{b_t}^{(k)}(n) = p_{\mathbf{t}}(n),$$

while for every $n \geq 2^k$,

$$\mathbf{b_t}^{(k)}(n) = \begin{cases} 3 \cdot 2^k - 3, & \text{if } n \equiv 0 \pmod{2^k}; \\ 3 \cdot 2^k - 4, & \text{otherwise.} \end{cases}$$

k-binomial complexity of Thue–Morse

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Since $\mathbf{t}$ is the fixed point of φ, we have

$$\mathbf{t} = \varphi(\mathbf{t}) = \varphi^2(\mathbf{t}) = \cdots = \varphi^k(\mathbf{t})$$

for all $k \in \mathbb{N}$.

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Therefore, every factor u of $\mathbf{t}$ can be written in the form

$$p\varphi^k(z)s,$$

where z is also a factor of $\mathbf{t}$ and where p (resp., s) is a proper suffix (resp., prefix) of $\varphi^k(0)$ or $\varphi^k(1)$.

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where z is also a factor of $\mathbf{t}$ and where p (resp., s) is a proper suffix (resp., prefix) of $\varphi^k(0)$ or $\varphi^k(1)$.

The pair (p, s) is called a **factorization of order k** (or k-factorization) of u.

Factorizations: an example

Example

We have

$$\begin{aligned} \mathbf{t} &= \varphi^3(0) \qquad \cdot \varphi^3(1) \qquad \cdot \varphi^3(1) \qquad \cdot \varphi^3(0) \qquad \cdots \\ &= 01101001 \cdot 10010110 \cdot 10010110 \cdot 01101001 \cdots \end{aligned}$$

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Let $u = 010011001011010$.

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Let $u = 0100110010110 10$.

A factorization of order 3 of u is

$$(01001, 10).$$

Application: computing $\mathbf{b}_\mathbf{t}^{(1)}$

For $k = 1$, we have $\mathbf{b}_\mathbf{t}^{(1)}(0) = 1$, $\mathbf{b}_\mathbf{t}^{(1)}(1) = 2$ and, for $n \geq 2$,

$$\mathbf{b}_\mathbf{t}^{(1)}(n) = \begin{cases} 3, & \text{if } n \equiv 0 \pmod 2; \\ 2, & \text{otherwise.} \end{cases}$$

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$$\mathbf{b}_{\mathbf{t}}^{(1)}(n) = \begin{cases} 3, & \text{if } n \equiv 0 \pmod 2; \\ 2, & \text{otherwise.} \end{cases}$$

Let us fix n and let u be a factor of length n of $\mathbf{t}$. There exists words p, z, s such that

$$u = p\varphi(z)s,$$

where $p, s \in \{\varepsilon, 0, 1\}$, $\varphi(z) \in \{01, 10\}^*$ and $|p| + 2|z| + |s| = n$.

Application: computing $\mathbf{b}_{\mathbf{t}}^{(1)}$ (continued)

Let $n = 2\ell + 1$ be an odd integer. Every factor of length n of $\mathbf{t}$ can be written

$$\varepsilon\varphi(z)0, \;\; \varepsilon\varphi(z)1, \;\; 0\varphi(z)\varepsilon \;\text{ or }\; 1\varphi(z)\varepsilon,$$

where $|z| = \ell$ and $\varphi(z) \in \{01, 10\}^{\ell}$.

Application: computing $\mathbf{b}_{\mathbf{t}}^{(1)}$ (continued)

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$$|\varepsilon\varphi(z)0|_0 = \ell + 1 = |0\varphi(z)\varepsilon|_0$$

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Since $u \sim_1 v$ iff $|u| = |v|$ and $|u|_0 = |v|_0$, we have

$$\mathbf{b}_{\mathbf{t}}^{(1)}(n) = 2$$

if n is odd.

Application: computing $\mathbf{b}_{\mathbf{t}}^{(1)}$ (continued)

Let now $n = 2\ell$ be an even integer. Every factor of length n of $\mathbf{t}$ can be written

$$\varepsilon\varphi(z)\varepsilon,\ \ 0\varphi(z')0,\ \ 0\varphi(z')1\ \ 1\varphi(z')0,\ \ \text{or}\ \ 1\varphi(z')1,$$

where $|z| = \ell, |z'| = \ell - 1$ and $\varphi(z) \in \{01, 10\}^{\ell}, \varphi(z') \in \{01, 10\}^{\ell-1}$.

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Therefore, if n is even, $\mathbf{b}_{\mathbf{t}}^{(1)}(n) = 3$.

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Therefore, if n is even, $\mathbf{b}_{\mathbf{t}}^{(1)}(n) = 3$.

Case where $k = 2$ can also be computed by hand. Let thus assume that $k \geq 3$.

Unicity of the factorization?

Is the factorization $p\varphi^k(z)s$ of a word in $\mathrm{Fac}_{\mathbf{t}}$ unique?

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Is the factorization $p\varphi^k(z)s$ of a word in $\mathrm{Fac}_{\mathbf{t}}$ unique?

No: the word 010 appears as a factor of $\mathbf{t}$ several times; it can be factorized as $0\varphi(1)$ or as $\varphi(0)0$.

$$\mathbf{t} = 01 \cdot 10 \cdot 10 \cdot 01 \cdot 10 \cdot 01 \cdot 01 \cdot 10 \cdots$$

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Proposition

Let $k \geq 3$ and let u be a factor of $\mathbf{t}$ of length at least $2^k - 1$.

- If u is a factor of $\varphi^{k-1}(010)$ or $\varphi^{k-1}(101)$
  - it has exactly two factorizations (p, s) and (p', s');
  - we have $||p| - |p'|| = ||s| - |s'|| = 2^{k-1}$.

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  - it has exactly two factorizations (p, s) and (p', s');
  - we have $||p| - |p'|| = ||s| - |s'|| = 2^{k-1}$.
- If u is not a factor of $\varphi^{k-1}(010)$ or $\varphi^{k-1}(101)$
  - it has a unique factorization.

Relation between factorizations of a word: an example

Example

Let us consider the factor $u = 01001011$, which is a subword of $\varphi^2(010)$.

$$\mathbf{t} = \varphi^3(\mathbf{t}) = 01101001 \cdot 1001011\textcolor{red}{0 \cdot 1001011}0 \cdot 01101001 \cdot$$
$$10010110 \cdot 011\textcolor{red}{01001 \cdot 011}01001 \cdots$$

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Hence, $(0, 1001011)$ and $(01001, 011)$ are the two 3-factorizations of u.

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$$(0, 1001011) = (0, \varphi^2(1)011)$$

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How can we deal with factors having two factorizations? Which one to choose?

k-binomial complexity of Thue–Morse

Dealing with two factorizations

Equivalence $\equiv_k$

Let (p_1, s_1), $(p_2, s_2) \in A^{<2^k} \times A^{<2^k}$. These two are equivalent for $\equiv_k$ if there exist $a \in A$, $x, y \in A^*$ such that one of these cases occurs:

1. $|p_1| + |s_1| = |p_2| + |s_2|$ and
   1. $(p_1, s_1) = (p_2, s_2)$;
   2. $(p_1, s_1) = (x\varphi^{k-1}(a), y)$ and $(p_2, s_2) = (x, \varphi^{k-1}(a)y)$;
   3. $(p_1, s_1) = (x, \varphi^{k-1}(a)y)$ and $(p_2, s_2) = (x\varphi^{k-1}(a), y)$;
   4. $(p_1, s_1) = (\varphi^{k-1}(a), \varphi^{k-1}(\bar{a}))$ and $(p_2, s_2) = (\varphi^{k-1}(\bar{a}), \varphi^{k-1}(a))$;
2. $\big|(|p_1| + |s_1|) - (|p_2| + |s_2|)\big| = 2^k$ and
   1. $(p_1, s_1) = (x, y)$ and $(p_2, s_2) = (x\varphi^{k-1}(a), \varphi^{k-1}(\bar{a})y)$;
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2. $\big|(|p_1| + |s_1|) - (|p_2| + |s_2|)\big| = 2^k$ and
   1. $(p_1, s_1) = (x, y)$ and $(p_2, s_2) = (x\varphi^{k-1}(a), \varphi^{k-1}(\bar{a})y)$;
   2. $(p_1, s_1) = (x\varphi^{k-1}(a), \varphi^{k-1}(\bar{a})y)$ and $(p_2, s_2) = (x, y)$.

Example (continuing)

The word $u = 01001011$ has the two 3-factorizations $(0, \varphi^2(1)011)$ and $(0\varphi^2(1), 011)$. This corresponds to case (1.3), where $x = 0$, $y = 011$.

Link between $\sim_k$ and $\equiv_k$

Proposition

If a word $u \in A^{\geq 2^k - 1}$ has two k-factorizations (p_1, s_1) and (p_2, s_2), then these two are equivalent for $\equiv_k$.

The equivalence class of the k-factorizations of u is called its **type of order k**.

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Theorem

Let u and v be two factors of $\mathbf{t}$ of length $n \geq 2^k - 1$. We have

$$u \sim_k v \Leftrightarrow (p_u, s_u) \equiv_k (p_v, s_v).$$

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Therefore, if $k \geq 3$ and $n \geq 2^k$, we have

$$\mathbf{b}_{\mathbf{t}}^{(k)}(n) = \#(\mathrm{Fac}_n(\mathbf{t})/\sim_k) = \#(\{(p_u, s_u) \,:\, u \in \mathrm{Fac}_n(\mathbf{t})\}/\equiv_k).$$

Computing this quantity

Let $n \geq 2^k$ and for all $\ell \in \{0, \dots, 2^{k-1} - 1\}$, define

$$P_\ell = \{(p_u, s_u) : u \in \mathrm{Fac}_n(\mathbf{t}), |p_u| = \ell \text{ or } |p_u| = 2^{k-1} + \ell\}.$$

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Example

Let $n = 15$, $k = 3$ and $\ell = 2$. We have

$$P_2 = \{(01, 10110), (01, 01001), (10, 10110), (10, 01001), \\ (101001, 0), (101001, 1), (010110, 0), (010110, 1)\}.$$

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Hence,

$$\{(p_u, s_u) \ : \ u \in \mathrm{Fac}_n(\mathbf{t})\} = \bigcup_{\ell=0}^{2^{k-1}-1} P_\ell.$$

Computing this quantity (continued)

If $(p_u, s_u) \in P_\ell$ and $(p_v, s_v) \in P_{\ell'}$ with $\ell \neq \ell'$, we know that

$$(p_u, s_u) \not\equiv_k (p_v, s_v).$$

Computing this quantity (continued)

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Therefore,

$$\mathbf{b}_{\mathbf{t}}^{(k)}(n) = \# \left(\left(\bigcup_{\ell=0}^{2^{k-1}-1} P_\ell \right) \Big/ \equiv_k \right) = \sum_{\ell=0}^{2^{k-1}-1} \#(P_\ell / \equiv_k).$$

Computing this quantity (continued)

There exists ℓ_0 such that

$$P_{\ell_0} = \{(p_u, s_u) : u \in \mathrm{Fac}_n(\mathbf{t}), |s_u| = 0 \text{ or } |s_u| = 2^{k-1}\}.$$

Computing this quantity (continued)

There exists ℓ_0 such that

$$P_{\ell_0} = \{(p_u, s_u) : u \in \mathsf{Fac}_n(\mathbf{t}), |s_u| = 0 \text{ or } |s_u| = 2^{k-1}\}.$$

Example

Let $n = 10$, $k = 3$ and $\ell = 2$. We have $\ell_0 = 2$ because

$$\begin{aligned} P_2 = \{&(01, \varepsilon), (10, \varepsilon), \\ &(101001, 0110), (101001, 1001), (010110, 0110), (010110, 1001)\}. \end{aligned}$$

Computing this quantity (continued)

There exists ℓ_0 such that

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$$\begin{aligned} P_2 = \{&(01, \varepsilon), (10, \varepsilon), \\ &(101001, 0110), (101001, 1001), (010110, 0110), (010110, 1001)\}. \end{aligned}$$

Denote by λ the quantity $n \bmod 2^k$. We have

$$0 = \ell_0 \Leftrightarrow \lambda = 0 \text{ or } \lambda = 2^{k-1}.$$

Computing this quantity (continued)

Moreover, we can show that

$$\#\left((P_0 \cup P_{\ell_0})/\equiv_k\right) = \begin{cases} 3, & \text{if } \lambda = 0; \\ 2, & \text{if } \lambda = 2^{k-1}; \\ 8, & \text{otherwise;} \end{cases}$$

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Hence, putting all the information together,

$$\begin{aligned} \#\left(\{(p_u, s_u) : u \in \mathrm{Fac}_n(\mathbf{t})\}/\equiv_k\right) &= \sum_{\ell=0}^{2^{k-1}-1} \#\left(P_\ell/\equiv_k\right) \\ &= \begin{cases} 3 \cdot 2^k - 3, & \text{if } \lambda = 0; \\ 3 \cdot 2^k - 4, & \text{otherwise.} \end{cases} \end{aligned}$$

It is also possible to show that, if u, v are two different factors of $\mathbf{t}$ of length less than 2^k, then $u \not\sim_k v$.
Therefore, $\mathbf{b}_{\mathbf{t}}^{(k)}(n) = p_{\mathbf{t}}(n)$ for all $n \leq 2^k - 1$.

Conclusion

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Finally, we obtain $\mathbf{b}_{\mathbf{t}}^{(k)}(n) = p_{\mathbf{t}}(n)$ for all $n \leq 2^k - 1$ and, for all $n \geq 2^k$,

$$\mathbf{b}_{\mathbf{t}}^{(k)}(n) = \begin{cases} 3 \cdot 2^k - 3, & \text{if } n \equiv 0 \pmod{2^k}; \\ 3 \cdot 2^k - 4, & \text{otherwise.} \end{cases}$$

To end with an open question...

Theorem (M. Rigo, P. Salimov, 2015)

For every $k \geq 1$ and for every fixed point of a Parikh-constant morphism $\mathbf{w}$, there exists a constant $C_{\mathbf{w},k} > 0$ such that, for every $n \in \mathbb{N}$,

$$\mathbf{b}_{\mathbf{w}}^{(k)}(n) \leq C_{\mathbf{w},k}.$$

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A **Parikh-constant morphism** is a morphism for which the images of all letters are equal up to a permutation.

Example

The morphism $\sigma : \{0,1,2\}^* \to \{0,1,2\}^* : \begin{cases} 0 & \mapsto & 0112 \\ 1 & \mapsto & 1021 \\ 2 & \mapsto & 2011 \end{cases}$ is

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Does it exist such an aperiodic word $\mathbf{w}$ such that $\mathbf{b}_{\mathbf{w}}^{(k)}(n) < \mathbf{b}_{\mathbf{t}}^{(k)}(n)$ for all large enough n?

Thank you!