# Computing the $k$-binomial complexity of the Thue–Morse word

July 08, 2019

Marie Lejeune (FNRS grantee)

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$$p_{\mathbf{t}} : n \mapsto \#\mathsf{Fac}_{\mathbf{t}}(n)$$

is not bounded by a constant while $k$-binomial complexity

$$\mathbf{b}_{\mathbf{t}}^{(k)} : n \mapsto \#(\mathsf{Fac}_{\mathbf{t}}(n)/\sim_k)$$

is bounded.

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# Factors and subwords

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We will deal with the last one.

# $k$-binomial equivalence

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Indeed, the words $u$ and $v$ are 1-abelian equivalent if

$$\binom{u}{a} = |u|_a = |v|_a = \binom{v}{a} \ \forall a \in A.$$

# $k$-binomial complexity

**Definition**

If $\mathbf{w}$ is an infinite word, we can define the function

$$\mathbf{b}_{\mathbf{w}}^{(k)} : \mathbb{N} \to \mathbb{N} : n \mapsto \#(\mathrm{Fac}_{\mathbf{w}}(n)/\sim_k),$$

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which is called the **k-binomial complexity** of $\mathbf{w}$.

We have an order relation between the different complexity functions:

$$\rho_{\mathbf{w}}^{ab}(n) \leq \mathbf{b}_{\mathbf{w}}^{(k)}(n) \leq \mathbf{b}_{\mathbf{w}}^{(k+1)}(n) \leq p_{\mathbf{w}}(n) \quad \forall n \in \mathbb{N}, k \in \mathbb{N}^+$$

where $\rho_{\mathbf{w}}^{ab}$ is the abelian complexity function of the word $\mathbf{w}$.

# A famous word...

Let us define the **Thue–Morse morphism**

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We have

$$\begin{aligned} \varphi(0) &= 01, \\ \varphi^2(0) &= 0110, \\ \varphi^3(0) &= 01101001, \\ &\ldots \end{aligned}$$

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We have

$$\begin{aligned} \varphi(0) &= 01, \\ \varphi^2(0) &= 0110, \\ \varphi^3(0) &= 01101001, \\ &\dots \end{aligned}$$

We can thus define the **Thue–Morse word** as one of the fixed points of the morphism $\varphi$ :

$$\mathbf{t} := \varphi^\omega(0) = 0110100110010110 \cdots$$

# $k$-binomial complexity of Thue–Morse

# About Morse–Hedlund theorem

A lot of properties about factor complexity are known.

Theorem (Morse–Hedlund)

Let $\mathbf{w}$ be an infinite word on an $\ell$-letter alphabet. The three following assertions are equivalent.

1. The word $\mathbf{w}$ is ultimately periodic: there exist finite words $u$ and $v$ such that $\mathbf{w} = u \cdot v^\omega$.
2. There exists $n \in \mathbb{N}$ such that $p_{\mathbf{w}}(n) < n + \ell - 1$.
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**Aperiodic words with minimal complexity**

A **Sturmian word** is an infinite word having, as factor complexity, $p(n) = n + 1$ for all $n \in \mathbb{N}$.

# Sturmian words vs. Thue–Morse word

Let $\mathbf{w}$ be a Sturmian word. We have, for every $n \geq 2$,

$$n < p_{\mathbf{w}}(n) < p_{\mathbf{t}}(n).$$

However, results are quite different when regarding the $k$-binomial complexity function.

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Theorem (M. Rigo, P. Salimov, 2015)
For every $k \geq 1$, there exists a constant $C_k > 0$ such that, for every $n \in \mathbb{N}$,

$$\mathbf{b}_{\mathbf{t}}^{(k)}(n) \leq C_k.$$

# The exact value of $\mathbf{b}_{\mathbf{t}}^{(k)}$

Theorem (M. L., J. Leroy, M. Rigo, 2018)

Let $k$ be a positive integer. For every $n \leq 2^k - 1$, we have

$$\mathbf{b}_{\mathbf{t}}^{(k)}(n) = p_{\mathbf{t}}(n),$$

while for every $n \geq 2^k$,

$$\mathbf{b}_{\mathbf{t}}^{(k)}(n) = \begin{cases} 3 \cdot 2^k - 3, & \text{if } n \equiv 0 \pmod{2^k}; \\ 3 \cdot 2^k - 4, & \text{otherwise.} \end{cases}$$

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Cases where $k = 1$ or $k = 2$ can be computed by hand. We will thus assume that $k \geq 3$.

# $k$-binomial complexity of Thue–Morse

# Factorizations

Since $\mathbf{t}$ is the fixed point of $\varphi$, we have

$$\mathbf{t} = \varphi(\mathbf{t}) = \varphi^2(\mathbf{t}) = \cdots = \varphi^k(\mathbf{t})$$

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Therefore, every factor $u$ of $\mathbf{t}$ can be written in the form

$$p\varphi^k(z)s,$$

where $z$ is also a factor of $\mathbf{t}$ and where $p$ (resp., $s$) is a proper suffix (resp., prefix) of $\varphi^k(0)$ or $\varphi^k(1)$.

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where $z$ is also a factor of $\mathbf{t}$ and where $p$ (resp., $s$) is a proper suffix (resp., prefix) of $\varphi^k(0)$ or $\varphi^k(1)$.

The pair $(p, s)$ is called a **factorization of order $k$** (or $k$-factorization) of $u$.

# Factorizations: an example

Example

We have

$$\begin{aligned}\mathbf{t} &= \varphi^3(0) \qquad \cdot \varphi^3(1) \qquad \cdot \varphi^3(1) \qquad \cdot \varphi^3(0) \qquad \cdots \\ &= 01101001 \cdot 10010110 \cdot 10010110 \cdot 01101001 \cdots\end{aligned}$$

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Let $u = {\color{red}010011001011010}$.

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Let $u = 010011001011010$.

A factorization of order 3 of $u$ is

$$(01001, 10).$$

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Is the factorization of a word in $\mathrm{Fac}_{\mathbf{t}}$ unique?

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**Proposition**

Let $u$ be a factor of $\mathbf{t}$ of length at least $2^k - 1$.

- If $u$ is a factor of $\varphi^{k-1}(010)$ or $\varphi^{k-1}(101)$
  - it has exactly two factorizations $(p, s)$ and $(p', s')$;
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  - it has exactly two factorizations $(p, s)$ and $(p', s')$;
  - we have $||p| - |p'|| = ||s| - |s'|| = 2^{k-1}$.
- If $u$ is not a factor of $\varphi^{k-1}(010)$ or $\varphi^{k-1}(101)$
  - it has a unique factorization.

# Relation between factorizations of a word: an example

Example

Let us consider the factor $u = 01001011$.

$$\mathbf{t} = \varphi^3(\mathbf{t}) = 01101001 \cdot 10010110 \cdot 10010110 \cdot 01101001 \cdot 10010110 \cdot 01101001 \cdot 01101001 \cdots$$

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$$(0, 1001011) = (0, \varphi^2(1)011)$$

and

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How can we deal with factors having two factorizations? Which one to choose?

# Dealing with two factorizations

**Equivalence $\equiv_k$**

Let $(p_1, s_1)$, $(p_2, s_2) \in A^{<2^k} \times A^{<2^k}$. These two are equivalent for $\equiv_k$ if there exist $a \in A$, $x, y \in A^*$ such that one of these cases occurs:

1. $|p_1| + |s_1| = |p_2| + |s_2|$ and
   1. $(p_1, s_1) = (p_2, s_2)$;
   2. $(p_1, s_1) = (x\varphi^{k-1}(a), y)$ and $(p_2, s_2) = (x, \varphi^{k-1}(a)y)$;
   3. $(p_1, s_1) = (x, \varphi^{k-1}(a)y)$ and $(p_2, s_2) = (x\varphi^{k-1}(a), y)$;
   4. $(p_1, s_1) = (\varphi^{k-1}(a), \varphi^{k-1}(\overline{a}))$ and $(p_2, s_2) = (\varphi^{k-1}(\overline{a}), \varphi^{k-1}(a))$;
2. $\big|(|p_1| + |s_1|) - (|p_2| + |s_2|)\big| = 2^k$ and
   1. $(p_1, s_1) = (x, y)$ and $(p_2, s_2) = (x\varphi^{k-1}(a), \varphi^{k-1}(\bar{a})y)$;
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**Example (continuing)**

The word $u = 01001011$ has the two factorizations $(0, \varphi^2(1)011)$ and $(0\varphi^2(1), 011)$. This corresponds to case (1.3), where $x = 0$, $y = 011$.

# Link between $\sim_k$ and $\equiv_k$

**Proposition**

If a word $u \in A^{\geq 2^k - 1}$ has two $k$-factorizations $(p_1, s_1)$ and $(p_2, s_2)$, then these two are equivalent for $\equiv_k$.

The equivalence class of the $k$-factorizations of $u$ is called its **type of order $k$**.

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**Theorem**

Let $u$ and $v$ be two factors of $\mathbf{t}$ of length $n \geq 2^k - 1$. We have

$$u \sim_k v \Leftrightarrow (p_u, s_u) \equiv_k (p_v, s_v).$$

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Therefore, if $k \geq 3$ and $n \geq 2^k$, we have

$$\mathbf{b}_{\mathbf{t}}^{(k)}(n) = \#(\mathrm{Fac}_n(\mathbf{t})/\sim_k) = \#(\{(p_u, s_u) \,:\, u \in \mathrm{Fac}_n(\mathbf{t})\}/\equiv_k).$$

# Computing this quantity

Let $n \geq 2^k$ and for all $\ell \in \{0, \ldots, 2^{k-1} - 1\}$, define

$$P_\ell = \{(p_u, s_u) : u \in \mathrm{Fac}_n(\mathbf{t}), |p_u| = \ell \text{ or } |p_u| = 2^{k-1} + \ell\}.$$

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Example

Let $n = 15$, $k = 3$ and $\ell = 2$. We have

$$P_2 = \{(01, 10110), (01, 01001), (10, 10110), (10, 01001), (101001, 0), (101001, 1), (010110, 0), (010110, 1)\}.$$

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Hence,

$$\{(p_u, s_u) \,:\, u \in \mathsf{Fac}_n(\mathbf{t})\} = \bigcup_{\ell=0}^{2^{k-1}-1} P_\ell$$

and

$$\mathbf{b}_{\mathbf{t}}^{(k)}(n) = \sum_{\ell=0}^{2^{k-1}-1} \#(P_\ell/\equiv_k).$$

# Computing this quantity (continued)

There exists $\ell_0$ such that

$$P_{\ell_0} = \{(p_u, s_u) : u \in \mathsf{Fac}_n(\mathbf{t}), |s_u| = 0 \ \text{ or } \ |s_u| = 2^{k-1}\}.$$

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Example

Let $n = 10$, $k = 3$ and $\ell = 2$. We have $\ell_0 = 2$ because

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# Computing this quantity (continued)

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Example

Let $n = 10$, $k = 3$ and $\ell = 2$. We have $\ell_0 = 2$ because

$$\begin{aligned} P_2 = \{&(01, \varepsilon), (10, \varepsilon), \\ &(101001, 0110), (101001, 1001), (010110, 0110), (010110, 1001)\}. \end{aligned}$$

Denote by $\lambda$ the quantity $n \bmod 2^k$. We have

$$\#\{0, \ldots, 2^{k-1} - 1\} \setminus \{0, \ell_0\} = \begin{cases} 2^{k-1} - 1, & \text{if } \lambda = 0 \text{ or } \lambda = 2^{k-1}; \\ 2^{k-1} - 2, & \text{otherwise.} \end{cases}$$

# Computing this quantity (continued)

Moreover, we can show that

$$\#\left((P_0 \cup P_{\ell_0})/\equiv_k\right) = \begin{cases} 3, & \text{if } \lambda = 0; \\ 2, & \text{if } \lambda = 2^{k-1}; \\ 8, & \text{otherwise;} \end{cases}$$

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Moreover, we can show that

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Hence, putting all the information together,

$$\begin{aligned} \#\left(\{(p_u, s_u) : u \in \mathrm{Fac}_n(\mathbf{t})\}/\equiv_k\right) &= \# \bigcup_{\ell=0}^{2^{k-1}-1} P_\ell \\ &= \begin{cases} 3 \cdot 2^k - 3, & \text{if } \lambda = 0; \\ 3 \cdot 2^k - 4, & \text{otherwise.} \end{cases} \end{aligned}$$

# Conclusion

It is also possible to show that, if $u, v$ are two different factors of $\mathbf{t}$ of length less than $2^k$, then $u \not\sim_k v$.
Therefore, $\mathbf{b}_{\mathbf{t}}^{(k)}(n) = p_{\mathbf{t}}(n)$ for all $n \leq 2^k - 1$.

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Finally, we obtain $\mathbf{b}_{\mathbf{t}}^{(k)}(n) = p_{\mathbf{t}}(n)$ for all $n \leq 2^k - 1$ and, for all $n \geq 2^k$,

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To end with an open question...
A **Parikh-constant morphism** is a morphism for which the images of all letters are equal up to a permutation.

Could we generalize the technique to compute the exact value of $\mathbf{b}^{(k)}$ for other fixed points of Parikh-constant morphisms?

*Thank you!*