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ORIGINAL PAPER

Some equivalent definitions of Besov spaces of generalized smoothness

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Abstract
In this paper, we present some alternative definitions of Besov spaces of generalized smoothness, defined via Littlewood–Paley-type decomposition, involving weak derivatives, polynomials, convolutions and generalized interpolation spaces.

KEYWORDS
Besov spaces, finite differences, generalized smoothness, real interpolation, Sobolev spaces

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1 | INTRODUCTION

Let $\mathcal{J}$ be the Bessel operator of order s:

$$\mathcal{J}^s f = \mathcal{F}^{-1}\Big(\big(1+|\cdot|^2\big)^{-s/2}\mathcal{F}f\Big) \qquad (s \in \mathbb{R},\ f \in \mathcal{S}'),$$

where standard notations have been used (see below). The fractional Sobolev space H^s_p (which can be seen as a generalization of the historical Sobolev space W^s_p with $s \in \mathbb{N}_0$) is defined by

$$H^s_p = \{ f \in \mathcal{S}' : \|\mathcal{J}^{-s} f\|_{L^p} < \infty \} \qquad (s \in \mathbb{R},\ 1 \le p \le \infty). \tag{1.1}$$

Such spaces play an important role in many fields of physics; in particular they take a prominent part in the study of the Navier–Stokes equations (see e.g. [14,30,37]). In this context, the usual Besov spaces naturally arise through the (real) interpolation theory: the Besov space $B^s_{p,q}$ is an interpolation space “which lies” in between the Sobolev spaces H^t_p and H^u_p with $s = (1-\alpha)t + \alpha u$. More precisely, we have

$$B^s_{p,q} = \big[H^t_p, H^u_p\big]_{\alpha,q} \tag{1.2}$$

(see e.g. [3,13,37,40]). The Besov spaces $B^s_{p,q}$ (with $s \in \mathbb{R}$ and $1 \le p, q \le \infty$) were introduced about sixty years ago [4,5] and many studies have been since devoted to such spaces (see e.g. [6,7,38–41]). They were generalized in the middle of the seventies by several authors in different contexts starting from different points of view. The variant, we will present here has been largely considered in [2,11,19,20,33] for example. They are still considered nowadays in connection with embeddings, limiting embedding, entropy numbers, probability theory and theory of stochastic processes for example (see e.g. [10,16,17,24,31,35] and references therein). More recently, such generalizations have been used to numerically detect the law of the iterated logarithm in signals [25,29,34].

A classical generalization of the usual Besov spaces was introduced in [12,32]. To obtain these spaces $B^f_{p,q}$, s is replaced by a function parameter g such that $f(x) = x^t/g\left(x^{t-u}\right)$ in the interpolation formula (1.2). These spaces can themselves be generalized using the Littlewood–Paley decomposition instead of the interpolation theory to define the spaces of generalized smoothness $B^{\sigma,\gamma}_{p,q}$, where σ and γ are two admissible sequences [20]. A sequence $\left(\sigma_j\right)_{j\in\mathbb{N}_0}$ of positive numbers is called admissible if both σ_j/σ_{j+1} and σ_{j+1}/σ_j are upper bounded (see Definition 2.1). One has $B^f_{p,q} = B^{\sigma,\gamma}_{p,q}$, with $\gamma = (2^j)_j$ and $\sigma = (f(2^j))_j$. In a way, these spaces provide a very general definition of the spaces of generalized smoothness [33]. This work can be seen as an intent to "close the circle" by defining a generalized interpolation method that allows to define the spaces $B^{\sigma,\gamma}_{p,q}$ starting from the usual Sobolev spaces. This interpolation method is quite different from the one introduced in [12].

In this paper, in the same spirit as in [27,28], we propose equivalent definitions of the spaces of generalized smoothness $B^{\sigma,\gamma}_{p,q}$. The first section is devoted to a brief review of the background material needed. Next, we give some preliminary results before looking at the links between these spaces and the weak derivatives of the elements of the historical Sobolev spaces W^k_p. We also give definitions involving Taylor expansion and polynomials before investigating how the generalized Besov spaces can be characterized in terms of convolution. Finally, we show that these spaces can be introduced using generalized interpolation of fractional or historical Sobolev spaces, as were the spaces $B^f_{p,q}$.

These developments are generalizations of well known results concerning the usual Besov spaces $B^s_{p,q}$ (see e.g. [39]) or the generalized Hölder spaces Λ^σ [27,28]. In a forthcoming work, we will show that these spaces provide a natural framework for a general multifractal formalism.

2 | DEFINITIONS

Let us recall the definitions that will be used in the sequel.

We shall use the standard notations for the functional spaces (see e.g. [39]): $\mathbb{R}^d$ is the d-dimensional Euclidian space and $B(x,r)$ is the open ball of $\mathbb{R}^d$ centered at x with radius $r > 0$. As usual, $\mathcal{S}$ is the Schwartz space of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$ equipped with the usual topology and $\mathcal{S}'$ denotes its topological dual, i.e., the space of tempered distributions on $\mathbb{R}^d$. If $f \in \mathcal{S}'$, then $\mathcal{F}f$ and $\mathcal{F}^{-1}f$ denote its Fourier transform and its inverse Fourier transform, respectively. For a given nonempty open set $\Omega \subset \mathbb{R}^d$ and $p \in [1,\infty]$, $L^p(\Omega)$ is the Lebesgue space of the measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f|^p\,dx\right)^{1/p} < \infty$$

if $p < \infty$ and

$$\|f\|_{L^\infty(\Omega)} := \sup_{x\in\Omega}\operatorname{ess}|f(x)| < \infty,$$

otherwise. One sets $L^p := L^p\left(\mathbb{R}^d\right)$. As usual, $\ell^p(\mathbb{K})$ (where $\mathbb{K}$ is either $\mathbb{N}$, $\mathbb{N}_0$ or $\mathbb{Z}$) is the Banach subspace of $\mathbb{R}^\mathbb{K}$ consisting of all sequences $\left(x_j\right)_j$ satisfying

$$\left\|\left(x_j\right)_j\right\|_{\ell^p(\mathbb{K})} := \left(\sum_{j\in\mathbb{K}} |x_j|^p\right)^{1/p} < \infty$$

if p is finite or $\left\|\left(x_j\right)_j\right\|_{\ell^\infty(\mathbb{K})} := \sup_{j\in\mathbb{K}} |x_j| < \infty$. One sets $\ell^p := \ell^p(\mathbb{N}_0)$. By $\mathcal{D}$ we denote the space of infinitely differentiable functions on $\mathbb{R}^d$ with compact support. Let $k \in \mathbb{N}_0$ and $p \in [1,\infty]$; the (historical) Sobolev space $W^k_p(\Omega)$ is defined by

$$W^k_p(\Omega) := \left\{f \in L^p(\Omega) :\ D^\alpha f \in L^p(\Omega)\,\forall\,|\alpha| \le k\right\},$$

where $D^\alpha f$ is the weak derivative of order α of f ($D^\alpha f$ will denote either the usual derivative or the weak derivative, depending on the context). Equipped with the norm

$$\|f\|_{W^k_p(\Omega)} := \sum_{|\alpha|\le k} \|D^\alpha f\|_{L^p(\Omega)},$$

$W_p^k(\Omega)$ is a Banach space (see e.g. [1,37]). We set $W_p^k := W_p^k(\mathbb{R}^d)$. Given $s \in \mathbb{R}$, let u_s be the tempered distribution defined by

$$\mathcal{F}u_s = \left(1 + |\cdot|^2\right)^{s/2}.$$

Of course, one has $u_{-s} * u_s = \delta$, where δ is the Dirac delta function. Given $s \in \mathbb{R}$ and $p \in [1, \infty]$, the (fractional) Sobolev space H_p^s is defined by

$$H_p^s := \left\{ f \in \mathcal{S}' : \|f\|_{H_p^s} = \|u_s * f\|_{L^p} < \infty \right\}.$$

Among the most common properties of these Sobolev spaces, the one that will be used mostly is maybe the continuous embedding $H_p^s \hookrightarrow H_p^r$, valid whenever $r \leq s$ [1,3,30,37,39]. Given $s \in \mathbb{N}_0$ and $1 < p < \infty$, one has $H_p^s = W_p^s$. Using Calderon–Zygmund theory, one can show that the fractional Sobolev spaces correspond to the Bessel potential spaces (1.1) (see e.g. [1,3]). Finally, $\mathbb{P}_n^d$ is the vector space of d-dimensional polynomials of degree at most n. Through this paper, we will implicitly use the classical properties of the tempered distributions (see e.g. [21]).

We will heavily use the finite differences in the sequel (see e.g. [8,23]). Given a function f defined on $\mathbb{R}^d$ and $x, h \in \mathbb{R}^d$, the finite difference $\Delta_h^n f$ of f is defined as follows

$$\Delta_h^1 f(x) = f(x+h) - f(x) \qquad \text{and} \qquad \Delta_h^{n+1} f(x) = \Delta_h^1 \Delta_h^n f(x),$$

for any $n \in \mathbb{N}$. It is easy to check that the following formula holds:

$$\Delta_h^n f(x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} f(x + (n-j)h). \tag{2.1}$$

The centered finite difference $\delta_h^n f$ is obtained in the same way:

$$\delta_h^1 f(x) = f(x+h/2) - f(x-h/2) \qquad \text{and} \qquad \delta_h^{n+1} f(x) = \delta_h^1 \delta_h^n f(x).$$

Since we have, $\delta_h^n f(x) = \Delta_h^n f(x - nh/2)$, these two notions will lead to the same definitions; for example, we obviously have

$$\left\|\Delta_h^n f\right\|_{L^p} = \|\delta_h^n f\|_{L^p},$$

for any $h \in \mathbb{R}^d$, any $n \in \mathbb{N}$, and any $p \in [1, \infty]$. If $f \in W_p^k$ ($k \in \mathbb{N}$, $p \in [1, \infty]$), for all $1 \leq n \leq k$, there exists a constant $C > 0$, not depending on the function f, such that

$$\left\|\Delta_h^n f\right\|_{L^p} \leq C|h|^n \sup_{|\alpha|=n} \|D^\alpha f\|_{L^p},$$

for all $h \in \mathbb{R}^d$.

The spaces of generalized smoothness are defined via admissible sequences.

Definition 2.1. A sequence $\sigma = \left(\sigma_j\right)_{j\in\mathbb{N}_0}$ of real positive numbers is called *admissible* if there exists a positive constant C such that

$$C^{-1}\sigma_j \leq \sigma_{j+1} \leq C\sigma_j,$$

for any $j \in \mathbb{N}_0$.

If σ is such a sequence, we set

$$\underline{\sigma}_j := \inf_{k\in\mathbb{N}_0} \frac{\sigma_{j+k}}{\sigma_k} \qquad \text{and} \qquad \overline{\sigma}_j := \sup_{k\in\mathbb{N}_0} \frac{\sigma_{j+k}}{\sigma_k}$$

and define the lower and upper Boyd indices as follows,

$$\underline{s}(\sigma) := \lim_j \frac{\log_2 \underline{\sigma}_j}{j} \qquad \text{and} \qquad \overline{s}(\sigma) := \lim_j \frac{\log_2 \overline{\sigma}_j}{j}.$$

Since $\left(\log_2 \underline{\sigma}_j\right)_j$ is a subadditive sequence, such limits always exist. If σ is an admissible sequence, let $\varepsilon > 0$; there exists a positive constant C such that

$$C^{-1} 2^{j(\underline{s}(\sigma)-\varepsilon)} \leq \underline{\sigma}_j \leq \frac{\sigma_{j+k}}{\sigma_k} \leq \overline{\sigma}_j \leq C 2^{j(\overline{s}(\sigma)+\varepsilon)}, \tag{2.2}$$

for any $j, k \in \mathbb{N}_0$. In the sequel, we will very often work with admissible sequences $\gamma = \left(\gamma_j\right)_j$ such that $\underline{\gamma}_1 > 1$. Such a sequence is strongly increasing (following [20]), i.e., there exists a number $k_0 \in \mathbb{N}$ such that

$$2\gamma_j \leq \gamma_k \text{ for all } j, k \in \mathbb{N}_0 \text{ s.t. } j + k_0 \leq k.$$

We can now introduce the Besov spaces of generalized smoothness $B^{\sigma,\gamma}_{p,q}$ that we will consider (see e.g. [20,33] and references therein). Let us recall some kind of generalization of the Littlewood–Paley decomposition (more details can be found in [20] for example). For an admissible sequence $\gamma = \left(\gamma_j\right)_j$ such that $\underline{\gamma}_1 > 1$, let $\rho \in \mathcal{D}$ be a positive function such that $\rho(t) = 1$ for all $|t| \leq 1$, ρ is decreasing for $t \geq 0$ and $\operatorname{supp}(\rho) \subset \{t \in \mathbb{R} : |t| \leq 2\}$ (where, as usual, $\operatorname{supp}(\rho)$ denotes the support of ρ). Given $J \in \mathbb{N}$, let us set

$$\varphi^{\gamma,J}_j := \rho\left(\gamma_j^{-1} |\cdot|\right) \quad \text{for } j \in \left\{0, \dots, Jk_0 - 1\right\}$$

and

$$\varphi^{\gamma,J}_j := \rho\left(\gamma_j^{-1} |\cdot|\right) - \rho\left(\gamma_{j-Jk_0}^{-1} |\cdot|\right) \quad \text{for } j \geq Jk_0.$$

Let us define $c_\varphi := Jk_0$ and

$$\widetilde{\varphi^{\gamma,J}_j} := \sum_{k=-(2J+1)k_0}^{(2J+1)k_0} \varphi^{\gamma,J}_{j+k} \quad \text{for } j \in \mathbb{N}_0,$$

where $\varphi^{\gamma,J}_{-1} = \cdots = \varphi^{\gamma,J}_{-(2J+1)k_0} = 0$. With such a system one has, for all $j \in \mathbb{N}_0$,

$$\widetilde{\varphi^{\gamma,J}_j} = c_\varphi \quad \text{on } \operatorname{supp}\left(\varphi^{\gamma,J}_j\right).$$

As a consequence, if we set, for any $f \in \mathcal{S}'$,

$$\Delta^{\gamma,J}_j f := \mathcal{F}^{-1}\left(\varphi^{\gamma,J}_j \mathcal{F} f\right) \qquad \text{and} \qquad \widetilde{\Delta^{\gamma,J}_j} f := \mathcal{F}^{-1}\left(\widetilde{\varphi^{\gamma,J}_j} \mathcal{F} f\right), \tag{2.3}$$

then

$$\widetilde{\Delta^{\gamma,J}_j} \Delta^{\gamma,J}_j f = c_\varphi \Delta^{\gamma,J}_j f.$$

Remark 2.2. From the Littlewood–Paley theory [18,36], $\Delta^{\gamma,J}_j f$ belongs to the space $C^\infty\left(\mathbb{R}^d\right)$.

Definition 2.3. Let σ and γ be two admissible sequences such that γ is strongly increasing (most often we will require $\underline{\gamma}_1 > 1$) and $p, q \in [1, \infty]$; the generalized Besov space $B^{\sigma,\gamma}_{p,q}$ is defined by

$$B^{\sigma,\gamma}_{p,q} := \left\{ f \in \mathcal{S}' : \|f\|_{B^{\sigma,\gamma}_{p,q}} = \left\| \left(\sigma_j \left\| \Delta^{\gamma,J}_j f \right\|_{L^p}\right)_j \right\|_{\ell^q} < \infty \right\}.$$

Remark 2.4. The space $B^{\sigma,\gamma}_{p,q}$ defined above does not depend on the particular decomposition chosen to represent the functions, in the sense of equivalent norms: two decompositions give rise to the same space (see Remark 3.1.3 in [20]). Indeed, such spaces can be defined with a general representation method which must satisfy conditions that are met by the decomposition given here [20].

The following characterization is given in [33]: Let $p, q \in [1, \infty]$, let $\sigma = \left(\sigma_j\right)_j$ and $\gamma = \left(\gamma_j\right)_j$ be two admissible sequences such that $\underline{\gamma}_1 > 1$ and $0 < \underline{s}(\sigma)\overline{s}(\gamma)^{-1}$. For any $n \in \mathbb{N}$ such that $\overline{s}(\sigma)\underline{s}(\gamma)^{-1} < n$, we have

$$B^{\sigma,\gamma}_{p,q} = \left\{ f \in L^p : \left(\sigma_j \sup_{|h| \le \gamma_j^{-1}} \left\| \Delta^n_h f \right\|_{L^p} \right)_j \in \ell^q \right\}. \tag{2.4}$$

In this framework, Besov spaces of generalized smoothness provide an obvious generalization of the usual Hölder spaces: if $\sigma = (2^{sj})_j$ $(s > 0)$ and $\gamma = (2^j)_j$, the space $B^{\sigma,\gamma}_{p,q}$ so defined is the usual Besov space $B^s_{p,q}$. One can therefore wonder if polynomials can play a role in the definition of the spaces $B^{\sigma,\gamma}_{p,q}$ (as it is the case for the Hölder spaces and their generalized version for example [26–28]). The Whitney theorem [9] states that for $f \in L^p$, $r > 0$ and $n \in \mathbb{N}$, there exists a constant $C > 0$ (which only depends on n and d) such that

$$\inf_{P \in \mathbb{P}^d_{n-1}} \|f - P\|_{L^p(B(x_0,r))} \le C \sup_{|h| \le r} \left\| \Delta^n_h f \right\|_{L^p(B_{nh}(x_0,r))},$$

where

$$B_{nh}(x_0, r) := \left\{ x \in B(x_0, r) : [x, x + nh] \subset B(x_0, r) \right\}.$$

It follows that, if $f \in B^{\sigma,\gamma}_{p,q}$, then, for n sufficiently large, there exists $(\varepsilon_j)_j \in \ell^q$ such that for all $x_0 \in \mathbb{R}^d$,

$$\sigma_j \inf_{P \in \mathbb{P}^d_{n-1}} \|f - P\|_{L^p\left(B\left(x_0,\gamma_j^{-1}\right)\right)} \le \varepsilon_j \quad \text{for all } j \in \mathbb{N}_0 . \tag{2.5}$$

The converse is not true unless $p = \infty$, as explained in Remark 2.5 below.

Remark 2.5. Now, suppose that $f \in L^\infty$ and let $\sigma, \gamma = \left(\gamma_j\right)_j$ be two admissible sequences satisfying $\overline{s}(\sigma)\underline{s}(\gamma)^{-1} < n$ with $n \in \mathbb{N}$. If γ is strongly increasing, given $x_0 \in \mathbb{R}^d$, we can claim that there exists $k_1 \in \mathbb{N}$ such that $n\gamma_k^{-1} \le \gamma_j^{-1}$ if $j + k_1 \le k$, which implies that if $|h| \le \gamma^{-1}_{j+k_1}$, then $x_0 + lh \in B\left(x_0, \gamma_j^{-1}\right)$ for all $l \in \{0, \dots, n\}$. Since, for any $P \in \mathbb{P}^d_{n-1}$, the following formula holds,

$$\Delta^n_h f(x_0) = \Delta^n_h (f - P)(x_0) = \sum_{l=0}^{n} (-1)^{n-l} \binom{n}{l} (f - P)(x_0 + lh),$$

we have

$$|\Delta^n_h f(x_0)| \le 2^n \inf_{P \in \mathbb{P}^d_{n-1}} \|f - P\|_{L^\infty\left(B\left(x_0,\gamma_j^{-1}\right)\right)}, \tag{2.6}$$

if $|h| \le \gamma^{-1}_{j+k_1}$, for almost every x_0. Therefore, if (2.5) holds with $p = \infty$, we have

$$\sigma_j \sup_{|h| \le \gamma_j^{-1}} \left\| \Delta^n_h f \right\|_{L^\infty} \le C\varepsilon_j \quad \text{for all } j \in \mathbb{N}_0$$

and thus $f \in B^{\sigma,\gamma}_{\infty,q}$. The inequality (2.6) is not sufficient to get such a conclusion in L^p with $p < \infty$.

Notation 2.6. Throughout this paper, we will use the letter C for a generic positive constant whose value may be different at each occurrence.

3 | PRELIMINARY RESULTS INVOLVING CONVOLUTIONS

Let us introduce some results that will be used in the sequel.

The following results about the convolution product are obtained using very classical proofs. We give them for the sake of completeness. If ϕ is a function defined on $\mathbb{R}^d$, given $\varepsilon \neq 0$ one sets

$$\phi_\varepsilon = \frac{1}{|\varepsilon|^d}\,\phi\Big(\frac{\cdot}{\varepsilon}\Big).$$

Proposition 3.1. *Let $n \in \mathbb{N}$, $p, q \in [1,\infty]$ and let $\big(\sigma_j\big)_j$, $\big(\gamma_j\big)_j$ be two sequences of positive real numbers. If $f \in L^p$ is such that*

$$\left(\sigma_j \sup_{|h|\le \gamma_j^{-1}} \left\|\Delta_h^n f\right\|_{L^p}\right)_j \in \ell^q,$$

then there exists $\phi \in \mathcal{D}$ such that

$$\left(\sigma_j \sup_{0<\varepsilon\le \gamma_j^{-1}} \|f * \phi_\varepsilon - f\|_{L^p}\right)_j \in \ell^q.$$

Proof. Without loss of generality, we can suppose that $n = 2m$ where m is an odd integer. Let $\rho \in \mathcal{D}$ be a radial function such that

- $\operatorname{supp}(\rho) \subseteq \overline{B(0,1)}$,
- $0 \le \rho \le 1$,
- $\|\rho\|_{L^1} = 1$

and set

$$\tilde{\phi} := \sum_{j=0}^{m-1} (-1)^j \binom{n}{j} \rho_{2j-n},$$

$c_n := \sum_{j=0}^{m-1}(-1)^j\binom{n}{j} = \binom{n}{m}/2$ and finally $\phi := \tilde{\phi}/c_n$. Obviously, $\phi \in \mathcal{D}$ and for all $x \in \mathbb{R}^d$ and $\varepsilon > 0$, we have

$$\begin{aligned} f * \phi_\varepsilon(x) - f(x) &= \int f(x-\varepsilon y)\phi(y)\,dy - f(x) \\ &= \frac{1}{c_n}\sum_{j=0}^{m-1}(-1)^j\binom{n}{j}\int f(x-\varepsilon y)\rho_{2j-n}(y)\,dy - f(x) \\ &= \frac{1}{c_n}\sum_{j=0}^{m-1}(-1)^j\binom{n}{j}\int f(x-\varepsilon(2j-n)t)\rho(t)\,dt - f(x). \end{aligned}$$

We get

$$\begin{aligned} \sum_{j=m+1}^{n}(-1)^j\binom{n}{j}\int f(x-\varepsilon(2j-n)t)\rho(t)\,dt &= \sum_{j=m+1}^{n}(-1)^j\binom{n}{j}\int f(x-\varepsilon(n-2j)y)\rho(y)\,dy \\ &= \sum_{j=0}^{m-1}(-1)^j\binom{n}{j}\int f(x-\varepsilon(2j-n)y)\rho(y)\,dy \end{aligned}$$

and

$$f * \phi_\varepsilon(x) - f(x) = \frac{1}{2c_n}\left(\sum_{\substack{j=0\\ j\neq m}}^{n}(-1)^j\binom{n}{j}\int f(x-\varepsilon(2j-n)t)\rho(t)\,dt - 2c_n f(x)\right)$$

$$= \frac{1}{2c_n}\int\left(\sum_{j=0}^{n}(-1)^j\binom{n}{j}f\left(x-2\varepsilon t\left(j-\frac{n}{2}\right)\right)\right)\rho(t)\,dt$$

$$= \frac{1}{2c_n}\int_{\overline{B(0,1)}}\delta^n_{2\varepsilon t}f(x)\rho(t)\,dt.$$

Using Hölder's inequality, we have

$$|f * \phi_\varepsilon(x) - f(x)| \leq C\|\rho\|_{L^q\left(\overline{B(0,1)}\right)}\|\delta^n_{2\varepsilon\cdot}f(x)\|_{L^p\left(\overline{B(0,1)}\right)} \leq C\|\Delta^n_{2\varepsilon\cdot}f(x)\|_{L^p\left(\overline{B(0,1)}\right)},$$

where q is the conjugate exponent of p. It follows, with the usual modification if $p=\infty$, that

$$\|f * \phi_\varepsilon - f\|_{L^p} \leq C\left(\iint_{\overline{B(0,1)}}\left|\Delta^n_{2\varepsilon t}f(x)\right|^p dt\,dx\right)^{1/p}$$

$$= C\left(\int_{\overline{B(0,1)}}\int\left|\Delta^n_{2\varepsilon t}f(x)\right|^p dx\,dt\right)^{1/p}$$

$$= C\left(\int_{\overline{B(0,1)}}\left\|\Delta^n_{2\varepsilon t}f\right\|^p_{L^p} dt\right)^{1/p}$$

$$\leq C\sup_{t\in\overline{B(0,1)}}\left\|\Delta^n_{2\varepsilon t}f\right\|_{L^p}$$

and finally, using a classical result for the last inequality [15, p. 45, formula (7.6)],

$$\sup_{0<\varepsilon\leq\gamma_j^{-1}}\|f * \phi_\varepsilon - f\|_{L^p} \leq C\sup_{|h|\leq\gamma_j^{-1}}\left\|\Delta^n_{2h}f\right\|_{L^p} \leq C\sup_{|h|\leq\gamma_j^{-1}}\left\|\Delta^n_{h}f\right\|_{L^p},$$

as desired □

Proposition 3.2. *Let $p,q\in[1,\infty]$, let $(\sigma_j)_j$ be an admissible sequence and let $(\gamma_j)_j$ be a sequence of positive real numbers such that there exists $d_0>0$ satisfying*

$$d_0\gamma_j \leq \gamma_{j+1} \quad \text{for all } j\in\mathbb{N}_0\,.$$

Let also $\phi\in\mathcal{D}$ and $f\in L^p$ satisfying

$$\left(\sigma_j\left\|f * \phi_{\gamma_j^{-1}} - f\right\|_{L^p}\right)_j \in \ell^q.$$

Then, for all $\alpha\in\mathbb{N}_0^d$,

$$\left(\sigma_j\gamma_j^{-|\alpha|}\left\|D^\alpha\left(f * \phi_{\gamma_j^{-1}} - f * \phi_{\gamma_{j-1}^{-1}}\right)\right\|_{L^p}\right)_{j\in\mathbb{N}} \in \ell^q(\mathbb{N}).$$

Proof. Let us write

$$f * \phi_{\gamma_j^{-1}} - f * \phi_{\gamma_{j-1}^{-1}} = \phi_{\gamma_j^{-1}} * \left(f * \phi_{\gamma_j^{-1}} - f * \phi_{\gamma_{j-1}^{-1}}\right) + \phi_{\gamma_j^{-1}} * \left(f - f * \phi_{\gamma_j^{-1}}\right) - \phi_{\gamma_{j-1}^{-1}} * \left(f - f * \phi_{\gamma_j^{-1}}\right). \tag{3.1}$$

Considering the first term on the right-hand side of this equality, we have, by Young's inequality,

$$\begin{aligned}\sigma_j\gamma_j^{-|\alpha|}\left\|D^\alpha\left(\phi_{\gamma_j^{-1}}*\left(f*\phi_{\gamma_j^{-1}}-f*\phi_{\gamma_{j-1}^{-1}}\right)\right)\right\|_{L^p}&=\sigma_j\gamma_j^{-|\alpha|}\left\|D^\alpha\phi_{\gamma_j^{-1}}*\left(f*\phi_{\gamma_j^{-1}}-f*\phi_{\gamma_{j-1}^{-1}}\right)\right\|_{L^p}\\&\le\sigma_j\gamma_j^{-|\alpha|}\left\|D^\alpha\phi_{\gamma_j^{-1}}\right\|_{L^1}\left\|f*\phi_{\gamma_j^{-1}}-f*\phi_{\gamma_{j-1}^{-1}}\right\|_{L^p},\end{aligned}$$

but, as $\left\|D^\alpha\phi_{\gamma_j^{-1}}\right\|_{L^1}=\gamma_j^{|\alpha|}\int|D^\alpha\phi(y)|\,dy$, we obtain, since σ is admissible,

$$\begin{aligned}\sigma_j\gamma_j^{-|\alpha|}\left\|D^\alpha\left(\phi_{\gamma_j^{-1}}*\left(f*\phi_{\gamma_j^{-1}}-f*\phi_{\gamma_{j-1}^{-1}}\right)\right)\right\|_{L^p}&\le C\sigma_j\left\|f*\phi_{\gamma_j^{-1}}-f\right\|_{L^p}+C\sigma_j\left\|f-f*\phi_{\gamma_{j-1}^{-1}}\right\|_{L^p}\\&\le C\sigma_j\left\|f*\phi_{\gamma_j^{-1}}-f\right\|_{L^p}+C'\sigma_{j-1}\left\|f-f*\phi_{\gamma_{j-1}^{-1}}\right\|_{L^p}.\end{aligned}$$

The conclusion comes by applying the same reasoning to the other terms of (3.1). □

4 | GENERALIZED BESOV SPACES AND WEAK DERIVATIVES

The spaces of generalized smoothness $B^{\sigma,\gamma}_{p,q}$ can be characterized using weak derivatives and finite differences.

We will need the following condition for a function to belong to W^k_p.

Proposition 4.1. *Let $k\in\mathbb{N}$, $p,q\in[1,\infty]$, let $(\sigma_j)_j$ be an admissible sequence and let $(\gamma_j)_j$ be a sequence of positive real numbers such that there exists $d_0>0$ satisfying*

$$d_0\gamma_j\le\gamma_{j+1}\quad\textit{for all } j\in\mathbb{N}_0\,.$$

Let us also suppose that the series

$$\sum_{j\in\mathbb{N}_0}\gamma_j^l\sigma_j^{-1}\tag{4.1}$$

converges for all $0\le l\le k$. If $f\in L^p$ is a function satisfying

$$\left(\sigma_j\sup_{|h|\le\gamma_j^{-1}}\left\|\Delta_h^k f\right\|_{L^p}\right)_j\in\ell^q,$$

then $f\in W^k_p$.

Proof. Let $\phi\in\mathcal{D}$ be a function as constructed in the proof of Proposition 3.1. Let us set

$$\psi_0:=\phi_{\gamma_0^{-1}},\qquad\psi_j:=\phi_{\gamma_j^{-1}}-\phi_{\gamma_{j-1}^{-1}}\quad\text{for all } j\in\mathbb{N},$$

and finally define

$$f_j:=f*\psi_j\quad\text{for all } j\in\mathbb{N}_0\,.$$

It follows from Proposition 3.2 that for all $\alpha\in\mathbb{N}_0^d$ satisfying $|\alpha|\le k$, there exists a constant $C_\alpha>0$ such that for all $j\in\mathbb{N}_0$, we have

$$\left\|D^\alpha f_j\right\|_{L^p}\le C_\alpha\gamma_j^{|\alpha|}\sigma_j^{-1}.$$

As a consequence, since (4.1) converges, the series $\sum_{j\in\mathbb{N}_0} D^\alpha f_j$ converges in L^p for all $|\alpha|\le k$. Let us denote its limit by f_α and show that $f_\alpha = D^\alpha f$ (with the derivative taken in the weak sense). It is clear that $f_0 = f$ since, by Proposition 3.1,

$$\left\|\sum_{j=0}^{J} f_j - f\right\|_{L^p} = \left\|f * \phi_{\gamma_J^{-1}} - f\right\|_{L^p} \le C\sigma_J^{-1} \to 0 \quad \text{as } J\to\infty.$$

Finally, for all $\varphi\in\mathcal{D}$ and $|\alpha|\le k$, we have

$$\begin{aligned}\int f(x)D^\alpha\varphi(x)\,dx &= \lim_{J\to\infty}\int\sum_{j=0}^{J} f_j(x)D^\alpha\varphi(x)\,dx\\ &= \lim_{J\to\infty}(-1)^{|\alpha|}\int\sum_{j=0}^{J} D^\alpha f_j(x)\varphi(x)\,dx\\ &= (-1)^{|\alpha|}\int f_\alpha(x)\varphi(x)\,dx,\end{aligned}$$

which is sufficient to conclude. □

We can now give necessary and sufficient conditions for a function to belong to $B^{\sigma,\gamma}_{p,q}$.

Theorem 4.2. *Let $p,q\in[1,\infty]$, let $\sigma=(\sigma_j)_j$ and $\gamma=(\gamma_j)_j$ be two admissible sequences such that $\underline{\gamma}_1>1$. Let the numbers $k,n\in\mathbb{N}_0$ be such that*

$$k<\underline{s}(\sigma)\overline{s}(\gamma)^{-1}\le\overline{s}(\sigma)\underline{s}(\gamma)^{-1}<n.$$

If $f\in B^{\sigma,\gamma}_{p,q}$, then $f\in W^k_p$ and for all $|\alpha|\le k$,

$$\left(\gamma_j^{-|\alpha|}\sigma_j\sup_{|h|\le\gamma_j^{-1}}\left\|\Delta_h^{n-|\alpha|}D^\alpha f\right\|_{L^p}\right)_{j\in\mathbb{N}}\in\ell^q,$$

which means that $D^\alpha f\in B^{\gamma^{-|\alpha|}\sigma,\gamma}_{p,q}$.

Conversely, if $f\in W^k_p$ satisfies

$$\left(\gamma_j^{-|\alpha|}\sigma_j\sup_{|h|\le\gamma_j^{-1}}\left\|\Delta_h^{n-|\alpha|}D^\alpha f\right\|_{L^p}\right)_j\in\ell^q\quad\text{for all } |\alpha|=k,$$

then $f\in B^{\sigma,\gamma}_{p,q}$.

Proof. Assume first that $f\in B^{\sigma,\gamma}_{p,q}$; using (2.4) and the convergence of (4.1) for $0\le l\le k$, which follows from the hypothesis on k and n, it is clear from the Proposition 4.1 that we have $f\in W^k_p$. Keeping the same notations used in the proof of Proposition 4.1, let us fix $J\in\mathbb{N}$; for all $|h|\le\gamma_J^{-1}$, we have

$$\begin{aligned}\left\|\Delta_h^{n-|\alpha|}D^\alpha f\right\|_{L^p} &\le \sum_{j=0}^{J} C|h|^{n-|\alpha|}\sup_{|\beta|=n}\left\|D^\beta f_j\right\|_{L^p} + \sum_{j=J+1}^{\infty} C|h|^{k-|\alpha|}\sup_{|\beta|=k}\left\|D^\beta f_j\right\|_{L^p}\\ &\le \sum_{j=0}^{J} C\gamma_J^{|\alpha|-n}\sup_{|\beta|=n}\left\|D^\beta f_j\right\|_{L^p} + \sum_{j=J+1}^{\infty} C\gamma_J^{|\alpha|-k}\sup_{|\beta|=k}\left\|D^\beta f_j\right\|_{L^p}.\end{aligned}$$

For all $|\beta|=k$, we also have, as $k<\underline{s}(\sigma)\overline{s}(\gamma)^{-1}$,

$$
\begin{aligned}
\left\| \left(\sum_{j=J+1}^{\infty} \gamma_J^{-k} \sigma_J \left\| D^\beta f_j \right\|_{L^p} \right)_J \right\|_{\ell^q} &\le \left\| \left(\sum_{j=J+1}^{\infty} \overline{\gamma}_{j-J}^k \underline{\sigma}_{j-J}^{-1} \gamma_j^{-k} \sigma_j \left\| D^\beta f_j \right\|_{L^p} \right)_J \right\|_{\ell^q} \\
&\le \sum_{j=1}^{\infty} \overline{\gamma}_j^k \underline{\sigma}_j^{-1} \left\| \left(\gamma_{j+J}^{-k} \sigma_{j+J} \left\| D^\beta f_{j+J} \right\|_{L^p} \right)_J \right\|_{\ell^q} \\
&\le C \sum_{j=1}^{\infty} \overline{\gamma}_j^k \underline{\sigma}_j^{-1} < \infty.
\end{aligned}
$$

Similarly, as $\overline{s}(\sigma)\underline{s}(\gamma)^{-1} < n$, for all $|\beta| = n$, we get

$$
\left\| \left(\sum_{j=0}^{J} \gamma_J^{-n} \sigma_J \left\| D^\beta f_j \right\|_{L^p} \right)_J \right\|_{\ell^q} < \infty,
$$

which allows us to conclude that

$$
\left\| \left(\gamma_J^{-|\alpha|} \sigma_J \sup_{|h| \le \gamma_J^{-1}} \left\| \Delta_h^{n-|\alpha|} D^\alpha f \right\|_{L^p} \right)_J \right\|_{\ell^q} < \infty.
$$

For the converse, assume $f \in W_p^k$; the desired conclusion follows directly from (2.4) and the fact that for all $|h| \le \gamma_j^{-1}$, we have, using classical inequalities (see [8] for example),

$$
\left\| \Delta_h^n f \right\|_{L^p} \le C|h|^k \sup_{|\alpha|=k} \left\| \Delta_h^{n-|\alpha|} D^\alpha f \right\|_{L^p} \le C \gamma_j^{-k} \sup_{|\alpha|=k} \left\| \Delta_h^{n-|\alpha|} D^\alpha f \right\|_{L^p}.
$$

□

As a corollary, we have the following alternative definition of $B_{p,q}^{\sigma,\gamma}$.

Corollary 4.3. *Let $p, q \in [1, \infty]$, let $\sigma = (\sigma_j)_j$ and $\gamma = (\gamma_j)_j$ be two admissible sequences such that $\underline{\gamma}_1 > 1$. Let the numbers $k, n \in \mathbb{N}_0$ be such that*

$$
k < \underline{s}(\sigma)\overline{s}(\gamma)^{-1} \le \overline{s}(\sigma)\underline{s}(\gamma)^{-1} < n.
$$

We have

$$
B_{p,q}^{\sigma,\gamma} = \left\{ f \in W_p^k : \left(\gamma_j^{-|\alpha|} \sigma_j \sup_{|h| \le \gamma_j^{-1}} \left\| \Delta_h^{n-|\alpha|} D^\alpha f \right\|_{L^p} \right)_j \in \ell^q \quad \forall |\alpha| = k \right\}.
$$

5 | GENERALIZED BESOV SPACES AND POLYNOMIALS

The following characterization is inspired from [22], where links between classical Besov spaces and related spaces are explored.

Theorem 5.1. *Let $p, q \in [1, \infty]$, let $\sigma = (\sigma_j)_j$ and $\gamma = (\gamma_j)_j$ be two admissible sequences such that $\underline{\gamma}_1 > 1$. Let the number $n \in \mathbb{N}$ be such that*

$$
n < \underline{s}(\sigma)\overline{s}(\gamma)^{-1} \le \overline{s}(\sigma)\underline{s}(\gamma)^{-1} < n + 1;
$$

the following assertions are equivalent:

(1) $f \in B_{p,q}^{\sigma,\gamma}$;

(2) $f \in W^n_p$ *and for all* $h \in \mathbb{R}^d$ *and almost every* $x \in \mathbb{R}^d$, *we have*

$$f(x+h) = \sum_{|\alpha|\le n} D^\alpha f(x)\frac{h^\alpha}{|\alpha|!} + R_n(x,h)\frac{|h|^n}{n!},$$

where

$$\left(\sigma_j \gamma_j^{-n} \sup_{|h|\le \gamma_j^{-1}} \|R_n(\cdot,h)\|_{L^p}\right)_j \in \ell^q;$$

(3) if, given $j \in \mathbb{N}_0$, π_j *is a net of* $\mathbb{R}^d$ *made of cubes of diagonal* γ_j^{-1}, *then for all* $j \in \mathbb{N}_0$, *there exists* g_{π_j} *such that*

- *the trace of* g_{π_j} *in each cube of* π_j *is a polynomial of degree at most n,*
- *one has* $\left(\sigma_j \|f - g_{\pi_j}\|_{L^p}\right)_j \in \ell^q$.

Proof. Let us first show that assertion (1) implies assertion (2). We know from Corollary 4.3 that $f \in W^n_p$; using the Taylor expansion with weak derivatives, we get

$$f(x+h) = \sum_{|\alpha|\le n-1} D^\alpha f(x)\frac{h^\alpha}{|\alpha|!} + \sum_{|\alpha|=n} h^\alpha \int_0^1 \frac{(1-t)^{(n-1)}}{(n-1)!} D^\alpha f(x+th)\,dt,$$

for all $h \in \mathbb{R}^d$ and a.e. $x \in \mathbb{R}^d$. Of course, we have

$$\begin{aligned}\int_0^1 \frac{(1-t)^{(n-1)}}{(n-1)!} D^\alpha f(x+th)\,dt &= \int_0^1 \frac{(1-t)^{(n-1)}}{(n-1)!} \Delta^1_{th} D^\alpha f(x)\,dt + \int_0^1 \frac{(1-t)^{(n-1)}}{(n-1)!} D^\alpha f(x)\,dt \\ &= \int_0^1 \frac{(1-t)^{(n-1)}}{(n-1)!} \Delta^1_{th} D^\alpha f(x)\,dt + \frac{D^\alpha f(x)}{n!}.\end{aligned}$$

Let us set

$$R_n(x,h) := \begin{cases} \dfrac{n!}{|h|^n} \displaystyle\sum_{|\alpha|=n} h^\alpha \int_0^1 \frac{(1-t)^{(n-1)}}{(n-1)!} \Delta^1_{th} D^\alpha f(x)\,dt & \text{if } h \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

Clearly, for all $h \in \mathbb{R}^d$ and a.e. $x \in \mathbb{R}^d$,

$$f(x+h) = \sum_{|\alpha|\le n} D^\alpha f(x)\frac{h^\alpha}{|\alpha|!} + R_n(x,h)\frac{|h|^n}{n!}.$$

Moreover, for any $h \neq 0$ such that $|h| \le \gamma_j^{-1}$, Hölder's inequality allows us to write

$$|R_n(x,h)| \le C \sum_{|\alpha|=n} \left\|\Delta^1_{\cdot h} D^\alpha f(x)\right\|_{L^p([0,1])}$$

and it follows that

$$\begin{aligned}\|R_n(\cdot,h)\|_{L^p} &\le C \sum_{|\alpha|=n} \left(\int_{\mathbb{R}^d}\int_0^1 \left|\Delta^1_{th} D^\alpha f(x)\right|^p dt\,dx\right)^{1/p} = C \sum_{|\alpha|=n} \left(\int_0^1\int_{\mathbb{R}^d} \left|\Delta^1_{th} D^\alpha f(x)\right|^p dx\,dt\right)^{1/p} \\ &= C \sum_{|\alpha|=n} \left(\int_0^1 \left\|\Delta^1_{th} D^\alpha f\right\|^p_{L^p} dt\right)^{1/p} \le C \sum_{|\alpha|=n} \sup_{|h|\le\gamma_j^{-1}} \left\|\Delta^1_h D^\alpha f\right\|_{L^p}.\end{aligned}$$

We can conclude this second point, using Corollary 4.3.

Now, let us show that assertion (2) implies (3). Let fix $j \in \mathbb{N}_0$ and let $\pi_j = (A_k)_k$ be a net of $\mathbb{R}^d$ with cubes of diagonal γ_j^{-1}. Set for all $k \in \mathbb{N}_0$,

$$P_k(x) := \frac{1}{\mathcal{L}(A_k)} \int_{A_k} \sum_{|\alpha| \leq n} D^\alpha f(y) \frac{(x-y)^\alpha}{|\alpha|!} \, dy,$$

where $\mathcal{L}$ is the Lebesgue measure. Of course, P_k is a polynomial of degree less or equal to n. Let us then define

$$g_{\pi_j} : \mathbb{R}^d \to \mathbb{R}^d, \quad x \mapsto P_k(x) \text{ if } x \in A_k \quad (k \in \mathbb{N}_0);$$

the trace of g_{π_j} in each cube of π_j is a polynomial of degree at most n and if $x \in A_k$, then $A_k \subset B\left(x, \gamma_j^{-1}\right)$. Moreover, if q is the conjugate exponent of p, using Hölder's inequality, we get, for $x \in A_k$,

$$\begin{aligned}
|f(x) - g_{\pi_j}(x)| &\leq \frac{1}{\mathcal{L}(A_k)} \int_{B\left(x,\gamma_j^{-1}\right)} \left| f(x) - \sum_{|\alpha| \leq n} D^\alpha f(y) \frac{(x-y)^\alpha}{|\alpha|!} \right| dy \\
&\leq C\gamma_j^d \int_{B\left(0,\gamma_j^{-1}\right)} \left| f(x) - \sum_{|\alpha| \leq n} D^\alpha f(x-h) \frac{h^\alpha}{|\alpha|!} \right| dh \\
&\leq C\gamma_j^d \gamma_j^{-d/q} \left\| f(x) - \sum_{|\alpha| \leq n} D^\alpha f(x - \cdot) \frac{\cdot^\alpha}{|\alpha|!} \right\|_{L^p\left(B\left(0,\gamma_j^{-1}\right)\right)}.
\end{aligned}$$

We thus can write

$$\begin{aligned}
\|f - g_{\pi_j}\|_{L^p} &\leq C\gamma_j^d \gamma_j^{-d/q} \left(\int_{B\left(0,\gamma_j^{-1}\right)} \int_{\mathbb{R}^d} \left| f(x) - \sum_{|\alpha| \leq n} D^\alpha f(x-h) \frac{h^\alpha}{|\alpha|!} \right|^p dx\, dh \right)^{1/p} \\
&= C\gamma_j^d \gamma_j^{-d/q} \left(\int_{B(0,\gamma_j^{-1})} \frac{|h|^{np}}{(n!)^p} \int_{\mathbb{R}^d} |R_n(x-h,h)|^p \, dx\, dh \right)^{1/p} \\
&\leq C\gamma_j^d \gamma_j^{-d\left(\frac{1}{p}+\frac{1}{q}\right)} \gamma_j^{-n} \sup_{|h| \leq \gamma_j^{-1}} \|R_n(\cdot,h)\|_{L^p} \\
&= C\gamma_j^{-n} \sup_{|h| \leq \gamma_j^{-1}} \|R_n(\cdot,h)\|_{L^p},
\end{aligned}$$

which procures the desired membership.

Finally, let us show that assertion (3) implies (1). As $\underline{\gamma}_1 > 1$, there exists $k_1 \in \mathbb{N}$ such that for any $x_0 \in \mathbb{R}^d$ and any $|h| \leq \gamma_{j+k_1}^{-1}$, we have $x_0 + kh \in B\left(x_0, \gamma_j^{-1}/3\sqrt{d}\right)$ for all $k \in \{0, \dots, n+1\}$. Let us fix $|h| \leq \gamma_{j+k_1}^{-1}$ and let $\pi_j = \left(A_k\right)_k$ be a net of $\mathbb{R}^d$ made of cubes of diagonal $\gamma_j^{-1}/3$ such that each vertex is the vertex of 2^d distinct cubes.

If $l \in \mathbb{N}$, then for all $x \in A_k$ and for all $l \in \{0, \dots, n+1\}$, $x + lh \in C_k$, where C_k is the cube of diagonal γ_j^{-1} whose center coincides with the center of A_k. Let π'_j be a net of $\mathbb{R}^d$ defined in the same way as above but made of cubes of diagonal γ_j^{-1} which contains C_k and let P_k be the polynomial which is the trace of $g_{\pi'_j}$ on C_k. As the degree of P_k is at most n, we have

$$\left\|\Delta_h^{n+1} f\right\|_{L^p}^p = \sum_{k \in \mathbb{N}_0} \left\|\Delta_h^{n+1} f\right\|_{L^p(A_k)}^p = \sum_{k \in \mathbb{N}_0} \left\|\Delta_h^{n+1} (f - P_k)\right\|_{L^p(A_k)}^p = \sum_{k \in \mathbb{N}_0} \left\|\Delta_h^{n+1} \left(f - g_{\pi'_j}\right)\right\|_{L^p(A_k)}^p,$$

with the usual modification if $p = \infty$. Let us remark that there exist $m := 3^d$ such nets with cubes of diagonal γ_j^{-1} whose centers are also center of some cube in π_j; let us denote by $\pi'_{j,1}, \dots, \pi'_{j,m}$ those nets. We have

$$\left\|\Delta_h^{n+1} f\right\|_{L^p}^p \leq \sum_{k\in\mathbb{N}_0} \sum_{l=1}^{m} \left\|\Delta_h^{n+1}\left(f - g_{\pi'_{j,l}}\right)\right\|_{L^p(A_k)}^p = \sum_{l=1}^{m} \left\|\Delta_h^{n+1}\left(f - g_{\pi'_{j,l}}\right)\right\|_{L^p}^p \leq C \sum_{l=1}^{m} \left\|\left(f - g_{\pi'_{j,l}}\right)\right\|_{L^p}^p .$$

Since we have

$$\left(\sigma_j \left\|\left(f - g_{\pi'_{j,l}}\right)\right\|_{L^p}\right)_j \in \ell^q,$$

for all $l \in \{1, \dots, m\}$ by hypothesis, we can write

$$\left(\sigma_j \sup_{|h|\leq\gamma_j^{-1}} \left\|\Delta_h^{n+1} f\right\|_{L^p}\right)_{j\in\mathbb{N}} \in \ell^q,$$

as desired. □

6 | GENERALIZED BESOV SPACES AND CONVOLUTION

The spaces of generalized smoothness $B^{\sigma,\gamma}_{p,q}$ can be defined in terms of convolutions.

The characterization relies on the following condition for a function to belong to $B^{\sigma,\gamma}_{p,q}$.

Proposition 6.1. *Let $p, q \in [1, \infty]$, let $\sigma = (\sigma_j)_j$ and $\gamma = (\gamma_j)_j$ be two admissible sequences such that $\underline{\gamma}_1 > 1$ and $\underline{s}(\sigma) > 0$. If $f \in L^p$ is such that there exists $\phi \in \mathcal{D}$ for which*

$$\left(\sigma_j \left\|f * \phi_{\gamma_j^{-1}} - f\right\|_{L^p}\right)_j \in \ell^q, \tag{6.1}$$

then $f \in B^{\sigma,\gamma}_{p,q}$.

Proof. Let $n \in \mathbb{N}$ be such that

$$0 < \underline{s}(\sigma)\overline{s}(\gamma)^{-1} \leq \overline{s}(\sigma)\underline{s}(\gamma)^{-1} < n.$$

As done before in the proof of Proposition 4.1 and having in mind that $\underline{s}(\sigma) > 0$, if (6.1) holds, we can build a sequence $(f_j)_j$ of infinitely differentiable functions belonging to L^p such that

$$f = \sum_{j\in\mathbb{N}_0} f_j$$

in L^p. It follows that, for any $J \in \mathbb{N}_0$ and any $|h| \leq \gamma_J^{-1}$, we have

$$\left\|\Delta_h^n f\right\|_{L^p} \leq \sum_{j=0}^{J} \|\Delta_h^n f_j\|_{L^p} + \sum_{j=J+1}^{\infty} \|\Delta_h^n f_j\|_{L^p} \leq C \sum_{j=0}^{J} \gamma_J^{-n} \sup_{|\alpha|=n} \left\|D^\alpha f_j\right\|_{L^p} + C \sum_{j=J+1}^{\infty} \|f_j\|_{L^p}.$$

Since, by Proposition 3.2, we also know that $\left(\sigma_j \gamma_j^{-|\alpha|} \left\|D^\alpha f_j\right\|_{L^p}\right)_j \in \ell^q$ for all $|\alpha| \leq n$, we can proceed as in the proof of Theorem 4.2, using the fact that $\underline{s}(\sigma) > 0$ and $\overline{s}(\sigma)\underline{s}(\gamma)^{-1} < n$, to conclude that the sequence $\left(\sigma_j \sup_{|h|\leq\gamma_j^{-1}} \left\|\Delta_h^n f\right\|_{L^p}\right)_j$ belongs to ℓ^q, and hence, by (2.4), $f \in B^{\sigma,\gamma}_{p,q}$. □

From Propositions 3.1 and 6.1, we have the following corollary.

Corollary 6.2. *Let $p, q \in [1, \infty]$, let $\sigma = (\sigma_j)_j$ and $\gamma = (\gamma_j)_j$ be two admissible sequences such that $\underline{\gamma}_1 > 1$ and $\underline{s}(\sigma) > 0$; we have*

$$B^{\sigma,\gamma}_{p,q} = \left\{ f \in L^p \,:\, \exists\, \phi \in \mathcal{D} \text{ such that } \left(\sigma_j \left\| f * \phi_{\gamma_j^{-1}} - f \right\|_{L^p} \right)_j \in \ell^q \right\}.$$

7 | GENERALIZED REAL INTERPOLATION METHODS

In the sequel, we will consider two normed vector spaces A_0 and A_1 which are continuously embedded in a Hausdorff topological vector space V. As a consequence, the spaces $A_0 \cap A_1$ and $A_0 + A_1$ are also normed vector spaces. Let us recall that the J-operator of interpolation is defined for $t > 0$ and $a \in A_0 \cap A_1$ by

$$J(t, a) := \max\left\{ \|a\|_{A_0}, t\|a\|_{A_1} \right\},$$

while the K-operator of interpolation is defined for $t > 0$ and $a \in A_0 + A_1$ by

$$K(t, a) := \inf\left\{ \|a_0\|_{A_0} + t\|a_1\|_{A_1} \,:\, a = a_0 + a_1 \right\}.$$

Definition 7.1. Let $\theta = (\theta_j)_{j\in\mathbb{Z}}$ and $\psi = (\psi_j)_{j\in\mathbb{Z}}$ be two sequences and let $q \geq 1$. We say that *a belongs to the (J, q)-generalized interpolation space* $[A_0, A_1]^{\theta,\psi}_{J,q}$ if there exists a sequence $(u_j)_{j\in\mathbb{Z}}$ of $A_0 \cap A_1$ such that $a = \sum_{j\in\mathbb{Z}} u_j$, with convergence in $A_0 + A_1$ and

$$\left\| \left(\theta_j J\left(\psi_j, u_j\right) \right)_j \right\|_{\ell^q(\mathbb{Z})} < \infty.$$

Definition 7.2. Let $\theta = (\theta_j)_{j\in\mathbb{Z}}$ and $\psi = (\psi_j)_{j\in\mathbb{Z}}$ be two sequences and let $q \geq 1$. We say that *a belongs to the (K, q)-generalized interpolation space* $[A_0, A_1]^{\theta,\psi}_{K,q}$ if $a \in A_0 + A_1$ and

$$\left\| \left(\theta_j K\left(\psi_j, a\right) \right)_j \right\|_{\ell^q(\mathbb{Z})} < \infty.$$

Remark 7.3. If one considers the admissible sequences $\left(\theta_j = 2^{-\alpha j}\right)_{j\in\mathbb{Z}}$ and $\left(\psi_j = 2^j\right)_{j\in\mathbb{Z}}$, the two preceding definitions correspond to the classical interpolation spaces $[A_0, A_1]_{\alpha,J,q}$ and $[A_0, A_1]_{\alpha,K,q}$ respectively.

As for the usual case, such interpolation methods often coincide; this result is a generalization of Proposition 11 in [28].

Theorem 7.4. *Let $r, s \in \mathbb{R}$ and let σ, γ be two admissible sequences such that $\underline{\gamma}_1 > 1$ and*

$$r < \min\left\{ \underline{s}(\sigma)\underline{s}(\gamma)^{-1}, \underline{s}(\sigma)\overline{s}(\gamma)^{-1} \right\} \leq \max\left\{ \overline{s}(\sigma)\underline{s}(\gamma)^{-1}, \overline{s}(\sigma)\overline{s}(\gamma)^{-1} \right\} < s. \tag{7.1}$$

We have

$$[A_0, A_1]^{\theta,\psi}_{J,q} = [A_0, A_1]^{\theta,\psi}_{K,q},$$

where the sequences $\theta = (\theta_j)_{j\in\mathbb{Z}}$ and $\psi = (\psi_j)_{j\in\mathbb{Z}}$ are defined by

$$\theta_j := \begin{cases} \gamma_{-j}^{-r}\sigma_{-j} & \text{if} \quad -j \in \mathbb{N}_0, \\ \gamma_j^r \sigma_j^{-1} & \text{if} \quad j \in \mathbb{N}, \end{cases}$$

and

$$\psi_j := \begin{cases} \gamma_{-j}^{-(s-r)} & \text{if} \quad -j \in \mathbb{N}_0, \\ \gamma_j^{(s-r)} & \text{if} \quad j \in \mathbb{N}. \end{cases}$$

Proof. Consider $f \in \left[A_0, A_1\right]^{\theta,\psi}_{J,q}$; we know that there exists a sequence $(f_l)_{l\in\mathbb{Z}}$ of $A_0 \cap A_1$ such that

$$f = \sum_{l\in\mathbb{Z}} f_l,$$

with convergence in $A_0 + A_1$ and

$$\left\| \left(\theta_l \max\left\{ \|f_l\|_{A_0}, \psi_l \|f_l\|_{A_1} \right\} \right)_l \right\|_{\ell^q(\mathbb{Z})} < \infty. \tag{7.2}$$

Set, for any $j \in \mathbb{Z}$,

$$b_j := \sum_{l=-\infty}^{j-1} f_l \qquad \text{and} \qquad c_j := \sum_{l=j}^{\infty} f_l.$$

Because of (7.1) and (7.2), we have $b_j \in A_0$, $c_j \in A_1$ and $f = b_j + c_j$. Let us prove that

$$\left\| \left(\theta_j \left(\|b_j\|_{A_0} + \psi_j \|c_j\|_{A_1} \right) \right)_j \right\|_{\ell^q(\mathbb{Z})} < \infty.$$

We have

$$\left\| \left(\theta_j \left(\|b_j\|_{A_0} + \psi_j \|c_j\|_{A_1} \right) \right)_j \right\|_{\ell^q} \leq \underbrace{\left\| \left(\theta_j \sum_{l=-\infty}^{j-1} \|f_l\|_{A_0} \right)_j \right\|_{\ell^q}}_{(A)} + \underbrace{\left\| \left(\theta_j \left(\sum_{l=j}^{\infty} \psi_j \|f_l\|_{A_1} \right) \right)_j \right\|_{\ell^q}}_{(B)}.$$

Using the triangle inequality, we obtain

$$\begin{aligned}
(A) &\leq \sum_{l=-\infty}^{0} \left\| \left(\theta_j \theta_{j+l-1}^{-1} \theta_{j+l-1} \|f_{l+j-1}\|_{A_0} \right)_j \right\|_{\ell^q} \\
&= \sum_{l=-\infty}^{0} \left(\sum_{j=-\infty}^{0} \left(\left(\frac{\gamma_{-j-l+1}}{\gamma_{-j}} \right)^r \left(\frac{\sigma_{-j-l+1}}{\sigma_{-j}} \right)^{-1} \theta_{j+l-1} \left\| f_{l+j-1} \right\|_{A_0} \right)^q \right. \\
&\quad + \sum_{j=1}^{1-l} \left(\gamma_j^r \sigma_j^{-1} \gamma_{-j-l+1}^r \sigma_{-j-l+1}^{-1} \theta_{j+l-1} \|f_{l+j-1}\|_{A_0} \right)^q \\
&\quad \left. + \sum_{j=2-l}^{\infty} \left(\left(\frac{\gamma_j}{\gamma_{j+l-1}} \right)^r \left(\frac{\sigma_j}{\sigma_{j+l-1}} \right)^{-1} \theta_{j+l-1} \|f_{l+j-1}\|_{A_0} \right)^q \right)^{\frac{1}{q}},
\end{aligned}$$

with the usual modification if $q = \infty$. If $r \geq 0$, there exists $\varepsilon > 0$ such that $r\overline{s}(\gamma) - \underline{s}(\sigma) + (r+1)\varepsilon < 0$ and (2.2) implies the existence of a constant $C > 0$ such that

$$\left(\frac{\gamma_{-j-l+1}}{\gamma_{-j}} \right)^r \left(\frac{\sigma_{-j-l+1}}{\sigma_{-j}} \right)^{-1} \leq \overline{\gamma}_{1-l}^r \underline{\sigma}_{1-l}^{-1} \leq C 2^{(1-l)(r\overline{s}(\gamma) - \underline{s}(\sigma) + (r+1)\varepsilon)}.$$

If $r < 0$, we can choose $\varepsilon > 0$ such that $r\underline{s}(\gamma) - \underline{s}(\sigma) + (1-r)\varepsilon < 0$ and find $C > 0$ such that

$$\left(\frac{\gamma_{-j-l+1}}{\gamma_{-j}} \right)^r \left(\frac{\sigma_{-j-l+1}}{\sigma_{-j}} \right)^{-1} \leq \underline{\gamma}_{1-l}^r \underline{\sigma}_{1-l}^{-1} \leq C 2^{(1-l)(r\underline{s}(\gamma) - \underline{s}(\sigma) + (1-r)\varepsilon)}.$$

Adapting this reasoning for the other terms, we can claim that there exists $\alpha < 0$ such that

$$(A) \leq C \sum_{l=-\infty}^{0} 2^{\alpha(1-l)} \left\| \left(\theta_{j+l-1} \|f_{l+j-1}\|_{A_0} \right)_j \right\|_{\ell^q} < \infty.$$

Similarly, there exists $\beta > 0$ such that

$$(B) \leq C \sum_{l=-\infty}^{0} 2^{\beta l} \left\| \left(\theta_{j-l} \psi_{j-l} \left\| f_{j-l} \right\|_{A_1} \right)_j \right\|_{\ell^q} < \infty.$$

Reciprocally, let us consider $f \in \left[A_0, A_1\right]^{\theta,\psi}_{K,q}$; for any $j \in \mathbb{Z}$ there exists $b_j \in A_0$ and $c_j \in A_1$ such that $f = b_j + c_j$ and

$$\left\| \left(\theta_j \left(\|b_j\|_{A_0} + \psi_j \|c_j\|_{A_1} \right) \right)_j \right\|_{\ell^q} < \infty. \tag{7.3}$$

Let us remark that, because of (7.1) and (7.3) , $b_0 = \sum_{j=-\infty}^{-1} (b_{j+1} - b_j)$ with convergence in A_0 and $c_0 = \sum_{j=0}^{\infty} (c_j - c_{j+1})$ with convergence in A_1. Now, let us set, for any $j \in \mathbb{Z}$,

$$f_j := b_{j+1} - b_j = c_j - c_{j+1}.$$

Clearly, $f_j \in A_0 \cap A_1$ for any $j \in \mathbb{Z}$ and $f = b_0 + c_0 = \sum_{j\in\mathbb{Z}} f_j$, with convergence in $A_0 + A_1$. Let us prove that

$$\left\| \left(\theta_j \max\left\{ \|f_j\|_{A_0}, \psi_j \|f_j\|_{A_1} \right\} \right)_j \right\|_{\ell^q} < \infty.$$

We have, as σ and γ are admissible,

$$\begin{aligned} \left\| \left(\theta_j \max\left\{ \|f_j\|_{A_0}, \psi_j \|f_j\|_{A_1} \right\} \right)_j \right\|_{\ell^q} &\leq \left\| \left(\theta_j \left(\|f_j\|_{A_0} + \psi_j \|f_j\|_{A_1} \right) \right)_j \right\|_{l^q} \\ &= \left\| \left(\theta_j \left(\|b_{j+1} - b_j\|_{A_0} + \psi_j \|c_j - c_{j+1}\|_{A_1} \right) \right)_j \right\|_{\ell^q} \\ &\leq C \left\| \left(\theta_j \left(\|b_j\|_{A_0} + \psi_j \|c_j\|_{A_1} \right) \right)_j \right\|_{\ell^q} \\ &< \infty, \end{aligned}$$

which allows to conclude. □

Definition 7.5. Given two admissible sequences σ and γ with $\underline{\gamma}_1 > 1$, let θ and ψ be the sequences defined as in Theorem 7.4 for some r, s as in (7.1); we define the space $\left[A_0, A_1\right]^{\sigma,\gamma}_q$ as follows:

$$\left[A_0, A_1\right]^{\sigma,\gamma}_q := \left[A_0, A_1\right]^{\theta,\psi}_{J,q} = \left[A_0, A_1\right]^{\theta,\psi}_{K,q}.$$

8 | GENERALIZED INTERPOLATION OF SOBOLEV SPACES

Let us show that the generalized Besov spaces $B^{\sigma,\gamma}_{p,q}$ can be defined from the usual Sobolev spaces W^s_p or H^s_p as generalized interpolation spaces, as it is the case with the usual Besov spaces $B^s_{p,q}$ and the classical real interpolation theory.

We need some auxiliary results. Roughly speaking, we aim at showing that there exist a constant $C > 0$ (depending on s) for which

$$C^{-1} \left\| \Delta^{\gamma,J}_j f \right\|_{H^s_p} \leq \gamma^s_j \left\| \Delta^{\gamma,J}_j f \right\|_{L^p} \leq C \|f\|_{H^s_p}.$$

Lemma 8.1. *Let γ be an admissible sequence such that $\underline{\gamma}_1 > 1$; given $s \in \mathbb{R}$ and $N \in \mathbb{N}_0$, there exists a constant $C_{s,N} > 0$ such that for all $j \in \mathbb{N}_0$,*

$$\left| \Delta^{\gamma,J}_j u_s \right| \leq C_{s,N} \gamma^{d+s}_j \left(1 + \gamma_j | \cdot |\right)^{-N}.$$

Proof. Let us fix $j \geq Jk_0$, the proof being similar for $0 \leq j \leq Jk_0 - 1$. As $\Delta_j^{\gamma,J} u_s = \mathcal{F}^{-1}\big(\varphi_j^{\gamma,J} \mathcal{F} u_s\big)$, we get

$$
\begin{aligned}
(2\pi)^d \left|\Delta_j^{\gamma,J} u_s\right| &\leq \int_{\mathbb{R}^d} \left|\rho\left(\gamma_j^{-1}|\xi|\right) - \rho\left(\gamma_{j-Jk_0}^{-1}|\xi|\right)\right| \big(1+|\xi|^2\big)^{s/2} \, d\xi \\
&= \gamma_j^d \int_{\mathbb{R}^d} \left|\rho(|y|) - \rho\left(\gamma_{j-Jk_0}^{-1}\gamma_j|y|\right)\right| \left(1+\gamma_j^2|y|^2\right)^{s/2} \, dy.
\end{aligned}
$$

Since the support of $\rho(|\cdot|) - \rho\left(\gamma_{j-Jk_0}^{-1}\gamma_j|\cdot|\right)$ is included in $\overline{\Omega}$, where Ω is defined by

$$
\Omega := B(0,2) \setminus B\left(0, \gamma_{-Jk_0}^{-1}\right),
$$

we have

$$
(2\pi)^d \left|\Delta_j^{\gamma,J} u_s\right| \leq \gamma_j^{d+s} \int_{\Omega} \left(|\rho(|y|)| + \left|\rho\left(\gamma_{-Jk_0}|y|\right)\right|\right) \left(\frac{1}{\gamma_j^2} + |y|^2\right)^{s/2} dy,
$$

and, as $0 < 1/\gamma_j^2 \leq 1/\gamma_0^2$, this implies the existence of a constant $C_{s,0} > 0$ such that

$$
\left|\Delta_j^{\gamma,J} u_s(x)\right| \leq C_{s,0}\gamma_j^{d+s}.
$$

Now, if $\alpha \in \mathbb{N}_0^d$ is a multi-index such that $|\alpha| \geq 1$, then

$$
(2\pi)^d \left|x^\alpha \Delta_j^{\gamma,J} u_s(x)\right| \leq \int_{\mathbb{R}^d} \left|D^\alpha\left(\varphi_j^{\gamma,J}(\xi)\big(1+|\xi|^2\big)^{s/2}\right)\right| d\xi,
$$

and similarly we get

$$
\left|x^\alpha \Delta_j^{\gamma,J} u_s(x)\right| \leq C_{s,|\alpha|}\gamma_j^{d+s-|\alpha|},
$$

for such an α, which is sufficient to conclude. □

Remark 8.2. Using the same proof as in Lemma 8.1, one can obtain the following result: *For all $s \in \mathbb{R}$ and $N \in \mathbb{N}_0$ there exists a constant $\widetilde{C_{s,N}} > 0$ such that for all $j \in \mathbb{N}_0$,*

$$
\left|\widetilde{\Delta_j^{\gamma,J} u_s}\right| \leq \widetilde{C_{s,N}}\gamma_j^{d+s}\big(1+\gamma_j|\cdot|\big)^{-N}.
$$

Proposition 8.3. *Let $s \in \mathbb{R}$ and $p \in [1,\infty]$; if $f \in H_p^s$, then there exists a constant $C_s > 0$ such that*

$$
\left\|\Delta_j^{\gamma,J} f\right\|_{L^p} \leq C_s\gamma_j^{-s}\|f\|_{H_p^s} \quad (j \in \mathbb{N}_0),
$$

where the notations defined by (2.3) *have been used.*

Proof. As $f = u_{-s} * u_s * f$, we get

$$
\Delta_j^{\gamma,J} f = \left(\Delta_j^{\gamma,J} u_{-s}\right) * (u_s * f).
$$

It follows from Young's inequality and Lemma 8.1 that

$$
\left\|\Delta_j^{\gamma,J} f\right\|_{L^p} \leq \left\|\Delta_j^{\gamma,J} u_{-s}\right\|_1 \|u_s * f\|_{L^p} \leq C_s\gamma_j^{-s}\|f\|_{H_p^s},
$$

for some constant C_s, which is the desired result. □

Proposition 8.4. *Let $s \in \mathbb{R}$, $p \in [1,\infty]$, and let $f \in \mathcal{S}'$. If, using the notations defined by* (2.3)*, $\Delta_j^{\gamma,J} f \in L^p$ (for some $j \in \mathbb{N}_0$), then there exists a constant $C_s > 0$ such that*

$$
\left\|\Delta_j^{\gamma,J} f\right\|_{H_p^s} \leq C_s\gamma_j^s\left\|\Delta_j^{\gamma,J} f\right\|_{L^p}.
$$

Proof. From $\widetilde{\Delta_j^{\gamma,J}}\Delta_j^{\gamma,J} f = c_\varphi \Delta_j^{\gamma,J} f$, we get

$$\mathcal{F}^{-1}\left(\mathcal{F}u_s\mathcal{F}\Delta_j^{\gamma,J} f\right) = \frac{1}{c_\varphi}\mathcal{F}^{-1}\left(\mathcal{F}u_s\mathcal{F}\left(\widetilde{\Delta_j^{\gamma,J}}\Delta_j^{\gamma,J} f\right)\right) = \frac{1}{c_\varphi}\mathcal{F}^{-1}\left(\widetilde{\varphi_j^{\gamma,J}}\mathcal{F}u_s\mathcal{F}\Delta_j^{\gamma,J} f\right) = \frac{1}{c_\varphi}\widetilde{\Delta_j^{\gamma,J}}u_s * \Delta_j^{\gamma,J} f.$$

It follows here again from the Young's inequality and Remark 8.2 that

$$\left\|\Delta_j^{\gamma,J} f\right\|_{H_p^s} \le \frac{1}{c_\varphi}\left\|\widetilde{\Delta_j^{\gamma,J}}u_s\right\|_1\left\|\Delta_j^{\gamma,J} f\right\|_{L^p} \le C_s\gamma_j^s\left\|\Delta_j^{\gamma,J} f\right\|_{L^p},$$

for a constant $C_s > 0$. □

We are now able to define the generalized Besov spaces $B_{p,q}^{\sigma,\gamma}$ from the Sobolev spaces using generalized interpolation.

Theorem 8.5. *Let $p, q \in [1,\infty]$, $r, s \in \mathbb{R}$, and let σ, γ be two admissible sequences such that $\underline{\gamma}_1 > 1$ and*

$$r < \min\left\{\underline{s}(\sigma)\underline{s}(\gamma)^{-1},\ \underline{s}(\sigma)\overline{s}(\gamma)^{-1}\right\} \le \max\left\{\overline{s}(\sigma)\underline{s}(\gamma)^{-1},\ \overline{s}(\sigma)\overline{s}(\gamma)^{-1}\right\} < s; \tag{8.1}$$

we have

$$B_{p,q}^{\sigma,\gamma} = \left[H_p^r, H_p^s\right]_q^{\sigma,\gamma}.$$

Proof. Let θ and ψ be the sequences defined as in Theorem 7.4.

Let us first suppose that $f \in B_{p,q}^{\sigma,\gamma}$ and set

$$u_j := \begin{cases} \Delta_{-j}^{\gamma,J} f & \text{if} \quad -j \in \mathbb{N}_0, \\ 0 & \text{if} \quad j \in \mathbb{N}. \end{cases}$$

From Proposition 8.4, for any $t \in \{r, s\}$ and $j \in -\mathbb{N}_0$, there exists a constant C_t such that

$$\|u_j\|_{H_p^t} \le C_t\gamma_{-j}^t\left\|\Delta_{-j}^{\gamma,J} f\right\|_{L^p},$$

which implies $u_j \in H_p^s$. Now, since $\left(\sigma_k\left\|\Delta_k^{\gamma,J} f\right\|_{L^p}\right)_{k\in\mathbb{N}_0}$ belongs to ℓ^q and (8.1) holds, we have $f = \sum_{j\in\mathbb{Z}} u_j$, with convergence in H_p^r. Moreover, for all j belonging to $-\mathbb{N}_0$, we get

$$\theta_j\|u_j\|_{H_p^r} \le C_r\sigma_{-j}\left\|\Delta_{-j}^{\gamma,J} f\right\|_{L^p} \qquad \text{and} \qquad \theta_j\psi_j\|u_j\|_{H_p^s} \le C_s\sigma_{-j}\left\|\Delta_{-j}^{\gamma,J} f\right\|_{L^p}.$$

From this, we can conclude that $\left(\theta_j J\left(\psi_j, u_j\right)\right)_{j\in\mathbb{Z}}$ belongs to $\ell^q(\mathbb{Z})$ and thus $f \in \left[H_p^r, H_p^s\right]_{J,q}^{\theta,\psi}$.

Let us now consider $f \in \left[H_p^r, H_p^s\right]_{J,q}^{\theta,\psi}$; there exists $(f_l)_{l\in\mathbb{Z}} \in H_p^s$ such that $f = \sum_{l\in\mathbb{Z}} f_l$ in H_p^r and $\left(\theta_l J(\psi_l, f_l)\right)_{l\in\mathbb{Z}}$ belongs to $\ell^q(\mathbb{Z})$. Now, for all $j \in \mathbb{N}_0$, Proposition 8.3 allows us to write

$$\begin{aligned}\left\|\Delta_j^{\gamma,J} f\right\|_{L^p} &\le \sum_{l\in\mathbb{Z}}\left\|\Delta_j^{\gamma,J} f_l\right\|_{L^p} \\ &\le C_r\sum_{l=-\infty}^{-j-1}\gamma_j^{-r}\|f_l\|_{H_p^r} + C_s\sum_{l=-j}^{0}\gamma_j^{-s}\|f_l\|_{H_p^s} + C_s\sum_{l=1}^{\infty}\gamma_j^{-s}\|f_l\|_{H_p^s}.\end{aligned}$$

It follows that

$$\begin{aligned}\left\|\left(\sigma_j\left\|\Delta_j^{\gamma,J} f\right\|_{L^p}\right)_j\right\|_{\ell^q} \le C&\left\|\left(\sum_{l=-\infty}^{-j-1}\sigma_j\gamma_j^{-r}\gamma_{-l}^{r}\sigma_{-l}^{-1}\theta_l\|f_l\|_{H_p^r}\right)_j\right\|_{\ell^q} + C\left\|\left(\sum_{l=-j}^{0}\sigma_j\gamma_j^{-s}\gamma_{-l}^{s}\sigma_{-l}^{-1}\theta_l\psi_l\|f_l\|_{H_p^s}\right)_j\right\|_{\ell^q} \\ &+ C\left\|\left(\sum_{l=1}^{\infty}\sigma_j\gamma_j^{-s}\gamma_l^{-s}\sigma_l\theta_l\psi_l\|f_l\|_{H_p^s}\right)_j\right\|_{\ell^q}.\end{aligned}$$

If $r \geq 0$, there exists $\varepsilon > 0$ such that $\alpha = r\overline{s}(\gamma) - \underline{s}(\sigma) + (r+1)\varepsilon < 0$ and (2.2) implies the existence of a constant $C > 0$ such that

$$\left(\frac{\gamma_{-l}}{\gamma_j}\right)^r \frac{\sigma_j}{\sigma_{-l}} \leq \overline{\gamma}^r_{-l-j}\underline{\sigma}^{-1}_{-l-j} \leq C2^{(-l-j)\alpha}.$$

Using the triangle inequality, we get

$$\left\| \left(\sum_{l=-\infty}^{-j-1} \sigma_j \gamma_j^{-r} \gamma_{-l}^r \sigma_{-l}^{-1} \theta_l \| f_l \|_{H_p^r} \right)_j \right\|_{\ell^q} \leq C \sum_{l=-\infty}^{-1} 2^{-\alpha l} \left\| \left(\theta_{l-j} \| f_{l-j} \|_{H_p^r} \right)_j \right\|_{\ell^q} < \infty.$$

If $r < 0$, we can choose $\varepsilon > 0$ such that $\beta = r\underline{s}(\gamma) - \underline{s}(\sigma) + (1-r)\varepsilon < 0$ and find $C > 0$ such that

$$\left(\frac{\gamma_{-l}}{\gamma_j}\right)^r \frac{\sigma_j}{\sigma_{-l}} \leq \underline{\gamma}^r_{-l-j}\underline{\sigma}^{-1}_{-l-j} \leq C2^{(-l-j)\beta}.$$

Again, we have

$$\left\| \left(\sum_{l=-\infty}^{-j-1} \sigma_j \gamma_j^{-r} \gamma_{-l}^r \sigma_{-l}^{-1} \theta_l \| f_l \|_{H_p^r} \right)_j \right\|_{\ell^q} < \infty.$$

The same reasoning can be applied to the other terms in order to obtain

$$\left\| \left(\sigma_j \left\| \Delta_j^{\gamma,J} f \right\|_{L^p} \right)_j \right\|_{\ell^q} < \infty,$$

which means that f belongs to $B^{\sigma,\gamma}_{p,q}$. □

If the admissible sequence γ is the usual sequence $(2^j)_j$, (8.1) can be written in a simpler way, which is given by Corollary 8.6.

Corollary 8.6. *Let $p, q \in [1, \infty]$, $r, s \in \mathbb{R}$ and let σ be an admissible sequence such that*

$$r < \underline{s}(\sigma) \leq \overline{s}(\sigma) < s; \tag{8.2}$$

we have

$$B^{\sigma}_{p,q} = \left[H^r_p, H^s_p\right]^{\sigma,(2^j)_j}_q.$$

The classical Besov spaces can be defined by interpolating the Sobolev spaces W^s_p even when $p = 1$ or $p = \infty$. Let us show that it is also the case in the generalized version.

Theorem 8.7. *Let $p, q \in [1, \infty]$, let $\sigma = \left(\sigma_j\right)_j$ and $\gamma = \left(\gamma_j\right)_j$ be two admissible sequences such that $\underline{\gamma}_1 > 1$. If $k, n \in \mathbb{N}_0$ are two numbers such that*

$$k < \underline{s}(\sigma)\overline{s}(\gamma)^{-1} \leq \overline{s}(\sigma)\underline{s}(\gamma)^{-1} < n,$$

we have

$$B^{\sigma,\gamma}_{p,q} = \left[W^k_p, W^n_p\right]^{\sigma,\gamma}_q.$$

Proof. Let us first suppose that $f \in B^{\sigma,\gamma}_{p,q}$; again, as in the proof of Proposition 4.1, there exists a sequence $\left(f_j\right)_j$ of infinitely differentiable functions belonging to L^p such that

$$D^\alpha f = \sum_{j \in \mathbb{N}_0} D^\alpha f_j$$

in L^p for all $|\alpha| \leq k$. Moreover, we have $\left(\sigma_j \gamma_j^{-|\alpha|} \|D^\alpha f_j\|_{L^p}\right)_j \in \ell^q$ for all $|\alpha| \leq n$. Let us then define the sequence

$$u_j := \begin{cases} f_{-j} & \text{if} \quad -j \in \mathbb{N}_0, \\ 0 & \text{if} \quad j \in \mathbb{N}. \end{cases}$$

We can write $f = \sum_{j\in\mathbb{Z}} u_j$ (with convergence in W_p^k); moreover, for all $j \in -\mathbb{N}_0$, we have

$$\theta_j \|u_j\|_{W_p^k} = \sum_{|\alpha|\leq k} \gamma_{-j}^{-k} \sigma_{-j} \left\| D^\alpha f_{-j} \right\|_{L^p} \leq C \sum_{|\alpha|\leq k} \gamma_{-j}^{-|\alpha|} \sigma_{-j} \left\| D^\alpha f_{-j} \right\|_{L^p}$$

and

$$\theta_j \psi_j \|u_j\|_{W_p^n} = \sum_{|\alpha|\leq n} \gamma_{-j}^{-n} \sigma_{-j} \left\| D^\alpha f_{-j} \right\|_{L^p} \leq C \sum_{|\alpha|\leq n} \gamma_{-j}^{-|\alpha|} \sigma_{-j} \left\| D^\alpha f_{-j} \right\|_{L^p},$$

which implies $\left(\theta_j J\left(u_j, \psi_j\right)\right)_j \in \ell^q$ and thus $f \in \left[W_p^k, W_p^n\right]_q^{\sigma,\gamma}$.

Now, let $f \in \left[W_p^k, W_p^n\right]_q^{\sigma,\gamma}$; there exists a sequence of functions $(u_l)_{l\in\mathbb{Z}}$ in W_p^n such that $f = \sum_{l\in\mathbb{Z}} u_j$ in W_p^k and $\left(\theta_l J\left(u_j, \psi_l\right)\right)_l \in \ell^q$. It follows that $D^\alpha f = \sum_{l\in\mathbb{Z}} D^\alpha u_l$ in L^p for all $|\alpha| \leq k$. Let us fix $|h| \leq \gamma_j^{-1}$ and $|\alpha| = k$; we have

$$\begin{aligned} \left\| \Delta_h^{n-k} D^\alpha f \right\|_{L^p} &\leq \sum_{l\in\mathbb{Z}} \left\| \Delta_h^{n-k} D^\alpha u_l \right\|_{L^p} \\ &\leq C \sum_{l=-\infty}^{-j-1} \|D^\alpha u_l\|_{L^p} + C \sum_{l=-j}^{\infty} \gamma_j^{k-n} \sup_{|\beta|=n} \left\| D^\beta u_l \right\|_{L^p} \\ &\leq C \sum_{l=-\infty}^{-j-1} \|u_l\|_{W_p^k} + C \sum_{l=-j}^{\infty} \gamma_j^{k-n} \|u_l\|_{W_p^n}. \end{aligned}$$

It follows, using the same arguments as before, that

$$\left(\gamma_j^{-k} \sigma_j \sup_{|h|\leq\gamma_j^{-1}} \left\| \Delta_h^{n-|\alpha|} D^\alpha f \right\|_{L^p} \right)_j \in \ell^q \quad \text{for all} \quad |\alpha| = k,$$

which implies $f \in B_{p,q}^{\sigma,\gamma}$, by Corollary 4.3. □

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REFERENCES

[1] R. A. Adams, *Sobolev spaces*, Pure Appl. Math., vol. 65, Academic Press, 1975.
[2] A. Almeida, *Wavelet bases in generalized Besov spaces*, J. Math. Anal. Appl. **304** (2005), 198–211.
[3] J. Bergh and J. Löfström, *Interpolation spaces: An introduction*, Springer-Verlag, 1976.
[4] O. V. Besov, *On a family of function spaces: Embedding theorems and applications*, Dokl. Akad. Nauk SSSR **126** (1959), 1163–1165.
[5] O. V. Besov, *On a family of functions spaces in connection with embeddings and extensions*, Trudy Mat. Inst. Steklov **60** (1961), 42–81.
[6] O. V. Besov, V. P. Il'in, and S. M. Nikol'skiĭ, *Integral representations of functions and imbedding theorems*, Scripta Series in Mathematics, John Wiley & Sons, 1979.
[7] O. V. Besov, L. D. Kudrjavcev, and P. I. Lizorkin, *Studies on the theory of spaces of differentiable functions of several variables*, Proc. Steklov Inst. Math. **182** (1990), 73–140.
[8] G. Boole and J. F. Moulton, *A treatise on the calculus of finite differences*, 2nd ed., Dover, 1960.
[9] Ju. A. Brudnyĭ, *A multidimensional analog of a theorem of Whitney*, Math. USSR Sb. **11** (1970), 157–170.
[10] V. Burenkov, *Extension theorems for Sobolev spaces*, The Maz'ya Anniversary Collection, Vol. 1 (J. Rossmann, P. Takáč, and G. Wildenhain, eds.), Oper. Theory Adv. Appl., vol. 109, Birkhäuser Verlag, 1999, pp. 187–200.

[11] A. M. Caetano and S. D. Moura, *Local growth envelopes of spaces of generalized smoothness: The critical case*, J. Inequal. Appl. **7** (2004), 573–606.

[12] F. Cobos and D. L. Fernandez, *Hardy–Sobolev spaces and Besov spaces with a function parameter*, Function Spaces and Applications (M. Cwikel, J. Peetre, Y. Sagher, and H. Wallin, eds.), Lectures Notes in Math., vol. 1302, Springer-Verlag, 1988, pp. 158–170.

[13] M. Cwikel and J. Peetre (eds.), *Interpolation spaces and allied topics in analysis*, Lectures Notes in Math., vol. 1070, Springer-Verlag, 1984.

[14] M. Cwikel, J. Peetre, Y. Sagher, and H. Wallin (eds.), *Function spaces and applications*, Lectures Notes in Math., vol. 1302, Springer-Verlag, 1988.

[15] R. A DeVore and G. G. Lorentz, *Constructive approximation*, Grundlehren Math. Wiss., Springer, 1993.

[16] D. E. Edmunds, P. Gurka, and B. Opic, *On embeddings of logarithmic Bessel potential spaces*, J. Funct. Anal. **146** (1997), 116–150.

[17] D. E. Edmunds and D. Haroske, *Spaces of Lipschitz type, embeddings and entropy numbers*, Dissertationes Math. (Rozprawy Mat.) **380** (1999).

[18] R. E. Edwards and G. I. Gaudry, *Littlewood–Paley and Multiplier theory*, Ergeb. Math. Grenzgeb., Band 90, Springer-Verlag, 1977.

[19] W. Farkas, Function spaces of generalised smoothness and pseudo-differential operators associated to a continuous negative definite function, Habilitation thesis, Universität München, 2002.

[20] W. Farkas and H.-G. Leopold, *Characterisations of function spaces of generalised smoothness*, Ann. Mat. Pura Appl. (4) **185** (2006), 1–62.

[21] I. M. Gel'fand and G. E. Shilov, *Generalized functions* 1–6, AMS Chelsea Publishing, 2016.

[22] A. Jonsson and H. Wallin, *Function spaces on subsets of* $\mathbb{R}^n$, Math. Rep., vol. 2, Harwood Academic Publishers, 1984.

[23] C. Jordan, *Calculus of finite differences*, AMS Chelsea Publishing, 3rd edition, 1965.

[24] G. A. Kalyabin and P. I. Lizorkin, *Spaces of generalized smoothness*, Math. Nachr. **133** (1987), 7–32.

[25] T. Kleyntssens and S. Nicolay, *A refinement of the* S^ν*-based multifractal formalism: Toward the detection of the law of the iterated logarithm*, (submitted).

[26] S. G. Krantz, *Lipschitz spaces, smoothness of functions, and approximation theory*, Exposition. Math. **1** (1983), no. 3, 193–260.

[27] D. Kreit and S. Nicolay, *Some characterizations of generalized Hölder spaces*, Math. Nachr. **285** (2012), no. 17–18, 2157–2172.

[28] D. Kreit and S. Nicolay, *Characterizations of the elements of generalized Hölder–Zygmund spaces by means of their representation*, J. Approx. Theory **172** (2013), 23–36.

[29] D. Kreit and S. Nicolay, *Generalized pointwise Hölder spaces defined via admissible sequences*, Hindawi, Journal of Function Spaces **2018**, article ID 8276258, available at https://doi.org/10.1155/2018/827658.

[30] G. Lemarié-Reusset, *Recent developments in the Navier–Stokes problem*, Research Notes in Mathematics, Chapman & Hall/CRC, 2002.

[31] H.-G. Leopold, *Embeddings and entropy numbers in Besov spaces of generalized smoothness*, Function Spaces (Poznań, 1998), Lectures Notes in Pure and Appl. Math., vol. 213, Dekker, 2000, pp. 323–336.

[32] C. Merucci, *Applications of interpolation with a function parameter to Lorentz, Sobolev and Besov spaces*, Interpolation Spaces and Allied Topics in Analysis (M. Cwikel and J. Peetre, eds.), Lectures Notes in Math., vol. 1070, Springer, 1984, pp. 183–201.

[33] S. D. Moura, *On some characterizations of Besov spaces of generalized smoothness*, Math. Nachr. **280** (2007), no. 8–9, 1190–1199.

[34] S. Nicolay and D. Kreit, *A refined method for estimating the global Hölder exponent*, Information Technology: New Generations (S. Latifi, ed.), Adv. Intell. Syst. Comput., vol. 448, Springer, 2016, pp. 949–958.

[35] B. Opic and W. Trebels, *Bessel potentials with logarithmic components and Sobolev-type embeddings*, Anal. Math. **26** (2000), 299–319.

[36] E. M. Stein, *Topics in harmonic analysis related to the Littlewood–Paley theory*, Ann. of Math. Stud., vol. 63, Princeton University Press, 1970.

[37] L. Tartar, *An introduction to Sobolev spaces and interpolation spaces*, Lect. Notes Unione Mat. Ital., Springer Verlag, 2007.

[38] H. Triebel, *Interpolation theory, function spaces, differential operators*, North Holland Publishing Company, 1978.

[39] H. Triebel, *Theory of function spaces I–III*, Birkhäuser Verlag, 1983.

[40] H. Triebel, *A note on wavelet bases in function spaces*, Banach Center Publ. **64** (2004), 193–206.

[41] H. Triebel, *Bases in function spaces, sampling, discrepancy, numerical integration*, EMS Tracts Math., vol. 11, 2010.

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