Analysis and Design of Telecommunications Systems: Manual of Exercises

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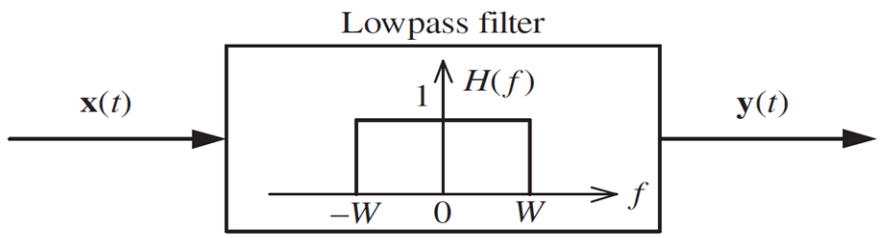
Introduction - Reminder

<u>Outline</u>:

1. Exercise 1: Random processes
2. Exercise 2: Digital modulation and matched filter
3. Exercise 3: Modulation
4. Exercise 4 : Radio systems

1. Random processes

The input noise $x(t)$ applied to the low-pass filter below is modeled as a WSS, white, Gaussian random process, with a zero mean and two-sided PSD $\frac{N_0}{2}$ [W/Hz]. Let $y(t)$ denote the random process at the output of the filter.



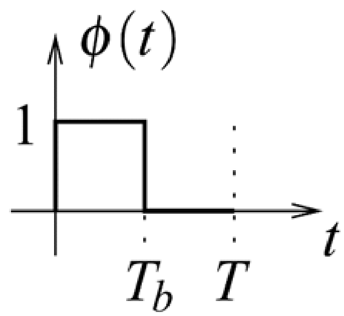
1. Find and sketch the power spectral density of $y(t)$.
2. Find and sketch the autocorrelation function of $y(t)$.
3. What are the average DC level and the average power of $y(t)$?
4. Does the correlation function of $y(t)$ depend on the fact that $x(t)$ is Gaussian? However, when the input is Gaussian, is $y(t)$ then Gaussian? Why?
5. Is $y(t)$, the LPF output, a correlated process?
6. Suppose that the output noise is sampled every T_s seconds to obtain the noise samples $y(kT_s)$ (where $k = 0, 1, 2, \cdots$). Find the smallest values of T_s so that the noise samples are statistically independent. Explain.
7. If the input noise is still white but not Gaussian anymore, does the noise samples $y(kT_s)$, with T_s chosen as in the previous question, remain statistically independent?

2. Digital modulation and matched filter

A PCM wave, obtained after quantization, is then modulated by a PAM-4 modulator with the following characteristics:

Symbol	Probability	Voltage [V]
00	0.15	-2.5
01	0.35	-1.0
10	0.35	1.0
11	0.15	2.5

the waveform is a rectangular signal with an unitary amplitude going from 0 to T_b



where T_b is the bit duration for the PCM wave and $T = 2T_b$.

1. When the symbol are not correlated, determine the PSD of the PAM-4 signal.
2. Compute the power of the modulated signal in [dBW] and [dBm].
3. When the signal arrive at the receptor, we decode 0111100010. Sketch the signal at the matched filter output (by integration and convolution) that allowed to build the given binary sequence.

Reminder: The PSD is $\gamma_g(f) = \|\Phi(f)\|^2 \frac{1}{T}\left[\sigma_A^2 + \mu_A^2 \sum_{m=-\infty}^{+\infty} \frac{1}{T}\delta\left(f - \frac{m}{T}\right)\right]$

3. Modulation

A particular version of AM stereo uses quadrature multiplexing. Specifically the carrier $A_c \cos(2\pi f_c t)$ is used to modulate the sum signal

$$m_1(t) = V_0 + m_l(t) + m_r(t)$$

where;

- V_0 is a DC offset included for the purpose of transmitting the carrier component,
- $m_l(t)$ is the left-hand audio signal,
- $m_r(t)$ is the right-hand audio signal.

The quadrature carrier $A_c \sin(2\pi f_c t)$ is used to modulate the difference signal

$$m_2(t) = m_l(t) - m_r(t)$$

1. Show that an envelope detector may be used to recover the sum $m_l(t) + m_r(t)$ from the quadrature-multiplexed signal. How would you minimize the signal distortion produced by the envelope detector?
2. Show that a coherent detector can recover the difference $m_l(t) - m_r(t)$?
3. (How are the desired left- and right-handed audio signal finally obtained?)

4. Radio systems

A geostationary satellite ($d = 36000$ [km]) exchange with a terrestrial station at 4 [GHz] frequency. This satellite uses a parabolic antenna whose diameter is equal to 50 [cm] and whose efficiency is equal to 0.6. The antenna misalignment at transmission α_T is equal to $\theta_{3[\text{dB}]}/2$. The antenna misalignment at the reception is neglected. The atmospheric losses are estimated to 0.4 [dB]. The losses in electric circuits at the transmission and reception are equal to 1.2 [dB]. The transmission power is equal to 100 [W].

1. Determine the free space losses.
2. Determine the minimum reception gain knowing that the receptor sensitivity is -140 [dB]. The sensitivity is the minimum signal value at the input of the receptor for this latter to work correctly.
3. Determine the 3 [dB] aperture angle and the effective area for the transmission antenna.
4. Define and determine the EIRP (*Equivalent Isotropic Radiating Power*).
5. Determine the maximum bandwidth usable if the signal to noise ratio at the receptor has to be equal to 10 [dB] minimum. The spectral density of noise estimated at the considered frequency is $\frac{N_0}{2} = 5 \times 10^{-24}$[W/Hz].

Reminder:

$$\theta_{3\,[\text{dB}]} = 70\frac{\lambda}{D} \qquad [^\circ]$$

$$G_{max} = \frac{4\pi}{\lambda^2}A_{eff}$$

$$L_{E,R} = 12\left(\frac{\alpha_{E,R}}{\theta_{3\,[\text{dB}]}}\right)^2 \qquad [\text{dB}]$$

Representation of Bandpass Signals

Outline:

1. HILBERT transform
2. Analytic signal
3. Complex envelope
4. Bandpass system
5. Exercises

1. Hilbert Transform

Direct transform:

$$\widetilde{g}(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{g(\tau)}{t-\tau}d\tau$$

Inverse transform :

$$g(t) = -\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\widetilde{g}(\tau)}{t-\tau}d\tau$$

Link with the FOURIER transform:

$$\widetilde{g}(t) = g(t) \otimes \frac{1}{\pi t}$$

But

$$\frac{1}{\pi t} \rightleftharpoons -j\,sign(f)$$

So, we derive

$$\widetilde{\mathscr{G}}(f) = -j\,sign(f)\,\mathscr{G}(f)$$

<u>Properties</u> :

1. Module
$$\left\| \widetilde{\mathscr{G}}(f) \right\| = \left\| \mathscr{G}(f) \right\|$$
2. Energy
$$\left\| \widetilde{\mathscr{G}}(f) \right\|^2 = \left\| \mathscr{G}(f) \right\|^2$$
3. Transform of the transform
$$\widetilde{\widetilde{g}}(t) = -g(t)$$
4. Orthogonality
$$\int_{-\infty}^{+\infty} g(t)\, \widetilde{g}(t)\, dt = 0$$

2. Analytic signal

Let $g(t)$ real and $g(t) \rightleftharpoons \mathscr{G}(f)$.

Definition of **analytic signal** :

$$g_a(t) = g(t) + j\widetilde{g}(t)$$

FOURIER transform:

$$\mathscr{G}_a(f) = \begin{cases} 2\mathscr{G}(f) & f > 0 \\ \mathscr{G}(0) & f = 0 \\ 0 & f < 0 \end{cases}$$

3. Complex envelope

Let $g(t)$ real and narrow-band, i.e.

$$\mathscr{G}(f) \begin{cases} \neq 0 & f_c - W < |f| < f_c + W \\ = 0 & \text{otherwise} \end{cases}$$

Definition of the **complex envelope** (baseband signal) of $g(t)$:

$$e_g(t) = g_a(t)\, e^{-2\pi j f_c t}$$

This could also be noted

$$e_g(t) = g_I(t) + j\, g_Q(t)$$

$g_I(t)$ = inphase component
$g_Q(t)$ = quadrature component

Canonical form of $g(t)$:

$$\begin{aligned} g(t) &= Re[g_a(t)] = Re\left[e_g(t)\, e^{2\pi j f_c t}\right] \\ &= g_I(t)\cos(2\pi f_c t) - g_Q(t)\sin(2\pi f_c t) \end{aligned}$$

Other form for the complex envelope :

$$e_g(t) = a(t)\, e^{j\phi(t)}$$

$a(t)$ and $\phi(t)$ are real and baseband.

$a(t)$ = natural envelope of the signal $g(t)$

$\phi(t)$ = signal phase

We may write

$$\begin{aligned} g(t) &= Re\left[g_a(t)\right] \\ &= Re\left[e_g(t)\, e^{j2\pi f_c t}\right] \\ &= Re\left[a(t)\, e^{j\phi(t)}\, e^{j2\pi f_c t}\right] \\ &= a(t)\cos\left[2\pi f_c t + \phi(t)\right] \end{aligned}$$

4. Bandpass systems

Let $x(t)$ a signal real and narrow-band :

$$x(t) = x_I(t)\cos(2\pi f_c t) - x_Q(t)\sin(2\pi f_c t)$$

Let $h(t)$ the impulse response of a narrow-band linear system :

$$h(t) = h_I(t)\cos(2\pi f_c t) - h_Q(t)\sin(2\pi f_c t)$$

Let $y(t)$ (bandpass) the signal at the system output. It can be shown that

$$e_y(t) = \frac{1}{2}\left[e_h(t) \otimes e_x(t)\right]$$

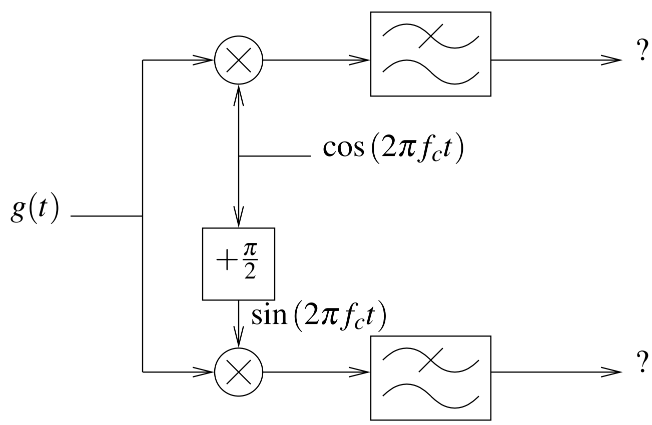
where $e_y(t)$ is the complex envelope of $y(t)$.

5. Exercises

1. Determine the HILBERT transform of the following signals:
   (a) $\delta(t)$
   (b) $\sin(2\pi f_c t)$

2. Determine the analytic signal $g_a(t)$ and the complex envelope $e_g(t)$ for the signal $g(t) = [1 + k\cos(2\pi f_m t)]\cos(2\pi f_c t)$.

3. Show that the following circuit is able to extract the inphase and quadrature components of the narrow-band signal $g(t)$:



Then, show that

$$\mathscr{G}_I(f) = \begin{cases} \mathscr{G}(f - f_c) + \mathscr{G}(f + f_c) & -W \le f \le W \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathscr{G}_Q(f) = \begin{cases} j[\mathscr{G}(f - f_c) - \mathscr{G}(f + f_c)] & -W \le f \le W \\ 0 & \text{otherwise} \end{cases}$$

4. The signal

$$x(t) = \begin{cases} A\cos(2\pi f_c t) & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

is supplied to a filter whose impulse response is given by

$$h(t) = x(T-t)$$

Assuming that $f_c \gg 1/T$, determine the filter response.

5. Show that the signal

$$s(t) = \frac{1}{2}A_c\, m(t)\cos(2\pi f_c t) - \frac{1}{2}A_c\,\widetilde{m}(t)\sin(2\pi f_c t)$$

correspond to the SSB modulation (we keep the upper lateral bands -> USB) of the modulating signal $m(t)$. How could you modify $s(t)$ to keep the lower lateral bands?

6. Let $s(t)$ the signal corresponding to the SSB modulation (USB) of a modulating signal $m(t)$. Show that

$$m(t) = \frac{2}{A_c}\left[s(t)\cos(2\pi f_c t) + \widetilde{s}(t)\sin(2\pi f_c t)\right]$$

And deducts from it a circuit allowing to demodulate a signal USB.

7. Let the following modulated signal

$$s(t) = A_c\cos(2\pi f_c t) + m(t)\cos(2\pi f_c t) - \widetilde{m}(t)\sin(2\pi f_c t)$$

Assuming that $A_c \gg |m(t)|$ and $A_c \gg |\widetilde{m}(t)|$, show that an ideal envelope detector delivers a good approximation of the modulating signal $m(t)$.

8. Let the modulating signal $m(t) = A_m\cos(2\pi f_m t)$. Determine the Hilbert transform of the corresponding FM-modulated signal.

Answers

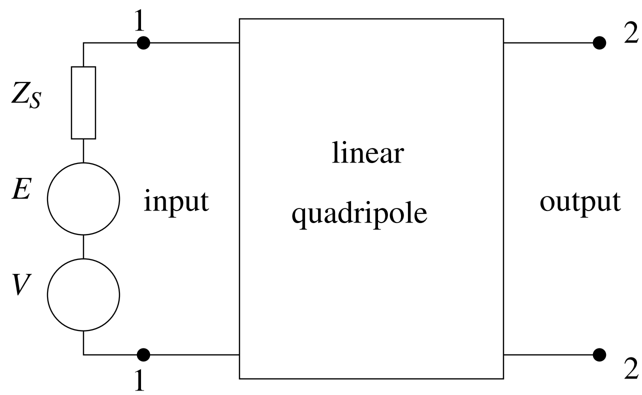
1. (a) $1/(\pi t)$.
   (b) $-\cos(2\pi f_c t)$.
2.
$$g_a(t) = e^{j2\pi f_c t}\,[1 + k\cos(2\pi f_m t)]$$
$$e_g(t) = 1 + k\cos(2\pi f_m t)$$
3. –
4.
$$\begin{cases} \frac{A^2 t}{2}\cos(2\pi f_c t) & \text{si } 0 \le t < T \\ \frac{A^2(2T-t)}{2}\cos(2\pi f_c t) & \text{si } T \le t < 2T \\ 0 & \text{otherwise} \end{cases}$$
5. –
6. –
7. Envelope detector output $\simeq A_c + m(t)$.
8. $\widetilde{s}(t) = A_c \sin[2\pi f_c t + \beta \sin(2\pi f_m t)]$.

Noise in telecommunication systems

Outline :

1. Noise figure
2. Noise temperature
3. Cascading two-port elements
4. Merit figure
5. Attenuator
6. Exercises

1. Noise figure



Spot noise figure:

$$F_0 = \frac{\gamma_{NO}(f)}{G(f)\,\gamma_{NS}(f)}$$

Matched impedance (maximum power transfer $Z_L = Z^*(f)$) :

$$Z^*(f) = R(f) + jX(f)$$

$$\begin{aligned} P_S(f) &= \left[\frac{V_0}{2R(f)}\right]^2 R(f) \\ &= \frac{V_0^2}{4R(f)} \end{aligned}$$

Power at the output of the two-port element:

$$P_O(f) = G(f)\,P_S(f)$$

$$\begin{aligned} F_0 &= \frac{P_S(f)\,\gamma_{NO}(f)\,W}{G(f)\,P_S(f)\,\gamma_{NS}(f)\,W} \\ &= \frac{P_S(f)\,\gamma_{NO}(f)\,W}{P_O(f)\,\gamma_{NS}(f)\,W} \\ &= \frac{\rho_S(f)}{\rho_O(f)} \end{aligned}$$

Signal to noise ratio:

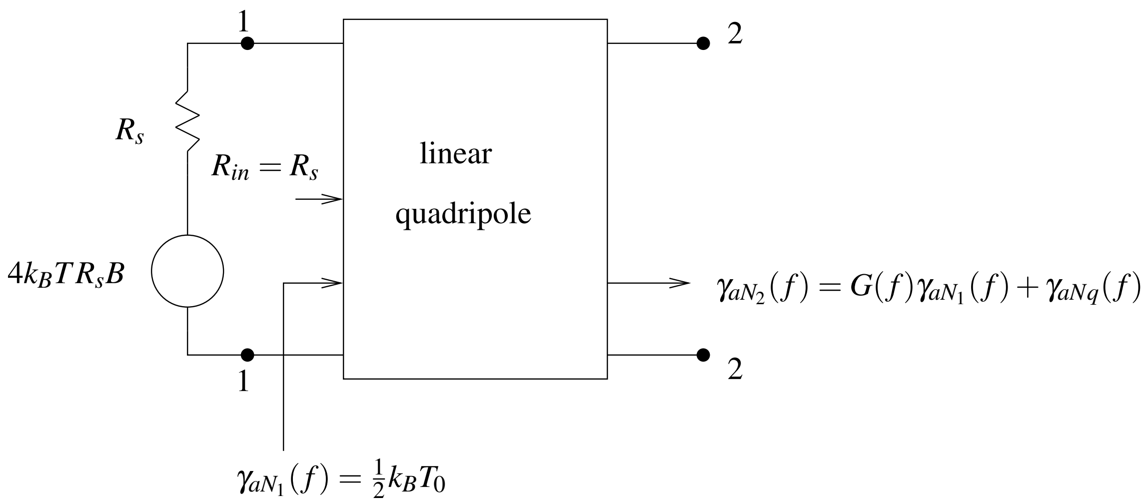
$$\rho_S(f) = \frac{P_S(f)}{\gamma_{NS}(f)\,W}$$

$$\rho_O(f) = \frac{P_O(f)}{\gamma_{NO}(f)\,W}$$

Mean noise figure:

$$F_{0m} = \frac{\int_{-\infty}^{+\infty} \gamma_{NO}(f)\,df}{\int_{-\infty}^{+\infty} G(f)\,\gamma_{NS}(f)\,df}$$

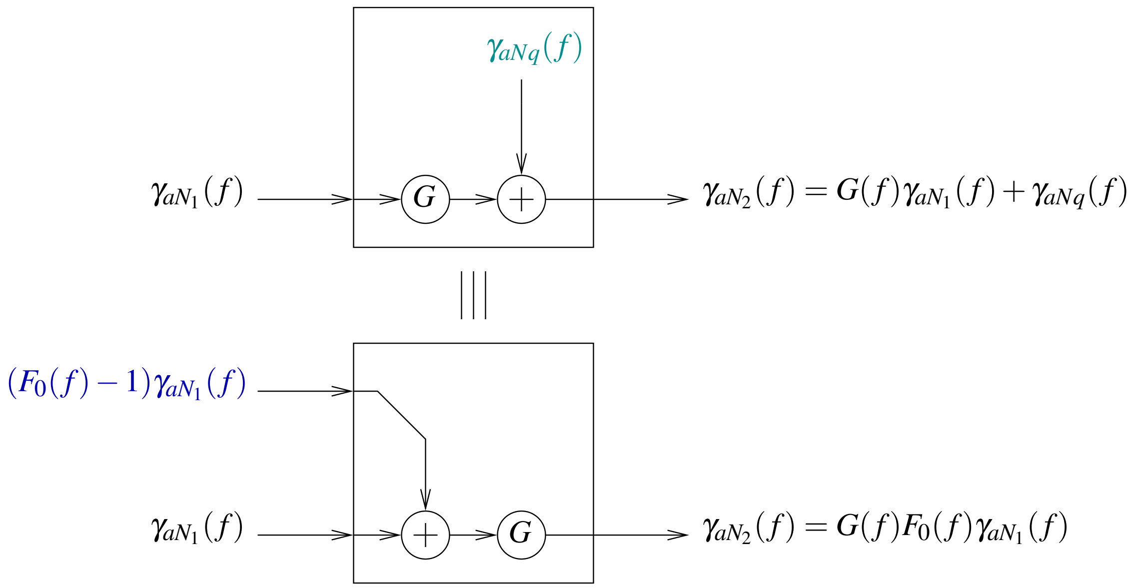
2.Noise temperature



$$F_0(f) = \frac{\gamma_{aN_2}(f)}{G(f)\gamma_{aN_1}(f)} = \frac{G(f)\gamma_{aN_1}(f) + \gamma_{aNq}(f)}{G(f)\gamma_{aN_1}(f)}$$

But we can model the internal noise at the entrance:

$$F_0(f) = \frac{G(f)\gamma_{aN_1}(f) + \gamma_{aNq}(f)}{G(f)\gamma_{aN_1}(f)} \Longleftrightarrow \gamma_{aNq}(f) = [(F_0(f) - 1)\,\gamma_{aN_1}(f)]\,G(f)$$


$\gamma_{aNq}(f)$
$\gamma_{aN_1}(f)$
G
+
$\gamma_{aN_2}(f) = G(f)\gamma_{aN_1}(f) + \gamma_{aNq}(f)$
$(F_0(f) - 1)\gamma_{aN_1}(f)$
$\gamma_{aN_1}(f)$
+
G
$\gamma_{aN_2}(f) = G(f)F_0(f)\gamma_{aN_1}(f)$

3. Merit figure

WARNING: Not normalized

If $T_s \neq T_0$

The internal noise of a two-port circuit is independent of the input temperature.

Link between F and F_0

$$\gamma_{aNq}(f) = (F_0 - 1)\frac{1}{2}k_B T_0 G(f) = (F - 1)\frac{1}{2}k_B T_s G(f)$$

$$F = 1 + \frac{T_0}{T_s}(F_0 - 1)$$

The effective noise temperature:

$$T_e = (F_0 - 1)T_0$$

is the additional temprature required for an input source to produce the same available power at the output.

4. Attenuator

Let an attenuation circuit described by an attenuation factor **L**

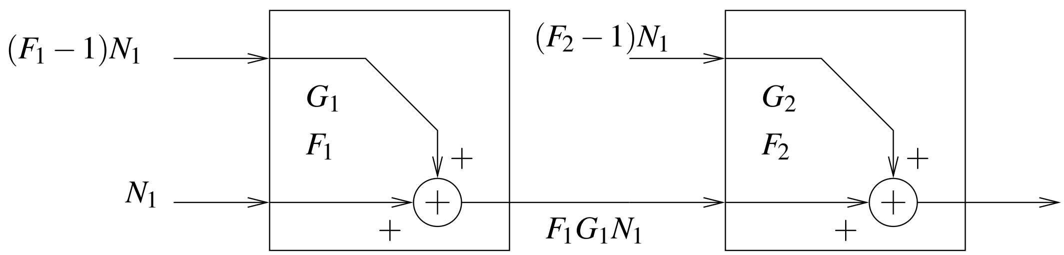
$$G = \frac{1}{L}$$

$$F_0 = F = L$$

$$T_e = (L - 1)\, T_s$$

For an attenuator with a factor L, the amount of noise is always unaffected.

5. Cascading two-port elements



$$F_0 = \frac{F_{01}G_1N_1G_2 + (F_{02}-1)N_1G_2}{N_1G_1G_2}$$

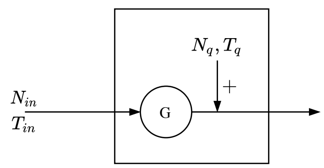
$$F_0 = F_{01} + \frac{F_{02}-1}{G_1}$$

$$F_0 = F_{01} + \frac{F_{02}-1}{G_1} + \frac{F_{03}-1}{G_1G_2} + \frac{F_{04}-1}{G_1G_2G_3} + \cdots$$

$$T_e = T_{e1} + \frac{T_{e2}}{G_1} + \frac{T_{e3}}{G_1G_2} + \frac{T_{e4}}{G_1G_2G_3} + \cdots$$

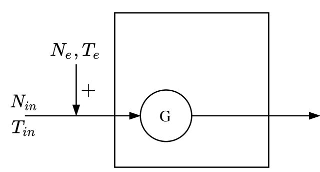
It is helpful to take $G_1 \gg$

6. Summary



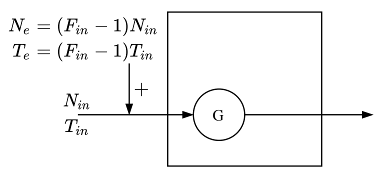
$N_{out} = GN_{in} + N_q$
$T_{out} = GT_{in} + T_q$

N_q indep. of N_{in}
T_q indep. of T_{in}



$N_{out} = G(N_{in} + N_e)$
$T_{out} = G(T_{in} + T_e)$

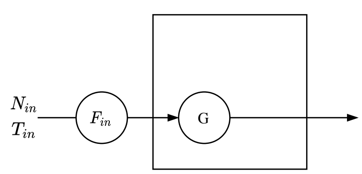
N_e indep. of N_{in}
T_e indep. of T_{in}



$N_{out} = F_{in}GNin$
$T_{out} = F_{in}GTin$

$\Rightarrow F_{in} = \dfrac{GNin + N_q}{GNin}$

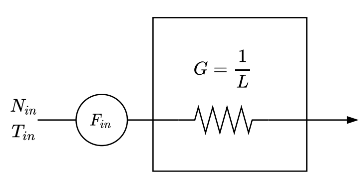
- depend of N_{in}
- depend of T_{in}



$N_{out} = F_{in}GN_{in}$
$T_{out} = F_{in}GT_{in}$

$T_e = (F_{in} - 1)T_{in} = (F_0 - 1)T_0$
$\Rightarrow F_{in} = 1 + \dfrac{T_0}{T_{in}}(F_0 - 1)$

Passive element at thermal equilibrium $\Rightarrow T_{in} = T_{out} = T_{phys}$



$N_{out} = F_{in}GN_{in}$
$T_{out} = F_{in}GT_{in}$

$\Rightarrow F_{in} = \dfrac{T_{out}}{GT_{in}} = L$

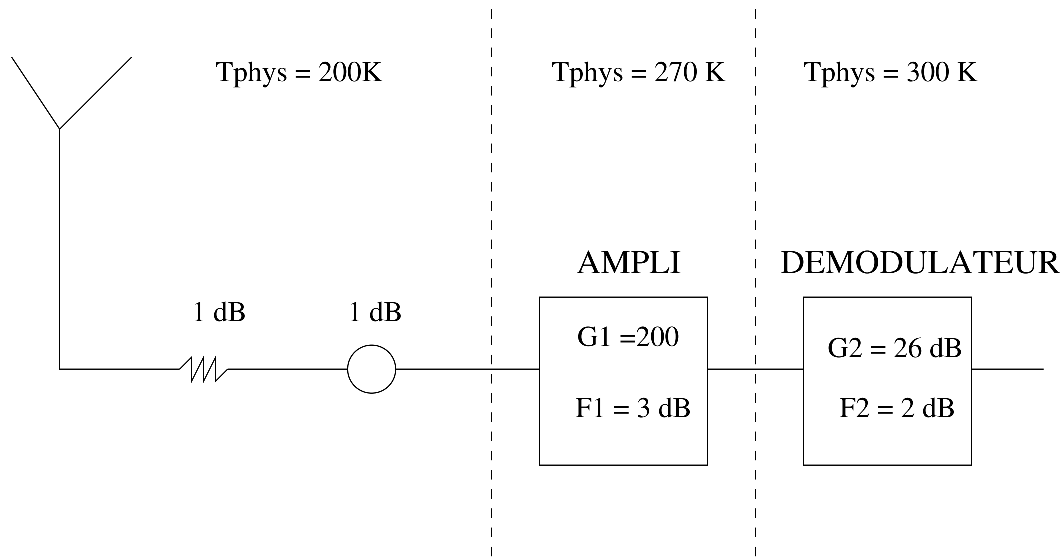
7. Exercises

1. Let a receiving antenna connected to a receiver. The receiver has a $10\,$[dB] noise figure, a $80\,$[dB] gain and a $6\,$[MHz] bandwidth. The input signal power, S_i, is $10^{-11}\,$[W].
   (a) The antenna noise temperature, T_a is equal to $150\,$[K]. Determine:
      i. the output noise temperature of the receiver,
      ii. the output noise temperature of the system (antenna + receiver),
      iii. the noise power at the output of the receiver and
      iv. the signal to noise ratio at the input and at the output of the receiver.
   (b) A pre-amplifier is inserted between the antenna and the receiver, in order to enhance the signal to noise ratio at the output. It has a $3\,$[dB] noise factor, a $13\,$[dB] gain and a $6\,$[MHz] bandwidth. Determine the effective noise temperature and the noise figure of the group (pre-amplifier & receiver), the effective noise temperature of the global system, the noise power at the output of the receiver and the signal to noise ratio at the output.
   (c) Repeat steps (a) and (b) when the effective noise temperature of the antenna T_a is now equal to $8000\,$[K].

2. A micro-wave receiver used for spatial telecommunications contains the following elements in sequence: the antenna, a MASER (microwave amplification by stimulated emission of radiation), a TWT (travelling wave tube) and a mixer ampli IF, described by the following parameters:
   - antenna temperature: $T_a = 14\,$[K]
   - MASER: $G = 30\,$[dB] and $T = 4\,$[K]
   - TWT: $G = 100$, $F_0 = 6\,$[dB]
   - mixer ampli IF: $G = 40\,$[dB], $F_0 = 12\,$[dB]

   Compute the noise figure of the sequence.

3. (May 2004) Let an antenna with a $200\,[\mathrm{K}]$ noise temperature. The receiving system consists of an antenna, an insulator, a wave guide, an amplifier and a demodulator. We suppose that the wave guide and the insulator may be considered as resistive attenuators only, with a $1\,[\mathrm{dB}]$ attenuation factor. the receiver bandwidth is $6\,[\mathrm{MHz}]$. The signal power at the input of the system is $-80\,[\mathrm{dBW}]$.



(a) Compute the noise temperature of the system.
(b) Compute the noise figure and the merit figure of the system.
(c) Compute the ratio between the noise power at the input and the noise power at the output of the system. Compute also the signal to noise ratio at the input and the output of the system.
(d) What do you think of the order of the amplifier and the demodulator in the chain? Is this order optimal? Explain your answer with computation.

Answer

1. (a) $T_{e\,\text{receiver}} = 2610\,[\text{K}]$, $T_{e\,\text{system}} = 2760\,[\text{K}]$, $N_{out} = 22,9\,[\mu\text{W}]$,

$$\left(\frac{S}{N}\right)_{in} = 29,1\,[\text{dB}]$$

$$\left(\frac{S}{N}\right)_{out} = 16,4\,[\text{dB}]$$

(b) $T_{e\,\text{pre-ampli+receiver}} = 419,5\,[\text{K}]$, $F_{\text{pre-ampli+receiver}} = 2,45$, $T_{e\,\text{system}} = 569,5\,[\text{K}]$, $N_{out} = 94,1\,[\mu\text{W}]$,

$$\left(\frac{S}{N}\right)_{out} = 23,3\,[\text{dB}]$$

(c) Part (a): $T_{e\,\text{system}} = 10610\,[\text{K}]$, $N_{out} = 87,8\,[\mu\text{W}]$,

$$\left(\frac{S}{N}\right)_{out} = 10,6\,[\text{dB}]$$

Part (b): $T_{e\,\text{systeme}} = 8419,5\,[\text{K}]$, $N_{out} = 1,4\,[m\text{W}]$,

$$\left(\frac{S}{N}\right)_{out} = 11,6\,[\text{dB}]$$

2. $F = 1,01794$.

Digital modulations (part 1)

Outline:

1. Digital modulations definition
2. Classic linear modulations
   - 2.1 Power spectral density
   - 2.2 Amplitude digital modulation (ASK)
   - 2.3 Phase digital modulation (PSK)
   - 2.4 Quadrature phase digital modulation (QPSK)
3. Exercises

1. Digital modulations definition

Let $m(t)$ a baseband signal (NRZ typically).

Modulated digital signal :

$$s(t) = Re\left\{\psi\left[m(t)\right] e^{j(2\pi f_c t + \varphi_c)}\right\}$$

where $\psi(.) = \psi_I(.) + j\,\psi_Q(.)$ defines the modulation type.

Other form for the modulated signal:

$$\begin{aligned} s(t) \quad &= \quad \psi_I\left[m(t)\right] \cos\left(2\pi f_c t + \varphi_c\right) \\ &\quad\quad -\psi_Q\left[m(t)\right] \sin\left(2\pi f_c t + \varphi_c\right) \end{aligned}$$

or

$$s(t) = \left\|\psi\left[m(t)\right]\right\| \cos\left(2\pi f_c t + \varphi_c + arg\,\psi\left[m(t)\right]\right)$$

Generally, we may distinguish two modulation types :

- **Linear modulations** :

$$\psi\left[m\left(t\right)\right] = \text{linear fonction of } m(t)$$

- **Angular modulations** :

$$\psi\left[m\left(t\right)\right] = e^{j\varphi[m(t)]}$$

where $\varphi\left[m\left(t\right)\right]$ = linear function of $m\left(t\right)$.

Linear digital modulations :

$$s\left(t\right) = Re\left\{e^{j(2\pi f_c t + \varphi_c)} \sum_{k=-\infty}^{+\infty} d_k\left(t\right) e^{j(\theta_k - 2\pi f_c kT)}\right\}$$

where the $d_k\left(t\right)$ signals contain the information to be transmitted and θ_k is a constant phase.

Two types of linear modulations :

- **Classic** modulations : $\theta_k = 2\pi f_c kT$
- **Offset** modulations : $\theta_k = 2\pi f_c kT + k\frac{\pi}{2}$

2. Classic linear modulations

Modulated signal :

$$s(t) = Re\left\{e_s(t)\, e^{j(2\pi f_c t + \varphi_c)}\right\}$$

Complex envelope of the modulated signal :

$$\begin{aligned} e_s(t) &= \sum_{k=-\infty}^{+\infty} d_k(t) \\ &= \sum_{k=-\infty}^{+\infty} D_k\, g_k(t - kT) \end{aligned}$$

where

- $g_k(t)$ = **real** modulating waveform signal. For the sake of simplicity, we will choose $g_k(t) = g(t), \forall k$.
- D_k = **complex** random variable which contains the digital information to be transmitted : $D_k = A_k + j B_k$

Where

$$e_s(t) = s_I(t) + j\, s_Q(t)$$

So,

$$\begin{aligned} s_I(t) &= \sum_{k=-\infty}^{+\infty} A_k\, g(t-kT) \\ s_Q(t) &= \sum_{k=-\infty}^{+\infty} B_k\, g(t-kT) \end{aligned}$$

Other form for the modulated signal :

$$s(t) = s_I(t)\cos(2\pi f_c t + \varphi_c) - s_Q(t)\sin(2\pi f_c t + \varphi_c)$$

Or

$$\begin{aligned} s(t) &= \left[\sum_{k=-\infty}^{+\infty} A_k\, g(t-kT)\right]\cos(2\pi f_c t + \varphi_c) \\ &\quad - \left[\sum_{k=-\infty}^{+\infty} B_k\, g(t-kT)\right]\sin(2\pi f_c t + \varphi_c) \end{aligned}$$

→ Quadrature modulation for two digital baseband signals (NRZ type).

2.1 Power spectral density (PSD)

Reminder : $X(t)$ is wide sense stationary (WSS) if :

- μ_X independent of t, and
- $\Gamma_{XX}(t,t-\tau) = E\{X(t)X^*(t-\tau)\}$ depends only on $\tau \rightarrow \Gamma_{XX}(\tau)$

PSD of the modulated signal $S(t)$

Such that, $s(t)$ non-stationary :

$$S(t) = Re\left\{M(t)\, e^{j2\pi f_c t}\right\}$$

We have to stationarize :

$$S(t) = Re\left\{M(t)\, e^{j(2\pi f_c t+\Theta)}\right\}$$

where Θ = Uniform random variable on $[0, 2\pi[$.

We showed that

$$\mu_S = 0$$

$$\Gamma_{SS}(t, t-\tau) = \frac{1}{2} Re\{\Gamma_{MM}(t, t-\tau)\, e^{j2\pi f_c \tau}\}$$

$\rightarrow$ If $M(t)$ is WSS, then $S(t)$ is WSS.

We can write

$$\Gamma_{SS}(\tau) = \frac{1}{4}\left[\Gamma_{MM}(\tau)\, e^{j2\pi f_c \tau} + \Gamma^*_{MM}(\tau)\, e^{-j2\pi f_c \tau}\right]$$

we deduce finally

$$\gamma_S(f) = \frac{\gamma_M(f - f_c) + \gamma^*_M(-f - f_c)}{4}$$

PSD of the complex envelope $M(t)$

$$M(t) = \sum_{k=-\infty}^{+\infty} D_k \, g(t - kT)$$

D_k is characterized by

- its mean : $\mu_D = E\{D_k\}$
- its variance : $\sigma_D^2 = E\{(D_k - \mu_D)(D_k - \mu_D)^*\} = E\{||D_k||^2\}$

If the random variables D_k are not correlated, then

$$\gamma_M(f) = \frac{\|\mathcal{G}(f)\|^2}{T} \left[\sigma_D^2 + ||\mu_D||^2 \sum_{m=-\infty}^{+\infty} \frac{1}{T} \delta\left(f - \frac{m}{T}\right) \right]$$

which is real and symmetric.

We can write

$$\gamma_S(f) = \frac{\gamma_M(f - f_c) + \gamma_M(f + f_c)}{4}$$

2.2 Amplitude digital modulation (ASK: Amplitude Shift Keying)

<u>Caracteristics</u>

- D_k purely real ($B_k \equiv 0$). So,

$$e_s(t) = s_I(t) = \sum_{k=-\infty}^{+\infty} A_k\, g(t - kT)$$

purely real ($s_Q(t) = 0$).

- Rectangular modulating waveform impulse :

$$g(t) = rect_{(0,T)}(t)$$

Envelope and phase of the modulated signal

We remind ourselves that,

$$e_s(t) = a(t)\, e^{j\varphi(t)}$$

We can then write

$$A_k = |A_k| e^{j\frac{\pi}{2}(1-sign(A_k))}$$

So,

$$\begin{aligned} a(t) &= \sum_{k=-\infty}^{+\infty} |A_k|\, rect_{(0,T)}(t-kT) \\ \varphi(t) &= \sum_{k=-\infty}^{+\infty} \frac{\pi}{2}(1-sign(A_k))\, rect_{(0,T)}(t-kT) \end{aligned}$$

Observations :

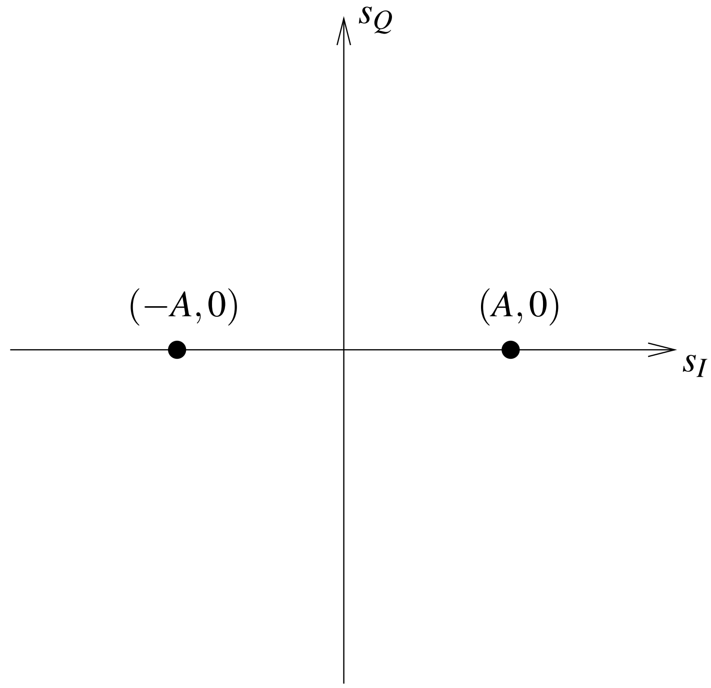
- The signal envelope is **not constant**.
- Phase jumps of $\pi \rightarrow$ **discontinuous** phase.

Exemple: ASK-2 Modulation

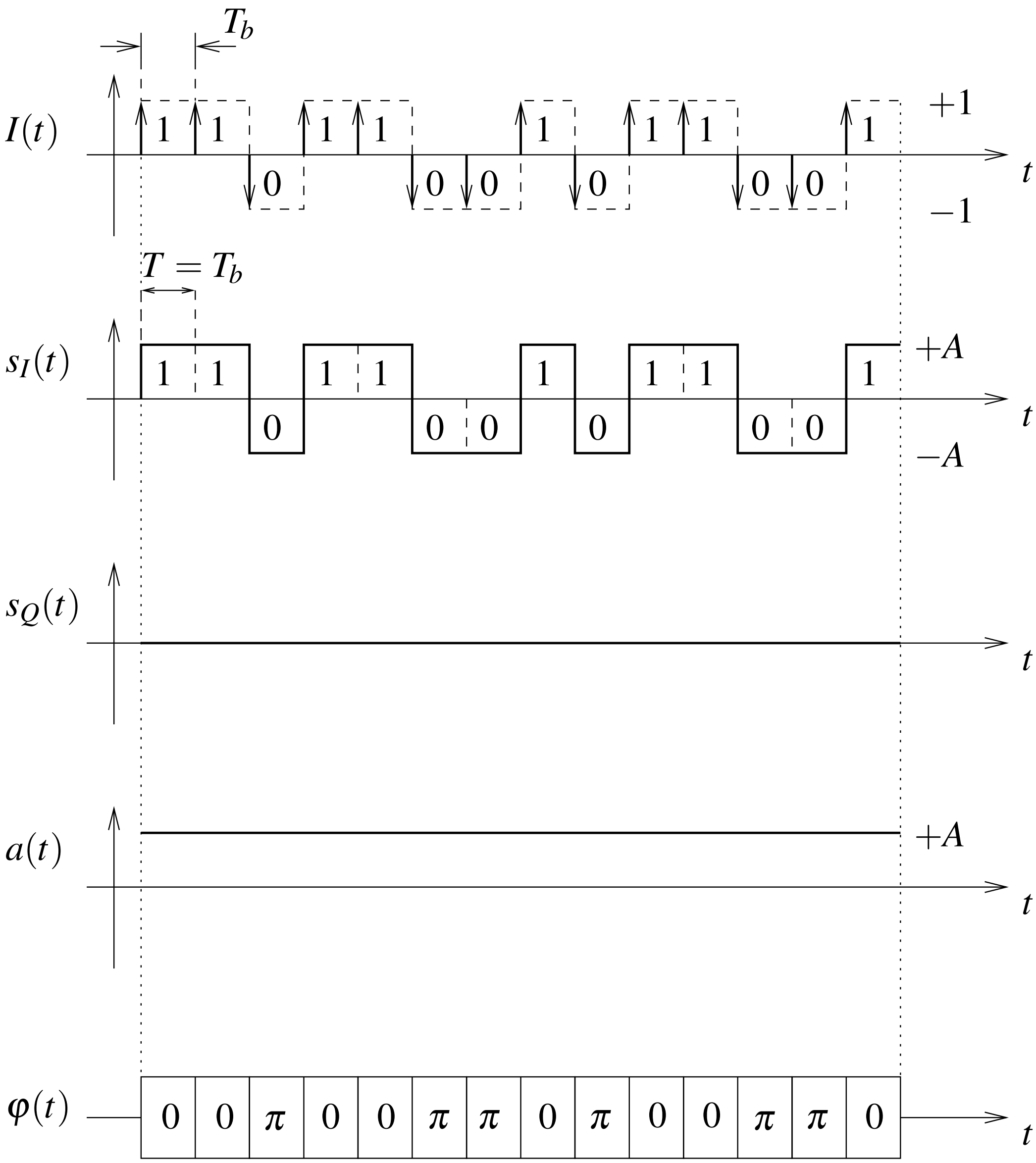
$$\begin{cases} A_k \in \{+A, -A\} \\ T = T_b \end{cases}$$

$\rightarrow$ constant envelope.

Constellation diagram $\equiv$ Complex plan of $e_s(t)$



Signals



T_b
$I(t)$
1 1 0 1 1 0 0 1 0 1 1 0 0 1
+1
−1
t
$T = T_b$
$s_I(t)$
+A
−A
$s_Q(t)$
$a(t)$
$\varphi(t)$
0 0 π 0 0 π π 0 π 0 0 π π 0

Power spectral density

Hypothesis : both signals $\pm A$ have an equal probability.

Mean

$$\mu_D = E\{D_k\} = 0$$

Variance

$$\sigma_D^2 = E\{||D_k||^2\} = E\{A_k^2\} = A^2$$

Modulating waveform signal

$$g(t) = rect_{(0,T_b)}(t) \rightleftharpoons \mathscr{G}(f) = e^{-j2\pi f \frac{T_b}{2}} T_b sinc(fT_b)$$

PSD of the complex envelope

$$\gamma_{e_s}(f) = A^2 T_b sinc^2(fT_b)$$

PSD of the modulated signal

$$\gamma_s(f) = \frac{A^2 T_b}{4} \left\{ sinc^2\left[(f-f_c)T_b\right] + sinc^2\left[(f+f_c)T_b\right] \right\}$$

2.3 Digital phase modulation (PSK: Phase Shift Keying)

Characteristics

General shape for the modulated signal

$$s(t) = A \sum_{k=-\infty}^{+\infty} rect_{(0,T)}(t-kT)\cos(2\pi f_c t + \varphi_c + \psi_k)$$

where $\psi_k =$ constant random variable on$[kT,(k+1)T[$:

$$\psi_k \in \left\{ \psi |\, \psi = \varphi_0 + i\frac{2\pi}{M}, i = 0, ..., M-1 \right\}$$

Shaping of the modulated signal

$$\begin{aligned} s(t) &= A\sum_{k=-\infty}^{+\infty} rect_{(0,T)}(t-kT) \\ &\quad [\cos(2\pi f_c t+\varphi_c)\cos\psi_k - \sin(2\pi f_c t+\varphi_c)\sin\psi_k] \\ &= \left[\sum_{k=-\infty}^{+\infty} A\cos\psi_k\, rect_{(0,T)}(t-kT)\right]\cos(2\pi f_c t+\varphi_c) \\ &- \left[\sum_{k=-\infty}^{+\infty} A\sin\psi_k\, rect_{(0,T)}(t-kT)\right]\sin(2\pi f_c t+\varphi_c) \end{aligned}$$

So,

$$\begin{aligned} e_s(t) &= s_I(t)+j\,s_Q(t) \\ &= A\sum_{k=-\infty}^{+\infty} rect_{(0,T)}(t-kT)\,(\cos\psi_k + j\sin\psi_k) \end{aligned}$$

→ **Classic linear** digital modulation with

$$\begin{aligned} D_k &= A\,e^{j\psi_k} \\ g(t) &= rect_{(0,T)}(t) \end{aligned}$$

Envelope and phase of the modulated signal

$$a(t) = A \sum_{k=-\infty}^{+\infty} rect_{(0,T)}(t-kT)$$

$$\varphi(t) = \sum_{k=-\infty}^{+\infty} \psi_k \, rect_{(0,T)}(t-kT)$$

Observations :

- **Constant** signal envelope.
- Phase jump $\rightarrow$ **discontinuous** phase.

Exemple: PSK-2 or BPSK modulations

$$\psi_k \in \{0, \pi\}$$

$$\rightarrow D_k \in \left\{A\, e^{j0}, A\, e^{j\pi}\right\}$$

BPSK $\equiv$ ASK-2 $\rightarrow$ Identical constellation diagram

Power spectral density

Identical to the ASK-2 modulation:

$$\gamma_s(f) = \frac{A^2 T_b}{4} \left\{sinc^2\left[(f - f_c)\, T_b\right] + sinc^2\left[(f + f_c)\, T_b\right]\right\}$$

2.4 Quadrature phase digital modulation (QPSK: Quadrature Phase Shift Keying)

Characteristics

- Phase modulation with 4 states (PSK-4) :

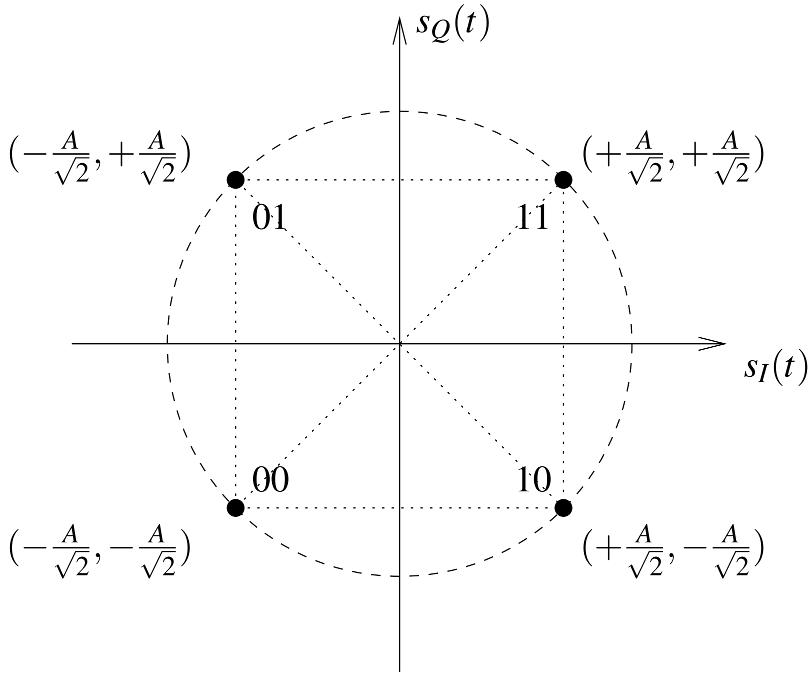
$$\psi_k \in \{-3\pi/4, -\pi/4, +\pi/4, +3\pi/4\}$$

$$\rightarrow D_k \in \left\{A\,e^{-j\frac{3\pi}{4}}, A\,e^{-j\frac{\pi}{4}}, A\,e^{j\frac{\pi}{4}}, A\,e^{j\frac{3\pi}{4}}\right\}$$

- Rectangular modulating waveform impulse :

$$g(t) = rect_{(0,T)}(t) = rect_{(0,2T_b)}(t)$$

Constellation diagram



Inphase and quadrature components

Let

$$I(t) = \sum_{k=-\infty}^{+\infty} I_k \, \delta(t - kT_b)$$

where

$$I_k = \begin{cases} +1 & \text{for the bit 1} \\ -1 & \text{for the bit 0} \end{cases}$$

We build the two sequences

$$\begin{aligned} s_I(t) &= \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} I_{2k} \, g(t - kT) = \sum_{k=-\infty}^{+\infty} A_k \, g(t - kT) \\ s_Q(t) &= \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} I_{2k+1} \, g(t - kT) = \sum_{k=-\infty}^{+\infty} B_k \, g(t - kT) \end{aligned}$$

where $T = 2T_b$, and

$$A_k = I_{2k} \frac{A}{\sqrt{2}} \;\rightarrow \text{even bits for the sequence } I_k$$

$$B_k = I_{2k+1} \frac{A}{\sqrt{2}} \;\rightarrow \text{odd bits of the sequence } I_k$$

Modulated signal envelope and phase

$$\begin{aligned} e_s(t) &= s_I(t) + j\,s_Q(t) \\ &= \sum_{k=-\infty}^{+\infty} (A_k + j\,B_k)\; rect_{(0,T)}(t-kT) \\ &= \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} (I_{2k} + j\,I_{2k+1})\; rect_{(0,T)}(t-kT) \end{aligned}$$

We can then write

$$\begin{aligned} a(t) &= \sqrt{s_I^2(t) + s_Q^2(t)} \\ &= \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} \sqrt{I_{2k}^2 + I_{2k+1}^2}\, rect_{(0,T)}(t-kT) \\ &= A \sum_{k=-\infty}^{+\infty} rect_{(0,T)}(t-kT) \end{aligned}$$

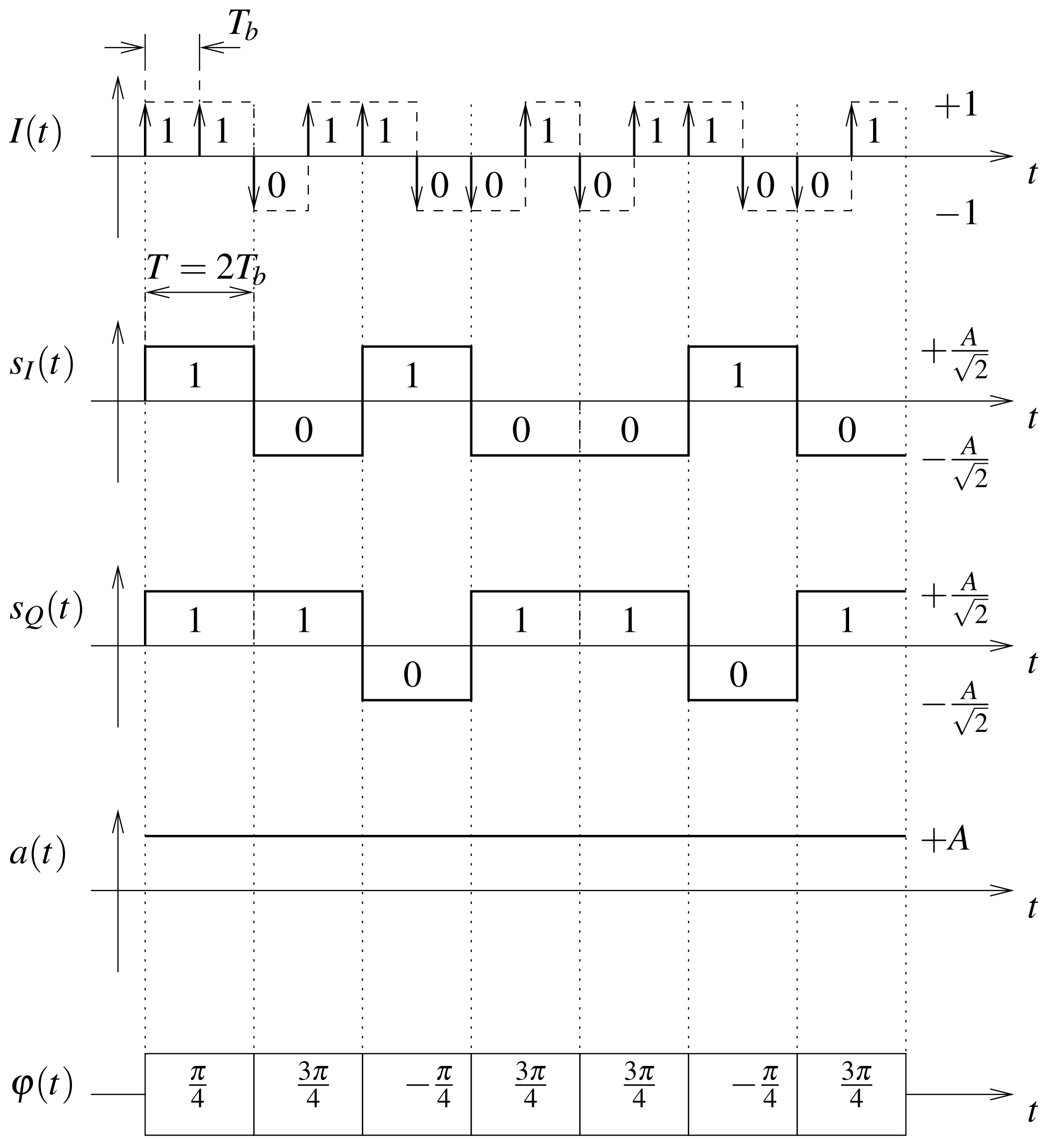
and

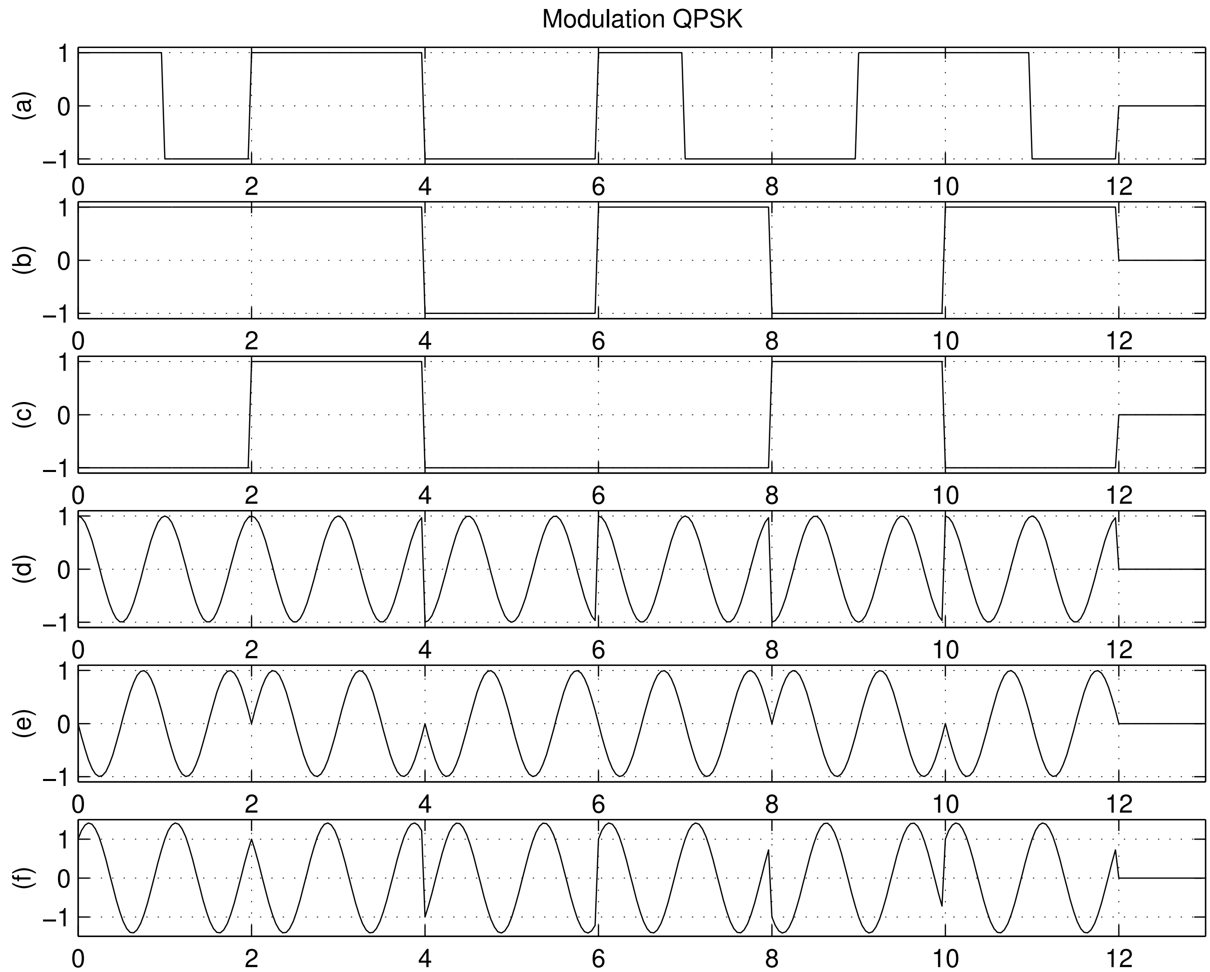
$$\varphi(t) = \sum_{k=-\infty}^{+\infty} rect_{(0,T)}(t-kT)\; \arctan\left(\frac{I_{2k+1}}{I_{2k}}\right)$$

Observations :

- **Constant**. signal envelope
- Phase jumps of π or $\pi/2 \rightarrow$ **discontinuous** phase.

Signals





(a) Binary sequence $I(t)$

(b) $s_I(t)$

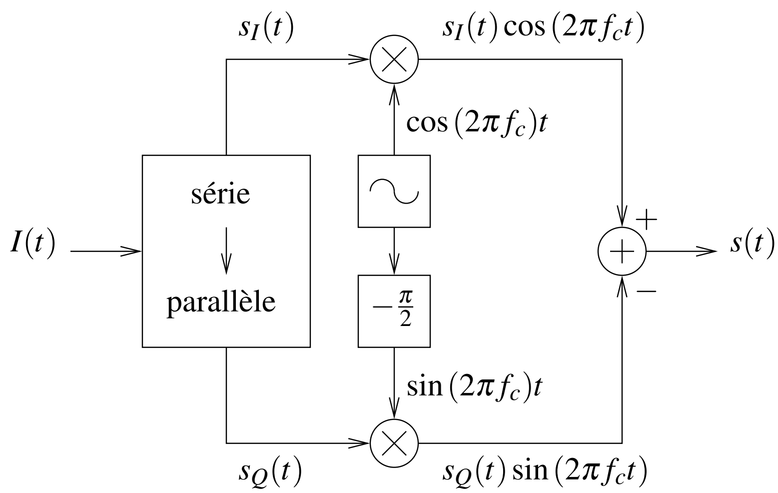
(c) $s_Q(t)$

(d) $s_I(t)\cos(2\pi f_c t)$

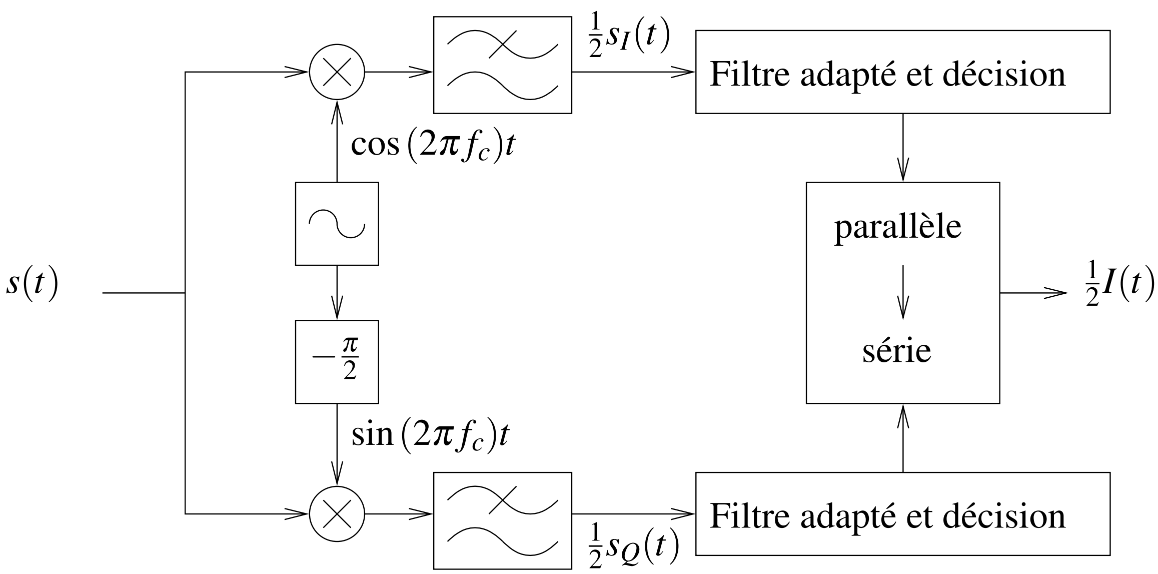
(e) $s_Q(t)\sin(2\pi f_c t)$

(f) Modulated signal $s(t)$

QPSK Modulator



QPSK Demodulator



Power spectral density

Hypothesis : The four states have an equal probability.

$$D_k = \left(\pm \frac{A}{\sqrt{2}}, \pm \frac{A}{\sqrt{2}} \right)$$

Mean

$$\mu_D = E\{D_k\} = 0$$

Variance

$$\sigma_D^2 = E\{||D_k||^2\} = A^2$$

Modulating waveform signal

$$g(t) = rect_{(0,2T_b)}(t) \rightleftharpoons \mathscr{G}(f) = 2T_b \, e_b^{-j2\pi f T_b} \, sinc(2fT_b)$$

Complex envelope signal

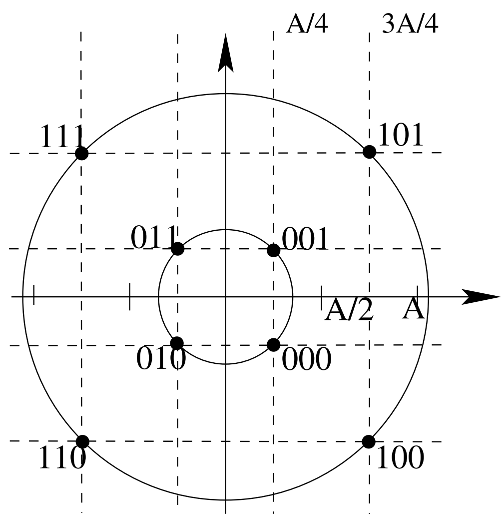
$$\gamma_{e_s}(f) = 2A^2 T_b \, sinc^2(2fT_b)$$

Modulated signal PSD

$$\gamma_s(f) = \frac{A^2 T_b}{2} \left\{ sinc^2\left[(f - f_c)\, 2T_b\right] + sinc^2\left[(f + f_c)\, 2T_b\right] \right\}$$

3. Exercises

1. Consider a classic linear modulation characterized by the following constellation diagram



The modulating waveform impulse is rectangular and extend from 0 to T. The carrier frequency is given by f_c.

(a) Represent graphically the temporal evolution of the inphase component, the quadrature component, the amplitude and the phase of the modulated signal for the binary sequence: 101111010011000.

(b) If the symbols beginning by 0 have a probability two times higher than those beginning by 1, compute the power spectral density of the modulated signal.

(c) If a bit has a $10\mu s$ duration, determine the bit rate R_b and the bandwidth of the modulated signal.

2. We achieve a classic linear digital modulation with a circuit comprising an 8-ways switch activated every $3T_b$ seconds depending on the binary sequence to be transmitted (T_b is the inverse of the bit rate R_b). The 8 inputs of the switch receive signals $s_{000}(t)$, $s_{001}(t)$, ... derived from the carrier $\cos(2\pi f_c t)$;

$$
\begin{aligned}
s_{000}(t) &= 2\cos\left(2\pi f_c t + \frac{5\pi}{6}\right) \\
s_{001}(t) &= \sqrt{3}\cos(2\pi f_c t - \pi) \\
s_{010}(t) &= -2\sin\left(2\pi f_c t - \frac{4\pi}{3}\right) \\
s_{011}(t) &= \sin(\pi - 2\pi f_c t) \\
s_{100}(t) &= 2\sin\left(\frac{2\pi}{3} - 2\pi f_c t\right) \\
s_{101}(t) &= -\sqrt{3}\sin\left(2\pi f_c t - \frac{\pi}{2}\right) \\
s_{110}(t) &= -2\cos\left(2\pi f_c t - \frac{5\pi}{6}\right) \\
s_{111}(t) &= 3\cos\left(2\pi f_c t + \frac{\pi}{2}\right)
\end{aligned}
$$

The switch delivers an output signal $s(t)$ which is the modulated digital signal.

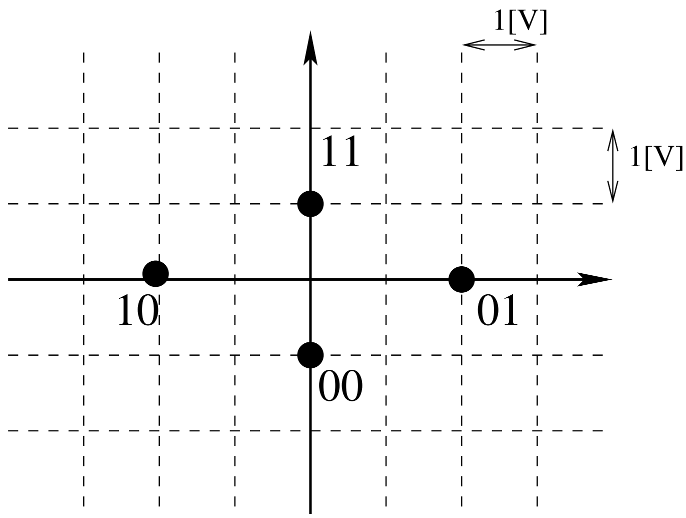
(a) Determine and draw the constellation diagram for this modulation. What is the number of states in this constellation?

(b) Draw the inphase component, the quadrature component, the envelope and the phase of the modulated signal for the following binary sequence : 011001101011000010111.

(c) Expressed, in terms of R_b, the bandwidth of the modulated signal.

(d) Determine the power spectral density of the modulated signal if the symbols have the same probability and are not correlated.

3. Let the classic linear modulation with the following constellation diagram



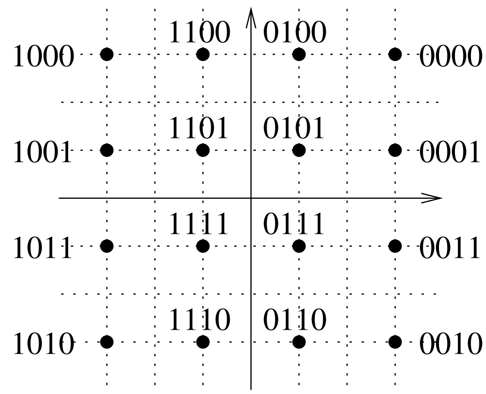
The emission probability for the symbols are $p(00) = p(11) = 1/6$, $p(10) = p(01) = 1/3$ and the symbols are not correlated. The modulating waveform is a rectangular impulse with an unit amplitude and a duration of $2T_b$ where T_b is the bit duration.
The resulting modulated signal can be written $s(t) = I(t) - Q(t)$ with

$$\begin{cases} I(t) = s_I(t)\cos(2\pi f_c t + \varphi) \\ Q(t) = s_Q(t)\sin(2\pi f_c t + \varphi) \end{cases}$$

where $s_I(t)$ and $s_Q(t)$ are respectively the inphase and quadrature components of the modulated signal and φ is a random variable with an uniform probability density function on the interval $[0, 2\pi]$.

(a) Compute the value of the inphase and quadrature components for the modulated signal for the following binary sequence: 01011110000110.
(b) Compute, in terms of the bit rate R_b ($=1/T_b$), the bandwidth of the modulated signal.
(c) Compute the power spectral density of the modulated signal $s(t)$.
(d) Compute the power spectral density of the signals $I(t)$ and $Q(t)$.
(e) With the help of the two previous points, determine the relation between the spectral density $\gamma_I(f)$, $\gamma_Q(f)$ and $\gamma_s(f)$. What's your conclusion concerning the correlation between the signals $I(t)$ and $Q(t)$?.

4. Let the classic linear digital modulation 16-QAM (or 16-QASK) whose constellation diagram is given by



The modulating waveform impulse is a $4T_b$ duration rectangular signal.

(a) For the binary sequence 1011001011011001, determine the inphase and quadrature components, the envelope and the phase of the modulated signal.

(b) Determine the power spectral density of the modulated signal (hypothesis : all symbols have an equal probability).

(c) Determine the bandwidth of $s(t)$ (in terms of of R_b) and the spectral efficiency η.

(d) Determine the type of modulation.

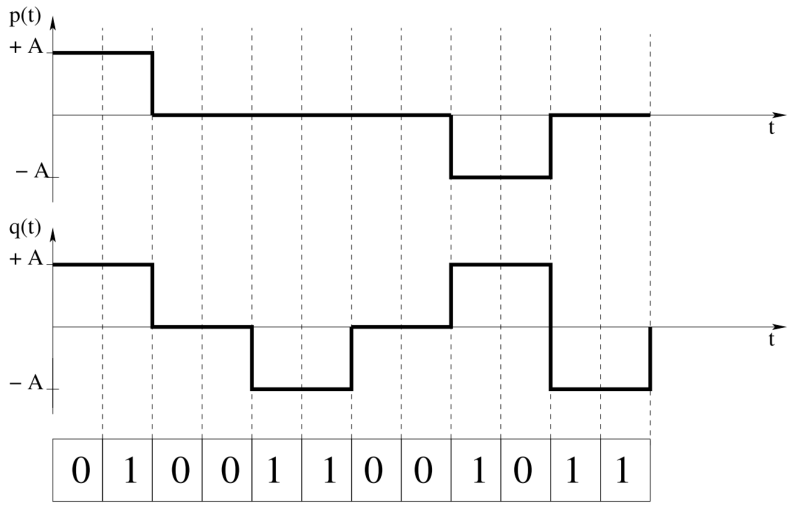
(e) What do you think of the repartition of the symbol in the constellation diagram? Is it wise in terms of bandwidth, power consumption and/or error probability?

(f) If the modulated signal is expressed by $s(t) = I(t) - Q(t)$ with

$$\begin{cases} I(t) = s_I(t)\cos(2\pi f_c t + \varphi) \\ Q(t) = s_Q(t)\sin(2\pi f_c t + \varphi) \end{cases}$$

What is the relation between $\gamma_I(f)$, $\gamma_Q(f)$ and $\gamma_s(f)$. Are they correlated?

5. The signal 010011001011 is transmitted with a classic linear modulation and the inphase ($p(t)$) and quadrature ($q(t)$) components are depicted in the following diagram:



(a) Draw the amplitude and phase of the modulated signal.
(b) Draw the constellation diagram of this modulation.
(c) Compute, in terms of the bit rate R_b ($=1/T_b$), the bandwidth of the modulated signal.
(d) If all symbols have the same probability, compute the power spectral density of the modulated signal $s(t)$.

Answer

1. (a) –
   (b)
   $$\gamma_s(f) = 10T_b \left\{sinc^2[4T_b(f-f_c)] + sinc^2[4T_b(f+f_c)]\right\}$$
   (c) $B = \frac{R_b}{4}$, $\eta = 4\,[\text{b/s/Hz}]$
   (d) Hybrid modulation
   (f)
   $$\gamma_I(f) = \gamma_Q(f) = 5T_b \left\{sinc^2[4T_b(f-f_c)] + sinc^2[4T_b(f+f_c)]\right\} = \frac{1}{2}\gamma_s(f)$$
   So $I(t)$ and $Q(t)$ are not correlated
2. (a) –
   (b)
   $$\gamma_s(f) = \frac{11A^2}{32}T_b \left\{sinc^2[3T_b(f-f_c)] + sinc^2[3T_b(f+f_c)]\right\}$$
   (c) $R_b = 100\,[\text{kb/s}]$, $B = 33.3\,[\text{kHz}]$
3. (a) 8 states
   (b) –
   (c) $B = \frac{R_b}{3}$
   (d)
   $$\gamma_M(f) = 3T_b sinc^2[3T_b f]\left[\frac{69}{16} + \frac{1}{16}\sum_{m=-\infty}^{+\infty}\frac{1}{3T_b}\delta\left(f - \frac{m}{3T_b}\right)\right]$$
   and
   $$\gamma_S(f) = \frac{\gamma_M(f-f_c) + \gamma_M(f+f_c)}{4}$$
4. (a) –
   (b) $B = \frac{R_b}{2}$
   (c)
   $$\gamma_s(f) = \frac{3T_b}{2}\left\{sinc^2[2T_b(f-f_c)] + sinc^2[2T_b(f+f_c)]\right\}$$
   (d)
   $$\gamma_I(f) = \frac{4T_b}{3}\left\{sinc^2[2T_b(f-f_c)] + sinc^2[2T_b(f+f_c)]\right\}$$
   $$\gamma_Q(f) = \frac{T_b}{6}\left\{sinc^2[2T_b(f-f_c)] + sinc^2[2T_b(f+f_c)]\right\}$$
   (e)
   $$\gamma_s(f) = \gamma_I(f) + \gamma_Q(f)$$
   So $\gamma_I(f)$ and $\gamma_Q(f)$ are not correlated.

Digital modulations (part 2)

<u>Outline</u>:

1. Reminder
2. Offset modulation
   2.1 Power spectral Density (PSD)
   2.2 OQPSK modulation
   2.3 MSK modulation
3. Exercises

1. Reminder

Linear digital modulation:

$$s(t) = Re\left\{ e^{j(2\pi f_c t + \varphi_c)} \sum_{k=-\infty}^{+\infty} d_k(t)\, e^{j(\theta_k - 2\pi f_c kT)} \right\}$$

where the $d_k(t)$ signals contain the information to be transmitted and θ_k is a constant phase.

Two types of linear modulation:

- **classic** modulations: $\theta_k = 2\pi f_c kT$
- **offset** modulations: $\theta_k = 2\pi f_c kT + k\frac{\pi}{2}$

2. Offset modulation

Modulated signal :

$$s(t) = Re\left\{e_s(t)\, e^{j(2\pi f_c t + \varphi_c)}\right\}$$

Complex envelope of the modulated signal:

$$\begin{aligned} e_s(t) &= \sum_{k=-\infty}^{+\infty} d_k(t)\, e^{jk\frac{\pi}{2}} \\ &= \sum_{k=-\infty}^{+\infty} D_k\, p_k(t-kT)\, e^{jk\frac{\pi}{2}} \end{aligned}$$

where

- $p_k(t)$ = **real** modulating waveform signal. For the sake of simplicity, we will choose $p_k(t) = p(t), \forall k$.
- D_k = random variable containing the digital information to be transmitted : $D_k = A_k \rightarrow$ **purely real**
- Choice : $T = T_b$

Consequently,

$$e_s(t) = \sum_{k=-\infty}^{+\infty} A_k\, p(t-kT_b)\, e^{jk\frac{\pi}{2}}$$

Inphase and quadrature components

$$\begin{aligned}
s(t) &= \sum_{k=-\infty}^{+\infty} A_k\, p(t-kT_b)\cos\left(2\pi f_c t + \varphi_c + k\frac{\pi}{2}\right) \\
&= \left[\sum_{k=-\infty}^{+\infty} A_k\, p(t-kT_b)\cos\left(k\frac{\pi}{2}\right)\right]\cos(2\pi f_c t + \varphi_c) \\
&- \left[\sum_{k=-\infty}^{+\infty} A_k\, p(t-kT_b)\sin\left(k\frac{\pi}{2}\right)\right]\sin(2\pi f_c t + \varphi_c)
\end{aligned}$$

So,

$$
\begin{aligned}
s_I(t) &= \sum_{k=-\infty}^{+\infty} A_k\, p(t-kT_b)\cos\left(k\frac{\pi}{2}\right) \\
&= \sum_{k=-\infty}^{+\infty} A_{2k}\,(-1)^k\, p(t-2kT_b)
\end{aligned}
$$

And

$$
\begin{aligned}
s_Q(t) &= \sum_{k=-\infty}^{+\infty} A_k\, p(t-kT_b)\sin\left(k\frac{\pi}{2}\right) \\
&= \sum_{k=-\infty}^{+\infty} A_{2k+1}\,(-1)^k\, p(t-(2k+1)\,T_b)
\end{aligned}
$$

→ $s_I(t)$ and $s_Q(t)$ are shifted by a duration of one bit T_b

→ **Offset** modulation.

2.1 Power spectral Density (PSD)

Modulated signal PSD $S(t)$

$$\gamma_S(f) = \frac{\gamma_M(f - f_c) + \gamma_M^*(-f - f_c)}{4}$$

where $\gamma_M(f)$ is the PSD of the complex envelope.

Complex envelope PSD $M(t)$

$$M(t) = \sum_{k=-\infty}^{+\infty} A_k \, p(t - kT_b) \, e^{jk\frac{\pi}{2}}$$

$\rightarrow$ impossible to compute directly the PSD of $M(t)$.

Complex envelope modulating waveform

$$
\begin{aligned}
s(t) &= Re\{e_s(t)\, e^{j(2\pi f_c t+\varphi_c)}\} \\
&= Re\{e_s(t)\, e^{-j2\pi\frac{t}{4T_b}}\, e^{j\left(2\pi\left(f_c+\frac{1}{4T_b}\right)t+\varphi_c\right)}\} \\
&= Re\{v(t)\, e^{j\left(2\pi f_c' t+\varphi_c\right)}\}
\end{aligned}
$$

where we let

$$
\begin{aligned}
v(t) &= e_s(t)\, e^{-j2\pi\frac{t}{4T_b}} \\
f_c' &= f_c + \frac{1}{4T_b}
\end{aligned}
$$

So,

$$
\gamma_s(f) = \frac{\gamma_v(f-f_c') + \gamma_v^*(-f-f_c')}{4}
$$

Modulating waveform for $v(t)$

$$\begin{aligned} v(t) &= e_s(t)\, e^{-j2\pi\frac{t}{4T_b}} \\ &= \sum_{k=-\infty}^{+\infty} A_k\, p\,(t-kT_b)\; e^{jk\frac{\pi}{2}}\, e^{-j2\pi\frac{t}{4T_b}} \\ &= \sum_{k=-\infty}^{+\infty} A_k\, p\,(t-kT_b)\; e^{-j\frac{\pi}{2T_b}(t-kT_b)} \\ &= \sum_{k=-\infty}^{+\infty} A_k\, h\,(t-kT_b) \end{aligned}$$

where we noted

$$h\,(t) = p\,(t)\; e^{-j\frac{\pi t}{2T_b}} \;\; (\text{ complex !})$$

$\rightarrow$ new modulating waveform :

$$\mathscr{H}\,(f) = \mathscr{P}\left(f + \frac{1}{4T_b}\right)$$

So,

$$\gamma_v\,(f) = \frac{||\mathscr{H}\,(f)||^2}{T_b}\left[\sigma_A^2 + \mu_A^2 \sum_{m=-\infty}^{+\infty} \frac{1}{T_b}\delta\left(f - \frac{m}{T_b}\right)\right]$$

2.2 OQPSK modulation (Offset Quadrature Phase Shift Keying)

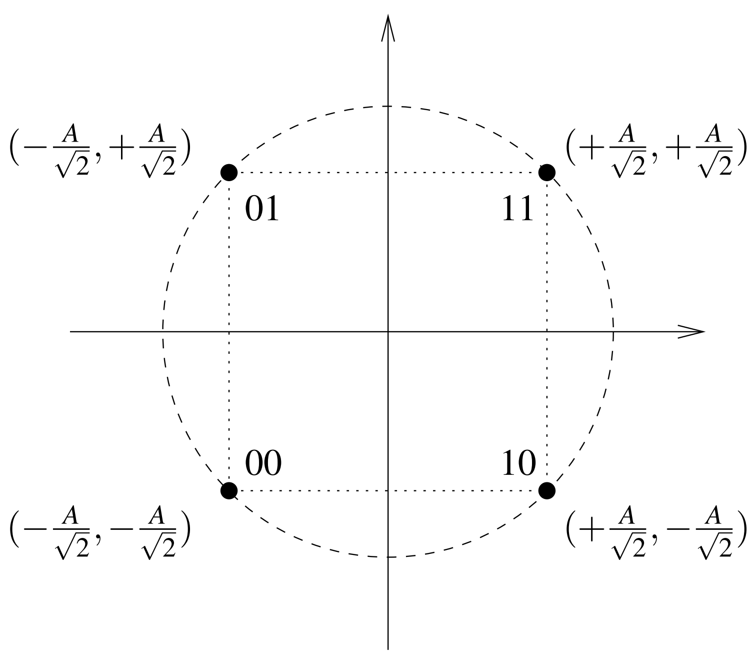
Caracteristics

- 4 states phase modulation → **offset version** of the QPSK modulation

- Rectangular modulating waveform :

$$p(t) = rect_{(0,T)}(t) = rect_{(0,2T_b)}(t)$$

Constellation diagram



Inphase and quadrature components

Let

$$I(t) = \sum_{k=-\infty}^{+\infty} I_k\, \delta(t - kT_b)$$

where

$$I_k = \begin{cases} +1 & \text{for the bit 1} \\ -1 & \text{for the bit 0} \end{cases}$$

We build the two sequences

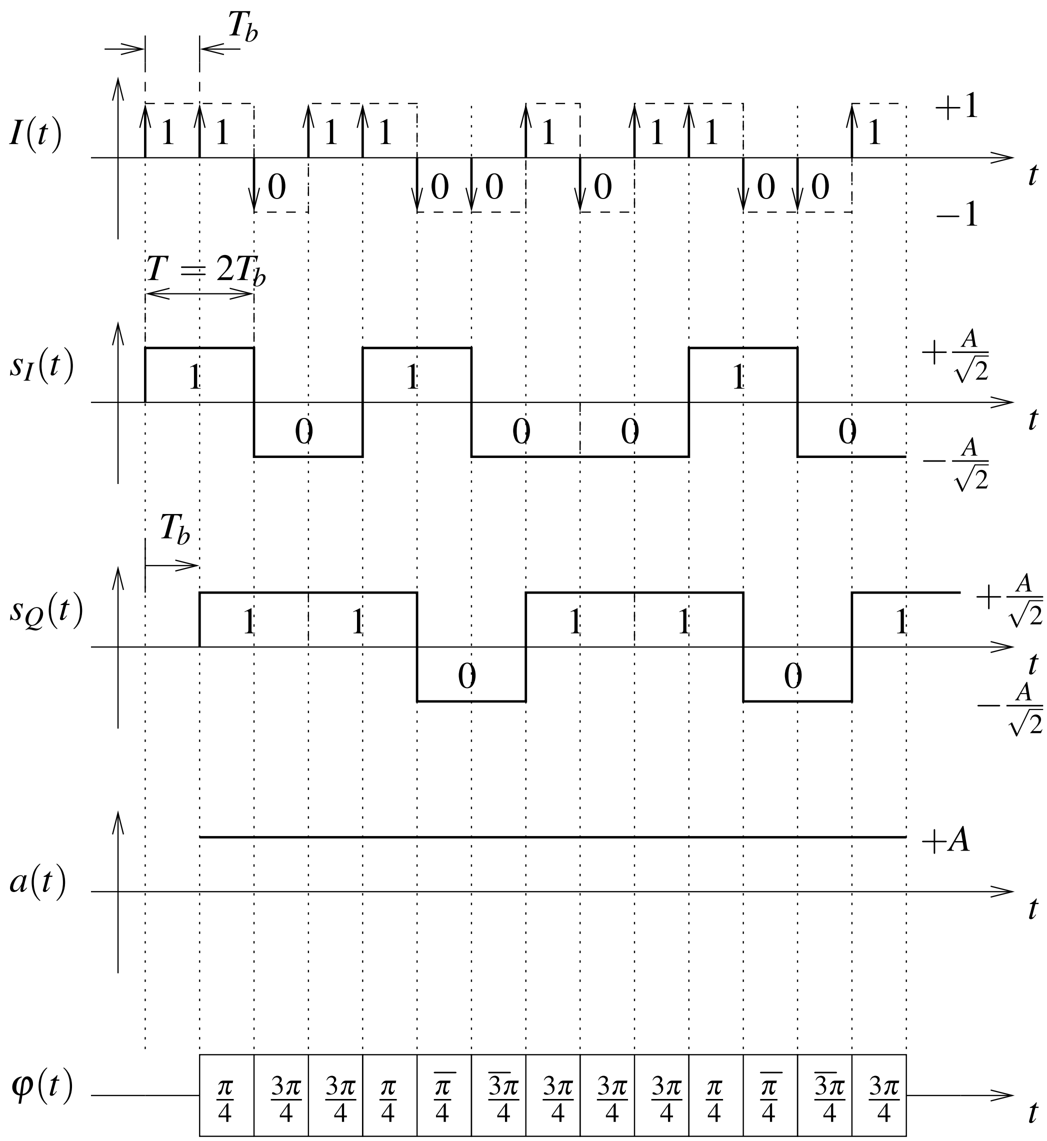
$$\begin{aligned} s_I(t) &= \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} I_{2k}\, g(t - 2kT_b) \\ &= \sum_{k=-\infty}^{+\infty} A_{2k} \,(-1)^k\, g(t - 2kT_b) \end{aligned}$$

$$\begin{aligned} s_Q(t) &= \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{+\infty} I_{2k+1}\, p(t - (2k+1)\,T_b) \\ &= \sum_{k=-\infty}^{+\infty} A_{2k+1} \,(-1)^k\, p(t - (2k+1)\,T_b) \end{aligned}$$

$$A_{2k} = (-1)^k I_{2k} \frac{A}{\sqrt{2}} \;\rightarrow \text{even bits of the sequence } I_k$$

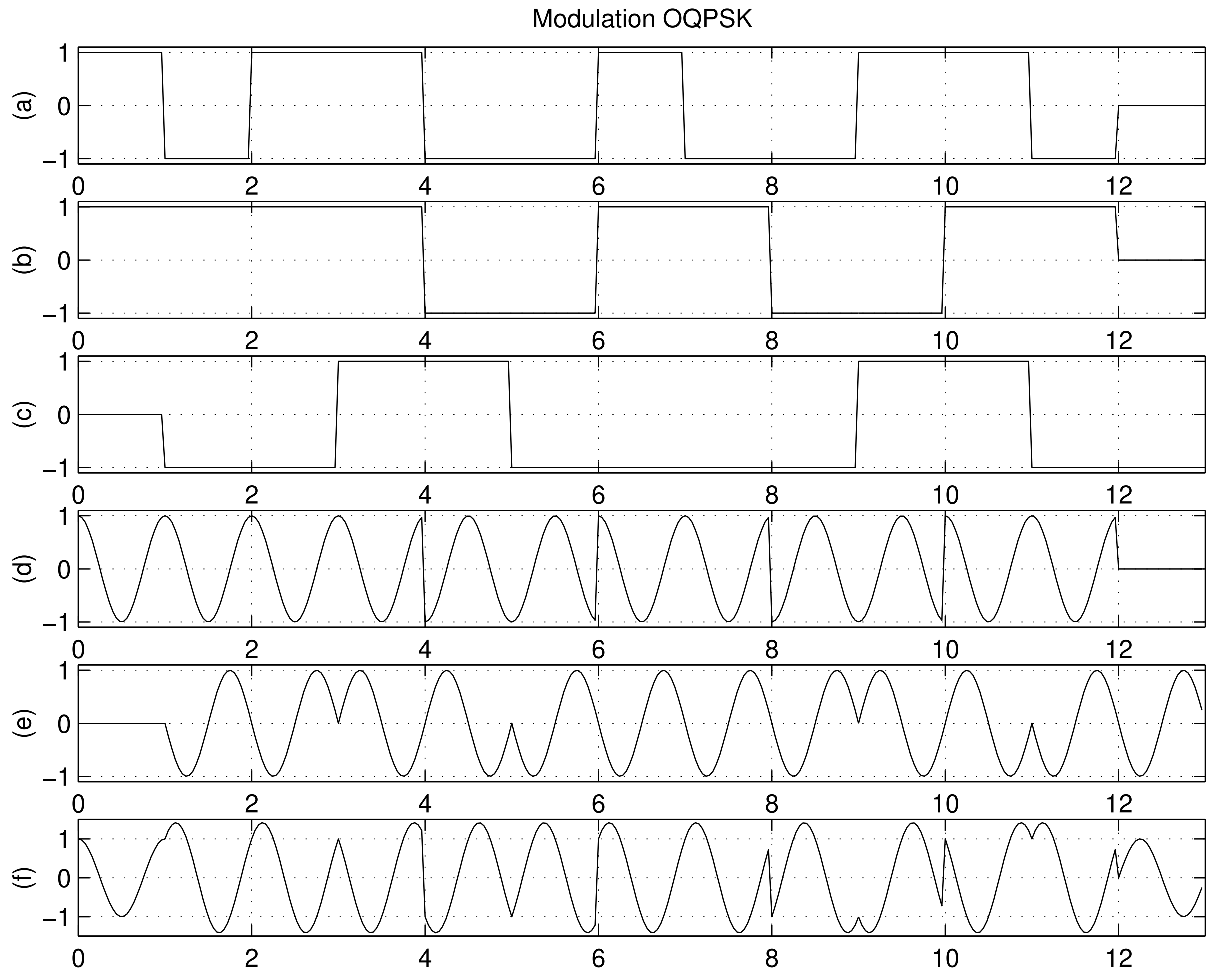
$$A_{2k+1} = (-1)^k I_{2k+1} \frac{A}{\sqrt{2}} \;\rightarrow \text{odd bits of the sequence } I_k$$

Signals



Observations :

- **Constant** signal envelope.
- Maximum $\pi/2$ phase jumps $\rightarrow$ **discontinuous** phase.



(a) Binary sequence $I(t)$

(b) $s_I(t)$

(c) $s_Q(t)$

(d) $s_I(t)\cos(2\pi f_c t)$

(e) $s_Q(t)\sin(2\pi f_c t)$

(f) Modulated signal $s(t)$

Power spectral density

Hypothesis : $A_k = \pm A/\sqrt{2}$ have equal probability.

Mean

$$\mu_A = E\{A_k\} = 0$$

Variance

$$\sigma_A^2 = E\{A_k^2\} = \frac{A^2}{2}$$

Modulating waveform

$$p(t) = rect_{(0,2T_b)}(t) \rightleftharpoons \mathscr{P}(f) = 2T_b\, e_b^{-j2\pi f T_b}\, sinc(2fT_b)$$

$$||\mathscr{H}(f)||^2 = ||\mathscr{P}(f + \frac{1}{4T_b})||^2 = 4T_b^2\, sinc^2\left[\left(f + \frac{1}{4T_b}\right)2T_b\right]$$

Complex envelope PSD

$$\gamma_v(f) = \sigma_A^2 \frac{||\mathscr{H}(f)||^2}{T_b} = 2A^2 T_b\, sinc^2\left[\left(f + \frac{1}{4T_b}\right)2T_b\right]$$

Modulated signal PSD

$$\gamma_s(f) = \frac{A^2 T_b}{2}\left\{sinc^2\left[(f - f_c)\,2T_b\right] + sinc^2\left[(f + f_c)\,2T_b\right]\right\}$$

$\rightarrow$ identical to the PSD of a QPSK modulated signal

2.3 MSK modulation (Minimum Shift Keying)

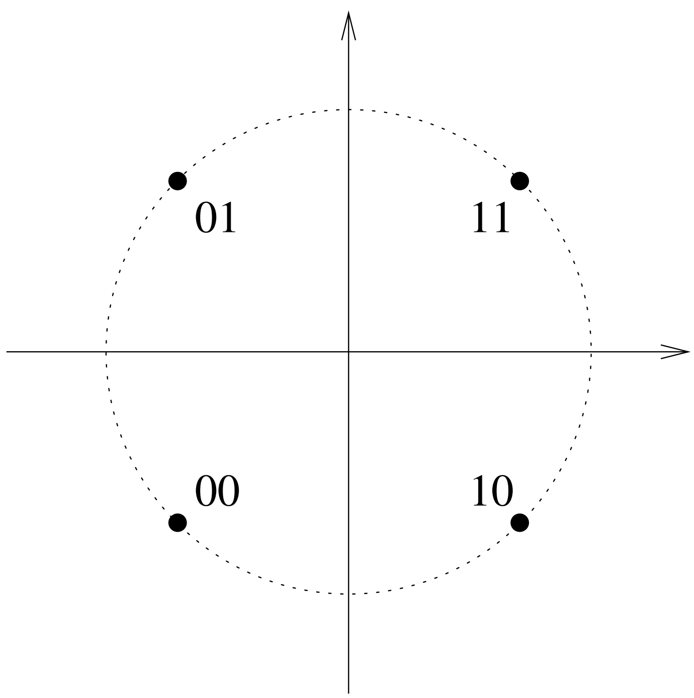
Characteristics

- Identical to the OQPSK modulation, excepted the modulating waveform :

$$p(t) = rect_{(0,2T_b)}(t)\sin\left(\frac{\pi t}{2T_b}\right)$$

- Continuous phase.

Constellation Diagram



Inphase and quadrature components

Let

$$I(t) = \sum_{k=-\infty}^{+\infty} I_k \, \delta(t - kT_b)$$

where

$$I_k = \begin{cases} +1 & \text{for the bit 1} \\ -1 & \text{for the bit 0} \end{cases}$$

We build the two sequences

$$\begin{aligned}
s_I(t) &= A \sum_{k=-\infty}^{+\infty} I_{2k} rect_{(0,2T_b)}(t - 2kT_b) \sin\left[\frac{\pi(t - 2kT_b)}{2T_b}\right] \\
&= \sum_{k=-\infty}^{+\infty} A I_{2k} (-1)^k \, rect_{(0,2T_b)}(t - 2kT_b) \, \sin\left(\frac{\pi t}{2T_b}\right) \\
&= \cos\left(\frac{\pi t}{2T_b} - \frac{\pi}{2}\right) \sum_{k=-\infty}^{+\infty} A_{2k} \, rect_{(0,2T_b)}(t - 2kT_b)
\end{aligned}$$

$$s_Q(t) = \sin\left(\frac{\pi t}{2T_b} - \frac{\pi}{2}\right) \sum_{k=-\infty}^{+\infty} A_{2k+1} rect_{(0,2T_b)}(t - (2k+1)T_b)$$

$$A_{2k} = (-1)^k I_{2k} A \;\; \rightarrow \text{even bits of the sequence } I_k$$

$$A_{2k+1} = (-1)^k I_{2k+1} A \;\; \rightarrow \text{odd bits of the sequence } I_k$$

Envelope and phase of the modulated signal

$$
\begin{aligned}
a(t) &= \sqrt{s_I^2(t) + s_Q^2(t)} \\
&= \sqrt{A^2 \sin^2\left(\frac{\pi t}{2T_b} - \frac{\pi}{2}\right) + A^2 \cos^2\left(\frac{\pi t}{2T_b} - \frac{\pi}{2}\right)} \\
&= A
\end{aligned}
$$

and

$$
\varphi(t) = \arctan\left[\frac{s_Q(t)}{s_I(t)}\right] = \arctan
$$

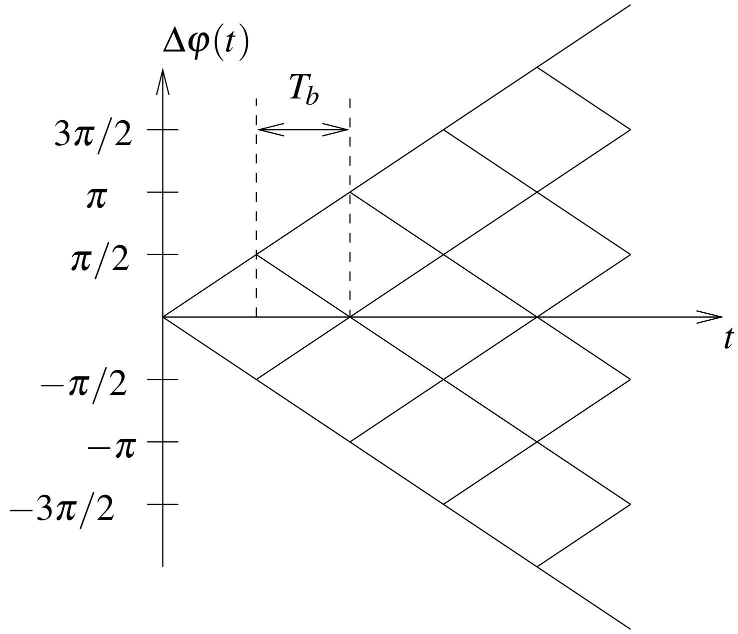
$$
\left\{\tan\left(\frac{\pi t}{2T_b} - \frac{\pi}{2}\right) \frac{\sum_{k=-\infty}^{+\infty} A_{2k}\, rect_{(0,2T_b)}(t - 2kT_b)}{\sum_{k=-\infty}^{+\infty} A_{2k+1}\, rect_{(0,2T_b)}(t - (2k+1)T_b)}\right\}
$$

Temporal variation of the phase:

$$
\Delta\varphi(t) = \pm\frac{\pi t}{2T_b}
$$

$\rightarrow$ The phase varies linearly of $\frac{\pi}{2}$ during the period T_b .

Phase trellis



MSK may be considered as a frequency modulation

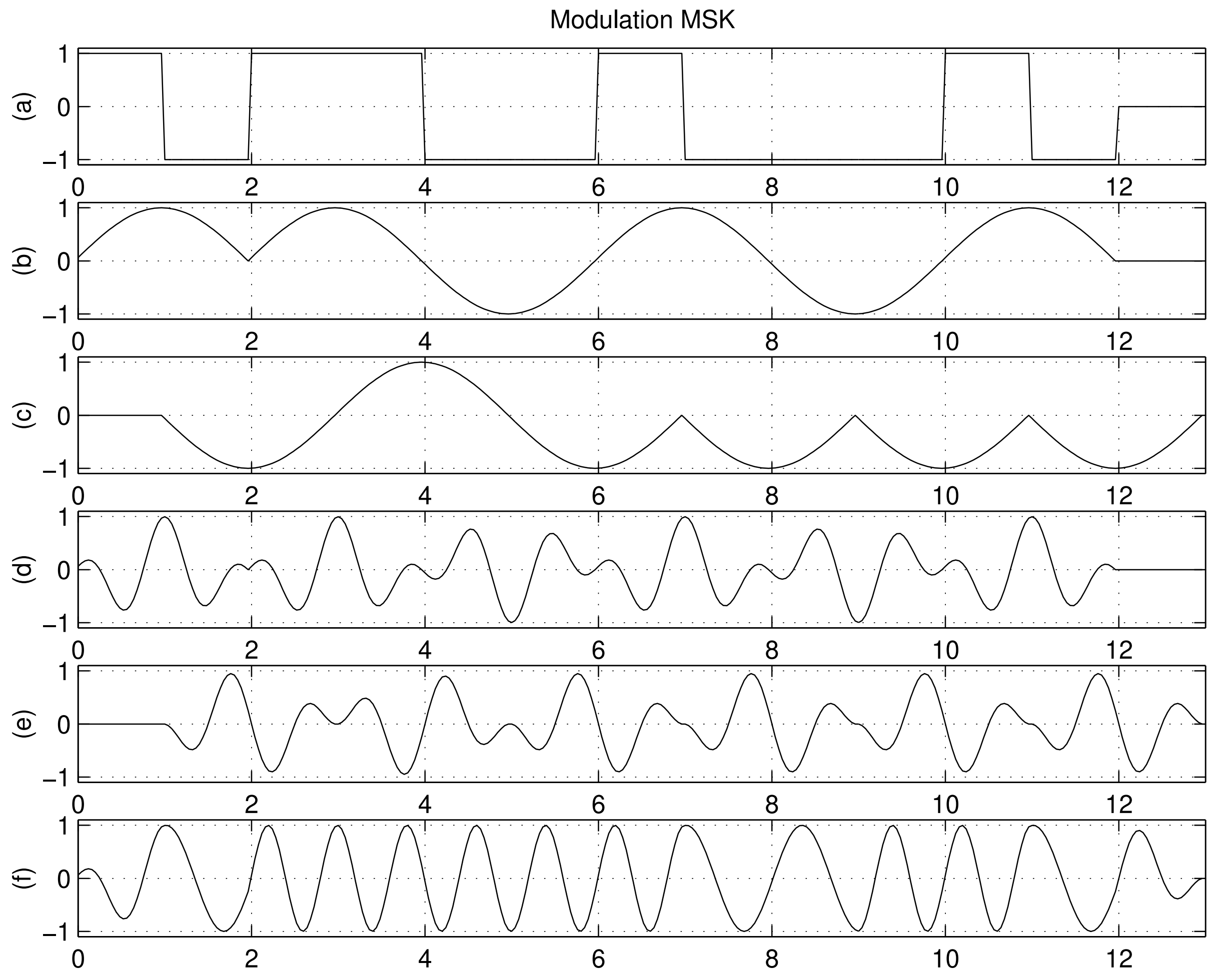
We may write

$$s(t) = a(t)\cos\left[2\pi f_c t + \varphi(t)\right]$$

During the period T_b , we get

$$\begin{aligned} s(t) &= A\cos\left(2\pi f_c t \pm \frac{\pi t}{2T_b}\right) \\ &= A\cos\left[2\pi\left(f_c \pm \frac{1}{4T_b}\right)t\right] \end{aligned}$$

Signals



(a) Binary sequence $I(t)$

(b) $s_I(t)$

(c) $s_Q(t)$

(d) $s_I(t)\cos(2\pi f_c t)$

(e) $s_Q(t)\sin(2\pi f_c t)$

(f) Modulated signal $s(t)$

Power spectral density

Hypothesis: $A_k = \pm A$ have an equal probability.

Mean

$$\mu_A = E\{A_k\} = 0$$

Variance

$$\sigma_A^2 = E\{A_k^2\} = A^2$$

Modulating waveform

$$p(t) = rect_{(0,2T_b)}(t)\,\sin\left(\frac{\pi t}{2T_b}\right)$$

$$||\mathscr{H}(f)||^2 = ||\mathscr{P}(f + \frac{1}{4T_b})||^2$$

Complex envelope PSD

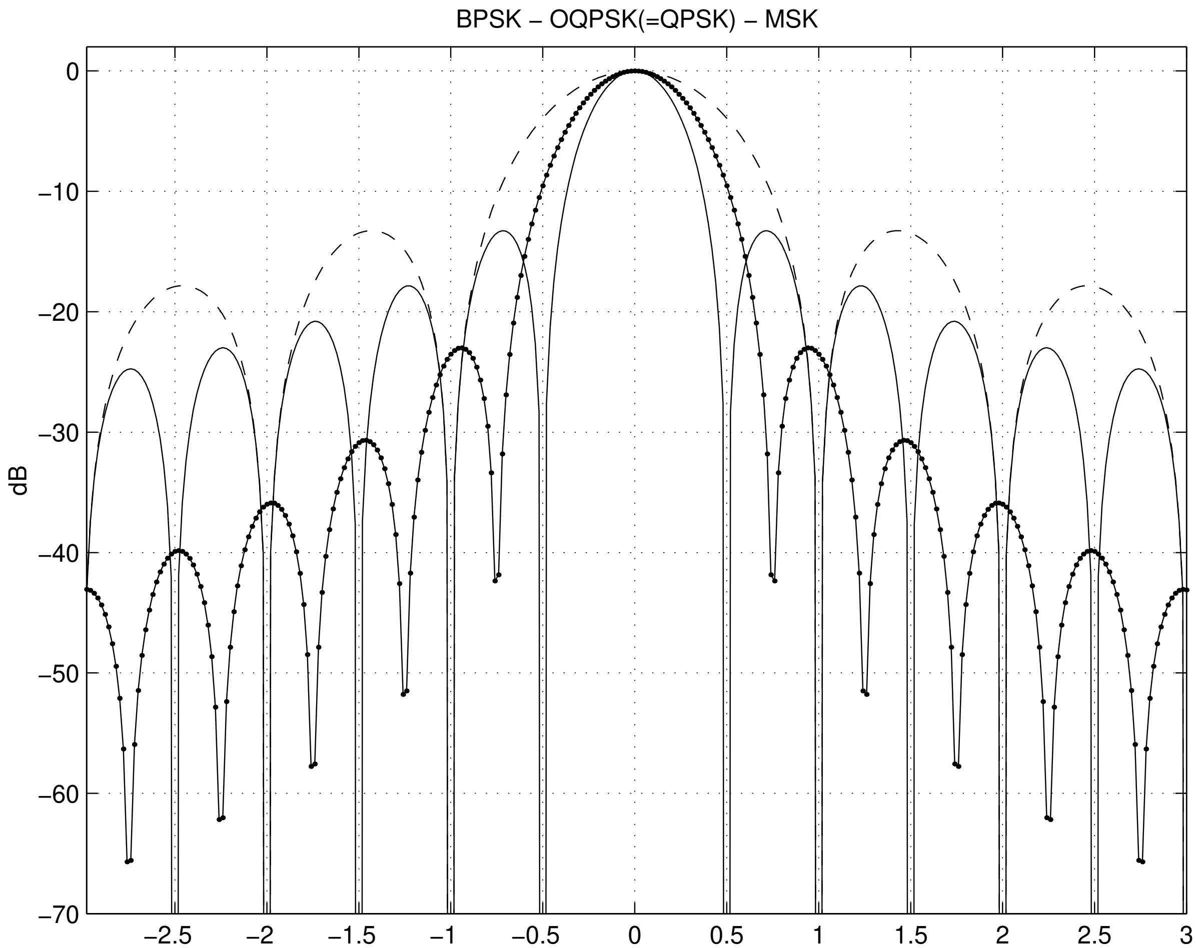
$$\gamma_v(f) = \sigma_A^2 \frac{||\mathscr{H}(f)||^2}{T_b} = \frac{16A^2T_b}{\pi^2}\left\{\frac{\cos\left[2\pi\left(f + \frac{1}{4T_b}\right)T_b\right]}{1 - 16\left(f + \frac{1}{4T_b}\right)^2 T_b^2}\right\}^2$$

modulated signal PSD

$$\gamma_s(f) = \frac{4A^2T_b}{\pi}\left\{\left(\frac{\cos\left[2\pi\left(f-f_c\right)T_b\right]}{1-16\left(f-f_c\right)^2T_b^2}\right)^2 + \left(\frac{\cos\left[2\pi\left(f+f_c\right)T_b\right]}{1-16\left(f+f_c\right)^2T_b^2}\right)^2\right\}$$

$\rightarrow$ decreasing in $1/f^4$ ($1/f^2$ for OQPSK).

PSD comparison



BPSK : dotted line

OQPSK = QPSK: solid line

MSK: solid line + points

3. Exercices

1. Consider the 2-4PSK modulation which corresponds exactly to the OQPSK modulation excepted that the duration of the rectangular modulating waveform is not equal to $2T_b$ anymore, but is equal to T_b .
   (a) For the binary sequence 101101001101, could you represent the inphase and quadrature components, the envelope and the phase of the modulated signal.
   (b) Draw the constellation diagrams (complex plan for $e_s(t)$).
   (c) Determine the power spectral density for the modulated signal (hypothesis: the symbols have equal probability).

Answer

1. (a) –
   (b) –
   (c)

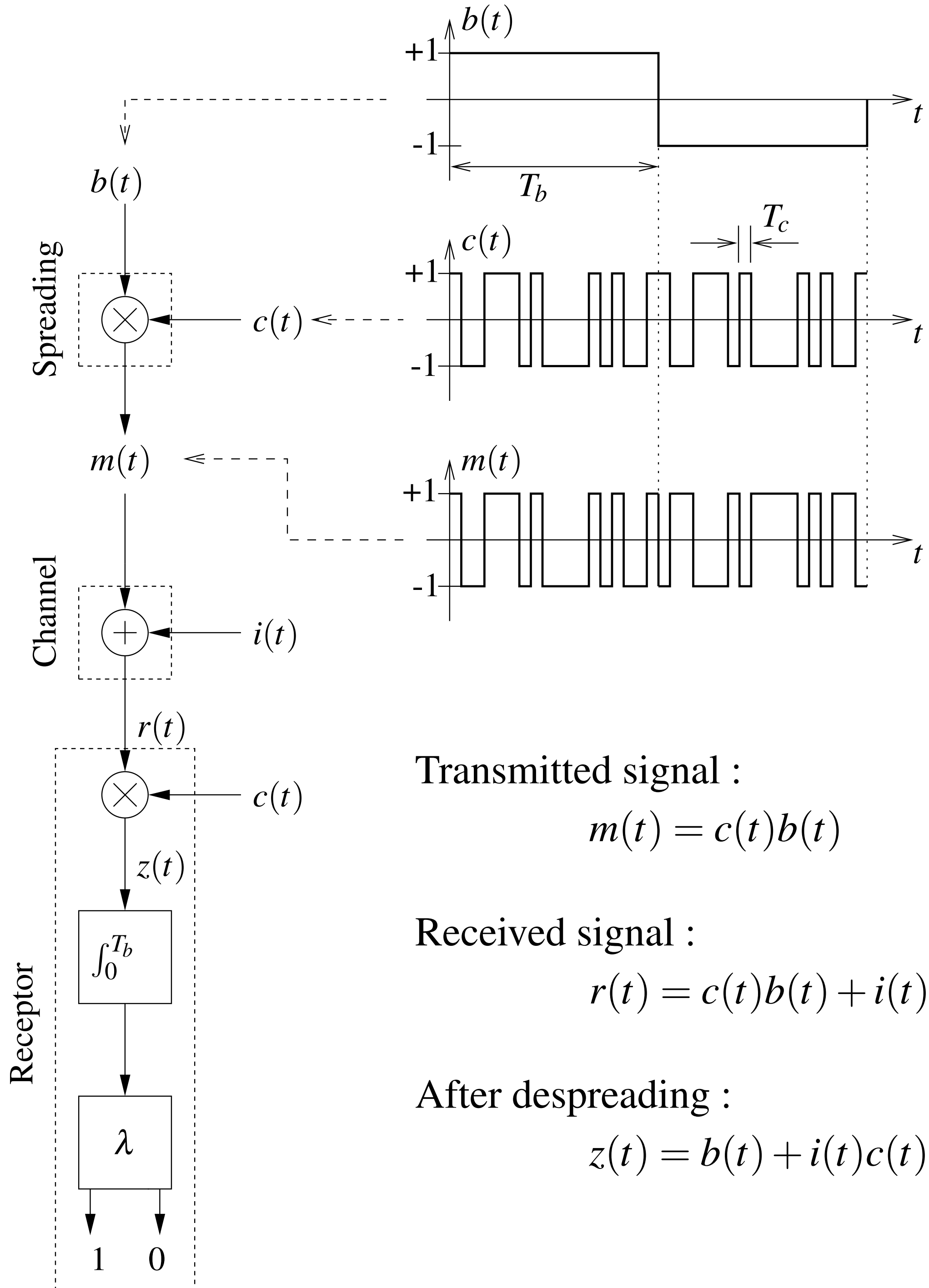
$$\gamma_s(f) = \frac{A^2 T_b}{4} \left\{ sinc^2 \left[(f - f_c) T_b \right] + sinc^2 \left[(f + f_c) T_b \right] \right\}$$

Spread spectrum

Outline:

1. Baseband
2. DS/BPSK Modulation
3. CDM(A) system
4. Multi-path
5. Exercises

1. Baseband



Transmitted signal :

$$m(t) = c(t)b(t)$$

Received signal :

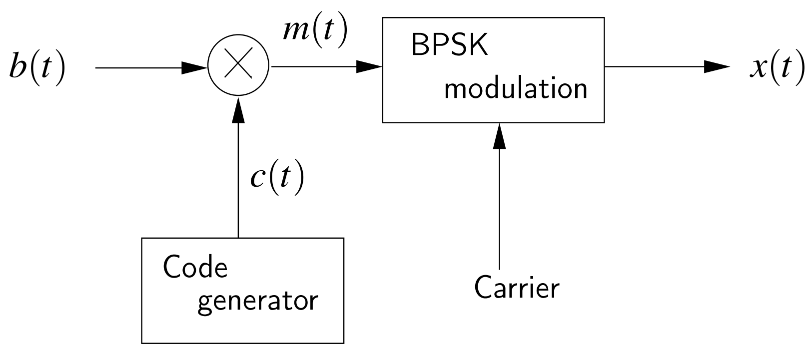
$$r(t) = c(t)b(t) + i(t)$$

After despreading :

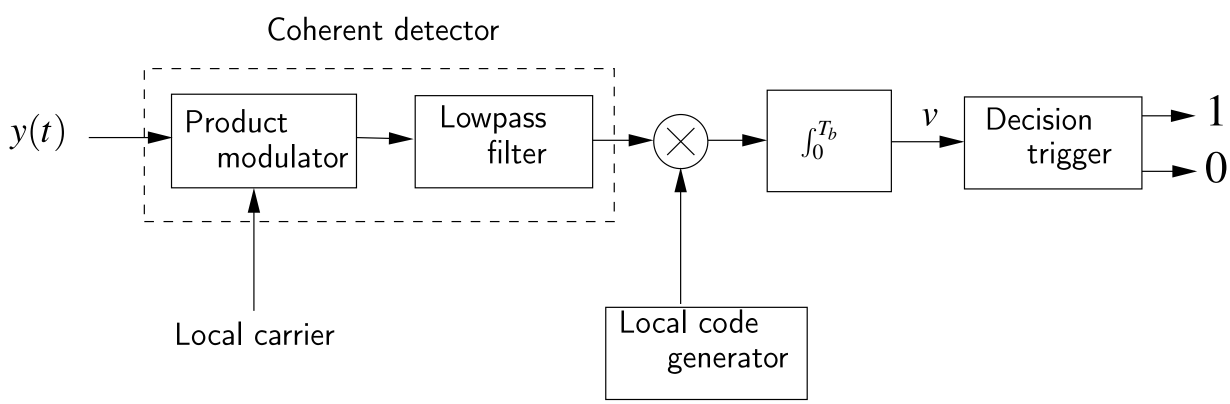
$$z(t) = b(t) + i(t)c(t)$$

2. DS/BPSK Modulation (Direct Sequence Spread Spectrum with coherent Binary Phase Shift Keying)

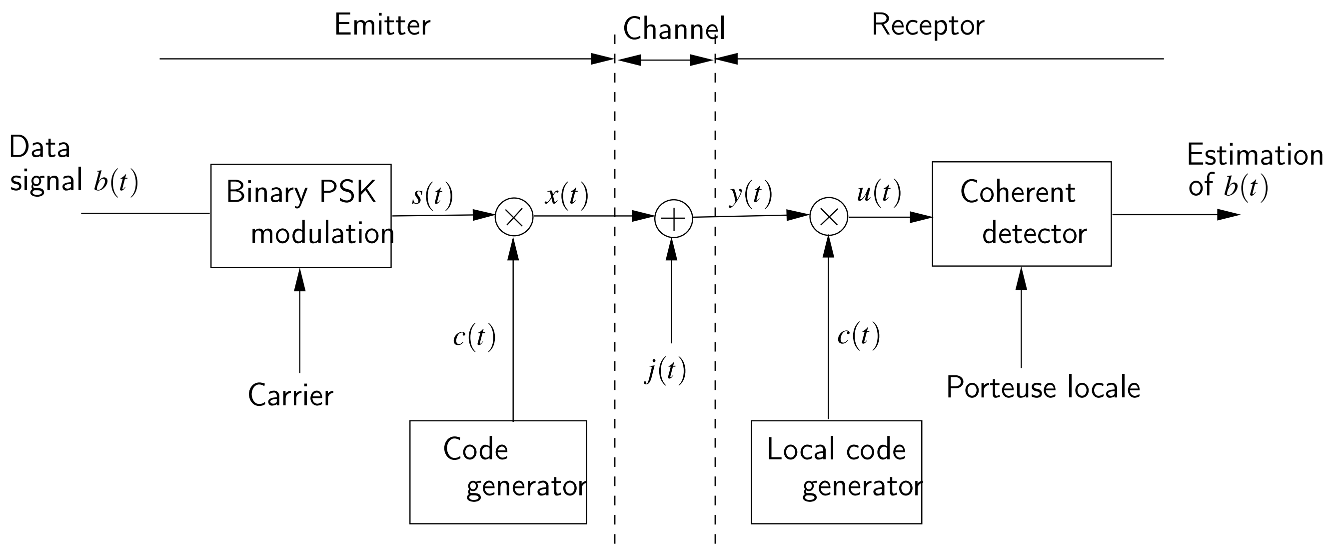
Modulator:



Demodulator:



The spreading and modulation operations are linear $\longrightarrow$ they may be permuted.



At the receptor input:

$$\begin{aligned} y(t) &= x(t) + j(t) \\ &= s(t)\,c(t) + j(t) \end{aligned}$$

where $s(t)$ = BPSK modulation of $b(t)$.

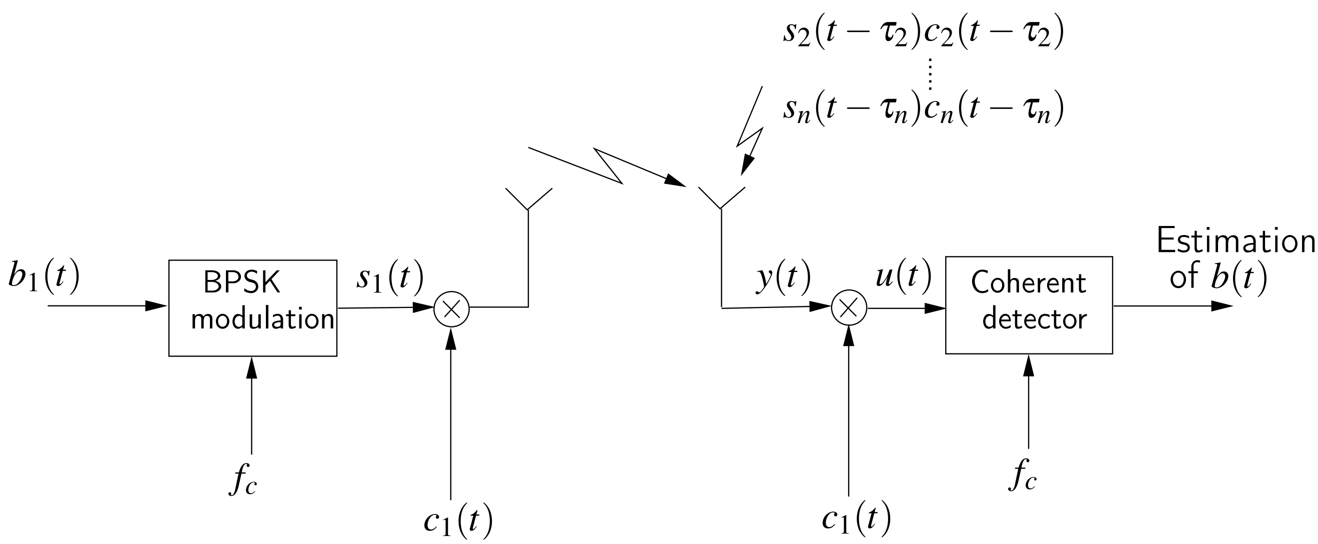
After despreading:

$$\begin{aligned} u(t) &= y(t)\,c(t) \\ &= s(t) + j(t)\,c(t) \end{aligned}$$

3. CDM(A) system

CDM = Code Division Multiplexing

CDMA = Code Division Multiple Access



Received signal:

$$\begin{aligned} y(t) &= s_1(t)\,c_1(t) + s_2(t-\tau_2)\,c_2(t-\tau_2) + \ldots \\ &\quad + s_n(t-\tau_n)\,c_n(t-\tau_n) \end{aligned}$$

After despreading:

$$\begin{aligned} u(t) &= y(t)\,c_1(t) \\ &= s_1(t) + s_2(t-\tau_2)\,c_2(t-\tau_2)\,c_1(t) + \ldots \\ &\quad + s_n(t-\tau_n)\,c_n(t-\tau_n)\,c_1(t) \end{aligned}$$

After BPSK demodulation:

$$\begin{aligned}\widetilde{b}_1(t) &= b_1(t) + b_2(t-\tau_2)\,c_2(t-\tau_2)\,c_1(t) + \ldots \\ &= +b_n(t-\tau_n)\,c_n(t-\tau_n)\,c_1(t)\end{aligned}$$

Through the matched filter:

$$\begin{aligned}v &= \int_0^{T_b} b_1(t)\,dt \\ &\quad + \int_0^{T_b} b_2(t-\tau_2)\,c_2(t-\tau_2)\,c_1(t)\,dt \rightarrow \pm T_b\,\Gamma_{12}(\tau_2) \\ &\quad + \ldots \\ &\quad + \int_0^{T_b} b_n(t-\tau_n)\,c_n(t-\tau_n)\,c_1(t)\,dt \rightarrow \pm T_b\,\Gamma_{1n}(\tau_2)\end{aligned}$$

$\rightarrow$ We are looking for spreading codes $c_i(t)$ which are almost uncorrelated. Ideally, we would like to have $\Gamma_{ij}(\tau) = 0$. So we use the GOLD sequences.

4. Multi-path

Received signal:

$$y(t) = s_1(t)\, c_1(t) + \alpha\, s_1(t-\tau)\, c_1(t-\tau)$$

After despreading:

$$\begin{aligned} u(t) &= y(t)\, c_1(t) \\ &= s_1(t) + \alpha\, s_1(t-\tau)\, c_1(t-\tau)\, c_1(t) \end{aligned}$$

After BPSK demodulation:

$$\widetilde{b}_1(t) = b_1(t) + \alpha\, b_1(t-\tau)\, c_1(t-\tau)\, c_1(t)$$

Through the matched filter:

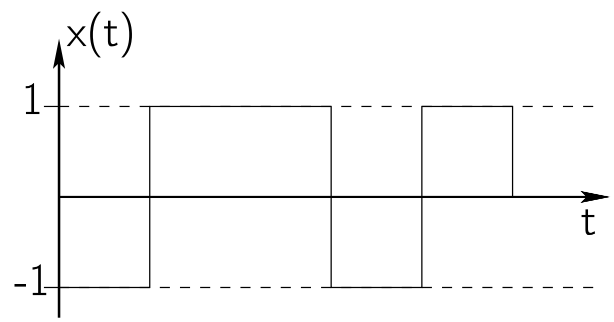
$$\begin{aligned} v &= \int_0^{T_b} b_1(t)\, dt \\ &\quad + \alpha \int_0^{T_b} b_1(t-\tau) c_1(t-\tau) c_1(t)\, dt \to \pm \alpha T_b \Gamma_{11}(\tau) \end{aligned}$$

5. Exercises

1. Let two given signals whose bandwidth are respectively equal to W and nW. Show that the product of these two signals gives a wide band signal.

2. In a spread spectrum communication system, the binary rate is $R_b = 1/T_b$ where $T_b = 4,095\,[ms]$. We use $T_c = 1\,[\mu\text{s}]$ and a BPSK modulation. In addition, the E_b/N_0 ratio leading to an error probability less than 10^{-5} is equal to 10. Determine the maximum number of simultaneous users and the bandwidth of the system.

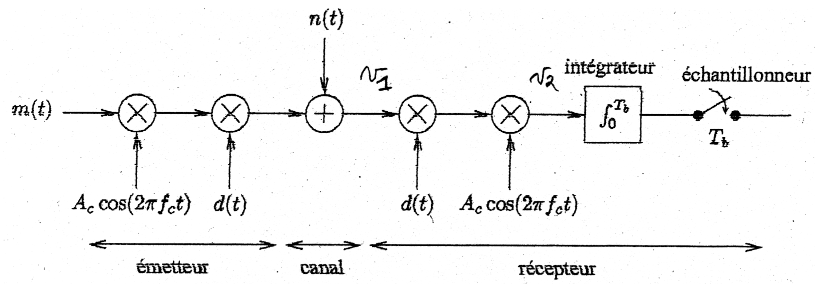
3. Given the bit sequence $0\,1\,1\,0\,1$. We will modulate it in baseband (NRZ modulation) with a rectangular modulating waveform of T_b duration and unitary amplitude. The voltage is respectively equal to $1\,[\text{V}]$ for a 1 and $-1\,[\text{V}]$ for a 0. The resulting signal $x(t)$ is then shaped like in in the drawing below and the message is transmitted at a speed of $75\,[\text{b/s}]$.



We then decide to use a spread spectrum method where the spreading signal $g(t)$ is generated by a 4 bits shift register whose initial sequence is $1\,1\,1\,1$. The clock frequency of this circuit is $1125\,[\text{Hz}]$.

(a) Determine the circuit diagram allowing the construction of the shift register of maximum length.(hint: use the $[4,1]$ feedback configuration)
(b) Draw the spread signal for the two first bits of $x(t)$.
(c) Determine the spread spectrum processing gain in [dB].
(d) If we then use a BPSK modulation and if the ratio between the energy per bit and the noise power is $5\,[\text{dB}]$, determine the maximum users number.
(e) Determine the spread signal bandwidth.

4. We consider the spread spectrum transmission system represented by the following diagram



where
- $m(t)$ is the useful binary signal; $m(t)$ is a NRZ signal with a $\pm V$ amplitude and a bit duration of $T_b = \frac{1}{f_b}$,
- $A_c \cos(2\pi f_c t)$ is the carrier,
- $d(t)$ is the spreading sequence; the bit duration is $T_d = \frac{T_b}{60}$,
- $n(t)$ is an additive noise.

This question has two parts and it is possible to answer almost of all the second part without having solved the first one.

<u>First part:</u> Here, we will try to find the analytic expression for the noise power spectral density at the integrator output. The noise signal is $n(t) = A_n \cos(2\pi f_c t + \Theta)$ where Θ is a zero mean random phase.

(a) What is the spreading factor?
(b) Give the analytic expression of the $v_1(t)$ signal at the receptor input.
(c) What is the $v_2(t)$ signal at the integrator input?
(d) If we take $f_c = \frac{600}{T_b}$, some terms in $v_2(t)$ will have a null contribution at the integrator output. What are these terms? (Hint: (1) you should develop the cosines, (2) the terms $\cos(\Theta)$ and $\sin(\Theta)$ do not depend on the time; they are constants on all the integration period).
(e) As all operations are linear, it is possible to neglect the terms with a null contribution starting from the integrator input. Then, what will be the simplified $v_3(t)$ signal derived from the expression of the $v_2(t)$ signal?
(f) What is the interference term in $v_3(t)$?
(g) What is the spectral density of the interference term at the integrator input?
(h) What is the spectral density of the interference term at the integrator output? In the computation, you may consider that the integrator will act as an ideal lowpass filter until the f_b frequency.

Second part: We would like to compute the bit error probability. We remind you that, in the case of a classical BPSK modulation, the bit error probability P_e is

$$P_e = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$

We will assume that the noise power spectral density is constant for $|f| \leq f_b$ and is equal to

$$\frac{V^2 E\left\{\cos^2(\Theta)\right\}}{\alpha f_d}$$

This spectral density is null outside the $[-f_b, f_b]$ interval. α is a constant.

(a) Compute the value of P_e. (Hint: replace E_b by its value)

(b) Which is the gain compared to the classical BPSK if we consider that Θ is a random variable uniformly distributed on the $[0, 2\pi]$ interval?

(c) Does the gain comes from the spreading?

Answers

1. –
2. Users number $= 410$. Bandwidth $= 1\,[\text{MHz}]$.

Intersymbol interference

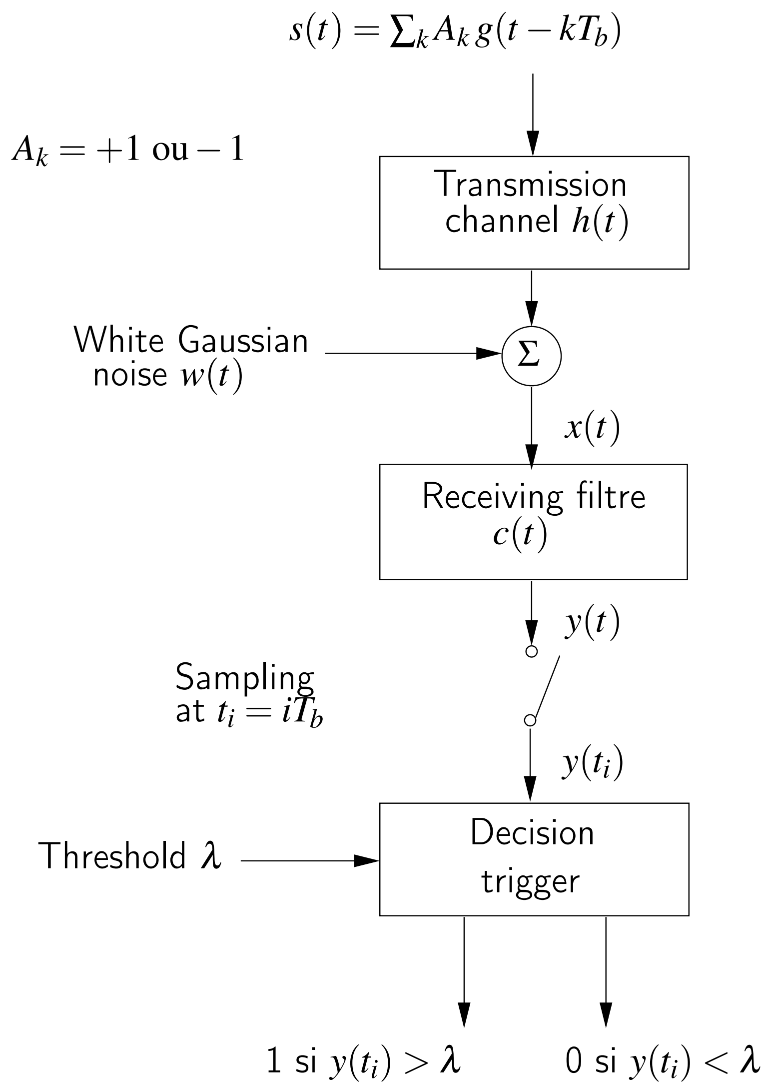
Outline:

1. Intersymbol interference definition (ISI)
2. NYQUIST Criteria
   2.1 NYQUIST ideal channel
   2.2 Raised cosine pulsed
3. Exercises

1. Intersymbol interference definition (ISI)

Origin: The transmission channel is dispersive.

Transmission scheme:



Transmitted signal:

$$s(t) = \sum_k A_k\, g(t - kT_b)$$

After reception and filtering:

$$y(t) = \mu \sum_k A_k \, p(t - kT_b) + n(t)$$

From the transmitted signal to the received and filtered one:

$$\mu P(f) = G(f) \, H(f) \, C(f)$$

where μ = normalization factor such that $p(0) = 1$.

After sampling:

$$\begin{aligned} y(t_i) &= y(iT_b) \\ &= \mu \sum_k A_k \, p\left[(i-k)T_b\right] + n(iT_b) \end{aligned}$$

$$= \underbrace{\mu A_i}_{\substack{\text{contribution} \\ \text{of the } i\text{th bit}}} + \underbrace{\sum_{k \neq i} A_k \, p\left[(i-k)T_b\right]}_{\text{ISI}} + \underbrace{n(iT_b)}_{\substack{\text{filtered and} \\ \text{sampled noise}}}$$

2. Nyquist Criteria

We are looking for $p(t)$ such that ISI=0. Consequently,

$$p\left[(i-k)T_b\right] = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

We consider the sampled signal

$$p_s(t) = \sum_{m=-\infty}^{+\infty} p(mT_b)\,\delta(t - mT_b)$$

$$P_s(f) = f_b \sum_{n=-\infty}^{+\infty} P(f - nf_b) \quad \text{where } f_b = 1/T_b$$

So,

$$\begin{aligned} P_s(f) &= \int_{-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \left[p(mT_b)\,\delta(t - mT_b)\right] e^{-j2\pi t f}\, dt \\ &= \int_{-\infty}^{+\infty} p(0)\,\delta(t)\, e^{-j2\pi t f}\, dt \\ &= 1 \end{aligned}$$

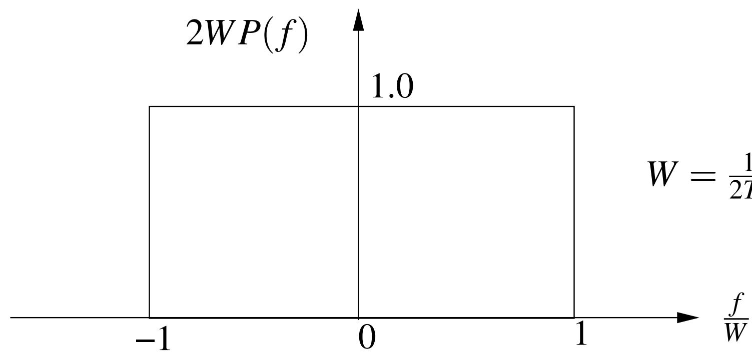
The NYQUIST criteria may be formulated as follows

$$\text{ISI} = 0 \;\text{ if }\; \sum_{n=-\infty}^{+\infty} P(f - nf_b) = T_b$$

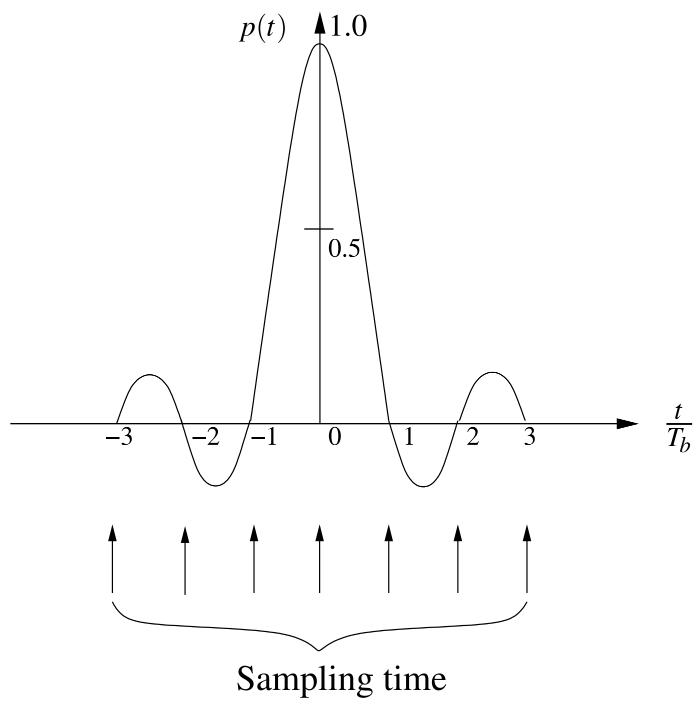
2.1 Nyquist ideal channel

$$P(f) = \begin{cases} \frac{1}{2W} & \text{if } -W < f < +W \\ 0 & \text{if } |f| > W \end{cases}$$

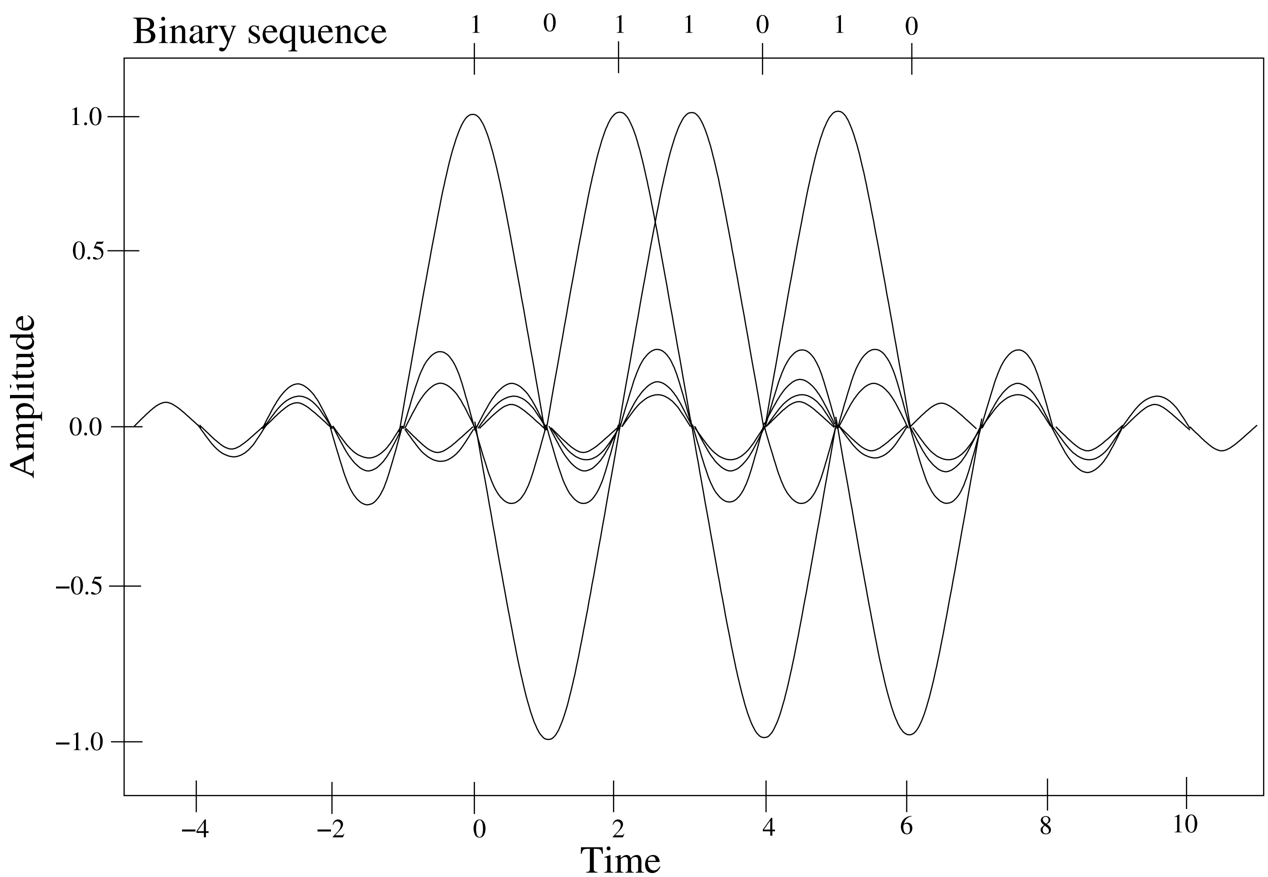
$$p(t) = sinc(2Wt)$$



$W = \frac{1}{2T_b} = \frac{R_b}{2}$



Superposition of the received impulses:



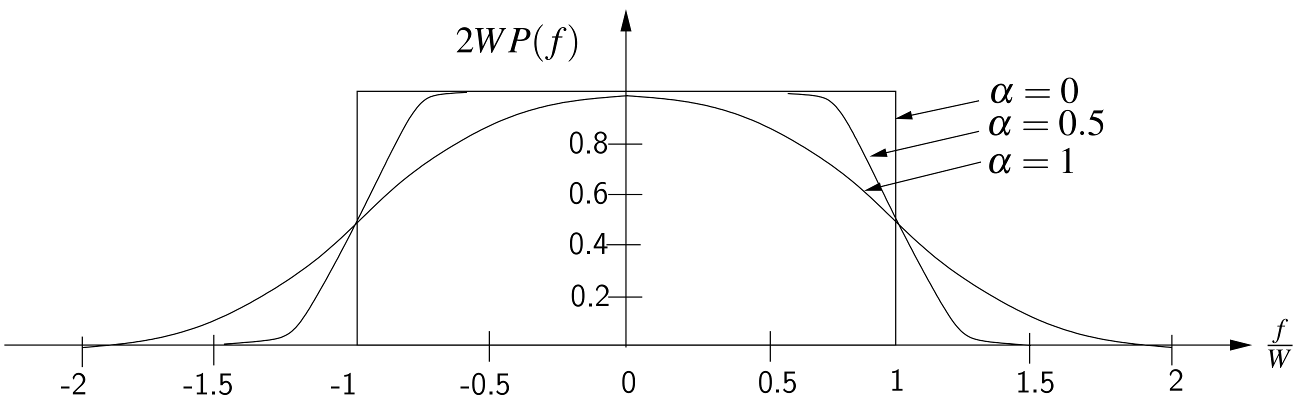
Drawbacks:

- Abrupt transitions in $\pm W \longrightarrow$ not physically achievable.
- $p(t)$ decreases in $1/|t| \longrightarrow$ too few error margin on the sampling time.

2.2 Raised cosine pulsed

In the frequency domain,

$$P(f) = \begin{cases} \frac{1}{2W} & 0 \le |f| < f_1 \\ \frac{1}{4W}\left\{1 - \sin\left[\frac{\pi(|f|-W)}{2W-2f_1}\right]\right\} & f_1 \le |f| < 2W - f_1 \\ 0 & |f| \ge 2W - f_1 \end{cases}$$



<u>**Transmission band**</u>:

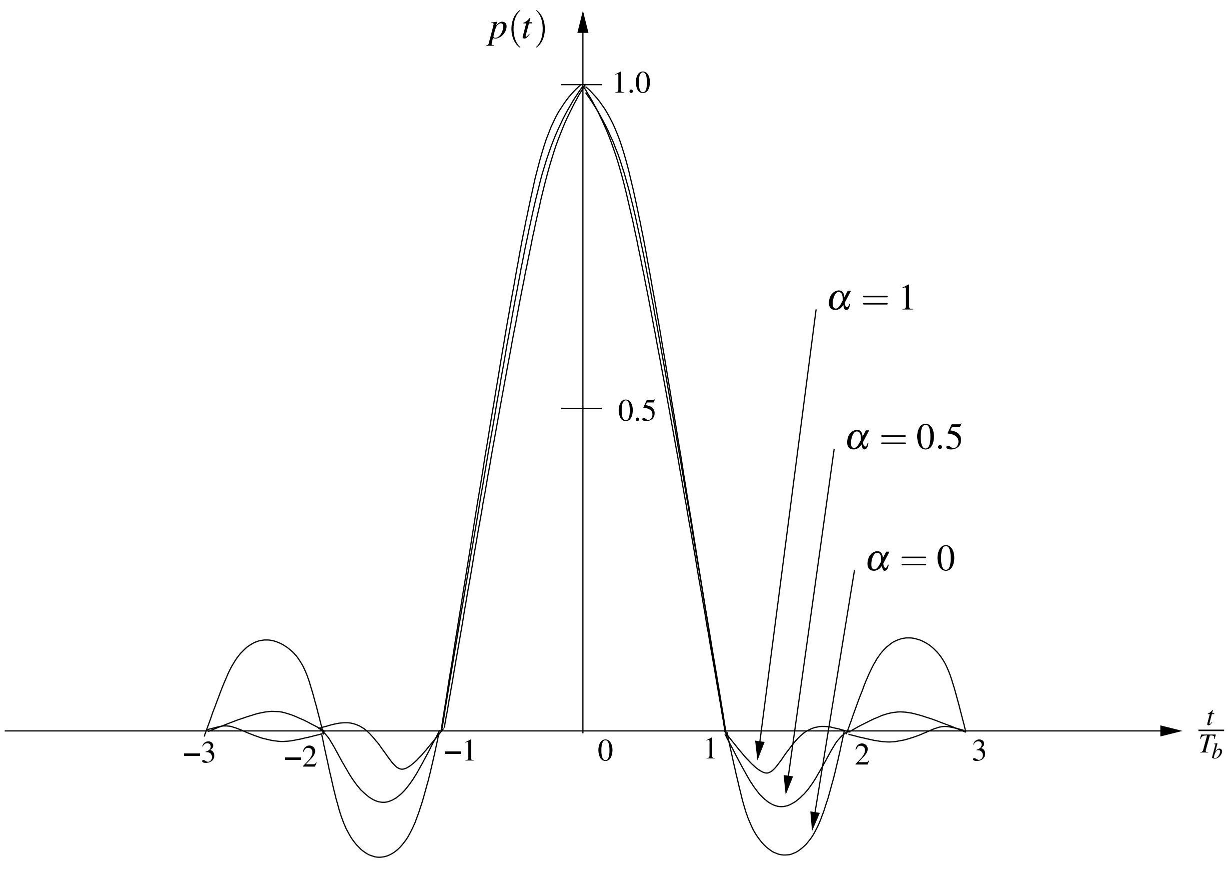
$$B_T = 2W - f_1 = W(1+\alpha)$$

where

$$\alpha = 1 - \frac{f_1}{W} \text{ (rolloff factor)}$$

In the temporal domain,

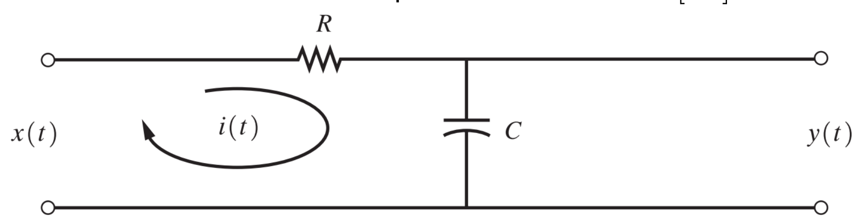
$$p(t) = sinc(2Wt)\left(\frac{\cos(2\pi\alpha Wt)}{1-16\alpha^2W^2t^2}\right)$$



$\longrightarrow$ decreases in $1/t^2$

3. Exercises

1. Plot the output waveform when a channel filters a unipolar NRZ signal. Assume that the overall filtering effect of the transmitter, channel, and the receiver is that of an RC low-pass filter where the $3\,[\text{dB}]$ bandwidth is $1\,[\text{Hz}]$.



   Assume that the unipolar NRZ input signal has a bit rate of $R_b = 1\,[\text{Hz}]$ and that the data on the unipolar NRZ signal is [1 0 0 1 0 1 1 0 1 0].

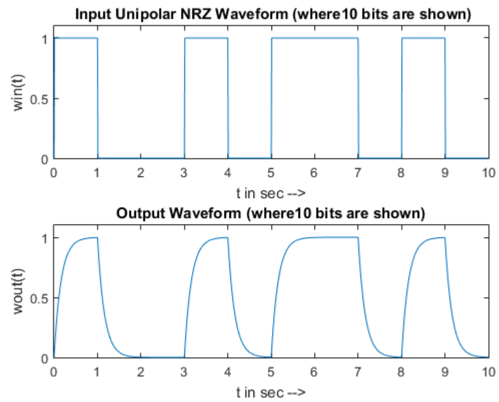
2. Determine the combined impulse response of the whole chain (transmitter, channel, receiver).
   (a) Compute and plot the waveform at the receiver output and observe the intersymbol interference

3. A computer send binary data at a $56\,[\text{kb/s}]$ rate. The base band transmission is achieved thanks to a PAM modulation with 2 tension levels, using a raised cosine impulse signal. Determine the necessary bandwidth for the transmission for $\alpha = 0.25, 0.5, 0.75, 1$. What become these bandwidth if we group the bits three by three in a PAM-8 modulation?

4. An analog signal is sampled, quantized and coded with a binary PCM. The quantification level number is 128. An error detection bit is added to each sample of the analog signal. The resulting PCM wave is transmitted in a $12\,[\text{kHz}]$ bandwidth channel, using a PAM-4 modulation and a raised cosine impulse ($\alpha = 1$).
   (a) Determine the transmission rate (in [b/s]) through the channel.
   (b) Determine the sampling frequency of the analog signal. What is the maximum possible frequency for the analog signal?

5. A PAM binary wave (2 tension levels) is transmitted in base band through a channel whose maximum bandwidth is $75\,[\text{kHz}]$. The bit duration is $10\,[\mu\text{s}]$. Determine the parameter α of a raised cosine impulse which verify these conditions.

Answers

1. 

2. The bandwidth are given by

α	B_T [kHz]
0.25	35
0.5	42
0.75	49
1	56

   If we group the bits 3 by 3, for the same transmission rate, the pulse may be 3 times longer; the transmission bands are all divided by 3.

3. The transmission rate is $24\,[\text{kb/s}]$. The analog signal is sampled at a $3\,[\text{kHz}]$ frequency. The maximum frequency of the signal may not be upper than $1.5\,[\text{kHz}]$.

4. $\alpha \leq 0.5$.

Frequency and time multiplexing

Outline:

1. Frequency Division Multiplexing (FDM)
   1.1 Multiplexing then digitization
   1.2 digitization then multiplexing
2. Time multiplexing (TDM)
3. Exercises

1. Frequency Division Multiplexing (FDM)

- <u>**Objectives of the multiplexing**</u>: to transmit several signals on the same channel.

- <u>**Characteristics of the frequency multiplexing**</u>:

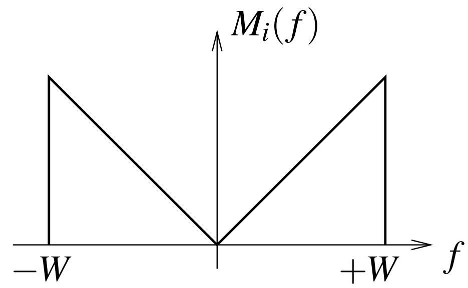
  - All signals are transmitted simultaneously.

  - Signals use different frequency bands.

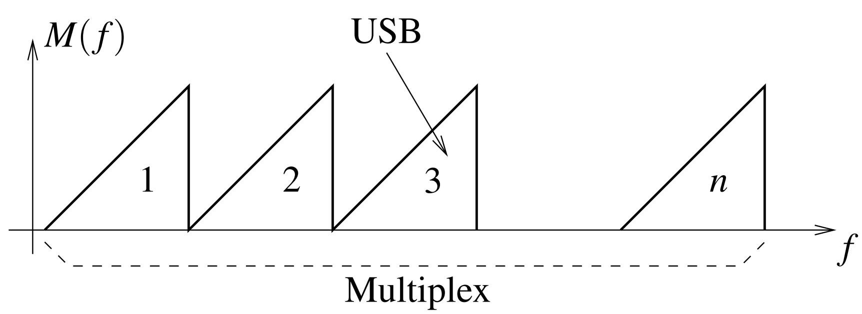
  - We may either multiplex then digitize (and modulate) or digitize (and modulate) then multiplex.

1.1 Multiplexing then digitization

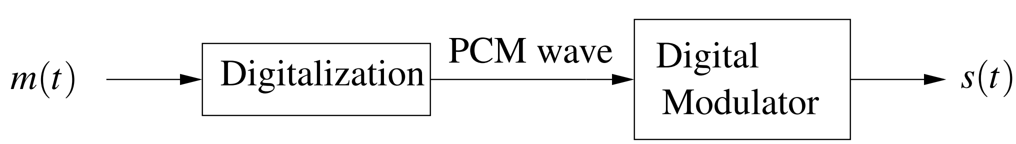
Shape for the spectrum of the transmitted signals $m_i(t)$:



Generation of an hybrid analog signal $m(t)$:



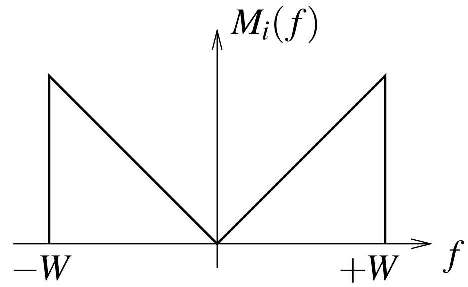
digitization and digital modulation of $m(t)$:



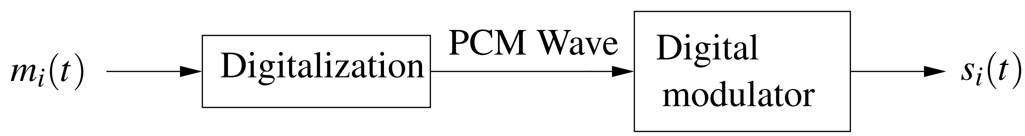
where $s(t)$ is a modulated digital signal (BPSK, QPSK, ...).

1.2 Digitization then multiplexing

Shape for the spectrum of the transmitted signals $m_i(t)$:

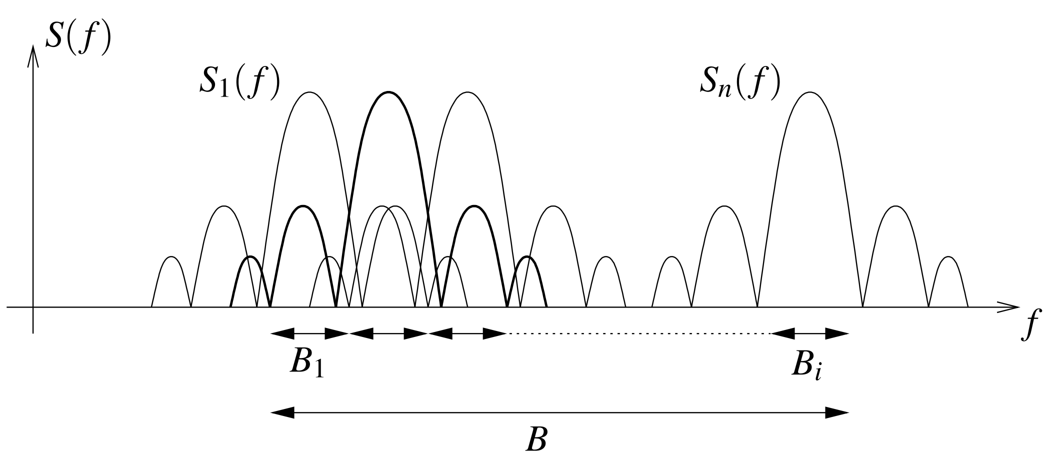


digitization and digital modulation of each $m_i(t)$:



where $s_i(t)$ is a modulated digital signal (BPSK, QPSK, ...).

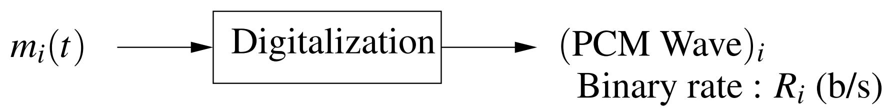
Frequency multiplexing of the digital signals $s_i(t)$:



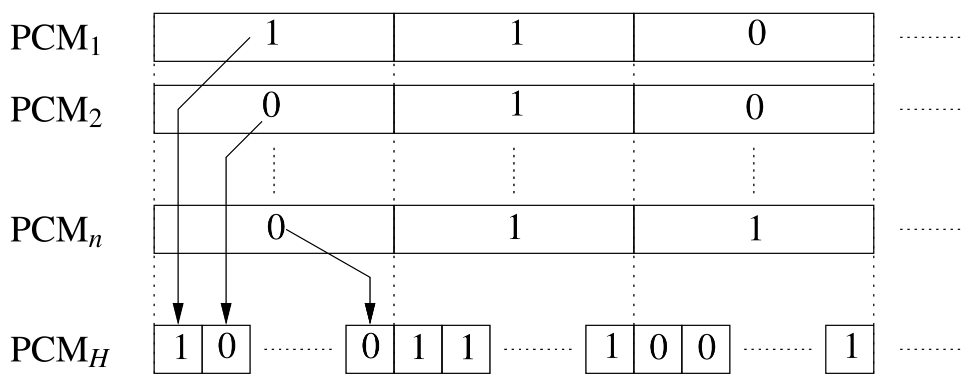
2. Temporal multiplexing (TDM: Time Division Multiplexing)

- All signals use the same frequency band.
- The signals are transmitted one by one.

Digitization of each $m_i(t)$:

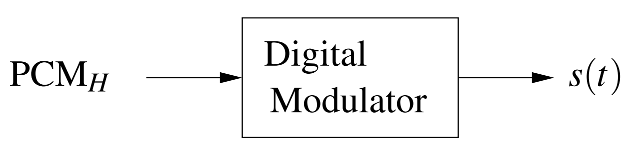


Generation of an hybrid PCM wave:



has a bit rate of $R_b = nR_i$ (if $R_i = R_j$).

Digital modulation of PCM_H :



where $s(t)$ is a modulated digital signal (BPSK, QPSK, ...).

3. Exercises

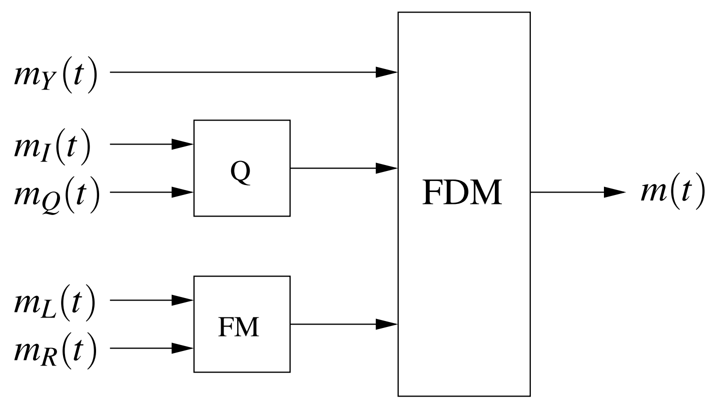
1. A basic CCITT group contains a set of 12 phone channels multiplexed by frequency division (FDM) between 12 and $60\,[\text{kHz}]$. Starting from the obtained multiplex, we generate a PCM wave, quantized on 8 bits. The PCM wave is transmitted thanks to a PAM modulation with 2 voltage levels.
   (a) Which sampling frequency should we use?
   (b) What is the binary rate?
   (c) What is the required bandwidth?

2. The bandwidth for an ECS satellite signal is $120\,[\text{MHz}]$. Then, we transmit the binary signals with a carrier modulated in QPSK. Knowing that the signals are quantized with 8 bits, how many phone streams can we transmit by frequency multiplexing (FDM) and by temporal multiplexing (TDM)?

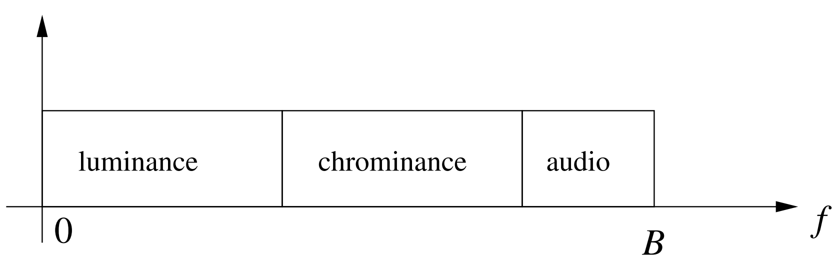
3. We generate a frequency multiplex comprising a group of FM modulated TV channels. This multiplex contains 5 sport channels and n entertainment channels. Given that the maximum power for the multiplex is $20\,[\text{W}]$ and that each channel has a peak voltage of $850\,[m\text{V}]$, determine the maximum value for n .

4. The IMAGE+ company decides to launch a new digital pay TV package distributed by satellite and containing 20 channels. Each channel comprises two kind of information:
   - ✘ **Video**: represented by a luminance signal $m_Y(t)$ with a 4.2 [MHz] bandwidth and two chrominance signals $m_I(t)$ and $m_Q(t)$ with 1.6 [MHz] and 0.6 [MHz] bandwidth respectively.
   - ✘ **Audio (stereo)**: containing two audio signals $m_L(t)$ and $m_R(t)$ with a 20 [kHz] bandwidth. This audio signal is generated in the same way than the FM radio signals.

   The 5 signals are aggregated with a FDM technique and form a composite signal $m(t)$ with a B bandwidth thanks to the following diagram



   where the Q block achieves the quadrature modulation of the chrominance signals at the f_1 carrier frequency and the FM block achieves the FM modulation (with $\Delta f = 75$ [kHz]) of the stereo signal at f_2 frequency. Graphically, the spectrum of the composite signal $m(t)$ corresponding to an unique channel has the following form



   The IMAGE+ company chooses a signal digitization at the practical sampling frequency and a 12-bits quantization. For the transmission, they use a satellite channel carrier at 11.5 [GHz] frequency. In order to optimize the required bandwidth, several multiplexing methods are considered. This question is aimed at studying the different possibilities.

   (a) Compute the minimal numerical value for B, and the corresponding numerical values for f_1 and f_2 (unlike analog television, we decide not to allow any covering between the luminance and chrominance signals).

(b) For each of the following multiplexing methods, give the bandwidth required for the aggregated multiplex and depending on B [Hz].

   i. First step: aggregation by FDM multiplexing of the signals obtained by USB modulation of the 20 channels, in the same way than a base group in analog telephony.
   Second step: digitization of the aggregated signal.
   Third step: QAM-16 modulation.

   ii. First step: individual digitization of the $m(t)$ signals related to each channels.
   Second step: OQPSK modulation for each generated binary data streams.
   Third step: FDM multiplexing of the obtained OQPSK signals.

   iii. First step: individual digitization of the $m(t)$ signals related to each channels.
   Second step: TDM multiplexing of the generated binary data streams.
   Third step: PSK-4 modulation.

   iv. First step: individual digitization of the $m(t)$ signals related to each channels.
   Second step: CDM multiplexing with DS/BPSK modulation with a spreading factor of 24 bits.

Answer

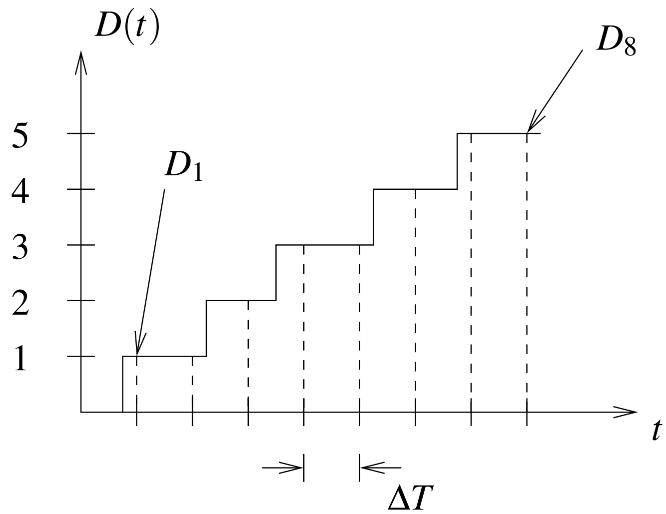
1. (a) $120\,[\text{kHz}]$
   (b) $960\,[\text{kb/s}]$
   (c) $480\,[\text{kHz}]$
2. 3750(whatever the multiplexing technique).
3. $n \leq 50$ if all channels are independents, $n \leq 30$ if the 5 sport channels transmit the same program.
4. (a) $B = 7.7$ [MHz], $f_1 = 5.8$ [MHz], $f_2 = 7.55$ [MHz]
   (b) (i) $132\ B$
   (b) (ii) $264\ B$
   (b) (iii) $264\ B$
   (b) (iv) $633.6\ B$

Traffic engineering

<u>Outline</u>:

1. Counting process
2. POISSON process
3. Application to the trafic engineering
4. Exercises

1. Counting process



Given:

- T = observation period, m = time intervals number containing no more than one event: $\Delta T = \frac{T}{m}$
- λ = average event number per time unit ($\rightarrow$ measure).
- p = probability that an event occurs during ΔT : $p = \lambda\, \Delta T$.
- D_i = Random variable representing the event number after $i\Delta T$.

We have,

$$P\left(D_m = n\right) = C_m^n p^n \left(1-p\right)^{m-n} = C_m^n \left(\frac{\lambda T}{m}\right)^n \left(1 - \frac{\lambda T}{m}\right)^{n-m}$$

for $n = 0, 1, \ldots, m \rightarrow D_m$ = binomial random variable with a mean of $mp = \lambda T$.

When $m \to \infty$ ($\Delta T \to 0$), binomial random variable $\to$ POISSON random variable

$$P(D = n) = \begin{cases} \frac{(\lambda T)^n}{n!} e^{-\lambda T} & n = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases}$$

D = Random variable representing the event number during the period T

$$E\{D\} = \lambda T = \alpha$$

2. Poisson process

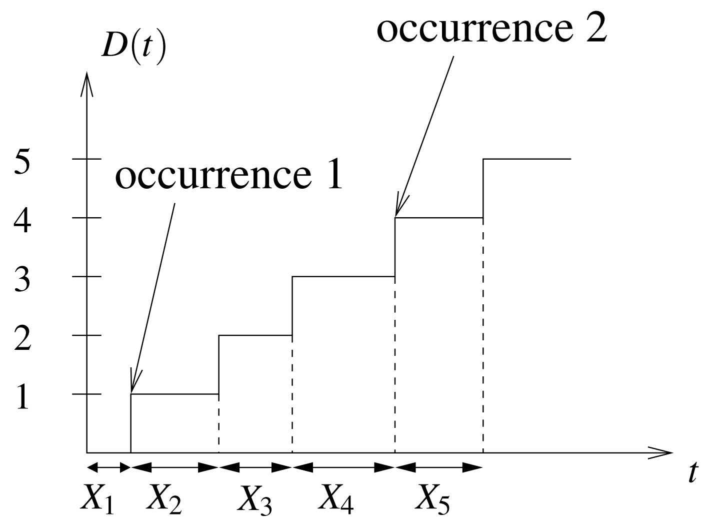
A counting process is a POISSON process if

- The event number during the interval $]t_0, t_1]$, $D(t_1) - D(t_0)$ is a Poisson random variable of mean $\lambda(t_1 - t_0)$.

- For all non-overlapping sub-intervals $]t_0, t_1]$ and $]t_0', t_1']$, $D(t_1) - D(t_0)$ and $D(t_1') - D(t_0')$ are independent random variables

$\rightarrow$ process without memory

Theorem



Given X_n the time period between the successive events n and $n-1$. Then,

$$P(X_n > x) = P\{[D(t_{n-1}+x) - D(t_{n-1})] = 0\} = e^{-\lambda x}$$

$$\begin{aligned} F_{X_n}(x) &= P(X_n \leq x) \\ &= 1 - P(X_n > x) \\ &= 1 - e^{-\lambda x} \end{aligned}$$

$$f_{X_n}(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

X_n is then an exponential random variable, and

$$E[X_n] = \frac{1}{\lambda}$$

3. Application to the traffic engineering

N_A = number of call attempts during ΔT

λ = mean number of call attempts / time unit

N_D = number of call stops during ΔT

η = mean number of call stops / time unit

$$\begin{aligned} P(N_A = n) &= \frac{(\lambda\,\Delta T)^n}{n!} e^{-\lambda\,\Delta T} \\ P(N_D = n) &= \frac{(\eta\,\Delta T)^n}{n!} e^{-\eta\,\Delta T} \end{aligned}$$

Load of a network link:

$$\begin{aligned} A &= (\text{call rate}).(\text{call duration}) \\ &= \frac{\lambda}{\eta} \end{aligned}$$

Given N physical lines and P_k the probability of having k busy lines. We show that

$$P_k = \frac{\frac{A^k}{k!}}{\sum_{k=0}^{N} \frac{A^k}{k!}}$$

Blocking probability (network congestion):

$$B = P_N = \frac{\frac{A^N}{N!}}{\sum_{k=0}^{N} \frac{A^k}{k!}}$$

→ **Erlang B** Formula

→ use tables

Tables for the Erlang B formula:

	B					B			
N	0.02	0.01	0.005	0.001	N	0.02	0.01	0.005	0.001
1	0.02	0.01	0.005	0.001	11	5.8	5.2	4.6	3.6
2	0.22	0.15	0.105	0.046	12	6.6	5.9	5.3	4.2
3	0.6	0.46	0.35	0.19	13	7.4	6.6	6.0	4.8
4	1.1	0.9	0.7	0.44	14	8.2	7.4	6.7	5.4
5	1.7	1.4	1.1	0.8	15	9.0	8.1	7.4	6.1
6	2.3	1.9	1.6	1.1	16	9.8	8.9	8.1	6.7
7	2.9	2.5	2.2	1.6	17	10.7	9.6	8.8	7.4
8	3.6	3.1	2.7	2.1	18	11.5	10.4	9.6	8.0
9	4.3	3.8	3.3	2.6	19	12.3	11.2	10.3	8.7
10	5.1	4.5	4.0	3.1	20	13.2	12.0	11.1	9.4

	B					B			
N	0.02	0.01	0.005	0.001	N	0.02	0.01	0.005	0.001
21	14.0	12.8	11.9	10.1	31	22.8	21.2	19.9	17.4
22	14.9	13.7	12.6	10.8	32	23.7	22.0	20.7	18.2
23	15.8	14.5	13.4	11.5	33	24.6	22.9	21.5	19.0
24	16.6	15.3	14.2	12.2	34	25.5	23.8	22.3	19.7
25	17.5	16.1	15.0	13.0	35	26.4	24.6	23.2	20.5
26	18.4	17.0	15.8	13.7	36	27.3	25.5	24.0	21.3
27	19.3	17.8	16.6	14.4	37	28.3	26.4	24.8	22.1
28	20.2	18.6	17.4	15.2	38	29.2	27.3	25.7	22.9
29	21.0	19.5	18.2	15.9	39	30.1	28.1	26.5	23.7
30	21.9	20.3	19.0	16.7	40	31.0	29.0	27.4	24.4

	B					B			
N	0.02	0.01	0.005	0.001	N	0.02	0.01	0.005	0.001
41	31.9	29.9	28.2	25.2	51	41.2	38.8	36.9	33.3
42	32.8	30.8	29.1	26.0	52	42.1	39.7	37.7	34.2
43	33.8	31.7	29.9	26.8	53	43.1	40.6	38.6	35.0
44	34.7	32.5	30.8	27.6	54	44.0	41.5	39.5	35.8
45	35.6	33.4	31.7	28.4	55	44.9	42.4	40.4	36.6
46	36.5	34.3	32.5	29.3	56	45.9	43.3	41.2	37.5
47	37.5	35.2	33.4	30.1	57	46.8	44.2	42.1	38.3
48	38.4	36.1	34.2	30.9	58	47.8	45.1	43.0	39.1
49	39.3	37.0	35.1	31.7	59	48.7	46.0	43.9	40.0
50	40.3	37.9	36.0	32.5	60	49.6	46.9	44.8	40.8

4. Exercises

1. In a printed circuits fabrication plant, n circuits are tested. A circuit is rejected with a probability p independently of the results of the other tests. Given K the random variable representing the number of rejections among the n tests. Determine the probability that $K = k$. Numeric application: $n = 10$, $p = 0,2$, $k = 4$.

2. Each time a modem transmit a bit, the receiving modem analyses the incoming signal and decides if the transmitted bit is 1 or 0. The decision error probability is p , independently of the decision concerning any other bit.
   (a) If X is the already transmitted bit number when the first error occurs, determine $P(X = 10)$ for $p = 0,1$.
   (b) Determine $P(X \geq 10)$ $(p = 0,1$).
   (c) On 100 transmitted bits, Y represents the number of actual errors, determine $P(Y = 2)$ for $p = 0,01$.
   (d) Determine $P(Y \leq 2)$ $(p = 0,01$).

3. The number of time a database B is accessed by a computer during any 10 seconds time period is a Poisson random variable. Its mean is 5 accesses during 10 seconds.
   (a) What is the probability that there are no access to B during a time period of 10 seconds?
   (b) What is the probability that there are at least 2 accesses to B during a time period of 2 seconds?

4. The data packets transmitted by a modem on a phone line form a Poisson process whose rate λ is equal to 10 packets/second. Given M_k the number of transmitted packets during the k th hour. Determine
   (a) The probability that M_k be equal to n .
   (b) The mean of M_k .

5. The arrival of cars, motorbikes and trucks to the roadworthiness tests center consists in 3 independent Poisson processes with the following rate; $\lambda_{\text{car}} = 1,2$ car/minute, $\lambda_{\text{moto}} = 0,9$ moto/minute and $\lambda_{\text{truck}} = 0,7$ truck/minute. Determine the probability of arrival of 20 vehicles (any of them) in 10 minutes.

6. When designing a telephone network, we use an observation period T of 15 minutes. Knowing that the blocking probability is equal to $0,005$, that the number of call attempts is equal to 100 and that the average duration of a call is 3 minutes,
   (a) Determine the required line number N .
   (b) If the average communication time increases to 12 minutes, what becomes the blocking probability?

7. A telephone link consists of 40 lines. Knowing that the average call duration is equal to 5 minutes and that the observation period is equal to 30 min, determine the maximum call number in order for the congestion probability to stay below $0,005$.

8. We accept a network congestion probability equal to $0,02$ on a telephone link consisting of 30 lines. The load allocated to each user is equal to $0,03$ [E].
   (a) Determine the number of users.
   (b) Determine the number of users when the rejected call attempts are renewed.

Answer

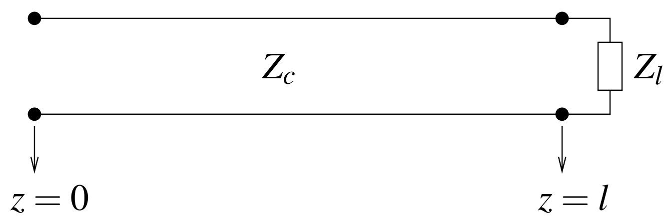
1. $P(K=4)=0,088$
2. (a) $0,0387$
   (b) $0,387$
   (c) $0,185$
   (d) $0,9207$
3. (a) $0,0067$
   (b) $0,264$
4. (a) –
   (b) 36000
5. $0,025$
6. (a) 32
   (b) $B \gg 0,02$
7. 164
8. (a) 730
   (b) 715

Crosstalk (NEXT - FEXT)

Outline:

1. Transmission lines reminder
2. Crosstalk and high-speed transmission
3. Power spectral density
4. Exercises

1. Transmission lines reminder



Signal (in volt) at a point of the line (steady state):

$$V(z,t) = V(z)\, e^{j\omega t}$$

where

$$\begin{aligned} V(z) &= V_i\, e^{-\gamma z} + V_r\, e^{\gamma z} \\ &= V_i \left(e^{-\gamma z} + \Gamma_l\, e^{\gamma z} \right) \end{aligned}$$

and Γ_l is the **reflection coefficient**:

$$\Gamma_l = \frac{Z_l - Z_c}{Z_l + Z_c}$$

and Z_c is the line **characteristic** impedance:

$$Z_c = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

Matching:

$$\Gamma_l = 0 \quad \longrightarrow \quad V(z) = V_i\, e^{-\gamma z}$$

Propagation coefficient:

$$\gamma = \alpha + j\beta$$

where α characterizes the attenuation in the line:

$$V(z) = V_i e^{-\alpha z} e^{-\beta z}$$

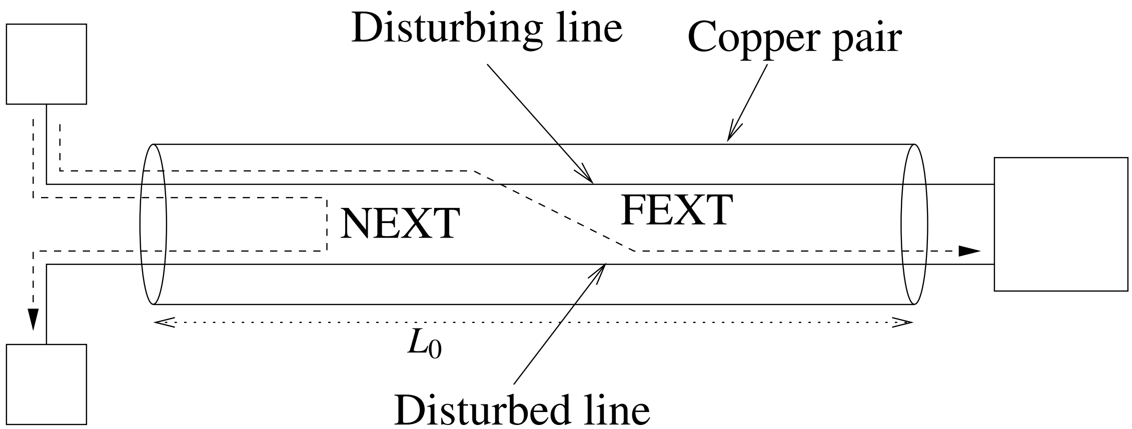
High frequency behavior ($\omega L \gg R$):

$$\gamma \simeq \frac{R}{2}\sqrt{\frac{C}{L}} + j\omega\sqrt{LC}$$

where L, C are independents of f and $R = R_0\sqrt{f}$ (skin effect). Therefore,

$$\alpha(f) \propto \sqrt{f}$$
$$\beta(f) \propto f \longrightarrow \text{no phase distorsion}$$

2. Crosstalk and high-speed transmission



NEXT: **Near-End Crosstalk** ('Paradiaphonie' in french)

FEXT: **Far-End Crosstalk** ('Telediaphonie' in french)

Power transfer function:

$$H_{\text{NEXT}}(f) = K_{\text{NEXT}}\, f^{3/2}$$

$\rightarrow$ independent of the line length L_0 .

$$H_{\text{FEXT}}(f) = K_{\text{FEXT}}\, f^2\, e^{-2\alpha(f)L_0}\, L_0$$

3. Power spectral density

If all disturbing signals have the same power spectral density (PSD),

$$\gamma_N(f) = \gamma_1(f)\, H_{\text{NEXT/FEXT}}(f)\, N$$

where

- $\gamma_1(f)$ = PSD of one disturbing signal.
- $\gamma_N(f)$ = global disturbing PSD.
- N = number of disturbing lines.

$\rightarrow$ over-evaluation of the disturbing power. Therefore, we use

$$\gamma_N(f) = \gamma_1(f)\, H_{\text{FEXT/NEXT}}(f)\, N^{0,6}$$

(UNGER empiric formula).

4. Exercices

1. A digital signal with a power of $5\,[\text{W}]$ should be sent from a transmitter to a receiver at a distance of $2\,[\text{km}]$. The signal bandwidth is equal to $100\,[\text{kHz}]$. Compare the received power levels for the two following transmission types:

   (a) Twisted pair, used bandwidth: $200-300\,[\text{kHz}]$, matched line, $\alpha(f) = 4,497.10^{-3}\sqrt{f}\,[\text{Np/km}]$.

   (b) Wireless connection, carrier frequency: $2,5\,[\text{GHz}]$, transmission antenna gain: $10\,[\text{dB}]$, reception antenna gain: $20\,[\text{dB}]$.

2. On a cable comprising 5 twisted pairs, we transmit 5 ISDN signals (PAM with 4 tension levels, $Z_c = 135\,[\Omega]$, $R_b = 160\,[\text{kb/s}]$, tension levels: $\pm 1[\text{V}], \pm 3[\text{V}]$). Determine the disturbing power spectral density coming from the NEXT effect on 1 of the 5 lines.

3. A telephone cable comprises 50 twisted pairs . The signals used for the transmission have a typical nominal power of $100\,[m\text{W}]$.
   We measure a NEXT power transfer function of $-53\,[\text{dB}]$ at $10\,[\text{kHz}]$, for a cable length of $375\,[\text{m}]$.

   (a) What is the value, in [dB], of the NEXT power transfer function when;
      i. we shorten the cable to half its length (when $f = 10\,[\text{kHz}]$)
      ii. we double the frequency

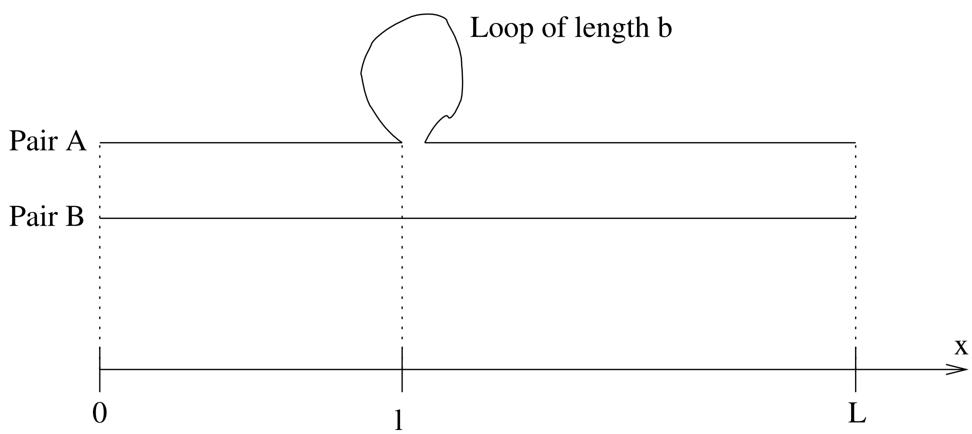
   (b) Estimate the useful signal power at a receiving end when $f = 10\,[\text{kHz}]$.

   (c) If all copper pairs are used, what is the harvested power by Near-End crosstalk on the central pair of the cable.

   (d) The computation result for the Near-End crosstalk does not hold for a $10\,[\text{cm}]$ cable used at a $1\,[\text{MHz}]$ frequency. Which computation hypothesis should you adapt to obtain a correct transfer function.

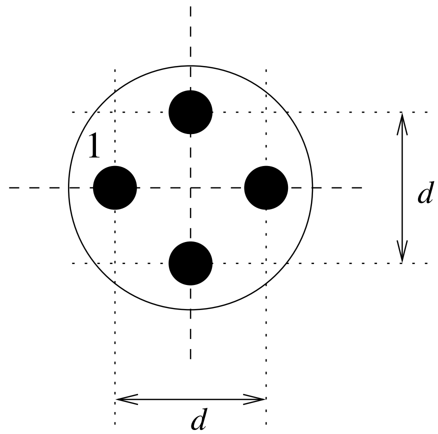
   (e) We would like to build a cable tester. This tester enable the simultaneous injection of disturbing signals on all copper pairs. To measure the Near-End crosstalk, is it better to inject correlated signals or uncorrelated signals ? Explain!

4. Consider the following system, comprised of two twisted pairs of length L.
We note that the line A comprises an additional loop of length B. Historically, this situation sometimes happens in the network for the sake of simplicity of connection.
We will study here the impact of the loop on the Near-End crosstalk.



(a) What is the NEXT power at $x = 0$ if we consider that the line A is the useful line and B is the disturbing line?
(b) For $l = L/2$, what becomes the NEXT effect if the cable attenuation in the loop is much more important than the attenuation in the other parts of the line.
(c) What is the NEXT power at $x = 0$ if we consider that the line B is the useful line and A is the disturbing line?
(d) If you have the choice between line A and B, which one would you choose? Explain!

5. A transmission cable consists of two lines composed respectively of the conductors 1-2 and 3-4.



(a) Suggest a configuration of the conductors 1, 2, 3 and 4 in the cable in such a way that the NEXT and FEXT completely vanish. Explain!
(*hint:* $a_{31} = a_{13} = -C_{13} + C_{14} + C_{23} - C_{24}$).
(b) The line length is 30 [km]. The attenuation coefficient α is $0,151$ [Np/km/km] at the working frequency. The temperature is 290 [K]. The input signal power is 100 [mW] and the bandwidth is 2 [MHz]. Compute the noise factor for one line of this cable.
(c) Compute the thermal noise power at the input of the system.
(d) Compute the signal to noise ratio at the input and at the output.
(e) Does the previous results stay valid when NEXT or FEXT effects are presents? Explain!

Answers

1. (a) 6.5×10^{-4} [W]
   (b) 1.1×10^{-7}[W]
2. $K_{\text{NEXT}}\, f^{3/2}\, 4^{0,6}\, (4.63 \times 10^{-7})\, sinc^2(12.5 \times 10^{-6} f)$
3. (a) i) -53 [dB] ii) -48.48 [dB]
   (b) 89.3 [mW]
   (c) 5.177×10^{-6}[W]
   (d) $E\{P_2(f)\} = \frac{R_L \omega^2 V_0^2(f) k}{-4\alpha(f)} \left(e^{-4\alpha(f)L} - 1\right)$
   (e) uncorrelated signals are the most probable case and correlated signals are the worst case
4. (a) $E\{P_2(f)\} = \frac{R_L \omega^2 V_0^2(f) k}{-4\alpha(f)} \left[e^{-4\alpha(f)l} - 1 + e^{-2\alpha(f)b}\left(e^{-4\alpha(f)L} - e^{-4\alpha(f)l}\right)\right]$
   (b) $E\{P_2(f)\} = \frac{R_L \omega^2 V_0^2(f) k}{-4\alpha(f)} \left(e^{-2\alpha(f)L} - 1\right)$
   (c) same as (a)
   (d) from a crosstalk point of view, both solution are equivalent. But the line B produces a shorter delay and a lower attenuation
5. (a)
   3
   1 2
   4
   (b) $F_0 = 39.35$ [dB]
   (c) $N_{in} = 8.004\,10^{-15}$ [W]
   (d) $({S}/{N})_{in} = 1.25\,10^{13} = 130.97$ [dB], $({S}/{N})_{out} = 91.62$ [dB]
   (e) No

Channel capacity

Outline:

1. Source entropy
2. Discrete memoryless channel
3. Mutual information
4. Channel capacity
5. Exercises

1. Source entropy

Given X a memoryless symbol source.

The source alphabet: J different symbols

$$x_0, x_1, ..., x_{J-1}$$

Each symbol is associated with an emission probability:

$$p(x_0), p(x_1), ..., p(x_{J-1})$$

$$\sum_{j=0}^{J-1} p(x_j) = 1$$

To each symbol, we associate its specific information:

$$i(x_j) = -\log_2 p(x_j)$$

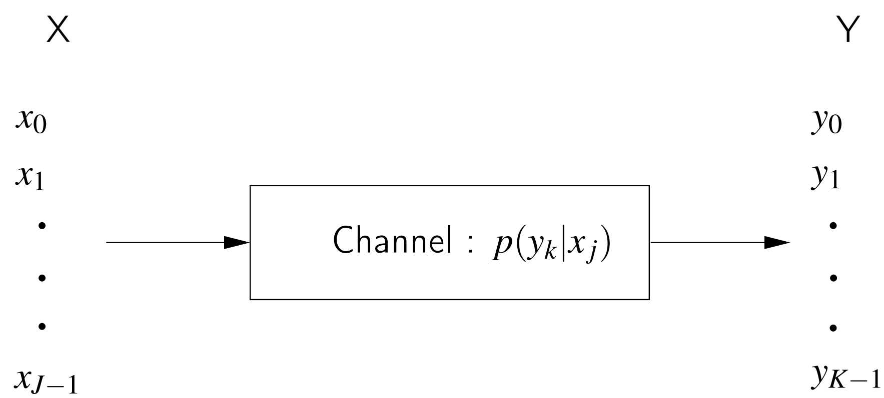
The source **entropy** is then defined by:

$$\begin{aligned} H(X) &= -\sum_{j=0}^{J-1} p(x_j) \log_2 p(x_j) \\ &= \text{average information per symbol} \end{aligned}$$

expressed in bit/symbol.

$$\text{ENTROPY} \propto \text{UNCERTAINTY} \propto \text{INFORMATION}$$

2. Discrete memoryless channel



- The noise on the channel $\longrightarrow$ the source and destination alphabets might be different.
- $p(y_k|x_j)$: transition probabilities.

3. Mutual information

We observe $Y = y_k$. Which uncertainty remains on X ?

We define the entropy of X conditionally to $Y = y_k$:

$$H(X|Y = y_k) = -\sum_{j=0}^{J-1} p(x_j|y_k) \log_2 p(x_j|y_k)$$

We take the average value of Y :

$$\begin{aligned} H(X|Y) &= \sum_{k=0}^{K-1} p(y_k) H(X|Y = y_k) \\ &= -\sum_{k=0}^{K-1}\sum_{j=0}^{J-1} p(x_j|y_k)\, p(y_k) \log_2 p(x_j|y_k) \\ &= -\sum_{k=0}^{K-1}\sum_{j=0}^{J-1} p(x_j, y_k) \log_2 p(x_j|y_k) \end{aligned}$$

The **average mutual information** is defined by

$$I(X;Y) = H(X) - H(X|Y)$$

$$I(X;Y) = H(X) - H(X|Y)$$

Two particular cases:

1. Channel without noise:

$$H(X|Y) = 0 \longrightarrow I(X;Y) = H(X)$$

$\rightarrow$ the channel convey only the useful information.

2. Very noisy channel:

$$H(X|Y) = H(X) \longrightarrow I(X;Y) = 0$$

$\rightarrow$ the channel does not convey any useful information.

Remark: The mutual information is symmetric

$$I(X;Y) = I(Y;X)$$

4. Channel capacity

Definition:

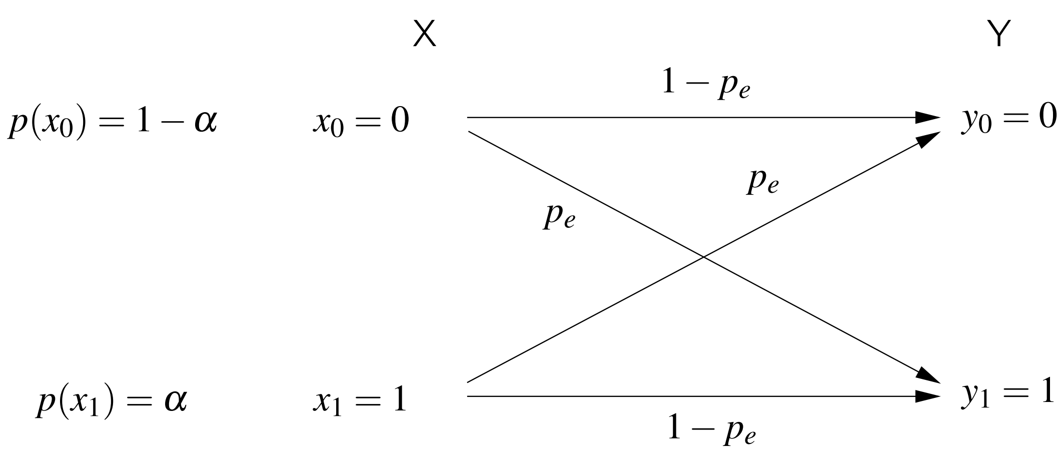
$$C_s = \max_{p(x_j)} I(X;Y)$$

expressed in bit/symbol. If s = symbol transmission rate (symbol/s),

$$C = sC_s$$

is the channel capacity in bit/s.

Binary symmetric channel case



$J = K = 2$. The mutual information is given by

$$I(X;Y) = H(Y) - H(Y|X)$$

Computation of $H(Y|X)$:

$$\begin{aligned} H(Y|X) &= -\sum_{k=0}^{1}\sum_{j=0}^{1} p(x_j)\,p(y_k|x_j)\,\log_2 p(y_k|x_j) \\ &= -(1-\alpha)\,(1-p_e)\,\log_2(1-p_e) \\ &\quad -(1-\alpha)\,p_e\,\log_2 p_e \\ &\quad -\alpha\,(1-p_e)\,\log_2(1-p_e) \\ &\quad -\alpha\,p_e\,\log_2 p_e \\ &= -(1-p_e)\,\log_2(1-p_e) - p_e\,\log_2 p_e \end{aligned}$$

→ independent of the $p(x_j)$.

→ may be considered as a channel entropy.

Therefore,

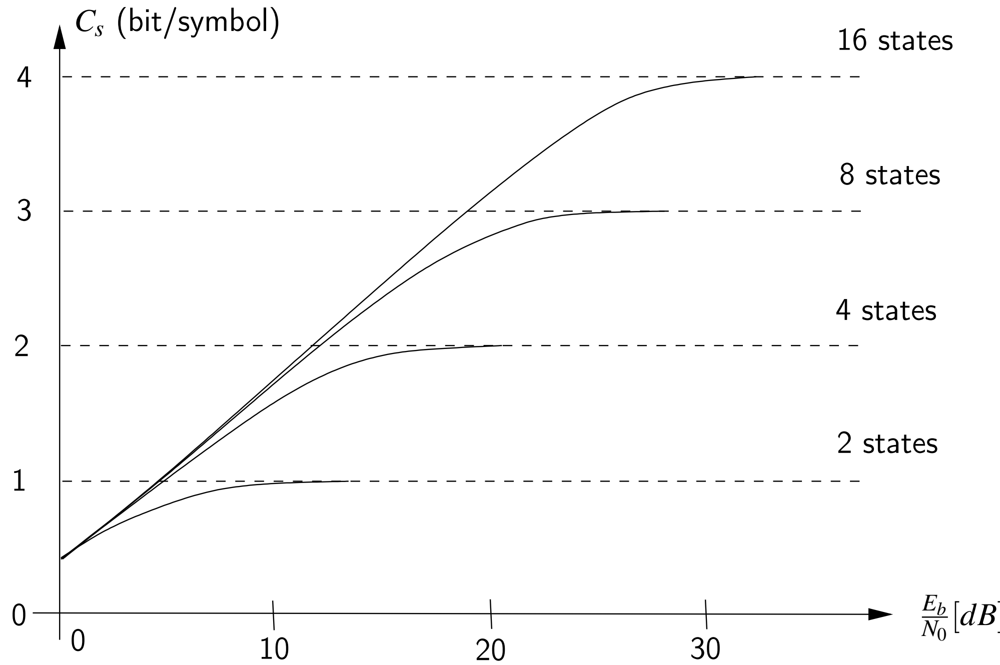
$$I(X;Y) = H(Y) + (1-p_e)\,\log_2(1-p_e) + p_e\,\log_2 p_e$$

and

$$\begin{aligned} C_s &= \max_{p(x_j)} I(X;Y) \\ &= 1 + (1-p_e)\,\log_2(1-p_e) + p_e\,\log_2 p_e \end{aligned}$$

NRZ baseband transmission case:

$$p_e = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$



Shannon Theorem

Continuous input and output alphabets. Example:

$$Y = X + N(0, \sigma_N^2)$$

Then

$$C_s = \frac{1}{2}\log_2\left(1 + \frac{\sigma_X^2}{\sigma_N^2}\right) \ [bit/symbole]$$

where $\sigma_X^2 =$ input power.

If the channel bandwidth is equal to B , its capacity is given by

$$C = B\log_2\left(1 + \frac{\sigma_X^2}{\sigma_N^2}\right) \ [bit/second]$$

(Shannon-Hartley relation).

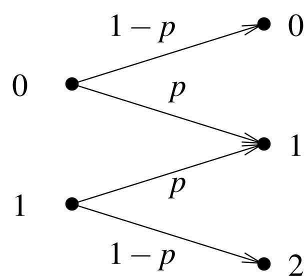
Information rate:

$$R = sH(X)$$

If $R < C$, we can find a source and channel encoding which give rise to a perfect transmission.

5. Exercises

1. Determine the capacity of the discrete channel whose transition probabilities are given by



2. Two binary symetric transmission channel of error probability p are cascaded. Determine the global channel capacity.

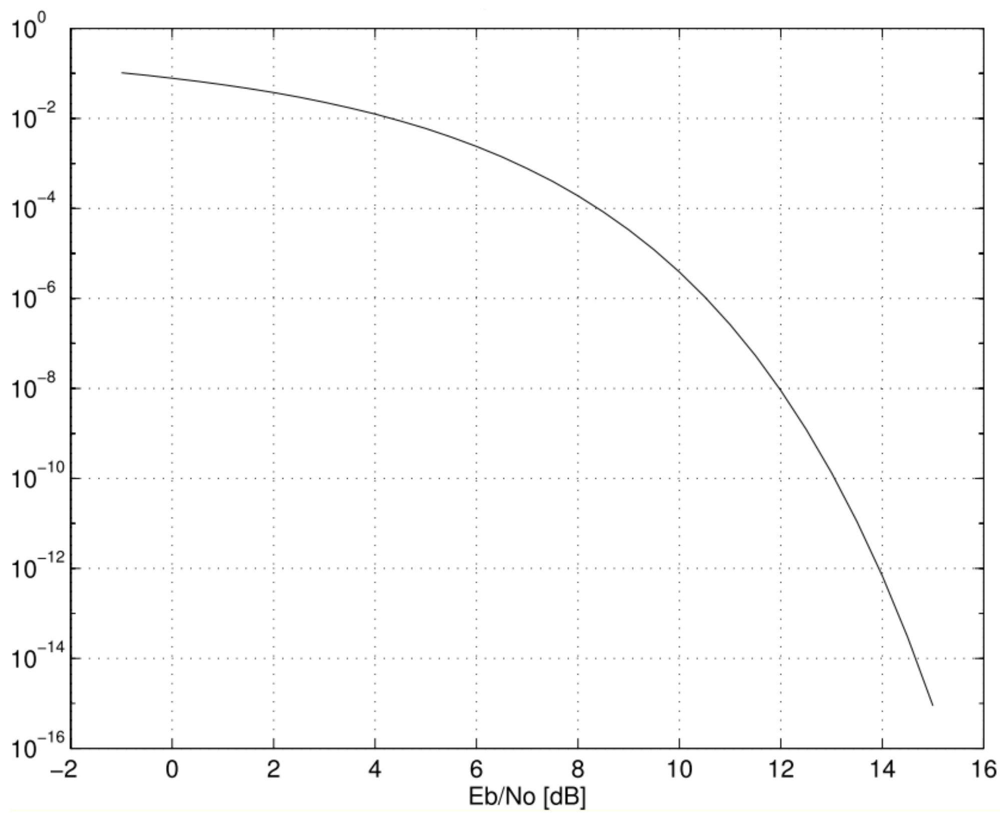
3. We consider a channel with some white additive Gaussian noise whose bandwidth is equal to $4\,$[kHz] and the noise power spectral density is equal to $N_0/2 = 10^{-12}\,$[W/Hz] . The required signal power at the receiver is equal to $0.1\,[m\text{W}]$. Compute the channel capacity.

4. An analog signal with a bandwidth of $4\,$[kHz] is sampled at 1.25 times the NYQUIST frequency, each sample is quantized into 256 levels of equal probability. We assume that the samples are statistically independents.
   (a) What is the source information rate?
   (b) Is it possible to transmit without errors the signals from this source on a channel subject to a Gaussian additive white noise with a bandwidth of $10\,$[kHz] and a signal to noise ratio of $20\,$[dB]?
   (c) Compute the required signal to noise to ensure a transmission without errors in the conditions edicted in (b).
   (d) Compute the required bandwidth to transmit without errors the signals from the same source through a channel with a Gaussian additive white noise to ensure a signal to noise ratio of $20\,$[dB].

5. The problem is to design a transmission system for packets comprising 1500 bytes. We impose the usage of a two states digital phase modulation (PSK-2) and that 99% of the packets be entirely corrects at the receiver (meaning that the packet error rate should be less than 1%).
   (a) If the noise density $\frac{N_0}{2}$ is 10^{-2} [W/Hz], what is the energy per bit E_b?
   (b) Determine the maximum theoretical value of the channel capacity!
   (c) Determine the real value of the channel capacity in the conditions of this question!

Remark:

Error probability for a bipolar NRZ signal



Answer

1. $(1-p)$
2. $1+2p(1-p)\log_2[2p(1-p)]+(1-2p+2p^2)\log_2(1-2p+2p^2)$
3. $54.44\,$[kb/s]
4. (a) $80\,$[kb/s]
   (b) $C = 66.6\,$[kb/s]. It is not possible to have a transmission without errors.
   (c) $24.1\,$[dB]
   (d) $12\,$[kHz]
5. (a) $E_b = 0.252\,$[J]
   (b) $C_{s,max} = 1.88\,[\text{b}/symbol]$
   (c) $C_s = 0.919\,[\text{b}/symbol]$

Mobile radiocommunications

<u>Outline</u>:

1. Mobile (receiver) sensitivity
2. General propagation model

2.1. Empirical model

   2.2. Shadowing

   2.3. Fading

3. Exercises

1. Mobile (receiver) sensitivity

- The transmitter is characterized by its power.
- The receiver is characterized by its sensitivity.

After demodulation:

$$\frac{C}{N} = \frac{E_b W}{N_0 W} = \frac{E_b}{N_0}$$

Sensitivity = minimum value of C such that

$$\frac{E_b}{N_0} > \left(\frac{E_b}{N_0}\right)_{\text{threshold}}$$

Therefore,

$$S = \left(\frac{E_b}{N_0}\right)_{\text{threshold}} + N$$

For the **mobile device**,

$$W = 271\,[\text{kHz}] \;\longrightarrow\; N = kT_0W = -120\,[\text{dBm}]$$

for $T_0 = 290\,[\text{K}]$, to which we add $10\,[\text{dB}]$ (noise generated by the input amplifier).

In addition,

$$\left(\frac{E_b}{N_0}\right) = 8\,[\text{dB}]$$

It comes

$$S = 8 - 120 + 10 = -102\,[\text{dBm}]$$

For a **base station**, we use generally

$$S = -104\,[\text{dBm}]$$

→ Two communicating entities (base station, mobile device) may demonstrate different transmission power and sensitivity.

Summary table (typical values):

Receiver type	Sensitivity [dBm]
Base station	-104
Mobile $8\,[\text{W}]$	-104
Mobile $2\,[\text{W}]$	-102
Two-band mobile	-102

2. General propagation model

Generally speaking,

$$P_R = P_T - L_T + G_T - L + G_R - L_R$$

where

- L_T = losses in the transmission circuits.
- L_R = losses in the receiving circuits.
- L = free space losses.

In the case of an unique direct path:

$$L = \left(\frac{4\pi d}{\lambda}\right)^2$$

$\rightarrow$ FRIIS equation. In this case,

$$L = 32,5 + 20\log f\,[\text{MHz}] + 20\log d\,[\text{km}]$$

But in practical situations:

- **Fading**: Multipath, due to reflections and diffractions by present objects.
- **Doppler effect**, due to the mobile device movements.
- **Shadowing**: attenuation due to present objects.
- The transmission channel is constantly changing due to the mobile movement.

2.1. Empirical models

$\rightarrow$ provide reliable order of magnitude for a lot of reference configuration.

Environment types

Models example:

	BS	L at 925 [MHz]	L at 1795 [MHz]
Rural	100	$90,9 + 31,8 \log d$	$97,0 + 31,8 \log d$
Suburban	100	$95,9 + 31,8 \log d$	$102,0 + 31,8 \log d$
Urban	50	$123,6 + 33,8 \log d$	$133,1 + 33,8 \log d$

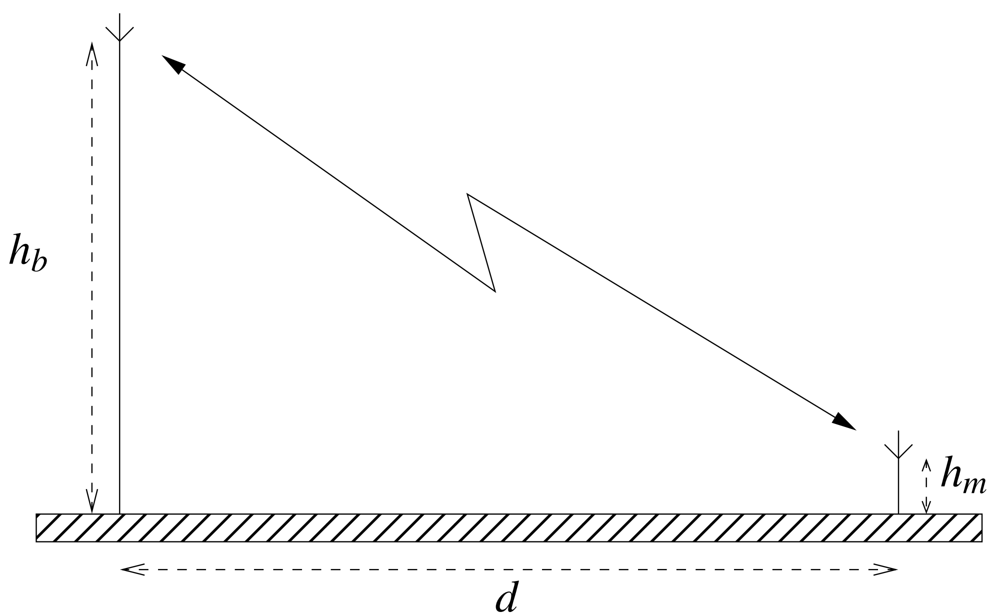
where

- BS = Base station antenna height (in [m]).
- d is expressed in kilometers.

Cell types

- Macro cell: within a radius of a few dozen of [km] , in rural environment.
- Small cell: within a radius of a few [km] , urban environment.
- Micro cell: within a radius of a few hundred [m] , dense urban environment.
- Pico cell: within a radius of a few dozen of [m] , indoor environment.

Influence of the antenna height



$$L \simeq -20\log h_b\,[\mathrm{m}] - 20\log h_m\,[\mathrm{m}] + 40\log d\,[\mathrm{km}]$$

Macro cells models

Formula of COST 231-HATA:

$$
\begin{aligned}
L &= 46.33 + 33.9 \log f\,[\text{MHz}] \\
&\quad -13.82 \log h_b[\text{m}] - a(h_m[\text{m}]) \\
&\quad +(44.9 - 6.55 \log h_b[\text{m}]) \log d\,[\text{km}] + C_m
\end{aligned}
$$

with

$$a(h_m) = (1.1 \log f - 0,7)h_m - (1.56 \log f - 0.8)$$

for a middle-sized tow, and

$$C_m = \begin{cases} 0\,[\text{dB}] & \rightarrow \text{ middled-size towns} \\ 3\,[\text{dB}] & \rightarrow \text{ large cities} \end{cases}$$

$\longrightarrow$ Formula correct in urban environment for cells of radius larger or equal to 1 [km] and for frequency from 1500 to 2000 [MHz] .

Indoor propagation

Two types of propagation "outdoor-indoor":

- Soft Indoor: fading in places close to the front of the building, typically $10\,$[dB] of additional fading.
- Deep Indoor: fading in places located deep inside the buildings, typically $20\,$[dB] of additional fading.

2.2. Shadowing

- Due to **obstruction by objects**.
- Also called **slow fading**.

If we assume the presence of N obstacles. Then

$$L_{\text{shadowing}}[\text{dB}] = L_1 + L_2 + \ldots + L_N$$

with $L_i\,(i = 1, 2, \ldots, N)$ = RV of same characteristics. Then L follow a normal law (central limit theorem):

$$L_{\text{shadowing}}\,[\text{dB}] = N\left(L_{50\%}, \sigma_s^2\right)$$

where $L_{50\%}$ is the median value of the attenuation (given by the empirical models coming from the experimental data).

Impact on the covering zone (cell border):

$$L_{\text{shadowing}}\,[\text{dB}] = L_{50\%}\,[\text{dB}] + L_s\,[\text{dB}]$$

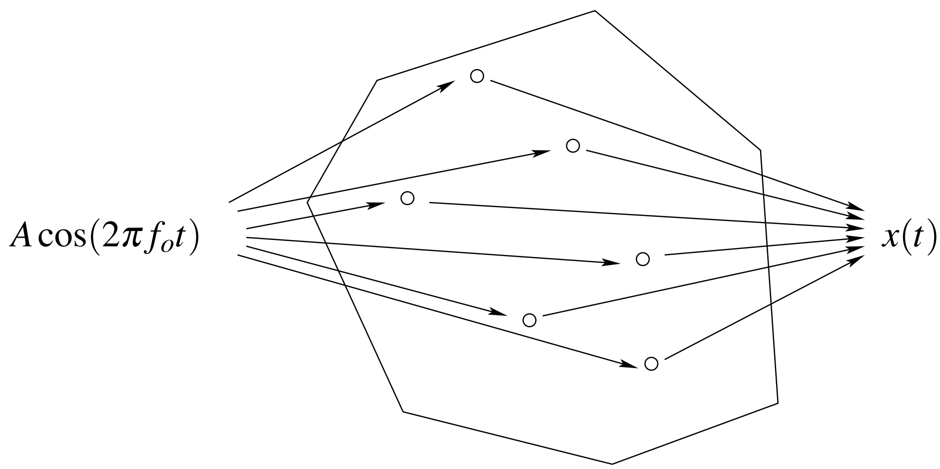
In practice, we add a security margin M_s to the transmitted power. The covering probability is given by $p(L_s < M_s)$ and is then a function of M_s .

Covering probability (in percent)

Margin in [dB]

2.3. Fading

- Due to **multipaths**:



- Generally, there is no direct path.

Received signal:

$$X(t) = \sum_i C_i \cos(2\pi f_0 t + \theta_i)$$

which may be written in the following form

$$X(t) = X_I(t)\cos(2\pi f_0 t) - X_Q(t)\sin(2\pi f_0 t)$$

with

$$X_I(t) = \sum_i C_i \cos\theta_i \text{ et } X_Q(t) = \sum_i C_i \sin\theta_i$$

$\rightarrow X_I(t)$ and $X_Q(t)$ = Gaussian and centered RV.

Therefore,

$$X(t) = R(t)\cos\left(2\pi f_0 t + \Phi(t)\right)$$

with

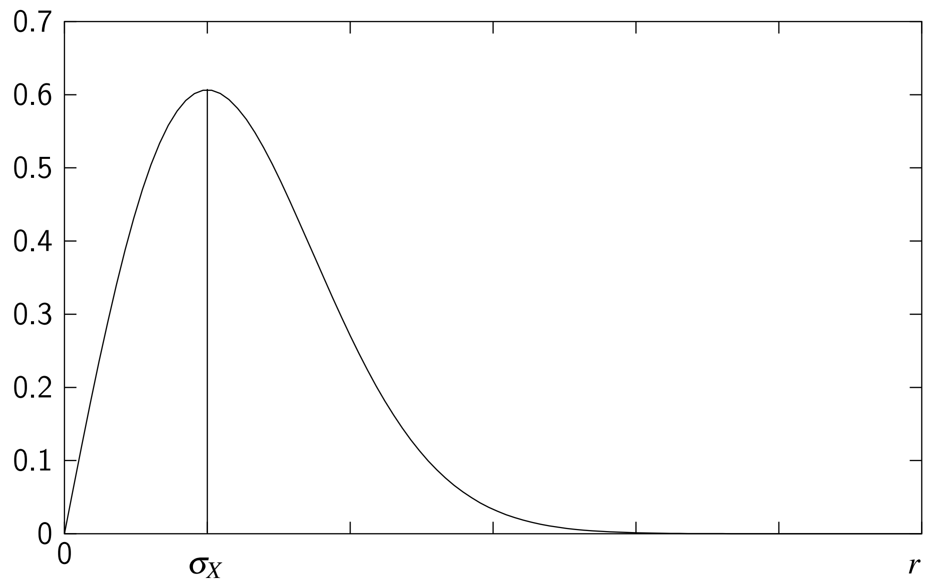
$$R(t) = \sqrt{X_I(t) + X_Q(t)}$$

and

$$\Phi(t) = \tan^{-1}\frac{X_Q(t)}{X_I(t)},\ \Phi(t) \in [0, 2\pi[$$

Probability density of $R(t)$:

$$f_{R(t)}(r) = \begin{cases} \frac{r}{\sigma_X^2} e^{-\frac{r^2}{2\sigma_X^2}} & \text{if } r \geq 0 \\ 0 & \text{if } r < 0 \end{cases}$$



Special characteristics:

$$E\{R(t)\} = 1.253 \times \sigma_X$$

$$\sigma^2_{R(t)} = 0.429 \times \sigma_X$$

Probability density of $\Phi(t)$:

$$f_{\Phi(t)}(\phi) = \begin{cases} \frac{1}{2\pi} & \text{if } \phi \in [0, 2\pi[\\ 0 & \text{otherwise} \end{cases}$$

3. Exercises

1. A base station must communicate to a telecommunication mobile subscriber using a mobile phone receiving in the 1800 [MHz] band, functioning at a nominal power of 2 [W] and comprising an isotropic antenna. The equivalent isotropic radiated power of the base station is 50 [dBW] .
   (a) Considering that the communication take place in a middle-sized town, by using the model COST 231-HATA, compute the maximum radius of the cell covered by the base station if its height is 40 [m] and by neglecting the effect due to the height of the mobile.
   (b) We would like to manage the shadowing effect. Determine the value of the additional margin if we require a covering ratio of 90%.
   (c) With this margin, and in case of Soft Indoor communications, what is the new value of the maximum cell radius. We will also consider 3 [dB] losses due to human bodies.

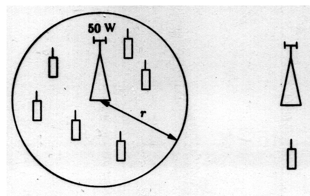
   Remarks:

   - Following the COST 231-HATA model, the fading L_u in urban environment is, in [dB],

   $$L_u = 46.33 + 33.9 \log(f) - 13.82 \log(h_b) - a(h_m) + [44.9 - 6.55 \log(h_b)] \log(d) + C_m$$

     with
     - f the frequency, d the distance, h_b, h_m the heights; these values are respectively expressed in [MHz], [km] and [m].
     - $a(h_m) = (1.1 \log(f) - 0.7) h_m - (1.56 \log(f) - 0.8)$ for a middle-sized town.
     - $C_m = 0$ [dB] for middle-sized towns and suburbs, and $C_m = 3$ [dB] for large cities

2. A GSM service provider would like to deploy a cellular network in a large city and is interested in the modelization of a circular cell. We would like to determine the maximum radius of the cell, knowing that the transmission power of the base station (BTS) is equal to $50\,[\mathrm{W}]$ and that the used frequency is $1800\,[\mathrm{MHz}]$.



(a) Determine the maximum radius of the cell by using the COST 231-HATA model by neglecting the effects depending on the mobile height. The height of the base station is $40\,[\mathrm{m}]$.

(b) We would like to guard ourselves from shadowing effects. Determine the additional margin value if we require a covering percentage of $90\,\%$, if we wish to communicate in Soft Indoor and if we also consider $3\,[\mathrm{dB}]$ of losses dues to human bodies? Determine again the maximum radius of the cell in these conditions.

(c) In a second time, we focus our attention on the design in terms of the traffic of the cell. We will arbitrarily choose a cell radius of $0.5\,[\mathrm{km}]$.
Knowing that:
- the service provider covers 500 $\left[clients/\mathrm{km}^2\right]$,
- $10\,\%$ of the covered customers in the cell have established a communication during the observation period of 15 $[min]$ and
- the average call duration is 5 $[min]$,

determine the simultaneous communication number that the base station have to support if we suppose a blocking probability of 0.02.

(d) Determine the minimum spectral occupancy, knowing that each carrier is able to transport a maximum of 8 calls.

Remarks:
- The receiving and transmitting antenna are assumed isotropic
- The margin values are assumed independent of the frequency
- Following the COST 231-HATA model, the fading L_u in urban environment is, in [dB],

$$L_u = 46.33 + 33.9\log(f) - 13.82\log(h_b) - a(h_m) + [44.9 - 6.55\log(h_b)]\log(d) + C_m$$

with
- f the frequency, d the distance, h_b, h_m the heights; these values are respectively expressed in [MHz], [km] and [m].

- $a(h_m) = (1.1\log(f) - 0.7)h_m - (1.56\log(f) - 0.8)$ for a middle-sized town.
- $C_m = 0\,[\text{dB}]$ for middle-sized towns and suburbs, and $C_m = 3\,[\text{dB}]$ for large cities

3. A mobile service provider analyzes, in a middle-sized town, the effect of the cell size on the power received by the mobile devices.

   (a) We assume that we work only in the $1800\,$[MHz] band and that the COST 231-HATA model is valid. Compute the maximum radius of a cell.
   Note: for the computation, we consider that the base height is $30\,$[m] and we neglect the effects dues to the mobile height. The transmitting antenna has a power of $100\,$[W] and a transmission gain of $5\,$[dB]. We are interested in a Deep Indoor covering.

   (b) If we double the radius of the (circular) cells; analyze the effect of such a modification on the transmission gain, when the other parameters stay unchanged!

   (c) To guard ourselves from some shadowing effects, the service provider decides to multiply the EIRP by 3. What do you think of this solution? What is the covering percentage?
   Note: We consider the same conditions that in the point (b).

Remarks:

- Following the COST 231-HATA model, the fading L_u in urban environment is, in [dB],

$$L_u = 46.33 + 33.9\log(f) - 13.82\log(h_b) - a(h_m) + [44.9 - 6.55\log(h_b)]\log(d) + C_m$$

  with
  - f the frequency, d the distance, h_b, h_m the heights; these values are respectively expressed in [MHz], [km] and [m].
  - $a(h_m) = (1.1\log(f) - 0.7)\,h_m - (1.56\log(f) - 0.8)$ for a middle-sized town.
  - $C_m = 0\,$[dB] for middle-sized towns and suburbs, and $C_m = 3\,$[dB] for large cities

4. A GSM service provider would like to cover a middle-sized town on an area of 20 $\left[\text{km}^2\right]$ with a number N of omnidirectional antennas having a power of 80 [W], a gain of 5 [dB] and a height of 40 [m].
   This service provider enforces a Deep Indoor covering with 90 % of covering percentage.
   (a) What is the spectral efficiency of this GSM system?
   (b) Determine, at 900 [MHz], the minimum number N of antennas required to cover the mentioned area if we suppose that these antennas cover the whole area without holes nor covering. We assume that the losses dues to human bodies are equal to 3 [dB]. We will use the COST 231-HATA model by neglecting the effects dependent on the mobile device height.
   (c) Compare the result obtained above with the result when the frequency is equal to 1800 [MHz]. Comment your answer.
   (d) If the omnidirectional antennas are replaced by trisectorial antennas having the same maximum gain than the omnidirectional antennas, could we place more antennas? What would be the advantages?

   Remarks:

   - The receiving antenna is supposed isotropic
   - The margin values are identical at 900 [MHz] and 1800 [MHz].
   - Following the COST 231-HATA model, the fading L_u in urban environment is, in [dB],

   $$L_u = 46.33 + 33.9 \log(f) - 13.82 \log(h_b) - a(h_m) + [44.9 - 6.55 \log(h_b)] \log(d) + C_m$$

     with
     - f the frequency, d the distance, h_b, h_m the heights; these values are respectively expressed in [MHz], [km] and [m].
     - $a(h_m) = (1.1 \log(f) - 0.7) h_m - (1.56 \log(f) - 0.8)$ for a middle-sized town; this correction factor depend on the mobile device antenna height but also on the environment type.
     - $C_m = 0$ [dB] for middle-sized towns and suburbs, and $C_m = 3$ [dB] for the large cities

5. The presence of a water surface influences the radio budget link from a base station to a mobile device. We usually consider that such a link comprises the sum of the direct path wave with the two reflected waves (by the water surface and by the ground surface close to the mobile device). We take into account the wave reflected by the the water surface because this wave doesn't hit any obstacle in the neighborhood of the water and has then a high power level.

   To compute the radio budget link, we assume the following hypothesis (fullfilled in practical situations):
   1. the power level of the reflected and direct waves are more or less the same,
   2. the reflected waves experience small phase offsets ($\Delta\phi_1$ and $\Delta\phi_2$).

   (a) Compute the general expression for the coefficient Γ modifying the electric field.
   (b) What is the value of the received power P_R?
   (c) Simplify the expression of P_R based on the hypothesis 2. [Hint: use the expression of the squared norm]
   (d) In almost all practical cases $\Delta\phi_1 + \Delta\phi_2 < 1$. Use this information to simplify further the expression of the power P_R.
   (e) Discuss the dependence in d (d being the distance between the transmitter and the receiver) for the received power compared to the same power:
   - in free space,
   - by taking into account the presence of the ground (and then the height of the antennas).

   What could you infer for the practical situations?

Answers

1. (a) $23.9\,[\text{km}]$
   (b) $8\,[\text{dB}]$
   (c) $5.87\,[\text{km}]$