State Complexity of the Multiples of the Thue-Morse Set

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Introduction

Definition

Let $b \in \mathbb{N}_{\geq 2}$. A subset X of $\mathbb{N}$ is *b-recognizable* if $\text{rep}_b(X)$ is regular.

It is equivalent to work with $0^* \text{rep}_b(X)$.

Theorem

Let $b \in \mathbb{N}_{\geq 2}$ and $m \in \mathbb{N}$. If $X \subseteq \mathbb{N}$ is b-recognizable, so is mX.

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Theorem [Alexeev, 2004]

The state complexity of the language $0^* \operatorname{rep}_b(m\mathbb{N})$ is

$$\min_{N \geq 0} \left\{ \frac{m}{\gcd(m, b^N)} + \sum_{n=0}^{N-1} \frac{b^n}{\gcd(b^n, m)} \right\}$$

Thue-Morse

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The Thue-Morse set is the set

$$\mathscr{T} = \{n \in \mathbb{N} : |\operatorname{rep}_2(n)|_1 \in 2\,\mathbb{N}\}.$$

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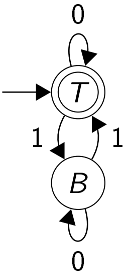

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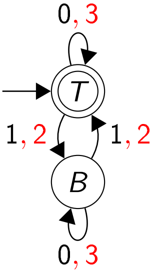

Definition

Let $p, q \in \mathbb{N}_{\geq 2}$. We say that p and q are *multiplicatively independent* if

$$p^a = q^b \Rightarrow a = b = 0.$$

They are said *multiplicatively dependent* otherwise.

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Theorem [Cobham, 1969]

- Let b, b' two multiplicatively dependent bases. A subset of $\mathbb{N}$ is b-recognizable iff it is b'-recognizable.
- Let b, b' two multiplicatively independent bases. A subset of $\mathbb{N}$ is both b-recognizable and b'-recognizable iff it is a finite union of arithmetic progressions.

The result

$$\mathscr{T} = \{n \in \mathbb{N} : |\operatorname{rep}_2(n)|_1 \in 2\,\mathbb{N}\}$$

Theorem

Let $m \in \mathbb{N}$ and $p \in \mathbb{N}_{\geq 1}$.
Then the state complexity of the language $0^* \operatorname{rep}_{2^p}(m\mathscr{T})$ is

$$2k + \left\lceil \frac{z}{p} \right\rceil$$

if $m = k2^z$ with k odd.

The method

Automaton	Language accepted
$\mathscr{A}_{\mathscr{T},2^p}$	$(0,0)^* \operatorname{rep}_{2^p}(\mathscr{T} \times \mathbb{N})$

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$\mathscr{A}_{\mathscr{T},2^p} \times \mathscr{A}_{m,2^p}$	$(0,0)^* \, \text{rep}_{2^p}(\{(t, mt) : t \in \mathscr{T}\})$

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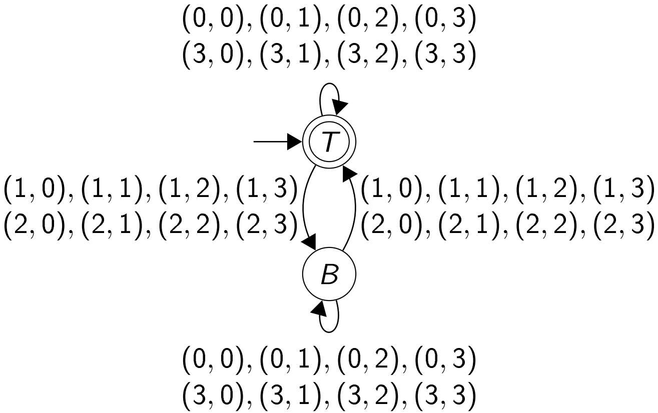
$\mathscr{A}_{\mathscr{T},2^p} \times \mathscr{A}_{m,2^p}$	$(0,0)^* \operatorname{rep}_{2^p}(\{(t, mt) : t \in \mathscr{T}\})$
$\pi\left(\mathscr{A}_{\mathscr{T},2^p} \times \mathscr{A}_{m,2^p}\right)$	$0^* \operatorname{rep}_{2^p}(m\mathscr{T})$

The automaton $\mathscr{A}_{\mathscr{T},4}$

$(0,0)^* \{\mathrm{rep}_{2^p}(t,n) : t \in \mathscr{T}, n \in \mathbb{N}\}$

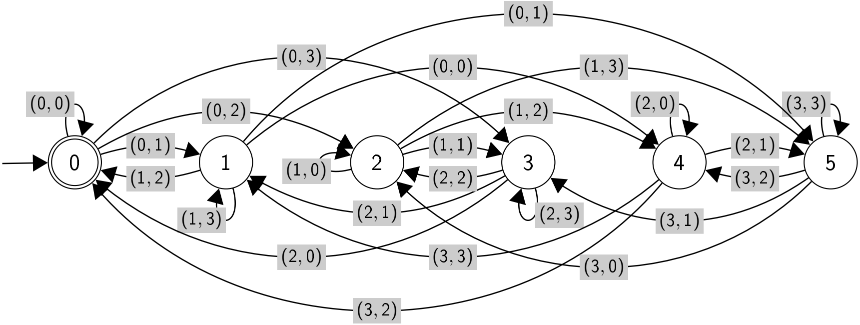


The automaton $\mathscr{A}_{m,b}$

$(0,0)^* \{\mathrm{rep}_b(n, mn) : n \in \mathbb{N}\}$

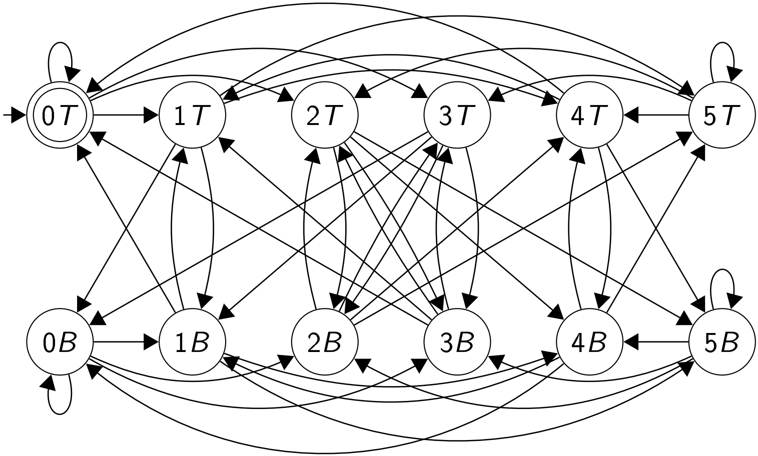
States	$0, \ldots, m-1$
Initial state	0
Final states	0
Alphabet	$\{0, \ldots, b-1\}^2$
Transitions	$\delta_{m,b}(i,(d,e)) = j \Leftrightarrow bi + e = md + j$

The automaton $\mathscr{A}_{6,4}$



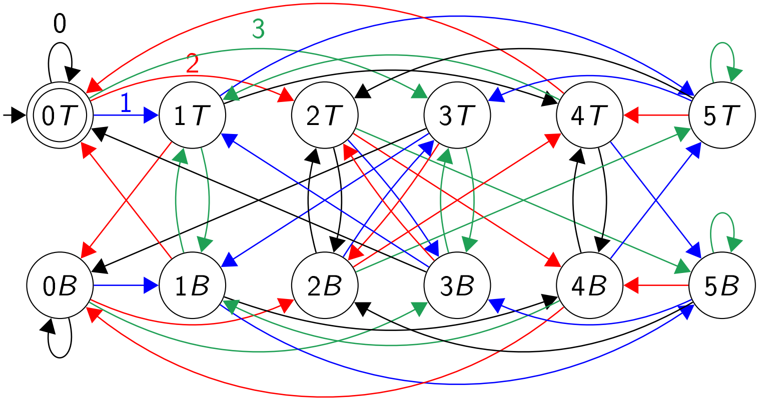
The product automaton $\mathscr{A}_{6,4} \times \mathscr{A}_{\mathscr{T},4}$

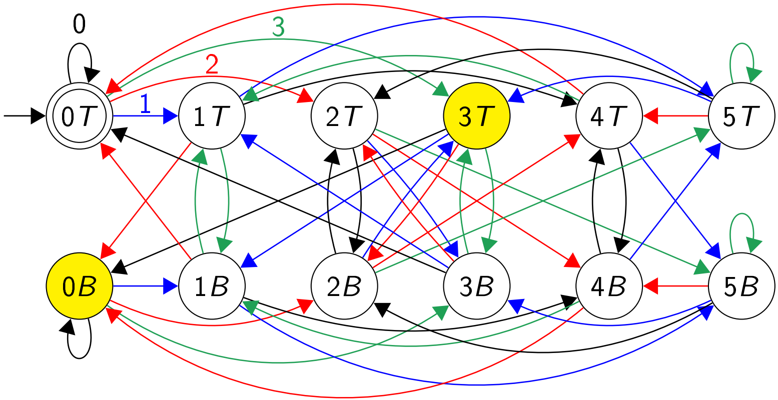
$(0,0)^* \{\text{rep}_{2^p}(t, mt) : t \in \mathscr{T}\}$



The projected automaton $\pi\left(\mathscr{A}_{6,4} \times \mathscr{A}_{\mathscr{T},4}\right)$

$0^* \operatorname{rep}_{2^p}(m\mathscr{T}) = 0^* \{\operatorname{rep}_{2^p}(mt) : t \in \mathscr{T}\}$

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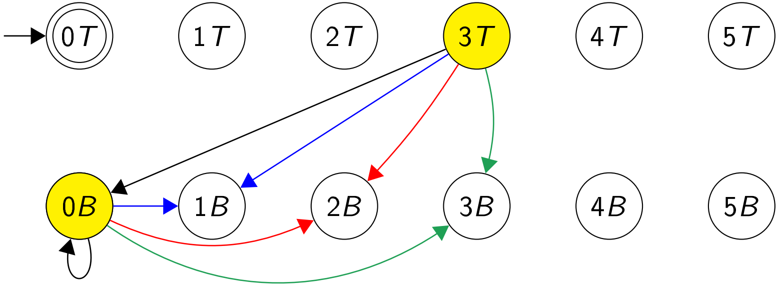
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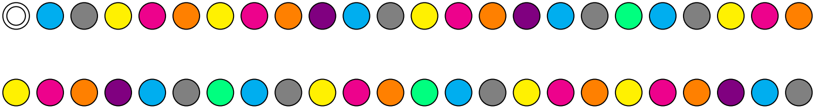
The automaton $\pi(\mathscr{A}_{24,4} \times \mathscr{A}_{\mathscr{T},4})$



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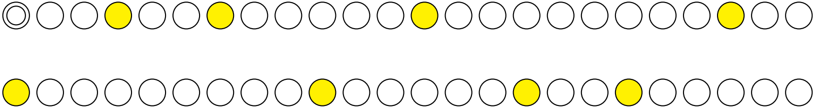

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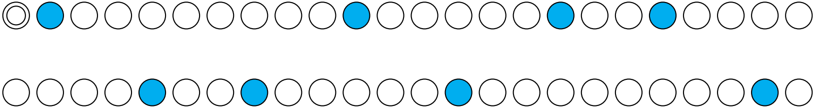

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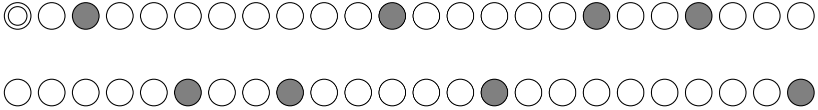

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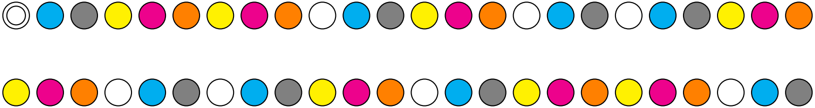


Recall

For $m \in \mathbb{N}, p \in \mathbb{N}_{\geq 1}$, we write $m = k2^z$ with z odd.

For all $n \in \mathbb{N}$, we set

$$T_n := \begin{cases} T \text{ if } n \in \mathscr{T} \\ B \text{ else.} \end{cases}$$

Definition

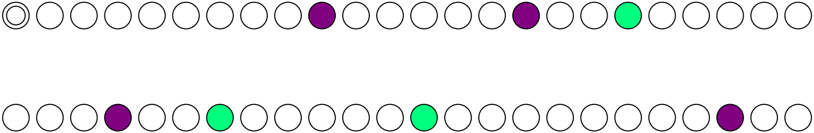
For all $j \in \{1, \ldots, k-1\}$, we set

$$[(j, T)] := \{(j + k\ell, T_\ell) : 0 \leq \ell \leq 2^z - 1\}$$
$$[(j, B)] := \left\{\left(j + k\ell, \overline{T_\ell}\right) : 0 \leq \ell \leq 2^z - 1\right\}.$$

We also set

$$[(0, T)] := \{(0, T)\} \text{ and } [(0, B)] := \left\{\left(k\ell, \overline{T_\ell} : 0 \leq \ell \leq 2^z - 1\right)\right\}.$$

The automaton $\pi(\mathscr{A}_{24,4} \times \mathscr{A}_{\mathscr{T},4})$



Definition

For all $\alpha \in \{0, \ldots, z-1\}$, we set

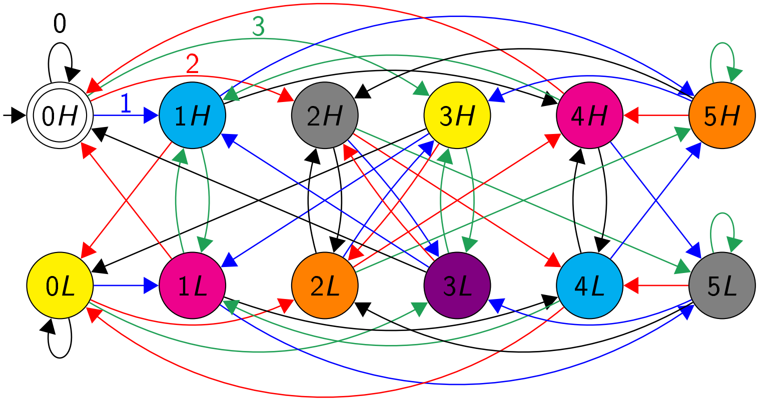
$$C_\alpha := \left\{ \left(k2^{z-\alpha-1} + k2^{z-\alpha}\ell, \overline{T_\ell} \right) : 0 \leq \ell \leq 2^\alpha - 1 \right\}.$$

For all $\beta \in \left\{0, \ldots, \left\lceil \frac{z}{p} \right\rceil - 2\right\}$, we set

$$\Gamma_\beta := \bigcup_{\alpha \in \{\beta p, \ldots, (\beta+1)p-1\}} C_\alpha.$$

We also set

$$\Gamma_{\left\lceil \frac{z}{p} \right\rceil - 1} := \bigcup_{\alpha \in \left\{ \left(\left\lceil \frac{z}{p} \right\rceil - 1 \right) p, \ldots, z-1 \right\}} C_\alpha.$$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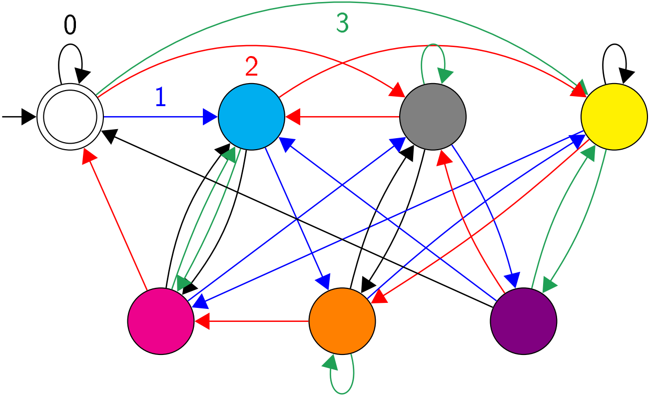
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Theorem

Let $m \in \mathbb{N}$ and $p \in \mathbb{N}_{\geq 1}$. Then the state complexity of the language $0^* \operatorname{rep}_{2^p}(m\mathscr{T})$ is equal to

$$2k + \left\lceil \frac{z}{p} \right\rceil$$

if $m = k2^z$ with k odd.

$2 \times 3 + \left\lceil \frac{1}{2} \right\rceil = 7$

Corollary

Given any 2^p-recognizable set Y (via a finite automaton $\mathscr{A}$ recognizing it), it is decidable whether $Y = m\mathscr{T}$ for some $m \in \mathbb{N}$. The decision procedure can be run in time $O(N^2)$ where N is the number of states of the given automaton $\mathscr{A}$.