Finite Element Formulations for Systems with High-Temperature Superconductors

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Abstract—In this paper, we consider finite element models for high-temperature superconductors and compare two dual formulations, either magnetic-field conforming or magnetic-flux-density conforming. The electrical resistivity of superconductors is described by a power law and is strongly nonlinear. We compare the accuracy and the efficiency of the dual formulations by starting from simple considerations on the concavity/convexity of the constitutive law involved in each case. We then study the numerical behavior of each formulation in 1D, 2D, and 3D problems and compare their results against benchmarks. We draw general recommendations for the choice of a formulation, an iteration scheme for treating the corresponding linearized constitutive law, and a time-stepping extrapolation scheme. This approach is extended to soft ferromagnetic materials with a saturation law. Since the outcome of our analysis shows that recommended formulations for treating ferromagnets are just the opposite of those for treating superconductors, we suggest a coupled formulation for systems where both types of materials are present. The coupled formulation is shown to be accurate and more efficient than single formulations applied indistinctly to all materials.

Index Terms—Finite element analysis, high-temperature superconductors, magnetic materials, nonlinear equations.

I. INTRODUCTION

Modeling accurately and efficiently the magnetic response of high-temperature superconductors (HTS) is of high importance in numerous applications, such as energy transport and storage [1], trapped-field magnets, magnetic shields, and levitating devices [2]. Among the existing numerical methods, the finite element (FE) method allows one to tackle complex geometries and is commonly used for the design and analysis of engineering devices [3]. The FE method expresses the problem in a weak form and different formulations of the same problem are available. More specifically in the framework of magnetodynamics, two classes of dual formulations exist: h-conform formulations, that respect the continuity conditions for magnetic fields, and b-conform ones, that guarantee the continuity of magnetic flux. These two classes involve the HTS nonlinear laws in inverse ways, and thus may exhibit different numerical behaviors. In parallel, several benchmark problems have been developed over the years to serve as reference problems [10]. However, as pointed out in [3], there are still few comparisons based on simulations which are carried out within the same environment and are validated against benchmarks, so that it is difficult to draw general conclusions on the performance of the different approaches.

The purpose of this work is to implement and study two FE formulations, a magnetic-field conforming formulation (here an h-φ-formulation) and a magnetic-flux-density conforming formulation (here an a-φ-formulation), and compare their performance using the same numerical environment. No a priori is made on their behavior and several time-stepping schemes, as well as different iterative techniques, are systematically investigated.

The manuscript is organized as follows: the different FE models and formulations are presented in Sec. III. In Sec. IV, we start from simple considerations on the treatment of the nonlinear constitutive law by iterative techniques (Picard and Newton-Raphson). We then discuss the advantages and drawbacks of each formulation based on their accuracy and efficiency in 1D, 2D, and 3D problems, together with a validation against known solutions and benchmarks. In Sec. V, we adopt a similar approach for soft ferromagnetic materials with a magnetic saturation law. As it turns out that the a-φ-formulation is the best performing one for ferromagnets, whereas the h-φ-formulation is preferred for superconductors, we suggest a coupled formulation for situations where both types of materials are present. We show that such an approach yields accurate results with a better efficiency than single formulations. Our final conclusions are drawn in Sec. VI.

Model files for the main test cases are available on the website www.life-hts.uliege.be.

II. MODEL AND FE FORMULATIONS

A. Physical model

We consider the magnetic response of a system containing type-II superconductors with strong pinning that are cooled in a cryogenic fluid or placed in the cold chamber of a measurement device.
Maxwell’s equations are solved in the magnetodynamic approximation [17], where the displacement current is neglected,

\[
\begin{aligned}
\text{div} \; b &= 0, \\
\text{curl} \; h &= j, \\
\text{curl} \; e &= -\partial_t b,
\end{aligned}
\]

where \( b, h, j, e \) are the magnetic flux density (T), the magnetic field (A/m), the electric current density (A/m²), and the electric field (V/m), respectively. The system is composed of non-conducting materials (but possibly magnetic ones), for which \( b = \mu(h)h \) and \( j = 0 \), together with superconductors, described by \( b = \mu_0 h \) (assuming \( \| h \| \gg H_c \)) and the \( e \)-\( j \) power law [13],

\[
\frac{\partial_j}{\partial t} \left( \frac{\| e \|}{c_e} \right)^{(1-n)/n} e,
\]

where \( c_e \) (V/m) is a threshold electric field defining the critical current density \( j_c \) (A/m²). By convention, \( c_e \) is often chosen to be 10⁻⁴ V/m. The dimensionless exponent \( n = U_0/k_B T \) describes the sharpness of the transition to flux flow, where \( U_0 \) is the pinning energy barrier and \( k_B T \) the thermal energy (\( k_B = 1.38 \times 10^{-23} \) J/K is the Boltzmann constant).

### B. Finite Element Formulations

The system is represented by a domain \( \Omega \) that is decomposed into a conducting domain, \( \Omega_c \), defined as an open set and containing the superconducting materials, and its complementary domain, \( \Omega^C_c \), a closed set containing the other, electrically insulating, materials. Boundary conditions are applied on the external boundary, \( \partial \Omega \), which is decomposed into two complementary domains: \( \Gamma_c \), where the normal component of \( b \) or the tangential component of \( e \) is imposed, and \( \Gamma_h \), where the tangential component of \( h \) is imposed.

The non-conducting domain \( \Omega^C_c \) can be simply or multiply connected. Multiplied connected geometries typically occur in the presence of an external source. They can be made simply connected by introducing cuts [19], as illustrated in Fig. 1 for the case of a tape with an imposed current.

![Fig. 1: Example of a problem domain (tape case of section III-B) scale not respected) with a current imposed across \( \Omega_c \). The problem is 2D and contains a single conducting region. The cut makes \( \Omega^C_c \) simply connected. The external boundary belongs to \( \Gamma_c \) as \( b \cdot n |_{\Gamma_c} = 0 \) can be assumed provided \( \Gamma_c \) is far enough from the conductor.](image)

An injected current intensity \( I \) and an applied voltage \( V \) are associated to each connected region of \( \Omega_e \) and are thus defined as global quantities. Exactly one of these quantities is fixed in each region (coupling with circuit equations is not considered here). If the domain \( \Omega_e \) is made up of \( N \) connected parts \( \Omega_{e,i} \), with \( i \in C = \{1, 2, \ldots, N\} \), the current intensity is imposed in a subset \( C_I \) of \( C \) and the voltage is imposed in the complementary set \( C_V \).

A finite element mesh is generated to discretize the domain. Nodes, edges and facets of the conducting domain boundary belong to the closed set \( \Omega^C_c \).

**Magnetic field conform formulation (h-formulation):** The chosen \( h \)-conform formulation expresses the magnetodynamic problem in terms of \( h \) discretized with Whitney elements [20] as follows:

\[
h = \sum_{e \in \Omega_e} h_e \psi_e + \sum_{n \in \Omega^C_c} \phi_n \text{grad} \psi_n + \sum_{i \in C} I_i \mathbf{c}_i,
\]

with \( \psi_e \) the edge function of edge \( e \), \( \psi_n \) the node function of node \( n \), and \( \mathbf{c}_i \) a discontinuous shape function associated with the cut related to conducting region \( i \), defined on a transition layer [21]. Coefficients \( h_e, \phi_n \) and \( I_i \) are the degrees of freedom of the problem. The decomposition in (3) confers two properties to \( h \). First, using gradients in \( \Omega^C_c \) ensures that \( h \) is curl-free in the non-conducting domain. No spurious current density is introduced, and there is no need to consider a large electrical resistivity in these regions. Second, the shape functions \( c_i \) directly implement the current constraints in the function space, rather than through an additional integration constraint.

**Essential conditions:** \( n = \mathbf{h} \times n |_{\Gamma_c} \) and \( I_i \) for \( i \in C_I \), are strongly imposed by fixing the corresponding degrees of freedom: \( h_e \) for \( e \in \Gamma \), \( \phi_n \) for \( n \in \Gamma_h \), and \( I_i \) for \( i \in C_I \). The resulting space for \( h \), after imposing essential conditions, is denoted by \( S_h \).

Test functions \( h' \) are chosen in the same space (Galerkin scheme), with coefficients \( h'_e, \phi'_n \) and \( I'_i \), but however with vanishing coefficients where essential conditions are imposed. The space of test functions is denoted as \( S_{h'} \).

The resulting weak formulation reads [21], [22]:

From an initial solution at time \( t = 0 \), find \( h \in S_{h'}^1 \) such that, for \( t > 0 \),

\[
(\partial_t (\mu(h)h), h')_\Omega + (\rho \text{curl} h \cdot \text{curl} h', \Omega) - (\mathbf{e} \cdot n, h')_{\Gamma_c} + \sum_{i \in C} V_i \mathcal{I}_i(h') = 0, \quad \forall h' \in S_{h'}^1,
\]

where the following notations are used for volume and surface integrals:

\[
(\mathbf{A}, \mathbf{B})_\Omega = \int_\Omega \mathbf{A} \cdot \mathbf{B} \, d\Omega, \quad (\mathbf{A}, \mathbf{B})_\Gamma = \int_\Gamma \mathbf{A} \cdot \mathbf{B} \, d\Gamma,
\]

with \( \mathbf{A} \) and \( \mathbf{B} \) two scalar or vector fields and the scalar multiplication or the dot product, respectively, and with \( \mathcal{I}_i(h) = I_i \).

For example, when \( h' = c_i \) is chosen \((i \in C_V, I'_i = 1)\), one obtains the circuit equation

\[
(\partial_t (\mu(h)c_i), c_i)_\Omega + (\rho \text{curl} h \cdot \text{curl} c_i)_\Omega = -V_i.
\]

The formulation (4) is referred to as the \( h-\phi \)-formulation or, equivalently, the \( h \)-formulation. It is a weak form of Faraday’s...
equation. The continuity of the tangential component of \( h \) is enforced by the considered shape functions. The natural constraints, involving the tangential component of the electric field, \( e \times n|_{\Gamma_e} \) and the potentials \( V_i \) for \( i \in C_V \), are weakly imposed.

In practice, the \( h \)-formulation can be solved in one of two ways. Either the total magnetic field \( h \) is considered as the unknown, leading to the total \( h \)-formulation, or \( h \) is decomposed into a known source field \( h_s \) and an unknown reaction field \( h_r \), \( h = h_s + h_r \), leading to the reaction \( h \)-formulation.

Note that other \( h \)-conform formulations exist, such as the \( t-\omega \)-formulation. Because the nonlinear constitutive law is involved in the same manner (resistivity), the main numerical behavior is expected to be similar.

**Magnetic flux density conform formulation (a-formulation):** Gauss’s law, \( \text{div} \; b = 0 \), implies that the magnetic flux density can be expressed in terms of a vector potential \( a \) such that \( b = \text{curl} \; a \). Faraday’s law can then be rewritten as \( \text{curl} \; (e + \partial_t a) = 0 \) so that \( e = -\partial_t a - \text{grad} \; v \), with \( v \) an electric scalar potential. A gauge condition is necessary to guarantee the uniqueness of the vector potential.

We first consider 3D geometries and 2D geometries with an in-plane current density (for instance, a long bar in a parallel magnetic field). The potentials \( a \) and \( v \) are discretized with Whitney elements as follows:

\[
a = \sum_{e \in \Omega} a_e \; \psi_e \quad \text{and} \quad v = \sum_{i \in C} V_i \nu_i, \quad (7)
\]

with \( \psi_e \) the edge function of edge \( e \) and \( \nu_i \) a discontinuous shape function associated with a cross-section of conducting region \( i \), known as the generalized source potential [23]. Coefficients \( a_e \) and \( V_i \) are the degrees of freedom. The shape functions \( \nu_i \) allow one to introduce voltage constraints directly in the basis function, rather than through an integral constraint. Here, the function \( \psi_e \) has a support limited to a transition layer, so that \( v \) is not the usual electrostatic potential, see [23].

Essential conditions, \( e \times n|_{\Gamma_e} \) and \( V_i \) for \( i \in C_V \), are strongly imposed by fixing the corresponding degrees of freedom, \( a_e \) for \( e \in \Gamma_e \) and \( V_i \) for \( i \in C_V \). The resulting space for \( a \), after imposing essential conditions, is denoted by \( S^1 \). The space for \( v \), after imposing essential conditions, is denoted by \( S^0 \). The potential \( a \) is unique in \( \Omega_a \), it is a primitive for \( e \), \( e = -\partial_t a \), but it is not unique in \( \Omega_e \), where \( e \) is not evaluated. A gauge must then be introduced in \( \Omega_e \) and the resulting space is renamed as \( S^1 \). One convenient gauge, named the co-tree gauge, consists in (strongly) imposing vanishing coefficients on edges that belong to an appropriate tree [24]. Another possibility is to (weakly) impose a Coulomb gauge [24].

Test functions \( a' \) and \( v' \) are chosen in the spaces defined in (7), with coefficients \( a'_e \) and \( V'_i \), respectively, but however with vanishing coefficients where essential conditions are imposed. The spaces of test functions are denoted as \( \tilde{S}^1 \) and \( \tilde{S}^0 \), respectively.

Given the nonlinear conductivity \( \sigma = \sigma(-\partial_t a - \text{grad} \; v) \) and reluctivity \( \nu = \nu(\text{curl} \; a) \), the weak formulation reads as follows [23], [25]:

\[
\begin{align*}
(\nu \, \text{curl} \; a , \; \text{curl} \; a')_{\Omega_a} &+ (\sigma \partial_t a , a')_{\Gamma_e} + (\sigma \text{grad} \; v , a')_{\Gamma_e} = 0, \\
\forall a' \in \tilde{S}^1_{\Omega_e}, \quad \text{and} \\
(\sigma \partial_t a , \; \text{grad} \; v')_{\Omega_e} + (\sigma \text{grad} \; v , \; \text{grad} \; v')_{\Omega_e} - \sum_{i \in C} I_i V_i (v') = 0, \quad \forall v' \in S^0_{\Omega_e},
\end{align*}
\]

with \( V_i(v) = V_i \). For example, when \( v' = v_i \) is chosen (\( i \in C_I \), \( V'_i = 1 \)), one obtains the circuit equation

\[
(\sigma \partial_t a , \; \text{grad} \; v_i)_{\Omega_e} + (\sigma \text{grad} \; v , \; \text{grad} \; v_i)_{\Omega_e} = I_i. \quad (9)
\]

The formulation (8) is referred to as the generalized modified vector potential formulation, or \( a \)-formulation. It is a weak form of Ampere’s equation. The continuity of the normal component of \( b \) is enforced by the considered shape functions while the natural constraints, \( h \times n|_{\Gamma_e} \) and \( I_i \) for \( i \in C_I \) are weakly imposed.

Note that it is also possible to define \( v \) everywhere in the conducting region (instead of defining it only in a transition layer) and extend the gauge condition for \( a \) to \( \Omega_a \) as well. In all tested situations, solutions obtained with this \( a-\nu \)-formulation are identical to those obtained with the \( a \)-formulation.

In practice, the \( a \)-formulation can be solved in one of two ways. Either the total vector potential \( a \) is considered as the unknown, leading to the total \( a \)-formulation, or \( a \) is decomposed into a known source potential \( a_s \) and an unknown reaction potential \( a_r \), \( a = a_s + a_r \), leading to the reaction \( a \)-formulation.

We now turn to 2D geometries with the current density along the invariant direction. Now, \( \text{grad} \; v \) is perpendicular to the plane and constant in each conducting domain [26]. The vector potential is chosen in the same direction so that the Coulomb gauge is implicit (\( \text{div} \; a = 0 \)). Fields \( a \) and \( \text{grad} \; v \) are discretized with Whitney elements as follows:

\[
a = \sum_{n \in \Omega} a_n \; \psi_n \quad \text{and} \quad \text{grad} \; v = \sum_{i \in C} U_i z_i, \quad (10)
\]

with \( \psi_n \) a perpendicular edge function associated to node \( n \) and \( z_i \), a piecewise-constant shape function, nonzero in \( \Omega_{e_i} \) only. Coefficients \( a_n \) and \( U_i \) (voltage per unit length) are the degrees of freedom. The resulting formulation is analogous to formulation (8), but with the different function spaces and unknowns described in (10).

**C. Time Integration**

The spatial discretization yields a semi-discrete nonlinear system of equations. To obtain a fully discrete system that can be solved numerically, time must also be discretized. In this work, an implicit Euler method has been chosen. From the initial solution at time \( t = t_0 \), the solution is successively sought at discrete time instants \( t_1, t_2, \ldots, t_n \), not necessarily equidistant. In the equation for time \( t_n, \; n \in \mathbb{N}_0 \), all terms are
evaluated at \( t_n \), with the time derivative of any quantity \( u \) at time \( t_n \) being approximated by the finite difference
\[
\frac{du}{dt}\bigg|_{t_n} \approx \frac{u(t_n) - u(t_{n-1})}{t_n - t_{n-1}}. \quad (11)
\]

**D. Iterative Techniques**

After complete discretization, the original problem takes the form of a system of nonlinear algebraic equations,
\[
A(x)x = b,
\]
where \( x \) is the vector of degrees of freedom at the considered time step. This nonlinear system cannot be solved directly and iterative techniques are necessary.

In this work, we consider Picard and Newton-Raphson iterative techniques.

**a) The Picard technique:** The Picard technique \cite{27} belongs to fixed point methods. From an iterate \( x_i \), the next iterate \( x_{i+1} \) is sought by solving the linear system,
\[
A(x_i)x_{i+1} = b, \quad i = 0, 1, \ldots \quad (13)
\]
In general, this method does not exhibit a high convergence rate. However, even if in some cases the iterates may enter cycles and fail to converge, this is usually a robust method in the sense that it converges for a wide range of initial guesses \( x_0 \). An illustration of one iteration of this method in the case of a single unknown is given in Fig. 2a.

**b) The Newton-Raphson technique:** The Newton-Raphson technique \cite{27} consists in approximating the nonlinear term \( A(x)x \) to first order with a limited Taylor development, yielding a linear system. From an iterate \( x_i \), the next iterate \( x_{i+1} \) is computed by solving the following linear system,
\[
A(x_i)x_i + J(x_i)(x_{i+1} - x_i) = b, \quad i = 0, 1, \ldots \quad (14)
\]
with \( J \) the Jacobian matrix defined by
\[
J(x) = \frac{\partial}{\partial x} (A(x)x). \quad (15)
\]
The Jacobian matrix can be evaluated numerically using finite differences but if the analytical expression of the matrix is known, it can be used. This is the case in this work. The involved analytical expressions are given in the appendix.

This method is usually less robust than the Picard technique in the sense that if the initial iterate is not sufficiently close to the solution, the method may diverge. However, when the initial guess is sufficiently close to the solution the convergence can be very fast, with a quadratic rate of convergence. An example of one iteration of this method in the case of a single unknown is given in Fig. 2b.

**c) Choosing the initial guess:** Iterations must start with an initial guess, or a predictor. The simplest possibility is to consider the solution at the previous time step. This method is referred to as the zeroth-order extrapolation method. Several previous solutions can also be extrapolated and a class of polynomial extrapolation methods can be defined. In this work, we consider extrapolations up to order two (that involves the solution at the last three time steps).

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**Fig. 2:** One iteration of the Picard and Newton-Raphson iterative techniques in the case of a single degree of freedom.

### III. Comparison of the Methods

The comparison of the different methods is conducted progressively. First, the application of the iterative techniques on the nonlinear laws is analyzed in a case with a single-degree-of-freedom. As shown below, the conclusions drawn in the simple case extend to 1D, 2D and 3D problems. Then, the accuracy of the different methods is evaluated and their efficiency is compared in two benchmarked 2D problems. Finally, the methods are illustrated on a benchmarked 3D problem.

In this section, no magnetic material is considered and \( b = \mu_0 h \) everywhere in the domain. Simulations are performed by the GetDP free software \cite{28}, \cite{29} and meshes are generated by Gmsh \cite{30}. Model files for the main test cases are available on the website www.life-hts.uliege.be.

#### A. Behavior of the Iterative Techniques

An important difference between the \( h \)- and \( a \)-formulations is that they involve the nonlinear electric constitutive law in a different manner, either through the electrical resistivity or the electrical conductivity. The shape of this constitutive law has a strong influence on the behavior of the iterative techniques, as we now illustrate for simple equations with a single unknown.

Consider a superconducting ring subjected to a time-varying external magnetic flux \( \Phi \). The ring can be modelled by lumped elements: a nonlinear resistor \( R = V_c/I_c(|I|/I_c)^{n-1}, \) that mimics the electrical resistivity law, in series with a linear inductor \( L \), yielding \( R(I)I + LL\dot{I} = \Phi \). Solving the problem for the current intensity \( I \) flowing in the ring gives rise to an equation of the form (after time discretization)
\[
f(x) = |x|^{n-1}x + x = b,
\]
to be solved at each time step. By contrast, solving for the voltage across the modeling resistor gives rise to an equation of the form
\[
f(x) = |x|^{(1-n)/n}x + x = b.
\]

The shape of \( f \) in both cases is depicted in Fig. 3. The first case is analogous to the \( h \)-formulation because it involves a resistivity-like parameter whereas the second one is analogous to the \( a \)-formulation with a conductivity-like component.

Applying the iterative techniques on the two nonlinear shapes yields the following results. The first equation, \( f(x) = |x|^{n-1}x + x = b \) (\( h \)-formulation), is more efficiently solved with the Newton-Raphson technique, as the Picard iterations easily cycle and fail to converge. An example of cycle is shown in Fig. 4a. Conversely, the second equation, \( f(x) = |x|^{(1-n)/n}x + x = b \) (\( a \)-formulation), can hardly be
solved with the Newton-Raphson technique whereas the Picard technique shows a good stability. With the Newton-Raphson technique, iterations also enter cycles as illustrated in Fig. 4b. Note that for each case in Fig. 4 the cycling behavior can be circumvented by using relaxation factors, which however requires several trials to determine the most efficient values to be used.

As shown below, the observed behaviors can be generalized to 1D, 2D and 3D problems: the Picard technique does not work directly with the $h$-formulation whereas the Newton-Raphson technique fails to converge when used to linearize the $a$-formulation. The origin of the corresponding convergence issues was observed to be identical to that in the above simple problems, as in some regions of the mesh, the degrees of freedom enter cycles. Again, it is sometimes possible to overcome these cycles by using finely tuned relaxation factors, a method which however requires trial-and-error.

### B. Comparison of 2D Models

The numerical study is conducted on two different problems. The first problem consists in a superconducting bulk cylinder subjected to an external applied field. The geometry is axisymmetric and only one slice of one radian is modelled. This test case will be referred to as the bulk case and is comparable to benchmark 4 of the HTS modeling website [16].

The second problem consists in a superconducting thin tape transporting an imposed AC current intensity. The geometry is assumed two-dimensional and only the cross-section of the tape of modelled. This test case will be referred to as the tape case and is comparable to benchmark 1 of the HTS modeling website [16].

Following the conclusions of the previous section, the $h$-formulation is solved with the Newton-Raphson technique whereas the $a$-formulation is solved with the Picard technique.

#### a) Convergence criterion

The chosen convergence criterion for the iterative techniques is based on an estimate $P$ of the instantaneous electromagnetic power. For the $h$-formulations, $P$ is expressed as

$$P = (\partial_t (\mu \mathbf{h}) \cdot \mathbf{h})_\Omega + (\rho \mathbf{j} \cdot \mathbf{j})_\Omega,$$

with $\mathbf{j} = \text{curl} \mathbf{h}$. For the $a$-formulations, $P$ is given as

$$P = (\nu \partial_t \mathbf{b} \cdot \mathbf{b})_\Omega + (\sigma \mathbf{e} \cdot \mathbf{e})_\Omega,$$

with $\mathbf{b} = \text{curl} \mathbf{a}$ and $\mathbf{e} = -\partial_t \mathbf{a} - \text{grad} \mathbf{a}$.

The criterion requires the relative change of $P$ between two iterations to be smaller than a given tolerance $\varepsilon$. This choice of convergence criterion is motivated by the observation that similar residuals do not correspond to similar accuracies in each formulations whereas quantity $P$ is comparable between the formulations.

In the following, we first evaluate the accuracy of the numerical solutions. Then, we compare the efficiency of the methods that lead to accurate results.

#### 1) Accuracy of the formulations

**Bulk case:** The bulk superconductor has a radius $R = 12.5$ mm and a height $H = 10$ mm. The critical current density is $j_c = 3 \times 10^8$ A/m$^2$. Cylindrical coordinates $(r, \theta, z)$ are introduced. The cylinder is subjected to an external field applied parallel to its axis (the $z$-axis), so that the problem is axisymmetric. The time evolution of the applied field is illustrated in Fig. 5 with $b_{\text{max}} = 1$ T and $t_1$, $t_2$, and $t_3$ equal to 5, 10, and 15 s, respectively.

![Fig. 5: Time evolution of the external applied field in the bulk test case.](image)

Fig. 5 shows the azimuthal current density obtained with both formulations at the three instants $t_1$, $t_2$, and $t_3$, whereas Fig. 7 shows the corresponding vertical components of the magnetic flux density 2 mm above the cylinder. With mesh and time refinement, the solutions of the $h$- and $a$-formulations

1The relative change is defined as $|(P_i - P_{i-1})/P_{i-1}|$, with $P_i$ the value of the convergence indicator at iteration $i$. If the value of $P_{i-1}$ is smaller than $10^{-7}$, the relative change is replaced by the absolute increment divided by $10^{-7}$ W/m. This scheme is followed to avoid convergence issues when $P_{i-1}$ is too small and is only motivated by the tested situations.
are seen to approach each other (both globally and locally). The global convergence is illustrated in Fig. 8.

![Image](image_url)

**Fig. 6:** Current density distribution in a slice of the bulk cylindrical geometry at the instants \( t_1 \), \( t_2 \) and \( t_3 \) for a maximum applied field of 1 T and \( n = 20 \). White areas are free of current, light gray areas are crossed by out-of-plane current densities and dark gray areas are crossed by opposite current densities. Top: \( a \)-formulation. Bottom: \( h \)-formulation.

![Image](image_url)

**Fig. 7:** Distribution of the \( z \)-component of the magnetic flux density 2 mm above the cylinder at the instants \( t_1 \), \( t_2 \) and \( t_3 \) for a maximum applied field of 1 T and \( n = 20 \). Solid lines: \( h \)-formulation. Dashed lines: \( a \)-formulation.

![Image](image_url)

**Fig. 8:** Change of AC losses \( \int_{cycle} (j \cdot e) \, dt \), relative to an accurate solution, for various numbers of spatial degrees of freedom, for the two formulations, with constant time step \( (\Delta t = t_1/50 \text{ s}) \), and \( n = 25 \). The reference value is determined as the arithmetic mean of the solutions obtained with the finest mesh, in the \( h \)- and \( a \)-formulations (4.9495 J/m-cycle).

Note that the quality of the solution is not always good for coarser meshes and/or large time steps. For instance, the fields \( h \) obtained with the \( h \)-formulation and \( a \) obtained with the \( a \)-formulation both exhibit small amplitude oscillations over a few elements, located ahead of the sharp flux penetration front. This results in oscillations in the other fields also. Because the current density is expressed through the power law in the \( a \)-formulation, the oscillations are strongly amplified. In the \( h \)-formulation, the current density is obtained through the curl of \( h \) and the oscillations are thus not as much amplified, as shown in Fig. 8. In both formulations, the error is localized in a few elements and thus becomes negligible when the mesh is refined (see the lower part of Fig. 8 and also Fig. 6). As discussed below, the choice of the time step also has an important influence on the solution accuracy.

![Image](image_url)

**Fig. 9:** Current density profiles, as a function of the distance \( r \) to the symmetry axis, at the mid-height of the cylinder, at time \( t_3 \) for a maximum applied field of 1 T and \( n = 20 \). Upper figure: coarse mesh of 1661 nodes. Lower figure: finer mesh with 36989 nodes.

**Tape case:** The tape has a large aspect ratio: its height is \( H = 1 \mu \text{m} \) and its width is \( W = 12 \text{ mm} \). The critical current density is \( j_c = 2.5 \times 10^{10} \text{ A/m}^2 \). Cartesian coordinates are introduced with the \( x \)-axis along the width of the tape and the \( y \)-axis along its height. A sine-wave current intensity \( I(t) = I_{\text{max}} \sin(2\pi ft) \) is imposed, with a frequency \( f = 50 \text{ Hz} \) and an amplitude \( I_{\text{max}} = FI_c \), where \( I_c = HWj_c \) is the critical current intensity and \( F \in [0, 1] \) a constant. The superconducting domain is meshed with one layer of first-order quadrangular elements.

Fig. 10 shows the current density and magnetic flux density distributions after one fourth of a period obtained with both formulations. Again, with mesh and time refinement, the solutions of the \( h \)- and \( a \)-formulations converge to each other and the solutions are reliable, see Fig. 11. Note that the current density profile is still very sensitive to the \( a \) field with the \( a \)-formulation, especially with large \( n \) values. The convergence criterion must be strong enough if accurate current density profiles are sought with the \( a \)-formulation.

**Reaction and total formulations:** The solutions above were obtained with the total \( h \)- and \( a \)-formulations. Numerical solutions of the reaction formulations are exactly identical to
Influence of the time step: The choice of the time step is important for both the solution accuracy and the convergence properties. The motivation for studying this influence is based on the observation that, in some situations, the iterative techniques converge even with very large time steps, while providing reliable magnetic flux density distributions. With the $a$-formulation and the Picard linearization technique, it has even been observed that the larger the time step, the easier the convergence. A single-time-step method has already been discussed by Lousberg in [11], [18]. To some extent, such an approach is also possible with the $h$-formulation and the Newton-Raphson technique. It is thus tempting to use very large time steps to accelerate the simulations. This possibility should however be exploited with care as not all quantities are reliable when large time steps are used.

To check the validity of the results, a simple one-dimensional case is considered. It consists in a 25-mm-wide slab subjected to a parallel field increasing linearly over time at 5 T/s. Note that in the limit of fine meshes and small time steps, both $h$- and $a$-formulations accurately reproduce the analytical results of [31]. Magnetic flux and current penetration profiles are compared for two time discretization levels: with a single-time-step, and with 100 time steps. Materials with $n = 10$ and $n = 100$ are chosen. Fig. 12 compares the corresponding numerical solutions for the $h$-formulation. Results from the $a$-formulation are similar. As can be seen in the figure, even if the method has converged in the four cases, there is a large error on the magnetic flux density distribution and the current density profile for the low power exponent $n = 10$. Clearly, the single-time-step approach is not reliable in that case. However, with a larger exponent, both time discretization levels provide accurate results.

These observations can be explained as follows. In the Bean model [22] limit ($n \to \infty$) and with a linear ramp of applied field, the magnetic flux density distribution is linear in space and the flux front propagates at constant speed. In the finite element model, time derivatives are estimated by a finite difference approximation (backward Euler method). This approximation amounts to replacing the instantaneous increase of the magnetic flux density by its average increase over the considered time step. This is illustrated in Fig. 13 in the extreme case of a single time step for the whole time interval, for a magnetization from a virgin state. In this situation, the time derivative is underestimated almost everywhere in the material.

If more time steps are considered, the underestimation error is localized near the front and its influence is thus reduced. A power law model with a finite exponent $n$ yields a different distribution but the finite difference still underestimates the
time derivative. The error on the time derivative is present both for small values of $n$ and in the Bean limit. However, the effect of this error on the current and flux distributions is found to be larger for smaller $n$, as previously observed in [11].

![Fig. 13: Illustration of the error induced by the finite difference approximation of the time derivative in the case of large time steps. Up: inside the superconductor, magnetic flux density distribution at several time instants for a linearly increasing source field in the Bean model limit. Down: time evolution of the magnetic flux density at several spatial points (bold gray curves) and illustration of the finite difference approximation (dash-dotted lines) in the case of a single numerical time step. Time instants $t_0, \ldots, t_4$ are equidistant.](image)

The error on time derivatives induces an error on $e$ (via Faraday’s law $\mathbf{curl} \ e = -\partial_t \mathbf{b}$ in the $h$-formulation and $e = -\partial_t \alpha - \mathbf{grad} \ \mathbf{v}$ in the $a$-formulation). With large $n$, the resulting error on $j$ is small because of the strong nonlinearity of the constitutive law. For decreasing $n$, the sensitivity of $e$ to $j$ increases for $\|j\|$ close to $j_c$ and the error on $j$ thus increases. Because $e$ is underestimated, $j$ is also underestimated and, to reach $\mathbf{h} = \mathbf{h}_s$ at the material-air interface, this implies a larger penetration distance, as observed in the simulations. Therefore, even though the use of very large time steps provides accurate current and field distributions when $n$ is large enough, time derivatives are always underestimated and this leads to non-negligible errors on the electric field. Power quantities and losses rely on time derivatives and are thus also underestimated. The use of large time steps is thus not appropriate for computing the instantaneous power dissipation or losses.

**Conclusions on the accuracy:** All formulations lead to accurate and reliable results provided that the time step, the mesh size, and the convergence criterion are sufficiently small. For coarser meshes, the $a$-formulation provides large oscillations of the current density in non-penetrated regions. There is a possibility to use large time steps to speed up the simulations, more specifically in the $a$-formulation, but the accuracy on time derivatives is reduced. This can nevertheless be exploited to obtain fast and reliable magnetic flux density and current distributions for large $n$.

2) **Efficiency of the formulations:** The convergence speed of all methods is not identical. Depending on the method and on the choice of the first iterate, or predictor, the iterative techniques may require very different numbers of iterations to converge. Because the simulation time is proportional to the total number of iterations, this affects the calculation speed. The accurate methods which were identified in the previous section (small time step, fine mesh, and strong convergence criterion) are now compared in terms of their efficiency.

To proceed, the bulk and the tape cases are considered with three discretization levels: coarse, medium, and fine, defined by a multiplier $\alpha$ equal to 4, 2, and 1, respectively. The power law exponent is $n = 25$ in both geometries and $F = 0.9$ for the tape. In the bulk case, the mesh size varies from 0.3$\alpha$ mm in the cylinder to 3$\alpha$ mm at the outer surface. The number of time steps from 0 to $t_3$ is 300/$\alpha$. In the tape case, the number of quadrangular elements along the width of the tape is 400/$\alpha$ and the number of time steps per one period is 100/$\alpha$.

In some situations, the iterative techniques diverge or do not converge in a reasonable number of iterations. To treat automatically these issues, an adaptive time step procedure is implemented. This is crucial for efficient resolutions with the Newton-Raphson technique: small time steps are typically necessary during the first penetration of magnetic flux but one can afford larger steps once the sample is saturated. The heuristic procedure is defined as follows (procedure from [33]): (1) if the number of iterations exceeds $i_{\text{max}}$ with a time step $\Delta t$ or if the iteration diverges, the current time integration step restarts with a smaller time step equal to $\gamma \Delta t$, with $\gamma < 1$, (2) if a step with time step $\Delta t$ converges in less than $i_{\text{fast}}$ iterations, the next time step is chosen equal to $\min(\beta \Delta t, \Delta t_{\text{max}})$, with $\beta > 1$ and a fixed $\Delta t_{\text{max}}$. Here, we choose $\gamma = 1/2$, $\beta = 2$, $\Delta t_{\text{max}}$ equal to the initial time step, $i_{\text{fast}} = i_{\text{max}}/4$. For the $h$- and $a$-formulations, we choose $i_{\text{max}} = 60$ and $i_{\text{max}} = 500$, respectively.

Table I shows the number of solved linear systems required in the bulk and tape cases, with the two formulations and different extrapolation orders for the first iterate. Clearly, for an identical time discretization, the slower convergence rate of the Picard technique used in the $a$-formulation is detrimental and the $h$-formulation is more efficient. This can be observed in Fig. 14, where the system residual does not decrease monotonously with the Picard technique whereas a quadratic convergence rate is obtained at the last iterations with the Newton-Raphson technique.

Results show that choosing the first iterate (the predictor) with an extrapolation from the previous time steps is efficient for reducing the total number of iterations. A good choice is the first-order extrapolation of the last two steps with the $h$-formulation and the second-order extrapolation of the last three steps for the $a$-formulation.

If a good accuracy is sought for all quantities, small time steps should be used and, in that case, the $h$-formulation reaches a solution faster. Typically, with identical discretization levels, the $h$-formulation is two to ten times faster than the $a$-formulation. On the contrary, in controlled situations where the use of large time steps is acceptable, e.g., with large $n$
TABLE I: Total number of solved linear systems for the simulation from 0 to $t_3$ in the bulk case and for $5/4t_f$ in the tape case, for three discretization levels, three extrapolation techniques with the $h$-formulation and the $a$-formulation. Relative tolerance for the convergence criterion: $10^{-6}$ (except for the tape with the $a$-formulation, $10^{-3}$). Simulation times for the fastest simulations of each line are, on a single Intel Xeon 2 GHz CPU with 4Gb RAM, from top to bottom, 12s, 1m55s, 1h38m, 20s, 2m50s, and 14m.

<table>
<thead>
<tr>
<th>Formulation</th>
<th>Total $h$</th>
<th>Newton-Raphson $0^b$</th>
<th>$1^a$</th>
<th>$2^a$</th>
<th>Picard $0^b$</th>
<th>$1^a$</th>
<th>$2^a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coarse Bulk</td>
<td>570</td>
<td>612</td>
<td>824</td>
<td></td>
<td>16382</td>
<td>10905</td>
<td>9182</td>
</tr>
<tr>
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<td>1287</td>
<td>1344</td>
<td>1808</td>
<td></td>
<td>34858</td>
<td>20935</td>
<td>16654</td>
</tr>
<tr>
<td>Fine</td>
<td>2671</td>
<td>2751</td>
<td>3780</td>
<td></td>
<td>74929</td>
<td>41467</td>
<td>27006</td>
</tr>
<tr>
<td>Coarse Tape</td>
<td>4124</td>
<td>836</td>
<td>455</td>
<td></td>
<td>2351</td>
<td>2338</td>
<td>1827</td>
</tr>
<tr>
<td>Medium Tape</td>
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<td>983</td>
<td>732</td>
<td></td>
<td>5665</td>
<td>5001</td>
<td>4222</td>
</tr>
<tr>
<td>Fine</td>
<td>17266</td>
<td>867</td>
<td>1058</td>
<td></td>
<td>13628</td>
<td>12039</td>
<td>8172</td>
</tr>
</tbody>
</table>

values and for determining the current and flux distributions, the $a$-formulation is much more stable and might be used to get fast results, as will be illustrated in the next section. In that case of large time steps, the $h$-formulation with the Newton-Raphson technique is less appropriate because the iterations diverge if the initial estimate is not sufficiently close to the solution.

C. Cube 3D Model

The conclusions shown above are now illustrated on a 3D problem: a superconducting cube subjected to an external applied field parallel to its side faces. The problem is similar to benchmark 5 of the HTS modeling website [34]. The cube side is $a = 10$ mm, materials properties are $j_c = 10^8$ A/m$^2$ and $n = 100$. The applied field is homogeneous and varies sinusoidally with time with a peak value of 200 mT and a frequency $f = 50$ Hz. One eighth of the cube is modelled. The superconducting domain is meshed with cubes ($16 \times 16 \times 16$ cubes) and the surrounding domain is meshed with tetrahedra and pyramids.

The magnetization curve is computed by two methods: the $a$-formulation with large time steps (20 steps only for the curve, calculation time 3h30m, same CPU as in Table 1) for a fast resolution, and the $h$-formulation with smaller time steps (300 steps for the curve, calculation time 10h) for an accurate evaluation of all quantities. The magnetization curves are given in Fig. 15, they both match the benchmark results. Even for large time steps, the $a$-formulation yields accurate magnetization values, due to the large $n$-value. On the contrary, to compute accurately time integrated quantities, it is necessary to use smaller time steps. In that case, the $h$-formulation is recommended. For illustration, a perspective view of the solution of both formulations is given in Fig. 16. Again, note the slight oscillations of the current density after the flux front from the $a$-formulation. If needed, these oscillations can be decreased by refining the mesh and strengthening the convergence criterion.

IV. EXTENSION TO SYSTEMS CONTAINING SOFT FERROMAGNETS

The models developed above are directly adapted for treating systems containing linear isotropic non-conducting ferromagnetic domains. To model saturation, a nonlinear law must be introduced and an appropriate linearization technique must be chosen. It appears that the approach followed for the superconductor can be transposed for the ferromagnet.

A simple model for an isotropic saturation law expresses the permeability $\mu = \mu(h)$ as a rational expression of first order-polynomials, leading to the following relationship:

$$b = \mu_0 \left( 1 + \left( \frac{1}{\mu_0} - \frac{\|h\|}{m_0} \right)^{-1} \right) h,$$

Fig. 15: Magnetization curve for the superconducting cube, obtained with two formulations. Gray dots represent the solutions at successive (large) time steps.

Fig. 16: Current density distribution on the boundaries of one eighth of the superconducting cube after the first ramp of $\mu_0 h_s = 0.2$ T. Left: $h$-formulation. Right: $a$-formulation. Scale is for the arrow color.
with \( \mu_0 \) the relative permeability at origin and \( m_0 \) (A/m) the saturation magnetic field. A graphical representation of this law is given in Fig. 17. The permeability law is involved in the \( h \)-formulation. The model can be explicitly inverted (see the appendix) to yield the reluctivity law required for the \( a \)-formulation.

![Diagram](image)

Fig. 17: Ferromagnetic constitutive law and representation of the physical parameters.

The behavior of the iterative techniques on this nonlinear law is found to be analogous to what was observed for the power law. On the one hand, the law \( b = \mu h \) has the same concavity than the law \( j = \sigma e \) and the application of the Newton-Raphson technique leads to iteration cycles, although the situation is less severe here because the permeability is finite at the origin. On the other hand, the law \( h = \nu b \) is comparable to the law \( e = \rho j \) and, in that case, the Picard technique is less appropriate.

Following the same reasoning as with the superconductor, it is therefore recommended to linearize the \( h \)-formulation with a Picard technique and the \( a \)-formulation with a Newton-Raphson technique when modeling soft ferromagnets only. The \( a \)-formulation is thus expected to be more efficient than the \( h \)-formulation because it will benefit from the high rate of convergence of the Newton-Raphson technique. This is what is actually observed. Note that we also found that switching from the Picard technique to the Newton-Raphson technique for \( b = \mu h \) after a given number of iterations also helps avoiding cycles. This however remains less efficient than using the \( a \)-formulation.

Remarkably, conclusions for the superconductor and the ferromagnet are antagonistic: the \( h \)-formulation is preferred for the former material whereas the \( a \)-formulation is more efficient for the latter. Using a single formulation for the whole domain is thus expected to be sub-optimal. Consequently, we investigated the benefits of using a coupled formulation in a 2D example.

### A. Coupled Formulation

Let the problem domain \( \Omega \) be decomposed into two parts: \( \hat{\Omega} \) containing the superconducting domain and its boundary and \( \bar{\Omega} \) containing the union of the ferromagnetic, the air (or cryogenic fluid) domains, and their boundaries. Thus, \( \hat{\Omega} \cup \bar{\Omega} = \Omega \). Their non-empty intersection \( \hat{\Omega} \cap \bar{\Omega} \) is a curve and is denoted by \( \Gamma_m \), as illustrated in Fig. 18 (note that \( \hat{\Omega} \) must not necessarily be enclosed by \( \bar{\Omega} \)). Surfaces \( \Gamma_m \), \( \Gamma_h \), \( \Gamma_e \) and \( \Gamma_h \) are defined in the associated domains. Note that the air domain could be placed in \( \hat{\Omega} \) instead.

Domain \( \Omega \) is solved with the \( h \)-formulation whereas domain \( \hat{\Omega} \) is solved with the \( a \)-formulation. The coupling surface \( \Gamma_m \) is treated differently in the two sets; degrees of freedom on this surface cannot be known \textit{a priori}, and thus cannot be strongly imposed. Instead, the coupling can be weakly imposed within the surface integrals in the weak formulations, as natural continuity constraints. Consequently, \( \Gamma_m \subset \Gamma_e \) and \( \Gamma_m \subset \Gamma_h \). It can be shown that this procedure weakly ensures the continuity of the tangential component of \( h \) and of the normal component of \( b \).

For conciseness, let \( \Gamma_c \) and \( \Gamma_h \) be equal to \( \Gamma_m \). The coupled formulation is then expressed as follows:

From an initial solution at time \( t = 0 \), find \( h \in \bar{S}_h^0(\bar{\Omega}) \) and \( a \in \bar{S}_a^1(\bar{\Omega}) \) such that, for \( t > 0 \),

\[
(\partial_t (\mu h), h')_{\bar{\Omega}} + \left( \rho \frac{\partial}{\partial t} h, h' \right)_{\bar{\Omega}} + \left( \partial_t a \times n_{\bar{\Omega}}, h' \right)_{\bar{\Omega}} + \sum_{\Gamma_m} G_i(h') = 0 ,
\]

\[
(\nu \nabla h, a')_{\bar{\Omega}} - (h \times n_{\bar{\Omega}}, a')_{\bar{\Omega}} = 0 ,
\]

\( \forall h' \in \bar{S}_h^1(\bar{\Omega}) \) and \( \forall a' \in \bar{S}_a^1(\bar{\Omega}) \).

Note that the coupling operates through the two surface integral terms. Similar coupling schemes are discussed in detail in [35] and [36], and have been introduced in another context in [37].

### B. Efficiency Comparison

The benefits of using the coupled formulation are illustrated on an induced current case. The geometry consists in a ferromagnetic cylinder (\( \mu_0 m_0 = 1.31 \) T, \( \mu_0 = 1700 \)) of radius \( R = 12.5 \) mm and height \( H_{\text{ferm}} = 5 \) mm placed on the top of a superconducting bulk cylinder (\( \sigma = 20 \), \( j_c = 3 \times 10^8 \) A/m\(^2\)) of radius \( R \) and height \( H_{\text{super}} = 5 \) mm. The system is subjected to an external applied field aligned with the cylinders axis (increasing from 0 T to the maximum applied field of 5 T with a rate of 25 mT/s, then decreasing to \(-5 \) T with the opposite rate). The accuracy of the formulation is successfully verified. The efficiency is summarized in Table 11 for different methods and discretization levels. The mesh size varies from 0.3\( \alpha \) mm in the cylinders to 50\( \alpha \) mm at the outer surface, with \( \alpha \) equal to 4, 2, and 1 in the coarse, medium and fine levels, respectively. For the coupled formulation for example, this gives rise to 840, 2800, and 10800 degrees of freedom, respectively. The minimum number of time steps is fixed to 400/\( \alpha \). The hybrid linearization technique consists in
V. CONCLUSION

In this work, we compared the accuracy and efficiency of two finite element formulations for modeling high-temperature superconductors in 1D, 2D, and 3D geometries. We concluded that the choice of the formulation essentially depends on the sought results. When accurate results are wanted, the \( h \)-formulation might be preferred, together with a Newton-Raphson linearization technique and an adaptive time step procedure. Moreover, efficiency is highly enhanced when an appropriate predictor is chosen, e.g., a linear extrapolation of the last two solutions. When one can afford less accuracy on quantities relying on time derivatives, very fast results can be obtained with the \( a \)-formulation together with the Picard linearization technique. This option is particularly interesting in the case of large exponent values in the conductivity power law to get a good approximation of the magnetic field and the current density.

We applied the same approach to soft ferromagnetic materials. We arrived at opposite conclusions, the \( a \)-formulation is the most efficient and we thus proposed a coupled formulation to model systems containing both high-temperature superconductors and soft ferromagnets.

In further works, others aspects such as thermal coupling, \( b \)-dependent \( j_c \), anisotropic properties or ferromagnetic hysteresis would be very interesting to investigate.

APPENDIX

The linearization of the system matrix by the Newton-Raphson technique requires the expressions of the derivatives of the nonlinear constitutive laws. As these laws are vector relations, the derivatives are second-order tensor expressions. In the following expressions, index notation is used.

The superconductor constitutive relation (2) diverges for \( \|e\| \to 0 \). In this work, the constitutive law has been regularized as follows,

\[
j = \frac{j_c}{e_c} \left( \frac{1}{\varepsilon_\sigma + (\|e\|/e_c)^{n-1/n}} \right) e = \sigma(\|e\|) e, \quad (20)
\]

with \( \varepsilon_\sigma = 10^{-8} \). Its tensor derivative is given by

\[
\frac{\partial j_i}{\partial e_j} = \sigma(\|e\|) \frac{\delta_{ij}}{n} - \frac{1}{n} \left( \frac{\sigma(\|e\|)}{j_c e_c} \right)^{2} \frac{e_c}{\|e\|} (\|e\|)^{(n+1)/n} e_1 e_j. \quad (21)
\]

Conversely, the inverse law (resistivity sense), given by

\[
e = e_c \frac{\|j\|}{j_c} \left( \frac{\|j\|}{\|e\|} \right)^{n-1} j = \rho(\|j\|) j, \quad (22)
\]

gives

\[
\frac{\partial e_i}{\partial j_j} = \rho(\|j\|) \delta_{ij} + (n - 1) \frac{\rho(\|j\|)}{\|j\|^2} j_i j_j. \quad (23)
\]

Using the law (18) for ferromagnetic materials yields,

\[
\frac{\partial b_i}{\partial h_j} = \frac{\mu_0}{m_0} \left[ 1 + \left( \frac{1}{\mu_0 - 1} + \frac{\|h\|}{m_0} \right)^{-1} \right] \delta_{ij} = \frac{\mu_0}{m_0} \left[ 1 + \left( \frac{\|h\|}{m_0} \right)^{-2} \right] h_i h_j. \quad (24)
\]

Conversely, the inverse law, given by

\[
h = \frac{1}{\mu} \left( \frac{\|b\|}{\mu_0} - \frac{\mu_0 m_0}{\mu_0 - 1} + s(\|b\|) \right) \frac{b}{\|b\|}, \quad (25)
\]
with
\[ s(|b|) = \sqrt{\frac{\mu_0 \mu_0 - 1}{\mu_0}} |b| + \frac{4\mu_0}{\mu_0 - 1} |b|, \]  
(26)
gives
\[ \frac{\partial h_i}{\partial b_j} = \frac{1}{2} \left( \frac{1}{\mu_0} - \frac{\mu_0 \mu_0 - 1}{\mu_0} |b| \right) s(|b|) \delta_{ij} + \frac{1}{2} \left( \frac{\mu_0 \mu_0 - 1}{\mu_0} |b| \right)^2 - \frac{1}{2} \frac{1}{\mu_0} \frac{s(|b|)}{|b|} \left( 2 - \frac{\mu_0 \mu_0 - 1}{\mu_0} |b| \right) b_i b_j. \]  
(27)

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