Combined use of finite volume and network modelling for Stokes flow and permeability tensor computation in porous media

N. Combaret$^{1,2}$  D. Bernard$^1$  E. Plougonven$^1$

$^1$ICMCB-CNRS
University of Bordeaux 1

$^2$VSG - Visualization Sciences Group

2nd Conference on 3D-Imaging of Materials and Systems 2010
3D image: direct numerical processing

Network modeling:
- Structured network model
- Unstructured network model
From 3D image to permeability computation

- 3D image: direct numerical processing
- Network modeling:
  - Structured network model
  - Unstructured network model
From 3D image to permeability computation

- 3D image: direct numerical processing
- Network modeling:
  - Structured network model
  - Unstructured network model
Electric resistances network equivalence?

Stokes equations

\[ \begin{align*}
\nabla \cdot \vec{v} &= 0 \\
\nabla^2 \vec{v} - \nabla p &= 0
\end{align*} \]

B.C.: \( \vec{v} = 0 \) on \( \mathcal{A}_{fs} \)

Linear system:

\[ \begin{align*}
\sum_{j=0}^{M} Q_{ij} &= 0 \\
P_i - P_j &= \lambda_{ij} \cdot Q_{ij}
\end{align*} \]
From the Stokes equations to a linear system

- Electric resistances network equivalence?
- Stokes equations

\[
\begin{align*}
\nabla \cdot \mathbf{v} &= 0 \\
\nabla^2 \mathbf{v} - \nabla p &= 0 \\
\text{B.C.: } \mathbf{v} &= 0 \text{ on } \mathcal{A}_{fs}
\end{align*}
\]

- Linear system:

\[
\begin{align*}
\sum_{j=0}^{M} Q_{ij} &= 0 \\
\lambda_{ij} \cdot Q_{ij} &= P_i - P_j
\end{align*}
\]
From the Stokes equations to a linear system

- Electric resistances network equivalence?
- Stokes equations

\[ \begin{align*}
\nabla \cdot \vec{v} &= 0 \\
\nabla^2 \vec{v} - \nabla p &= 0
\end{align*} \]

B.C.: \( \vec{v} = 0 \) on \( A_{fs} \)

- Linear system:

\[ \begin{align*}
\sum_{j=0}^{M} Q_{ij} &= 0 \\
P_i - P_j &= \lambda_{ij} \cdot Q_{ij}
\end{align*} \]
Graph construction: from the pore space to a graph relating pores
- Pore positioning: what is a pore?
- Pores separation: where are the pores?

Network equations and parameters: what are the relations between the pores?
- Stokes equations equivalence on a graph
- Direct numerical computation of the resistances

Conclusion & future work
Graph construction: from the pore space to a graph relating pores
  - Pore positioning: what is a pore?
  - Pores separation: where are the pores?

Network equations and parameters: what are the relations between the pores?
  - Stokes equations equivalence on a graph
  - Direct numerical computation of the resistances

Conclusion & future work
Outline

- Graph construction: from the pore space to a graph relating pores
  - Pore positioning: what is a pore?
  - Pores separation: where are the pores?
- Network equations and parameters: what are the relations between the pores?
  - Stokes equations equivalence on a graph
  - Direct numerical computation of the resistances
- Conclusion & future work
(Almost) three steps method to get the graph

- Skeletonisation: homotopic thinning
  - Digitisation artefact removal method
  - Boundary conditions
- Skeleton points characterisation: curves and intersection points
- Nodes fusion criterion: relative volume overlapping
(Almost) three steps method to get the graph

- Skeletonisation: homotopic thinning
  - Digitisation artefact removal method
  - Boundary conditions
- Skeleton points characterisation: curves and intersection points
- Nodes fusion criterion: relative volume overlapping
(Almost) three steps method to get the graph

- Skeletonisation: homotopic thinning
  - Digitisation artefact removal method
  - Boundary conditions
- Skeleton points characterisation: curves and intersection points
- Nodes fusion criterion: relative volume overlapping
(Almost) three steps method to get the graph

- Skeletonisation: homotopic thinning
  - Digitisation artefact removal method
  - Boundary conditions
- Skeleton points characterisation: curves and intersection points
- Nodes fusion criterion: relative volume overlapping
(Almost) three steps method to get the graph

- Skeletonisation: homotopic thinning
  - Digitisation artefact removal method
  - Boundary conditions

- Skeleton points characterisation: curves and intersection points

- Nodes fusion criterion: relative volume overlapping
(Almost) three steps method to get the graph

N. Combaret

Finite volume and network modelling for permeability
(Almost) three steps method to get the graph

N. Combaret

Finite volume and network modelling for permeability
(Almost) three steps method to get the graph

- Skeletonisation: homotopic thinning
  - Digitisation artefact removal method
  - Boundary conditions
- Skeleton points characterisation: curves and intersection points
- Nodes fusion criterion: relative volume overlapping
(Almost) three steps method to get the graph

- Skeletonisation: homotopic thinning
  - Digitisation artefact removal method
  - Boundary conditions
- Skeleton points characterisation: curves and intersection points
- Nodes fusion criterion: relative volume overlapping
(Almost) three steps method to get the graph

N. Combaret
Finite volume and network modelling for permeability
(Almost) three steps method to get the graph

N. Combaret
Finite volume and network modelling for permeability
Pore space partitioning

- Pore delimitation:
  - Pore throats positioning
  - Pore volume invasion: watershed
- Unexpected configurations
  ⇒ Skeleton: good for positioning the pores, bad for representing their interconnexions
Pore space partitioning

- Pore delimitation:
  - Pore throats positioning
  - Pore volume invasion: watershed
- Unexpected configurations
  ⇒ Skeleton: good for positioning the pores, bad for representing their interconnexions
Pore space partitioning

- Pore delimitation:
  - Pore throats positioning
  - Pore volume invasion: watershed

- Unexpected configurations

⇒ Skeleton: good for positioning the pores, bad for representing their interconnexions
Pore space partitioning

Finite volume and network modelling for permeability
Pore space partitioning

- Pore delimitation:
  - Pore throats positioning
  - Pore volume invasion: watershed

- Unexpected configurations

  ⇒ Skeleton: good for positioning the pores, bad for representing their interconnexions
Pore space partitioning

Finite volume and network modelling for permeability
Pore space partitioning

- Pore delimitation:
  - Pore throats positioning
  - Pore volume invasion: watershed

- Unexpected configurations

⇒ Skeleton: good for positioning the pores, bad for representing their interconnexions
Integration of the first Stokes equation

Integration on the volume of a pore $H_i$

\[
\int_{V_{H_i}} \nabla \cdot \vec{v} \ dV = 0
\]

\[
\int_{A_{H_i}} \vec{v} \cdot \vec{n}_{H_i} \ dS = 0
\]

\[
\int_{A_{H_i \cap H}} \vec{v} \cdot \vec{n}_{H_i} \ dS = 0
\]

\[
\sum_{j=1}^{M} \left[ \int_{A_{H_i \cap H_j}} \vec{v} \cdot \vec{n}_{H_i \cap H_j} \ dS \right] = 0
\]

\[
\sum_{j=1}^{M} Q_{ij} = 0
\]

N. Combaret
Finite volume and network modelling for permeability
Integration on the volume of a pore $H_i$

$$\int_{V_{H_i}} \nabla \cdot \vec{v} \, dV = 0$$

$$\int_{A_{H_i}} \vec{v} \cdot \vec{n}_{H_i} \, dS = 0$$

$$\int_{A_{H_i,H_j}} \vec{v} \cdot \vec{n}_{H_i,H_j} \, dS = 0$$

$$\sum_{j=1}^{M} \left[ \int_{A_{H_i,H_j}} \vec{v} \cdot \vec{n}_{H_i,H_j} \, dS \right] = 0$$

$$\sum_{j=1}^{M} Q_{ij} = 0$$
Integration of the first Stokes equation

Integration on the volume of a pore $H_i$

\[
\int_{V_{H_i}} \vec{\nabla} \cdot \vec{v} \, dV = 0
\]

\[
\int_{A_{H_i}} \vec{v} \cdot \vec{n}_{H_i} \, dS = 0
\]

\[
\int_{A_{H_i,H}} \vec{v} \cdot \vec{n}_{H_i,H} \, dS = 0
\]

\[
\sum_{j=1}^{M} \left[ \int_{A_{H_i,H_j}} \vec{v} \cdot \vec{n}_{H_i,H_j} \, dS \right] = 0
\]

\[
\sum_{j=1}^{M} Q_{ij} = 0
\]

N. Combaret

Finite volume and network modelling for permeability
Integration of the first Stokes equation

Integration on the volume of a pore $H_i$

\[ \int_{V_{H_i}} \vec{\nabla} \cdot \vec{v} \, dV = 0 \]

\[ \int_{A_{H_i}} \vec{v} \cdot \vec{n}_{H_i} \, dS = 0 \]

\[ \int_{A_{H_iH}} \vec{v} \cdot \vec{n}_{H_iH} \, dS = 0 \]

\[ \sum_{j=1}^{M} \left[ \int_{A_{H_iH_j}} \vec{v} \cdot \vec{n}_{H_iH_j} \, dS \right] = 0 \]

\[ \sum_{j=1}^{M} Q_{ij} = 0 \]
Integration of the first Stokes equation

Integration on the volume of a pore $H_i$

\[ \int_{V_{H_i}} \nabla \cdot \vec{v} \, dV = 0 \]
\[ \int_{A_{H_i}} \vec{v} \cdot \vec{n}_{H_i} \, dS = 0 \]
\[ \int_{A_{H_iH_j}} \vec{v} \cdot \vec{n}_{H_iH_j} \, dS = 0 \]

\[ \sum_{j=1}^{M} \left[ \int_{A_{H_iH_j}} \vec{v} \cdot \vec{n}_{H_iH_j} \, dS \right] = 0 \]

\[ \sum_{j=1}^{M} Q_{ij} = 0 \]
Integration of the first Stokes equation

What did we learn?

- We need something on the network to support the flow rate through the surfaces

⇒ one edge per surface separating two pores
What did we learn?

- We need something on the network to support the flow rate through the surfaces

⇒ one edge per surface separating two pores
Integration of the second Stokes equation

Integration along a path $C_{ij}$ between two pores $H_i$ and $H_j$

$$\int_{C_{ij}} \nabla p \cdot \vec{c} \, dC - \int_{C_{ij}} \nabla^2 \vec{v} \cdot \vec{c} \, dC = 0$$

1: $\int_{C_{ij}} \nabla p \cdot \vec{c} \, dC = p(H_j) - p(H_i)$

2: independent from the path (see 1)

- $Q \propto \vec{v} \propto \nabla^2 \vec{v}$

$$\Rightarrow \int_{C_{ij}} \nabla^2 \vec{v} \cdot \vec{c} \, dC = \lambda_{ij} Q_{ij}$$

$p(H_j) - p(H_i) - \lambda_{ij} Q_{ij} = 0$
Integration of the second Stokes equation

Integration along a path $C_{ij}$ between two pores $H_i$ and $H_j$

\[
\begin{align*}
\int_{C_{ij}} \nabla p \cdot \vec{c} \, dC - \int_{C_{ij}} \nabla^2 \vec{v} \cdot \vec{c} \, dC &= 0 \\
\text{(1)} & & \text{(2)}
\end{align*}
\]

\(1:\)

\[
\int_{C_{ij}} \nabla p \cdot \vec{c} \, dC = p(H_j) - p(H_i)
\]

\(2:\)

- independent from the path (see \(1\))
- $Q \propto \vec{v} \propto \nabla^2 \vec{v}$

\[
\Rightarrow \int_{C_{ij}} \nabla^2 \vec{v} \cdot \vec{c} \, dC = \lambda_{ij} Q_{ij}
\]

\[
p(H_j) - p(H_i) - \lambda_{ij} Q_{ij} = 0
\]
Integration of the second Stokes equation
Integration along a path $C_{ij}$ between two pores $H_i$ and $H_j$

\[
\int_{C_{ij}} \nabla p \cdot \vec{c} \, dC - \int_{C_{ij}} \nabla^2 \vec{v} \cdot \vec{c} \, dC = 0
\]

①: \[
\int_{C_{ij}} \nabla p \cdot \vec{c} \, dC = p(H_j) - p(H_i)
\]

②: independent from the path (see ①)

\[
Q \propto \vec{v} \propto \nabla^2 \vec{v}
\]

\[
\Rightarrow \int_{C_{ij}} \nabla^2 \vec{v} \cdot \vec{c} \, dC = \lambda_{ij} Q_{ij}
\]

\[
p(H_j) - p(H_i) - \lambda_{ij} Q_{ij} = 0
\]
Integration of the second Stokes equation

Integration along a path $C_{ij}$ between two pores $H_i$ and $H_j$

\[
\int_{C_{ij}} \vec{\nabla} p \cdot \vec{c} \, dC - \int_{C_{ij}} \nabla^2 \vec{v} \cdot \vec{c} \, dC = 0
\]

1: \[
\int_{C_{ij}} \vec{\nabla} p \cdot \vec{c} \, dC = p(H_j) - p(H_i)
\]

2: independent from the path (see 1)

\[
Q \propto \vec{V} \propto \nabla^2 \vec{V}
\]

\[
\Rightarrow \int_{C_{ij}} \nabla^2 \vec{v} \cdot \vec{c} \, dC = \lambda_{ij} Q_{ij}
\]

\[
p(H_j) - p(H_i) - \lambda_{ij} Q_{ij} = 0
\]
Integration of the second Stokes equation

Integration along a path $C_{ij}$ between two pores $H_i$ and $H_j$

$$\int_{C_{ij}} \nabla p \cdot \vec{c} \, dC - \int_{C_{ij}} \nabla^2 \vec{v} \cdot \vec{c} \, dC = 0$$

$\boxed{1}$:

$$\int_{C_{ij}} \nabla p \cdot \vec{c} \, dC = p(H_j) - p(H_i)$$

$\boxed{2}$:

- independent from the path (see $\boxed{1}$)
- $Q \propto \vec{v} \propto \nabla^2 \vec{v}$

$$\Rightarrow \int_{C_{ij}} \nabla^2 \vec{v} \cdot \vec{c} \, dC = \lambda_{ij} Q_{ij}$$

$$p(H_j) - p(H_i) - \lambda_{ij} Q_{ij} = 0$$

N. Combaret
Finite volume and network modelling for permeability
Integration of the second Stokes equation

Integration along a path $C_{ij}$ between two pores $H_i$ and $H_j$

\[
\begin{align*}
\int_{C_{ij}} \nabla p \cdot \vec{c} \, dC - \int_{C_{ij}} \nabla^2 \vec{v} \cdot \vec{c} \, dC &= 0 \\
\end{align*}
\]

\(1:\)
\[
\int_{C_{ij}} \nabla p \cdot \vec{c} \, dC = p(H_j) - p(H_i)
\]

\(2:\)
- independent from the path (see 1)
- $Q \propto \vec{v} \propto \nabla^2 \vec{v}$

\[
\Rightarrow \int_{C_{ij}} \nabla^2 \vec{v} \cdot \vec{c} \, dC = \lambda_{ij} Q_{ij}
\]

\[
p(H_j) - p(H_i) - \lambda_{ij} Q_{ij} = 0
\]
Integration of the second Stokes equation

What did we learn?

- We need something on the network to support the flow rate through the surfaces
  - one edge per surface separating two pores
- We need something on the pores to support the pressure values
  - one node per pore
Integration of the second Stokes equation

What did we learn?

- We need something on the network to support the flow rate through the surfaces
  ⇒ one edge per surface separating two pores
- We need something on the pores to support the pressure values
  ⇒ one node per pore
Local computation of the resistances

- Extracting the pores connected to a surface (↔ an edge)
- Finite volume computation of local Stokes flow
- Resistance determination from the second Stokes equation in a linear system:

\[ \lambda_{ij} = \frac{P_{i}^{local} - P_{j}^{local}}{Q_{ij}^{local}} \]

- Linear system solution: pressure value at nodes, flow rate through edges

N. Combaret
Finite volume and network modelling for permeability
Local computation of the resistances

N. Combaret

Finite volume and network modelling for permeability
Local computation of the resistances

N. Combaret

Finite volume and network modelling for permeability
Local computation of the resistances

N. Combaret

Finite volume and network modelling for permeability
Local computation of the resistances

- Extracting the pores connected to a surface (⇔ an edge)
- Finite volume computation of local Stokes flow
- Resistance determination from the second Stokes equation in a linear system:

\[ \lambda_{ij} = \frac{P_{i\text{local}} - P_{j\text{local}}}{Q_{ij\text{local}}} \]

- Linear system solution: pressure value at nodes, flow rate through edges

N. Combaret
Finite volume and network modelling for permeability
Local computation of the resistances

N. Combaret

Finite volume and network modelling for permeability
Local computation of the resistances

- Extracting the pores connected to a surface (↔ an edge)
- Finite volume computation of local Stokes flow
- Resistance determination from the second Stokes equation in a linear system:

\[ \lambda_{ij} = \frac{P_{i \text{ local}} - P_{j \text{ local}}}{Q_{ij \text{ local}}} \]

- Linear system solution: pressure value at nodes, flow rate through edges
Local computation of the resistances

N. Combaret

Finite volume and network modelling for permeability
Local computation of the resistances

- Extracting the pores connected to a surface (⇔ an edge)
- Finite volume computation of local Stokes flow
- Resistance determination from the second Stokes equation in a linear system:

\[ \lambda_{ij} = \frac{P_{i}\text{local} - P_{j}\text{local}}{Q_{ij}\text{local}} \]

- Linear system solution: pressure value at nodes, flow rate through edges
Conclusion

- **Strong graph construction method**
  - Digitisation artefacts removal prior to skeletonisation
  - Nodes merging criterion
  - Pores delimitation methods comparison

- **Network settings**
  - Linear system construction
  - Direct local computation of the resistance
Conclusion

- Strong graph construction method
  - Digitisation artefacts removal prior to skeletonisation
  - Nodes merging criterion
  - Pores delimitation methods comparison

- Network settings
  - Linear system construction
  - Direct local computation of the resistance
Conclusion

- Strong graph construction method
  - Digitisation artefacts removal prior to skeletonisation
  - Nodes merging criterion
  - Pores delimitation methods comparison
- Network settings
  - Linear system construction
  - Direct local computation of the resistance
Conclusion

- Strong graph construction method
  - Digitisation artefacts removal prior to skeletonisation
  - Nodes merging criterion
  - Pores delimitation methods comparison

- Network settings
  - Linear system construction
  - Direct local computation of the resistance
Conclusion

- Strong graph construction method
  - Digitisation artefacts removal prior to skeletonisation
  - Nodes merging criterion
  - Pores delimitation methods comparison

- Network settings
  - Linear system construction
  - Direct local computation of the resistance
Conclusion

- Strong graph construction method
  - Digitisation artefacts removal prior to skeletonisation
  - Nodes merging criterion
  - Pores delimitation methods comparison

- Network settings
  - Linear system construction
  - Direct local computation of the resistance
Conclusion

- Strong graph construction method
  - Digitisation artefacts removal prior to skeletonisation
  - Nodes merging criterion
  - Pores delimitation methods comparison

- Network settings
  - Linear system construction
  - Direct local computation of the resistance
Future work

- Compare the linear system solution to the real numerical computation
- Finalise the implementation of the permeability tensor computation

\[
\begin{align*}
\nabla \cdot \vec{D}_k &= 0 \\
\nabla^2 \vec{D}_k - \nabla d_k &= -\vec{I}_k
\end{align*}
\]

B.C.: \( \vec{D}_k = 0 \) on \( \mathcal{A}_{fs} \)

B.C.: \( \vec{D}_k \) and \( d_k \) are periodic
Future work

- Compare the linear system solution to the real numerical computation
- Finalise the implementation of the permeability tensor computation

\[
\begin{align*}
\nabla \cdot \vec{D}_k &= 0 \\
\nabla^2 \vec{D}_k - \nabla \vec{d}_k &= -\vec{l}_k
\end{align*}
\]

B.C.: \( \vec{D}_k = 0 \) on \( \mathcal{A}_{fs} \)

B.C.: \( \vec{D}_k \) and \( d_k \) are periodic