Aggregating Farrell efficiencies with private and public inputs*

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Abstract

Since the initial paper on aggregating Farrell efficiencies of entities into a group counterpart by Färe and Zelenyuk (2003) [Färe R., Zelenyuk V., 2003, "On aggregate Farrell efficiencies", *European Journal of Operational Research* 146, 615-620.], much attention has been given in the literature to the aggregation of efficiency and efficiency-related indexes and indicators for both static and dynamic settings. In this paper, we make a distinction between two types of inputs in the aggregation framework: private and public inputs. Private inputs are those used by each entity individually, while public inputs are used collectively. The obtained aggregation procedure, while remaining consistent with previous contributions, offers the advantage of considering more practical cases, and increases our understanding of aggregation techniques.

Keywords: data envelopment analysis; aggregation; Farrell efficiency; private inputs; public inputs.

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1 Introduction

Farrell efficiencies (after Farrell (1957)) have been widely used to benchmark entities (such as firms, plants, sectors, regions, countries) by comparing their individual performance. In practice, Farrell efficiencies are used to detect inefficiency behaviour, hence as a means to reduce cost and improve profit for the entities. Entities, in many contexts, form a group: in microeconomic contexts, this is the case of firms within a multinational company, cartel or any type of coalition; in macroeconomic contexts, we may think of national regions or sectors, countries within a union or free trade zone. For these many contexts, evaluating performance at group level provides additional valuable information. Färe and Zelenyuk (2003) suggested an aggregation procedure to obtain the group Farrell efficiencies as a weighted sum of the entity counterparts.[1] To derive this procedure, they make certain assumptions regarding the group's technology and prices. Attractively, the weights they obtained are non-trivial and consistent with economic intuition. Besides enhancing efficiency analysis in group contexts, their result has set the stage for an extensive literature about aggregating efficiency and efficiency-related indexes and indicators for both static and dynamic settings. See Färe, Grosskopf, and Zelenyuk (2004), Färe and Zelenyuk (2005, 2007, 2012), Zelenyuk (2006, 2011, 2016), Nesterenko and Zelenyuk (2007), Peyrache (2013, 2015), Mayer and Zelenyuk (2014), Färe and Karagiannis (2017), Rogge (2018), and Walheer (2018a, e) among others.

In this paper, we make a distinction between private and public inputs in the aggregation framework. Private inputs are those used by each entity separately. In other words, these inputs are rival and exclusive to the entities. Thus, they parallel private goods for consumers (This type of input has been considered in, for example, Färe and Grosskopf (2000), Färe, Grosskopf and Whittaker (2007), Tone and Tsutsui (2009), Walheer (2016, 2018e), Ding et al. (2017), and Silva (2018) in different efficiency models.). Examples of private inputs are employees, machines, or resources used exclusively by each entity. In fact, private inputs were those considered by Färe and Zelenyuk (2003) when defining their aggregation framework (see Section 2). Next, public inputs are those entities used simultaneously. That is, these inputs are non-rival and non-exclusive to the entities. In a sense, they parallel public goods for

[1]See Li and Ng (1995), Blackorby and Russell (1999), and Ylvinger (2000) for earlier works about aggregation of Farrell efficiencies.

consumers.[2] Public inputs are either quantitative, such as infrastructure, capital, or natural resources, or qualitative public inputs, such as scientific knowledge, human capital, patents, specific production secrets, or processes. Note that quantitative public inputs, on the whole, are a prime economic motivation to form a group as they enable entities to make economies of scope (see Panzar and Willig (1981) and Nehring and Puppe (2004)). In addition, public inputs can be seen as a natural condition to group entities together as their grouping supposes that they have something in common.[3]

Therefore, our aim is to allow for public inputs in the aggregation framework initiated by Färe and Zelenyuk (2003). So doing, we propose a new interpretation of their assumption about the technology and the prices. Beyond its theoretical significance, our extension is useful for practical works. There are many situations indeed where public inputs are involved in the production process. In macroeconomic contexts, this is the case of sectors or regions sharing some national inputs (such as the educational system, the tax system); or of countries sharing some super-national or international inputs (such as the legal system, trade partnership). In microeconomic contexts, this is the case of firms having access to the same natural resources, of university departments using the same administration or general management office, of hospital services using the same equipment, of firms in a cartel using the same brand advertising or research and development department, and of electricity utilities with different plants sharing the same infrastructure.[4] For all these cases, we may see the

[2]Recently, public inputs have also been considered within entity production processes. See, Cherchye et al. (2013), Cherchye, De Rock, and Walheer (2016), and Walheer (2018b, c, d). Their results can be combined with the aggregation procedure of this paper, giving the option to consider public inputs at group and entity level. Note as well that our methodology can be extended to other types of public inputs. For example, public inputs used only by a subset of the entities in the group, or public inputs also used by entities that do not belong to the group (see Negishi (1973) and Section 3 for more discussion about the different types of public inputs).

[3]Any criteria can be used to define the groups (e.g. ownership, geographical localization, economic infrastructure, resource endowments, social environment, operational settings). We suggest that an additional criterion may be whether entities use the same public inputs or not. In a sense, the efficiency evaluation process with groups may be summarized in four steps: (1) define the groups; (2) define the input types; (3) evaluate efficiency of the entities; and (4) evaluation efficiency of the groups. This paper essentially attempts to provide a theoretical justification to relate steps (3) and (4) when several input types are considered in step (2). Step (1) is, in general, based on a subjective judgment or a common practice.

[4]It can be argued that some of these inputs are not purely public or that they are rival or exclusive to the entities. These cases, however, can easily be accommodated in our approach. See Section 3 for more detail.

common use of public inputs as a natural condition to regroup the entities. All in all, we believe that considering both types of inputs in the aggregation framework, in a plurality of contexts indeed, yields useful and reliable efficiency measurements.

The rest of the paper is structured as follows. In Section 2, we define the aggregation of Farrell cost efficiency when both private and public inputs are considered. In Section 3, we discuss several extensions. In Section 4, we present our conclusions.

2 Farrell cost efficiency

We consider that we observe n entities that can be partitioned into J groups.[5] Each entity t in group j produces Q outputs, captured by the vector $\mathbf{y}_t^j \in \mathbb{R}_+^Q$, using two types of inputs: private and public inputs. Note that it is not needed that the J groups contain the same number of entities. Formally, there are n_j entities in group j such that $n_1 + \cdots + n_J = n$. Note also that we consider that the entities are partitioned in such a manner that they use the same public input quantities in each group, but not necessarily between groups. As such, the public inputs are used to form the groups. Our aim is to propose an intuitive and coherent way to relate the Farrell cost efficiencies of the entities and the groups in the tradition of Färe and Zelenyuk (2003).

An initial step is to relate the actual total cost at the entity and group levels. To do so, we first introduce several notations for the inputs and their prices. In particular, let $\mathbf{x}^j \in \mathbb{R}_+^P$ and $\mathbf{z}^j \in \mathbb{R}_+^R$ denote the private and public input quantities for group j. Their corresponding prices are denoted as $\mathbf{w}^j \in \mathbb{R}_+^P$ and $\mathbf{p}^j \in \mathbb{R}_+^R$. The actual total cost of group j is thus given by summing the actual costs of the private and public inputs, i.e. $\mathbf{w}^{j\prime}\mathbf{x}^j + \mathbf{p}^{j\prime}\mathbf{z}^j$ (where $\prime$ is used to denote the transpose of a vector).

Next, let us denote the private and public input quantities used by entity t in group j by $\mathbf{x}_t^j \in \mathbb{R}_+^P$ and $\mathbf{z}_t^j \in \mathbb{R}_+^R$, respectively; and their corresponding prices by $\mathbf{w}_t^j \in \mathbb{R}_+^P$ and $\mathbf{p}_t^j \in \mathbb{R}_+^R$, respectively. The actual total cost of entity t in group j

[5]We consider the case of multiple groups for two main reasons. First, it represents a more general setting from a theoretical point of view. Second, it is more likely that we observe several groups in practice. See our examples in the Introduction. Of course, it is always possible to consider one group as in the initial setting of Färe and Zelenyuk (2003) by letting $J = 1$. Note that the aggregation procedure for multiple groups has already been considered by Zelenyuk (2011) for the case of private inputs. We discuss how to find overall Farrell efficiencies (i.e. when considering all groups together) in Section 3.

is also obtained by summing the actual costs of the private and public inputs, i.e. $\mathbf{w}_t^{j'}\mathbf{x}_t^j + \mathbf{p}_t^{j'}\mathbf{z}_t^j$.

Several natural connections exist between the inputs and their prices for the entities and the groups. These connections enable us to relate the actual total costs at both levels. First, we can easily relate the private input quantities. Indeed, as these inputs are rival and exclusive, it means that a fraction of the group quantity $\mathbf{x}^j$ is used by each entity. We denote the fraction of $\mathbf{x}^j$ used by entity t by $\boldsymbol{\omega}_t^j \in \mathbb{R}_+^P$. Clearly, we have that $\boldsymbol{\omega}_t^j \geq 0, \forall t$, and $\sum_{t=1}^{n_j} \boldsymbol{\omega}_t^j = \mathbf{1}$. It implies that $\mathbf{x}_t^j = \boldsymbol{\omega}_t^j \odot \mathbf{x}^j$ (where $\odot$ stands for the element-by-element product). We thus obtain the following connection between the private inputs at the entity and group levels:

$$\sum_{t=1}^{n_j} \mathbf{x}_t^j = \mathbf{x}^j. \tag{1}$$

It means that by summing the inputs used by the entities, we recover the group input quantities.[6] This is intuitive as each entity uses a different fraction of the group inputs. Note that it offers the advantage of considering reallocating the inputs across entities (see Nesterenko and Zelenyuk (2007), Pachkova (2009), and Mayer and Zelenyuk (2014) for related discussion; and Section 3).

Also, as the private inputs are the same across entities (but in different quantities), so are their price:

$$\forall t \in \{1, \ldots, n_j\} : \mathbf{w}_t^j = \mathbf{w}^j. \tag{2}$$

The assumption of common prices was introduced by Färe and Zelenyuk (2003) when defining their aggregation procedure for Farrell efficiencies. It is a necessary condition for their procedure to hold true. For many settings and applications, it is not a strong assumption. From a theoretical point of view, this assumption is coherent with many economic models satisfying perfect competition. In practice, the common prices can be seen as a benchmark or as reference prices for the group (Kuosmanen, Cherchye and Sipilainen (2006)). Finally, note that it does not mean that the prices have to be observed, only that they are the same. Different strategies have been suggested when prices are unknown. More discussion about this in Färe and Zelenyuk (2003) and in Section 3.

Second, as the public inputs are non-rival and non-exclusive they are not allocated

[6] $\sum_{t=1}^{n_j} \mathbf{x}_t^j = \sum_{t=1}^{n_j} (\boldsymbol{\omega}_t^j \odot \mathbf{x}^j) = (\sum_{t=1}^{n_j} \boldsymbol{\omega}_t^j) \odot \mathbf{x}^j = \mathbf{1} \odot \mathbf{x}^j = \mathbf{x}^j$.

between the entities (contrary to the private inputs). Thus, the public input quantities at both group and entity level must correspond. Formally, we obtain the following condition:

$$\forall t \in \{1, \ldots, n_j\} : \mathbf{z}_t^j = \mathbf{z}^j. \tag{3}$$

This equation means that the entities have access to the same public input quantities. It does not mean that they use the public inputs in the same intensity. This is captured by the public input prices at the entity level. In particular, we suggest using the following connection for the public input prices:

$$\sum_{t=1}^{n_j} \mathbf{p}_t^j = \mathbf{p}^j. \tag{4}$$

Intuitively, the entity-level prices are conceptually similar to Lindahl's prices for public goods for consumers. In the context of public goods, these prices represent the various consumers' willingness to pay. Here then, we may interpret these prices as the various entities' willingness to buy. The greater the willingness to buy, the greater the intensity of using the public inputs. Pareto efficient provision of public goods requires that the Lindahl's prices sum to the aggregate prices. As such, (4) parallels this condition in the production world: entity-specific public input prices have to sum to the group public input price. This connection thus represents a necessary condition to derive an aggregation result when public inputs are involved in the production process. Finally, note that it also implies that the prices of the group for the public inputs are equal to the sum of the entity-level prices. The higher the prices for the entity, the higher the price for the group, and reciprocally.[7]

[7] A similar condition was introduced by Cherchye et al. (2013) and Cherchye, De Rock, and Walheer (2016) in the context of multiple divisional firms. In some situations, we may wish to assume that the public inputs are not provided in a Pareto efficient way. In other words, there exists an inefficiency behaviour coming from the provision of the public inputs. This may occur when entities choose not to cooperate in each group. In that case, (4) does not hold true and has to be replaced by another condition. Building on the work of Cherchye et al. (2014) for multiple divisional firms, we suggest using the following condition in that case: $\max_{1,\ldots,n_j} \mathbf{p}_t^j = \mathbf{p}^j$. That is, the entity with the greatest willingness to pay is the only one that purchases the public inputs. As the public inputs are non-rival and non-exclusive, it implies that the other entities can use the public inputs freely. This is analogous to the well-known issue of free-riding in the consumer world. Unfortunately, in that case, we cannot obtain a simple connection between the actual and minimal total costs between the entities and the groups.

Combining the previous equations, we can relate the actual private and public costs for the entity and the group as follows:

$$\sum_{t=1}^{n_j} \mathbf{w}_t^{j\prime} \mathbf{x}_t^j = \sum_{t=1}^{n_j} \mathbf{w}^{j\prime} \mathbf{x}_t^j = \mathbf{w}^{j\prime} \left(\sum_{t=1}^{n_j} \mathbf{x}_t^j \right) = \mathbf{w}^{j\prime} \mathbf{x}^j. \tag{5}$$

$$\sum_{t=1}^{n_j} \mathbf{p}_t^{j\prime} \mathbf{z}_t^j = \sum_{t=1}^{n_j} \mathbf{p}_t^{j\prime} \mathbf{z}^j = \left(\sum_{t=1}^{n_j} \mathbf{p}_t^j \right)' \mathbf{z}^j = \mathbf{p}^{j\prime} \mathbf{z}^j. \tag{6}$$

To put it in words, the group-level actual cost is equal to the sum of the entity-level actual costs for both the private and public inputs. At this point, we notice that (5) is also present in Färe and Zelenyuk (2003) when defining their aggregation procedure (they also impose a specific structure about the technology, see (11)).[8] As such, they consider the case when only private inputs are used by entities for their aggregation procedure.

By summing the actual costs of the private and public inputs, we obtain our connection for the actual total cost between the entities and the groups:

$$\sum_{t=1}^{n_j} (\mathbf{w}_t^{j\prime} \mathbf{x}_t^j + \mathbf{p}_t^{j\prime} \mathbf{z}_t^j) = \sum_{t=1}^{n_j} \mathbf{w}_t^{j\prime} \mathbf{x}_t^j + \sum_{t=1}^{n_j} \mathbf{p}_t^{j\prime} \mathbf{z}_t^j = \mathbf{w}^{j\prime} \mathbf{x}^j + \mathbf{p}^{j\prime} \mathbf{z}^j. \tag{7}$$

Our next step is to show that such a connection also holds true for the minimal total costs. To define the concept of minimal total costs, we first have to specify the technology for the entities and the groups. As we consider the cost side of the production process, we can rely on input requirement sets. Specifically, for entity t in group j it is defined as follows:

$$I_t^j(\mathbf{y}_t^j) = \{(\mathbf{x}_t, \mathbf{z}) \in \mathbb{R}_+^{P+R} \mid (\mathbf{x}_t, \mathbf{z}) \text{ can produce } \mathbf{y}_t^j\}. \tag{8}$$

$I_t^j(\mathbf{y}_t^j)$ contains the combination of private and public inputs that can produce the output quantity $\mathbf{y}_t^j$. We assume that those sets fulfill standard regularity conditions to perform a cost evaluation.[9] Note also that the technology is different for each

[8] In fact, (5) and (11) are present in Färe et al. (2004) that have considered the aggregation of Farrell cost efficiency; Färe and Zelenyuk (2003) have considered Farrell revenue efficiency.

[9] See Färe and Primont (1995), and, in particular, Varian (1984) and Tulkens (1993) for discussion of the cost setting. Note that convexity is not required at this stage, but it is needed when considering technical efficiency. See Section 3.

entity t and group j. We provide graphical illustrations of these sets in Figures 1, 2, and 3.

Building on these sets, we can define the minimal total cost for entity t in group j, i.e. the minimal costs for producing $\mathbf{y}_t^j$, as follows:

$$C_t^j(\mathbf{y}_t^j, \mathbf{w}^j, \mathbf{p}_t^j) = \min_{(\mathbf{x}_t, \mathbf{z}) \in I_t^j(\mathbf{y}_t^j)} \mathbf{w}^{j\prime}\mathbf{x}_t + \mathbf{p}_t^{j\prime}\mathbf{z}. \tag{9}$$

$C_t^j(\mathbf{y}_t^j, \mathbf{w}^j, \mathbf{p}_t^j) = \mathbf{w}^{j\prime}\mathbf{x}_t^j + \mathbf{p}_t^{j\prime}\mathbf{z}^j$ means that the outputs are produced with minimal total cost, while $C_t^j(\mathbf{y}_t^j, \mathbf{w}^j, \mathbf{p}_t^j) < \mathbf{w}^{j\prime}\mathbf{x}_t^j + \mathbf{p}_t^{j\prime}\mathbf{z}^j$ reflects potential cost savings for entity t in group j.

Similar steps can be followed to obtain the minimal total cost for the groups. First, we define the group-level input requirement sets. In a general way, we can define the input requirement set of group j as follows:

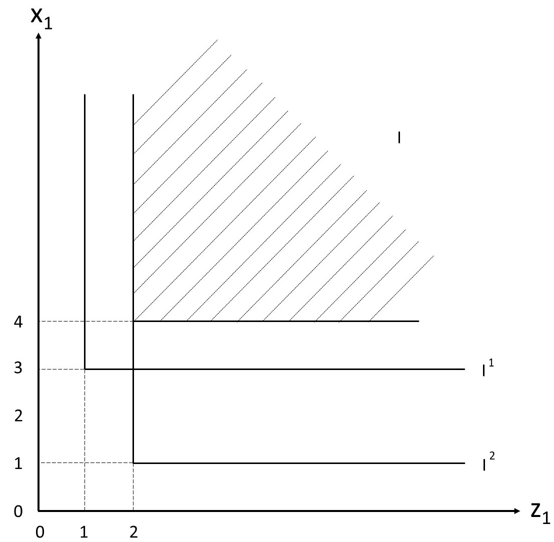
$$I^j(\mathbf{y}^j) = \left\{ (\mathbf{x}, \mathbf{z}) \in \mathbb{R}_+^{P+R} \mid \mathbf{x} = \sum_{t=1}^{n_j} \mathbf{x}_t \text{ and } \forall t \in \{1, \ldots, n_j\} : (\mathbf{x}_t, \mathbf{z}) \in I_t^j(\mathbf{y}_t^j) \right\}. \tag{10}$$

That is, $(\mathbf{x}, \mathbf{z})$ can produce $\mathbf{y}^j$ if, and only if, every entity $t \in \{1, \ldots, n_j\}$ can produce $\mathbf{y}_t^j$ using $(\mathbf{x}_t, \mathbf{z})$, where $\mathbf{x} = \sum_{t=1}^{n_j} \mathbf{x}_t$. It is important to note that $\mathbf{y}^j$ is not the sum of the entity-level output vectors (contrary to $\mathbf{x}^j$), but it is a matrix that collects the entity-specific output vectors. Formally, for any group j, we have that $\mathbf{y}^j = (\mathbf{y}_1^j, \ldots, \mathbf{y}_t^j, \ldots, \mathbf{y}_{n_j}^j)$. An illustration of the input requirement sets for a setting with two firms that use 1 public input and 1 private input is provided in Figure 1.

The minimal level of the public input z_1 is one for entity 1 and two for entity 2. As the public input quantity has to be the same for all entities (and also for the group), we obtain that the minimal public input quantity for the group is two. For the private input x_1, the minimal quantity is three for entity 1 and one for entity 2. To obtain the group-level quantity, it suffices to sum up the entity-level counterparts. This is why we obtain four for the minimal private input quantity.

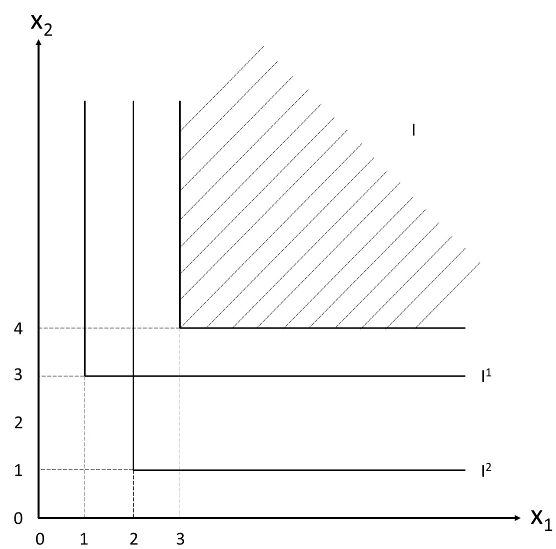
Before defining the minimal total cost for group j, we briefly discuss two interesting extreme cases. The first is when no public inputs are used in the production process. This is the setting considered in Färe and Zelenyuk (2003). It is illustrated with the case of two firms using 2 private inputs in Figure 2. A similar illustration can be found in Färe, Grosskopf, and Zelenyuk (2004). As explained before for Figure 1, it

Figure 1: example with 1 private input and 1 public input



suffices to sum up the entity-level private input quantities to obtain the group-level counterparts. For x_1, we obtain three (one for entity 1 and two for entity 2) for the minimal quantity; while for x_2, we obtain four (three for entity 1 and one for entity 2).

Figure 2: example with 2 private inputs



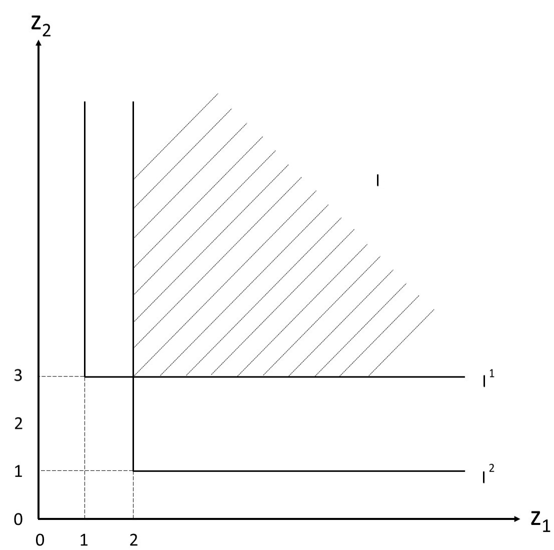
In that case, the connection between the group- and entity-level input requirement sets takes a specific form:

$$I^j(\mathbf{y}^j) = \sum_{t=1}^{n_j} I^t(\mathbf{y}_t^j). \tag{11}$$

That is, $I^j(\mathbf{y}^j)$ is the (Minkowski) sum of the entity-level input requirement sets. This is graphically illustrated in Figure 2.

The second extreme case is when there are no private inputs in the production process. An illustration with two firms using 2 public inputs is provided in Figure 3.

Figure 3: example with 2 public inputs



The features of the public inputs imply that the group-level quantity has to be equal to the entity-level counterparts. It turns out that the minimal level for z_1 is two (one for entity 1 and two for entity 2) and for z_2 is three (three for entity 1 and one for entity 2).

Formally, we obtain the following particular connection:

$$I^j(\mathbf{y}^j) = \bigcap_{t=1}^{n_j} I_t^j(\mathbf{y}_t^j). \tag{12}$$

In words, the group input requirement set is defined as the intersection of the entity counterparts. Intuitively, this is explained as these inputs have to be used in the same proportion by the entities (see (3)). This property is also illustrated in Figure 3.

The minimal total cost for group j, that is to produce the (matrix of) outputs $\mathbf{y}^j$, is defined as:

$$C^j(\mathbf{y}^j, \mathbf{w}^j, \mathbf{p}^j) = \min_{(\mathbf{x},\mathbf{z}) \in I^j(\mathbf{y}^j)} \mathbf{w}^{j\prime}\mathbf{x} + \mathbf{p}^{j\prime}\mathbf{z}. \tag{13}$$

The group minimal total cost has to be interpreted as the entity minimal total cost. That is $C^j(\mathbf{y}^j, \mathbf{w}^j, \mathbf{p}^j) = \mathbf{w}^{j\prime}\mathbf{x}^j + \mathbf{p}^{j\prime}\mathbf{z}^j$ means that the outputs are produced with minimal total cost, while $C^j(\mathbf{y}^j, \mathbf{w}^j, \mathbf{p}^j) < \mathbf{w}^{j\prime}\mathbf{x}^j + \mathbf{p}^{j\prime}\mathbf{z}^j$ reflects potential cost savings for group j.

Building on our previously established connections for the inputs, the input prices, and the input requirement sets between the entities and the groups, we obtain the following relationship between the minimal total costs at the entity and group levels:

$$\sum_{t=1}^{n_j} C_t^j(\mathbf{y}_t^j, \mathbf{w}^j, \mathbf{p}_t^j) = C^j(\mathbf{y}^j, \mathbf{w}^j, \mathbf{p}^j). \tag{14}$$

This relationship is very similar to the one obtained for the actual costs in (7), but applies here to the minimal (instead of the actual) total costs. A detailed proof of this is given in the Appendix.

We now have all the connections needed to establish a relationship between the Farrell cost efficiencies. First, let us define the Farrell efficiencies at both levels. For entity t in group j, it is given by:

$$CE_t^j(\mathbf{y}_t^j, \mathbf{x}_t^j, \mathbf{z}^j, \mathbf{w}^j, \mathbf{p}_t^j) = \frac{C_t^j(\mathbf{y}_t^j, \mathbf{w}^j, \mathbf{p}_t^j)}{\mathbf{w}^{j\prime}\mathbf{x}_t^j + \mathbf{p}_t^{j\prime}\mathbf{z}^j}. \tag{15}$$

$CE_t^j(\mathbf{y}_t^j, \mathbf{x}_t^j, \mathbf{z}^j, \mathbf{w}^j, \mathbf{p}_t^j)$ is by construction bounded from above by 1. When it is 1, it implies that the outputs are produced cost efficiently, i.e. when $C_t^j(\mathbf{y}_t^j, \mathbf{w}^j, \mathbf{p}_t^j) = \mathbf{w}^{j\prime}\mathbf{x}_t^j + \mathbf{p}_t^{j\prime}\mathbf{z}^j$. When it is smaller than 1, it implies the opposite, i.e. $C_t^j(\mathbf{y}_t^j, \mathbf{w}^j, \mathbf{p}_t^j) < \mathbf{w}^{j\prime}\mathbf{x}_t^j + \mathbf{p}_t^{j\prime}\mathbf{z}^j$

Similarly, Farrell cost efficiency for group j is given by:

$$CE^j(\mathbf{y}^j, \mathbf{x}^j, \mathbf{z}^j, \mathbf{w}^j, \mathbf{p}^j) = \frac{C^j(\mathbf{y}^j, \mathbf{w}^j, \mathbf{p}^j)}{\mathbf{w}^{j\prime}\mathbf{x}^j + \mathbf{p}^{j\prime}\mathbf{z}^j}. \tag{16}$$

Again, $CE^j(\mathbf{y}^j, \mathbf{x}^j, \mathbf{z}^j, \mathbf{w}^j, \mathbf{p}^j) = 1$ implies cost efficient behaviour for group j, while $CE^j(\mathbf{y}^j, \mathbf{x}^j, \mathbf{z}^j, \mathbf{w}^j, \mathbf{p}^j) < 1$ reveals potential cost savings for group j.

Building on our results for the actual total cost in (7) and for the minimal total cost in (14), we can redefine the Farrell cost efficiency measurement for group j as follows:

$$CE^j(\mathbf{y}^j, \mathbf{x}^j, \mathbf{z}^j, \mathbf{w}^j, \mathbf{p}^j) = \frac{C^j(\mathbf{y}^j, \mathbf{w}^j, \mathbf{p}^j)}{\mathbf{w}^{j\prime}\mathbf{x}^j + \mathbf{p}^{j\prime}\mathbf{z}^j} = \frac{\sum_{t=1}^{n_j} C_t^j(\mathbf{y}_t^j, \mathbf{w}^j, \mathbf{p}_t^j)}{\sum_{t=1}^{n_j}(\mathbf{w}_t^{j\prime}\mathbf{x}_t^j + \mathbf{p}_t^{j\prime}\mathbf{z}_t^j)}. \tag{17}$$

In words, this means that we can rewrite the group Farrell cost efficiency in such a manner that it only depends on (minimal and actual) entity-specific total costs. While this new definition is attractive since it implies that the group-level Farrell cost efficiency for the groups is known when the entity-specific total costs are known, it does not give an immediate connection between the Farrell cost efficiency of the groups and the entities. We can obtain such a connection by multiplying top and bottom of (17) by the entity-specific actual total cost (i.e. $\mathbf{w}^{j\prime}\mathbf{x}_t^j + \mathbf{p}_t^{j\prime}\mathbf{z}^j$):

$$\begin{aligned}
CE^j(\mathbf{y}^j, \mathbf{x}^j, \mathbf{z}^j, \mathbf{w}^j, \mathbf{p}^j) =& \frac{\sum_{t=1}^{n_j} C_t^j(\mathbf{y}_t^j, \mathbf{w}^j, \mathbf{p}_t^j)}{\sum_{t=1}^{n_j}(\mathbf{w}^{j\prime}\mathbf{x}_t^j + \mathbf{p}_t^{j\prime}\mathbf{z}^j)} \times \frac{\mathbf{w}^{j\prime}\mathbf{x}_t^j + \mathbf{p}_t^{j\prime}\mathbf{z}^j}{\mathbf{w}^{j\prime}\mathbf{x}_t^j + \mathbf{p}_t^{j\prime}\mathbf{z}^j}, \\
=& \sum_{t=1}^{n_j} \frac{\mathbf{w}^{j\prime}\mathbf{x}_t^j + \mathbf{p}_t^{j\prime}\mathbf{z}^j}{\sum_{t=1}^{n_j}(\mathbf{w}^{j\prime}\mathbf{x}_t^j + \mathbf{p}_t^{j\prime}\mathbf{z}^j)} \times \frac{C_t^j(\mathbf{y}_t^j, \mathbf{w}^j, \mathbf{p}_t^j)}{\mathbf{w}^{j\prime}\mathbf{x}_t^j + \mathbf{p}_t^{j\prime}\mathbf{z}^j}, \\
=& \sum_{t=1}^{n_j} \frac{\mathbf{w}^{j\prime}\mathbf{x}_t^j + \mathbf{p}_t^{j\prime}\mathbf{z}^j}{\sum_{t=1}^{n_j}(\mathbf{w}^{j\prime}\mathbf{x}_t^j + \mathbf{p}_t^{j\prime}\mathbf{z}^j)} \times CE_t^j(\mathbf{y}_t^j, \mathbf{x}_t^j, \mathbf{z}^j, \mathbf{w}^j, \mathbf{p}_t^j).
\end{aligned} \tag{18}$$

The weights have a nice economic interpretation: they represent the actual total cost share of entity t in group j. Therefore, these weights give the option to investigate how the entity contributes to the cost (in)efficient behaviour of the group. We also point out that the group can only be cost efficient provided that all entities are cost efficient.

In conclusion, the nature of the inputs does not alter the main result: the group Farrell cost efficiency is a weighted average of the entity-specific Farrell cost efficiencies, and the weights are defined as the relative actual total cost shares.

We end this part with a short illustration presenting how the aggregation procedure works in practice. Let us assume that we observe four entities: A, B, C, and D. Assume also that A and B are in group 1 (i.e. they have access to the same quantity of public input), while C and D are in group 2 (i.e. they have access to the same quantity of public input). This illustration may correspond to the case where we want to benchmark two sectors in two different countries having access to country-level inputs (e.g. education and legal systems), or two firms belonging to two different companies sharing company-level inputs (e.g human resource department or production process secret). For simplicity, we normalize the outputs to unity. The data for the inputs and their respective prices is provided in Table 1. The quantities of private inputs can differ between firms within groups, while the public input quantities have to be similar inside every group. The contrary applies to prices.

Different strategies can be used to estimate the cost efficiency scores. We choose to rely on Data Envelopment Analysis (DEA; after Charnes, Cooper, and Rhodes (1978)). DEA is a popular nonparametric method to estimate efficiency scores. One of the assumptions of this method is that the entities are homogeneous in terms of technology. In our context, this implies that the input requirement sets are similar for all entities. In practice, DEA is attractive since the efficiency scores are obtained by solving simple linear programmings. Refer for example to Färe, Grosskopf, and Lovell (1994).

Table 1: Entity data and results

	Group 1		Group 2	
	Entity A	Entity B	Entity C	Entity D
$\mathbf{y}_t^j$	1	1	1	1
$\mathbf{x}_t^j$	2	3	1	5
$\mathbf{w}^j$	2	2	3	3
$\mathbf{z}^j$	3	3	2	2
$\mathbf{p}_t^j$	1	2	3	2
$CE_t^j(\mathbf{y}_t^j, \mathbf{x}_t^j, \mathbf{z}^j, \mathbf{w}^j, \mathbf{p}_t^j)$	0.5714	0.5000	1	0.3684
$\frac{\mathbf{w}_t^{j'}\mathbf{x}_t^j + \mathbf{p}_t^{j'}\mathbf{z}_t^j}{\sum_{t=1}^{n_j}(\mathbf{w}_t^{j'}\mathbf{x}_t^j + \mathbf{p}_t^{j'}\mathbf{z}_t^j)}$	0.3684	0.6316	0.3214	0.6786

We find that Entity C is cost efficient while the three other entities may save cost without reducing their output production. Next, we compute the actual total cost shares for the four entities. Entities B and D are the most important entities in their respective groups. We notice that there is no specific connection between the cost efficiency scores and the actual total cost shares.

The data and results for the groups are provided in Table 2.

We find that Group 2 is more cost efficient than Group 2. The cost efficiency score of Group 1 is closer to Entity A and the cost efficiency score of Group 2 is closer to

Table 2: Group data and results

	Group 1	Group 2
$\mathbf{y}^j$	(1,1)	(1,1)
$\mathbf{x}^j$	5	6
$\mathbf{w}^j$	2	3
$\mathbf{z}^j$	3	2
$\mathbf{p}^j$	3	5
$CE^j(\mathbf{y}^j, \mathbf{x}^j, \mathbf{z}^j, \mathbf{w}^j, \mathbf{p}^j)$	0.5263	0.6011

Entity D. This is excepted since these entities have greater actual total cost share in their respective group.

Finally, several remarks have to be made about our illustration. First, the cost efficiency scores capture the (in)efficiency behaviour due to both the inputs and the prices. Usually, cost efficiency can be broken up into two dimensions: technical and allocative efficiency. Second, our simple illustration assumes that all prices are observed. In practice, it may be difficult to find price data or to rely on the existing price data. In principle, it is still possible to propose an interval or restrictions for the possible price values. Next, we may wish to know the overall cost efficiency score. That is the cost efficiency score when grouping all the entities. In that case, additional assumptions are required. Finally, inputs can be reallocated within or between groups. These various extensions are discussed in the next Section.

3 Extensions

As mentioned earlier, the initial result of Färe and Zelenyuk (2003) for Farrell efficiencies has been used to define aggregation techniques of various efficiency and efficiency-related indexes and indicators for both static and dynamic settings. Here, we present five main extensions of our aggregation result when considering both private and public inputs: technical efficiency, allocative efficiency, and distance function; cost efficiency when input prices are partiality or not observed; cost efficiency when considering other types of public inputs; overall Farrell efficiencies, and the potential gain when resource reallocation is possible.

Technical efficiency, allocative efficiency, and distance function. Technical efficiency (after Debreu (1951) and Farrell (1957)) is defined for entity t in group j

as follows:

$$TE^j_t(\mathbf{y}^j_t, \mathbf{x}^j_t, \mathbf{z}^j) = \inf\{\theta \mid (\theta\mathbf{x}^j_t, \theta\mathbf{z}^j) \in I^j_t(\mathbf{y}^j_t)\}. \tag{19}$$

Generally, $TE^j_t(\mathbf{y}^j_t, \mathbf{x}^j_t, \mathbf{z}^j)$ is smaller than 1, and a lower value indicates greater (input-oriented) technical inefficiency.[10] Farrell (1957) noticed that there is no particular reason why cost efficiency should coincide with technical efficiency, but he found a natural way to relate the two concepts. In particular, for entity t in group j, we obtain the following:

$$CE^j_t(\mathbf{y}^j_t, \mathbf{x}^j_t, \mathbf{z}^j, \mathbf{w}^j, \mathbf{p}^j_t) = TE^j_t(\mathbf{y}^j_t, \mathbf{x}^j_t, \mathbf{z}^j) \times AE^j_t(\mathbf{y}^j_t, \mathbf{x}^j_t, \mathbf{z}^j, \mathbf{w}^j, \mathbf{p}^j_t). \tag{20}$$

$AE^j_t(\mathbf{y}^j_t, \mathbf{x}^j_t, \mathbf{z}^j, \mathbf{w}^j, \mathbf{p}^j_t)$ stands for the (input-oriented) allocative efficiency and measures inefficiency due to non-optimal allocation of inputs given the input prices $\mathbf{w}^j$ and $\mathbf{p}^j_t$. As such, the two efficiency measurements coincide when $AE^j(\mathbf{y}^j_t, \mathbf{x}^j_t, \mathbf{z}^j, \mathbf{w}^j, \mathbf{p}^j_t)$=1, or in other words, when there is no inappropriate allocation of inputs.

Extending (20) at the group level, we naturally obtain for a particular group j:

$$CE^j(\mathbf{y}^j, \mathbf{x}^j, \mathbf{z}^j, \mathbf{w}^j, \mathbf{p}^j) = TE^j(\mathbf{y}^j, \mathbf{x}^j, \mathbf{z}^j, \mathbf{w}^j, \mathbf{p}^j) \times AE^j(\mathbf{y}^j, \mathbf{x}^j, \mathbf{z}^j, \mathbf{w}^j, \mathbf{p}^j). \tag{21}$$

Building on the aggregation result of $CE^j(\mathbf{y}^j, \mathbf{x}^j, \mathbf{z}^j, \mathbf{w}^j, \mathbf{p}^j)$ in (18), we obtain:

$$TE^j(\mathbf{y}^j, \mathbf{x}^j, \mathbf{z}^j, \mathbf{w}^j, \mathbf{p}^j) = \sum_{t=1}^{n_j} \frac{\mathbf{w}^{j\prime}\mathbf{x}^j_t + \mathbf{p}^{j\prime}_t\mathbf{z}^j}{\sum_{t=1}^{n_j}(\mathbf{w}^{j\prime}\mathbf{x}^j_t + \mathbf{p}^{j\prime}_t\mathbf{z}^j)} \times TE^j_t(\mathbf{y}^j_t, \mathbf{x}^j_t, \mathbf{z}^j), \tag{22}$$

$$AE^j(\mathbf{y}^j, \mathbf{x}^j, \mathbf{z}^j, \mathbf{w}^j, \mathbf{p}^j) = \sum_{t=1}^{n_j} \frac{\mathbf{w}^{j\prime}\mathbf{x}^{j*}_t + \mathbf{p}^{j\prime}_t\mathbf{z}^{j*}}{\sum_{t=1}^{n_j}(\mathbf{w}^{j\prime}\mathbf{x}^{j*}_t + \mathbf{p}^{j\prime}_t\mathbf{z}^{j*})} \times AE^j_t(\mathbf{y}^j_t, \mathbf{x}^j_t, \mathbf{z}^j, \mathbf{w}^j, \mathbf{p}^j_t), \tag{23}$$

where $\mathbf{x}^{j*}_t = \mathbf{x}^j_t \times TE^j_t(\mathbf{y}^j_t, \mathbf{x}^j_t, \mathbf{z}^j)$ and $\mathbf{z}^{j*} = \mathbf{z}^j \times TE^j_t(\mathbf{y}^j_t, \mathbf{x}^j_t, \mathbf{z}^j)$, are the efficient or potential levels of the private and public inputs for entity t in group j, respectively.

The (input-oriented) distance function (after Shephard (1953, 1970)) for entity t in group j is given as follows:

$$D^j_t(\mathbf{y}^j_t, \mathbf{x}^j_t, \mathbf{z}^j) = \sup\left\{\eta \mid \left(\frac{\mathbf{x}^j_t}{\eta}, \frac{\mathbf{z}^j}{\eta}\right) \in I^j_t(\mathbf{y}^j_t)\right\}. \tag{24}$$

[10] Note that when technical efficiency is of interest, convexity of the technology is, in general, assumed.

It turns out that

$$TE_t^j(\mathbf{y}_t^j, \mathbf{x}_t^j, \mathbf{z}^j) = \frac{1}{D_t^j(\mathbf{y}_t^j, \mathbf{x}_t^j, \mathbf{z}^j)}. \tag{25}$$

Therefore, we can rewrite (22) using distance functions instead of technical efficiencies:

$$TE^j(\mathbf{y}^j, \mathbf{x}^j, \mathbf{z}^j, \mathbf{w}^j, \mathbf{p}^j) = \sum_{t=1}^{n_j} \frac{\mathbf{w}^{j\prime}\mathbf{x}_t^j + \mathbf{p}_t^{j\prime}\mathbf{z}^j}{\sum_{t=1}^{n_j}(\mathbf{w}^{j\prime}\mathbf{x}_t^j + \mathbf{p}_t^{j\prime}\mathbf{z}^j)} \times \frac{1}{D_t^j(\mathbf{y}_t^j, \mathbf{x}_t^j, \mathbf{z}^j)}. \tag{26}$$

All in all, these results naturally extend those of Färe and Zelenyuk (2003) and Färe, Grosskopf, and Zelenyuk (2004) when considering both private and public inputs.

We end this part by providing the technical and allocative efficiency scores for the entities and groups of Table 1. As for the cost efficiency scores, we make use of DEA for the computation. The results are displayed in Table 3.

Table 3: Technical and allocative efficiency results

	Group 1		Group 2	
	Entity A	Entity B	Entity C	Entity D
$TE_t^j(\mathbf{y}_t^j, \mathbf{x}_t^j, \mathbf{z}^j)$	0.6667	0.6667	1	1
$AE_t^j(\mathbf{y}_t^j, \mathbf{x}_t^j, \mathbf{z}^j, \mathbf{w}^j, \mathbf{p}_t^j)$	0.8571	0.7500	1	0.3684
$TE^j(\mathbf{y}^j, \mathbf{x}^j, \mathbf{z}^j, \mathbf{w}^j, \mathbf{p}^j)$	0.6667		1	
$AE^j(\mathbf{y}^j, \mathbf{x}^j, \mathbf{z}^j, \mathbf{w}^j, \mathbf{p}^j)$	0.7894		0.6011	

It may seem surprising that the technical efficiency scores are the same for Entities A and B, and for Entities C and D. This is simply explained by how we define technical efficiency in (19). Here, indeed, we compute a radial technical efficiency measurement. It is clear that Entity C is the most technical efficient entity since it uses less private and public inputs (recall that the outputs are normalized to unity). It turns out that the other entities are compared to Entity C. Let us take the example of entity D, it uses more private input quantity than entity C, but the same quantity of the public input. Therefore, it is labelled as technical efficient even if it can reduce its private input by four units. A similar reasoning holds true for the other entities. This also shows that computing slacks may be important when public inputs are involved in the production process. Another option is to consider a more flexible efficiency measurement (e.g. a directional distance function). We find that Entity C

is allocative efficient, while the three other entities present potential improvement of their inputs given the prices. This is particularly true for Entity D.

At the group level, Group 2 is declared technical efficient, meaning that the only explanation for its cost inefficient behaviour is an allocative inefficient behaviour. Group 1 is both technical and allocative inefficient, but its allocative efficiency score is greater than the one of Group 2. We insist that the group-level technical and allocative efficiency scores are obtained by the aggregation procedures obtained in (22) and (23) and thus not computed by linear programmings.

Unobserved input prices. In practice, input prices are often partially if at all observed. For our aggregation procedure, the most difficult prices to observe are probably the public input prices at entity level. Let us first consider that the public input prices for the group, i.e. $\mathbf{p}^j$, are observed. Then, one option is to define the minimal total cost of the group by relying on the most favourable (i.e. the shadow) input prices that meet condition (4):

$$\widehat{C}^j(\mathbf{y}^j, \mathbf{w}^j, \mathbf{p}^j) = \max_{\mathbf{p}_1^j,\ldots,\mathbf{p}_{n_j}^j} \left\{ \sum_{t=1}^{n_j} \widehat{C}_t^j(\mathbf{y}_t^j, \mathbf{w}^j, \mathbf{p}_t^j) \mid \sum_{t=1}^{n_j} \mathbf{p}_t^j = \mathbf{p}^j \right\}. \tag{27}$$

$\widehat{C}^j(\mathbf{y}^j, \mathbf{w}^j, \mathbf{p}^j)$ defines an upper bound for the minimal total cost, i.e. $\widehat{C}^j(\mathbf{y}^j, \mathbf{w}^j, \mathbf{p}^j) \geq C^j(\mathbf{y}^j, \mathbf{w}^j, \mathbf{p}^j)$. Clearly, we can extend this procedure when $\mathbf{p}^j$ and/or $\mathbf{w}^j$ are unknown. In that case, we may need additional assumptions to compute the minimal total cost (e.g. to overcome non-linearity issue). Note that the realism of the computed prices can be improved by using any available information. In that case, we can rely on intervals or weight restrictions for the prices. In practice, such restrictions are added to the maximization programme in (27). Also, it is important to notice that the connections between the entity- and group- level prices ((2) and (4)) are used to improve the realism of the computed prices in that case. Formally, they represent an additional constraint for the computed prices in (27).

There exist other approaches to overcome the lack of price data. For example, another option that has been suggested by Färe and Zelenyuk (2003) and Zelenyuk (2006) for aggregation schemes with private inputs is to use the input-output data to construct the prices. It is fairly straightforward to adapt their technique to our case with public inputs. We do not discuss this case here for the sake of compactness, and refer to their paper for more detail.

We end this discussion with two important remarks. First, it should be clear that it is not always the entity-level prices that are difficult to observe. It clearly depends on the applications considered. We can easily adapt (27) to other cases. Second, we may see the suggested approach as an opportunity to compute unobserved prices. There is an ample illustration of computing shadow prices of undesirable outputs/inputs in the efficiency literature (e.g. greenhouse gas emissions, waste, pollutant production factor; refer, for example, to Zhou et al. (2008), Cherchye et al. (2015), and Dakpo et al. (2016) for recent literature reviews). The shadow cost approach in (27) can be used to investigate the shadow prices of public inputs for entities and/or for the groups.

Types of inputs. Extra categories of public inputs can be considered in the aggregation framework. For example, inputs used collectively by a subset of entities within the group. These are public inputs, but only for the subset of entities. As such, they are nonrival but exclusive for them, and hence analogous to club goods in consumer theory.[11] In fact, the setting with private and public inputs is general enough to include extra categories. For club inputs, our previous definition of public inputs can be adapted as follows:

$$(\mathbf{z}_t^j)_r = \begin{cases} (\mathbf{z})_r^j & \text{if input } r \text{ is used by entity } t \text{ in group } j, \\ 0 & \text{otherwise.} \end{cases} \tag{28}$$

Likewise, our previous definition of public input prices can be adapted as follows:

$$\begin{aligned} \sum_{t=1}^{n_j} \mathbf{p}_t^j =& \mathbf{p}^j, \\ (\mathbf{p}_t^j)_r =& 0 \text{ if input } r \text{ is not used by entity } t \text{ in group } j. \end{aligned} \tag{29}$$

Expressed in words, if entity t in group j does not use the club input r, then the quantity is naturally zero, and the willingness to buy is also zero.

[11]These types of inputs have been considered in Salerian and Chan (2005), Despic, Despic, and Paradi (2007), and Cherchye, De Rock, and Walheer (2015) in different efficiency models.

Overall Farrell efficiencies. Our aggregation scheme gives us Farrell efficiencies for every group $\{1, \ldots, J\}$. This is attractive since the entities are partitioned into these J groups. Nevertheless, a natural question is whether we can define overall Farrell efficiencies. That is, Farrell efficiencies when considering all entities in all groups. The answer is affirmative, but additional assumptions are needed. Following our derivation of the aggregation scheme in Section 2, two conditions need to be satisfied to define consistent overall Farrell efficiencies: (1) define the overall actual total costs, and (2) define the overall minimal total costs. Of course, they have to be defined in such a manner that they are related to the group-level counterparts (in a similar way that what has be done in Section 2 to relate the group-level concepts to the entity-level counterparts).

For condition (1), we need the private and public input prices to be similar for all groups (i.e. independent of j). For condition (2), we have to define the overall input requirement set in a similar fashion to what we did in (10). Clearly, these additional conditions are more restrictive and add structure to the technology, yet overall Farrell efficiencies cannot be derived without them.

Resource reallocation. When entities are regrouped, it is natural to analyze whether there are potential gains if resources (i.e. inputs in our context) can be reallocated. This question has been considered by Nesterenko and Zelenyuk (2007), Pachkova (2009), Mayer and Zelenyuk (2014) for static and dynamic settings. These works are built on the aggregation procedure initiated by Färe and Zelenyuk (2003). That is, when one group of entities uses private inputs to produce the outputs. In light of our extended framework with multiple groups and public inputs, it makes sensel to return to this question. It turns out that reallocation can be considered in more dimensions, namely, within and between groups, and for private and public inputs. At this point, we highlight that, in such a situation, it is crucial to clearly specify what reallocation means, i.e. when and where reallocation is possible. Related discussion can be found in Balk (2016) which provides another way of looking at the reallocation possibility (in constraint to the references cited before). Providing such concepts is left to further research.

4 Conclusion

In the wake of Färe and Zelenyuk's initial result for Farrell efficiencies, aggregation of efficiency and efficiency-related indexes and indicators for static and dynamic settings has gained momentum in the literature. Such aggregated indexes and indicators are important for contexts where entities form a group, as they provide additional information about the group performance.

In this paper, we make a distinction between two types of inputs in the aggregation framework: private and public inputs. Private inputs are those used by each entity individually, while public inputs are those used collectively. The obtained aggregation procedure, while remaining consistent with previous works, offers the opportunity to consider more practical cases, and increases our understanding of aggregation techniques. We also consider several extensions of our aggregation results and provide an illustration.

References

[1] Balk B. M., 2016, "Various approaches to the aggregation of economic productivity indices", *Pacific Economic Review* 21(4), 445-463.

[2] Blackorby C., Russell R. R., 1999 "Aggregation of efficiency indices", *Journal of Productivity Analysis* 12 (1), 5-20.

[3] Charnes A., Cooper W. W., Rhodes E., 1978, "Measuring the Efficiency of Decision Making Units", *European Journal of Operational Research* 2, 429-444.

[4] Cherchye L., De Rock B., Dierynck B., Roodhooft F., Sabbe J., 2013, "Opening the Black Box of Efficiency Measurement: Input Allocation in Multi-Output Settings", *Operations Research* 61, 1148-1165.

[5] Cherchye L., Demuynck T., De Rock B., De Witte K., 2014, "Non-parametric analysis of multi-output production with joint inputs", *The Economic Journal* 124, 735-775.

[6] Cherchye L., De Rock B., Walheer B., 2015, "Multi-output efficiency with good and bad outputs", *European Journal of Operational Research* 240, 872-881.

[7] Cherchye L., De Rock B., Walheer B., 2016, "Multi-Output Profit Efficiency And Directional Distance Functions", *Omega* 61, 100-109.

[8] Dakpo K. H., Jeanneaux P., Latruffe L., 2016, "Modelling pollution-generating technologies in performance benchmarking: Recent developments, limits and future prospects in the nonparametric framework ", *European Journal of Operational Research* 250, 347-359.

[9] Debreu G., 1951, "The Coefficient of Resource Utilization", *Econometrica* 19(3), 273-292.

[10] Despic O., Despic M., Paradi J., 2007, "DEA-R: ratio-based comparative efficiency model, its mathematical relation to DEA and its use in applications", *Journal of Productivity Analysis* 28, 33-44.

[11] Ding J., Dong W., Liang L., Zhu J., 2017, "Goal congruence analysis in multi-Division Organizations with shared resources based on data envelopment analysis", *European Journal of Operational Research* 263, 961-973.

[12] Färe R., Grosskopf S., 2000, "Network DEA", *Socio-Economic Planning Sciences* 34, 35-49.

[13] Färe R., Grosskopf S., Lovell C. A. K., 1994, "Production Frontier", *Cambridge University Press*.

[14] Färe R., Grosskopf S., Whittaker G., 2007, "Network DEA", *Modeling Data Irregularities and Structural Complexities in Data Envelopment Analysis, J. Zhu and W. Cook, Eds. Springer*.

[15] Färe R., Grosskopf S., Zelenyuk V., 2004, "Aggregation of Cost Efficiency: Indicators and Indexes Across Firms", *Academia Economic Papers* 32(3), 395-411.

[16] Färe R., Karagiannis G., 2017, "The denominator rule for share-weighting aggregation", *European Journal of Operational Research* 260, 1175-1180.

[17] Färe R., Primont D., 1995, "Multi-output Production and Duality: Theory and Applications", *Kluwer Academic Publishers* Boston.

[18] Färe R., Zelenyuk V., 2003, "On aggregate Farrell efficiencies", *European Journal of Operational Research* 146, 615-620.

[19] Färe R., Zelenyuk V., 2007, "Extending Färe and Zelenyuk (2003)", *European Journal of Operational Research* 179, 594-595.

[20] Färe R., Zelenyuk V., 2012, "Aggregation of scale elasticities across firms", *Applied Economics Letters* 19, 1593-1597.

[21] Farrell M., 1957, "The Measurement of Productive Efficiency", *Journal of the Royal Statistical Society* Series 1, General, 120, Part 3, 253-281.

[22] Kuosmanen T., Cherchye L., Sipilainen T., 2006, "The law of one price in data envelopment analysis: restricting weight flexibility across firms", *European Journal of Operational Research* 170 , 735-757 .

[23] Li S. K., Ng Y. C., 1995 "Measuring the productive efficiency of a group of firms", *International Advances in Economic Research* 1 (4), 377-390.

[24] Mayer A., Zelenyuk V., 2014, "Aggregation of Malmquist productivity indexes allowing for reallocation of resources", *European Journal of Operational Research* 238, 774-785.

[25] Mas-Colell M., Whinston D., Green J. R., 1995, "MicroeconomicTheory", *Oxford University Press* Oxford.

[26] Nehring K., Puppe C., 2004, "Modelling cost complementarities in terms of joint production", *Journal of Economic Theory* 118, 252-264.

[27] Negishi T., 1973, "The excess of public expenditures on industries", *Journal of Public Economics* 2, 231-240.

[28] Nesterenko V. , Zelenyuk V., 2007, "Measuring potential gains form reallocation of resources", *Journal of Productivity Analysis* 28, 107-116 .

[29] Pachkova E. V., 2009, "Restricted reallocation of resources", *European Journal of Operational Research* 196 , 1049-1057.

[30] Panzar J. C., Willig R. D., 1981, "Economies of Scope", *American Economic Review* 71(2), 268-272.

[31] Peyrache A., 2013, "Industry structural inefficiency and potential gains from mergers and break-ups: A comprehensive approach", *European Journal of Operational Research* 230, 422-430.

[32] Peyrache A., 2015, "Cost constrained industry inefficiency", *European Journal of Operational Research* 247, 996-1002.

[33] Rogge N., 2018, "On aggregating Benefit of the Doubt composite indicators", *European Journal of Operational Research* 264, 364-369.

[34] Salerian J., Chan C., 2005, "Restricting multiple-output multiple-input DEA models by disaggregating the output-input vector", *Journal of Productivity Analysis* 24, 5-29.

[35] Shephard R. W., 1953, "Cost and Production Functions", *Princeton University Press*.

[36] Shephard R. W., 1970, "Theory of Cost and Production Functions", *Princeton University Press*.

[37] Silva M. C. A., 2018, "Output-specific inputs in DEA: An application to courts of justice in Portugal", *Omega*79, 43-53.

[38] Tone K., Tsutsui M., 2009, "Network DEA: A slacks-based measure approach", *European Journal of Operational Research* 197, 243-252.

[39] Tulkens H., 1993, "On FDH analysis: Some methodological issues and applications to retail banking, courts and urban transit", *Journal of Productivity Analysis* 4, 183-210.

[40] Varian H. R., 1984, "The Non-Parametric Approach to Production Analysis", *Econometrica* 52, 579-598.

[41] Walheer B., 2016, "Growth and Convergence of the OECD countries: A Multi-Sector Production-Frontier Approach", *European Journal of Operational Research*, 252, 665-675.

[42] Walheer B., 2018a, "Aggregation of metafrontier technology gap ratios: the case of European sectors in 1995-2015", *European Journal of Operational Research* 269, 1013-1026.

[43] Walheer B., 2018b, "Cost Malmquist productivity index: an output-specific approach for group comparison", *Journal of Productivity Analysis* 49(1), 79-94.

[44] Walheer B., 2018c, "Disaggregation of the Cost Malmquist Productivity Index with joint and output-specific inputs", *Omega* 75, 1-12.

[45] Walheer B., 2018d, "Scale efficiency for multi-output cost minimizing producers: the case of the US electricity plants", *Energy Economics* 70, 26-36.

[46] Walheer B., 2018e, Scale, congestion, and technical efficiency of European countries: a sector-based nonparametric approach", *Empirical Economics* doi.org/10.1007/s00181-018-1426-7.

[47] Ylvinger, S. 2000, "Industry performance and structural efficiency measures: Solutions to problems in firm models", *European Journal of Operational Research* 121, 164-174.

[48] Zelenyuk V., 2006, "Aggregation of Malmquist productivity indexes", *European Journal of Operational Research* 174, 1076-1086.

[49] Zelenyuk V., 2011, "Aggregation of economic growth rates and of its sources", *European Journal of Operational Research*, 212, 190-198.

[50] Zelenyuk V., 2016, "Aggregation of scale efficiency", *European Journal of Operational Research*, 240, 269-277.

[51] Zhou P., Ang B. W., Poh K. L., 2008, "A survey of data envelopment analysis in energy and environmental studies", *European Journal of Operational Research* 189, 1-18.

Appendix

We give the proof that the minimal total cost of group j is equal to the sum of the minimal total cost of all entities in that group, i.e. (14). Note that the proof is directly derived from Färe, Grosskopf, and Zelenyuk (2004), which is itself a particular version of the proof of the aggregation of profit function of Mas-Colell, Whinston and Green (1995).

The proof contains two steps:

1. Let $(\mathbf{x}_t, \mathbf{z}) \in I_t^j(\mathbf{y}_t^j)$ be arbitrary, then since $\mathbf{x} = \sum_{t=1}^{n_j} \mathbf{x}_t$, $(\mathbf{x}_t, \mathbf{z}) \in I^j(\mathbf{y}^j)$ by (10). Next, by construction, minimal cost at the group level cannot exceed actual cost: $C^j(\mathbf{y}^j, \mathbf{w}^j, \mathbf{p}^j) \leq \mathbf{w}^{j\prime}\mathbf{x} + \mathbf{p}^{j\prime}\mathbf{z}$. By (7), we have that $\mathbf{w}^{j\prime}\mathbf{x} + \mathbf{p}^{j\prime}\mathbf{z} = \sum_{t=1}^{n_j}(\mathbf{w}^{j\prime}\mathbf{x}_t + \mathbf{p}_t^{j\prime}\mathbf{z})$. It implies that $C^j(\mathbf{y}^j, \mathbf{w}^j, \mathbf{p}^j) \leq \sum_{t=1}^{n_j}(\mathbf{w}^{j\prime}\mathbf{x}_t + \mathbf{p}_t^{j\prime}\mathbf{z})$. As $(\mathbf{x}_t, \mathbf{z})$ are arbitrary, we can choose the cost minimizer levels, which gives:

$$C^j(\mathbf{y}^j, \mathbf{w}^j, \mathbf{p}^j) \leq \sum_{t=1}^{n_j} C_t^j(\mathbf{y}_t^j, \mathbf{w}^j, \mathbf{p}_t^j). \tag{30}$$

2. Let $(\mathbf{x}, \mathbf{z}) \in I^j(\mathbf{y}^j)$ be arbitrary, then there exists $(\mathbf{x}_t, \mathbf{z}) \in I_t^j(\mathbf{y}_t^j)$ such that $\mathbf{x} = \sum_{t=1}^{n_j} \mathbf{x}_t$. Next, by construction, minimal cost at the entity level cannot exceed actual cost: $\mathbf{w}^{j\prime}\mathbf{x}_t + \mathbf{p}_t^{j\prime}\mathbf{z} \geq C_t^j(\mathbf{y}_t^j, \mathbf{w}^j, \mathbf{p}_t^j)$. By using (7) and (10): $\mathbf{w}^{j\prime}\mathbf{x} + \mathbf{p}^{j\prime}\mathbf{z} = \sum_{t=1}^{n_j}(\mathbf{w}^{j\prime}\mathbf{x}_t + \mathbf{p}_t^{j\prime}\mathbf{z}) \geq \sum_{t=1}^{n_j} C_t^j(\mathbf{y}_t^j, \mathbf{w}^j, \mathbf{p}_t^j)$. As $(\mathbf{x}, \mathbf{z})$ are arbitrary, we can choose the cost minimizer levels, which gives:

$$C^j(\mathbf{y}^j, \mathbf{w}^j, \mathbf{p}^j) \geq \sum_{t=1}^{n_j} C_t^j(\mathbf{y}_t^j, \mathbf{w}^j, \mathbf{p}_t^j). \tag{31}$$

By combining (30) and (31), we have:

$$\sum_{t=1}^{n_j} C_t^j(\mathbf{y}_t^j, \mathbf{w}^j, \mathbf{p}_t^j) \leq C^j(\mathbf{y}^j, \mathbf{w}^j, \mathbf{p}^j) \leq \sum_{t=1}^{n_j} C_t^j(\mathbf{y}_t^j, \mathbf{w}^j, \mathbf{p}_t^j). \tag{32}$$

That is:

$$C^j(\mathbf{y}^j, \mathbf{w}^j, \mathbf{p}^j) = \sum_{t=1}^{n_j} C_t^j(\mathbf{y}_t^j, \mathbf{w}^j, \mathbf{p}_t^j). \tag{33}$$