Genuinely entangled symmetric states with no N-partite correlations

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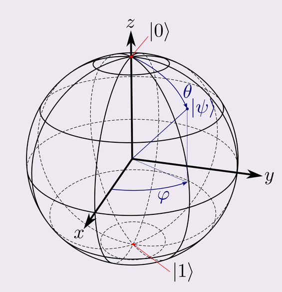
Geometrical description of spin-j states

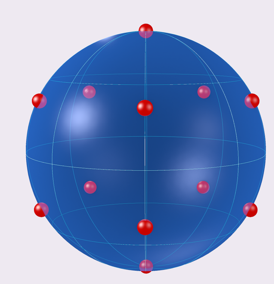
- $j = 1/2$: Bloch sphere representation (mixed or pure states)



- Any j: Majorana representation (pure states only)

– Convenient property: rigid rotation under SU(2)

– Many generalizations to mixed states of any spin j.
  In this work, we use the tensor representation [1]

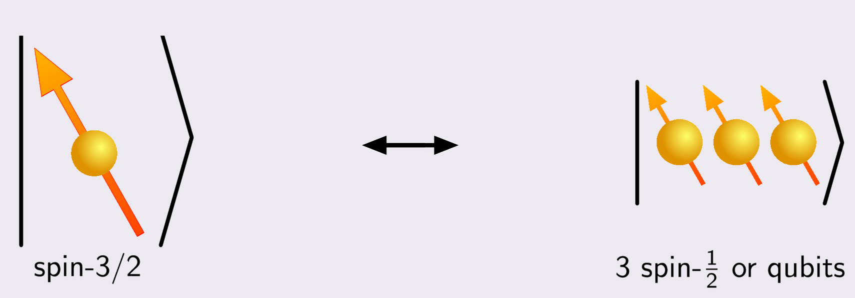


Tensor representation of spin-j states

- Expansion of ρ on an overcomplete set of 4^N Hermitian matrices $S_{\mu_1\mu_2\ldots\mu_N}$, invariant under permutation of indices $\mu_i = 0,1,2,3$ with $i = 1,\ldots,N \equiv 2j$
- Traceless matrices in the sense $g_{\mu_1\mu_2} S_{\mu_1\mu_2\ldots\mu_N} = 0$ with $g = \mathrm{diag}(-,+,+,+)$
- $j = 1/2$: $S_0 = \sigma_0$ and $S_a = \sigma_a$ with $a = 1,2,3$
  $\rho = \frac{1}{2}x_{\mu_1}S_{\mu_1} = \frac{1}{2} + \frac{1}{2}\sum_a x_a\sigma_a, \quad \|\mathbf{x}\| \leqslant 1 \quad \rightarrow$ Bloch sphere picture
  Properties: – x_a transforms as a vector under SU(2)
  – if ρ = coherent state $|\mathbf{n}\rangle\langle\mathbf{n}|$, then $x_a = n_a$
- $j = 1$: $S_{00} = J_0$, $S_{a0} = J_a$ and $S_{ab} = J_aJ_b + J_bJ_a - \delta_{ab}J_0$ with $a,b = 1,2,3$
  $\rho = \frac{1}{4}x_{\mu_1\mu_2}S_{\mu_1\mu_2} \quad \rightarrow$ generalization of the Bloch sphere picture
  Properties: – $x_{\mu_1\mu_2}$ transforms as a tensor under SU(2)
  – if ρ = coherent state $|\mathbf{n}\rangle\langle\mathbf{n}|$, then $x_{\mu_1\mu_2} = n_{\mu_1}n_{\mu_2}$
- $j = 3/2$: …

Spin-j states $\leftrightarrow$ N-qubit symmetric states

- Basis vectors $|j,m\rangle \leftrightarrow |D_N^{(k)}\rangle = \frac{1}{\sqrt{C_N^k}}\sum_\sigma |\underbrace{0\ldots0}_{N-k}\underbrace{1\ldots1}_{k}\rangle \equiv$ symmetric Dicke states with $k = j-m$ excitations and $N = 2j$ qubits
- Expansion of a symmetric N-qubit state: $|\psi_S\rangle = \sum_{k=0}^N d_k|D_N^{(k)}\rangle, \quad d_k \in \mathbb{C}$
- Spin-coherent states $\leftrightarrow$ symmetric separable states



Symmetric states with no N-partite correlations

Characterization in terms of tensor coefficients

Symmetric states with no N-partite correlations are defined in [2-3] as states ρ such that $\langle\sigma_{a_1}\otimes\cdots\otimes\sigma_{a_N}\rangle_\rho = 0$ for any a_i with $1 \le a_i \le 3$. We will denote by $\mathcal{SNC}_N$ the set of symmetric N-qubit states with no N-partite correlations.

Setting $x_{\mu_1\mu_2\ldots\mu_N} = \langle\sigma_{\mu_1}\otimes\cdots\otimes\sigma_{\mu_N}\rangle_\rho$, it can be expressed for odd N as the hierarchy of conditions

$$\begin{array}{rclcrcl} x_{a_1a_2a_3\ldots a_N} &=& 0 & \quad \forall a_i &=& 1,2,3, & 1 \le i \le N \\ x_{a_1a_2\ldots a_{N-2}00} &=& 0 & \quad \forall a_i &=& 1,2,3, & 1 \le i \le N-2 \\ \vdots & & & & \vdots & & \\ x_{a_100\ldots0} &=& 0 & \quad \forall a_1 &=& 1,2,3. & \end{array}$$

absence of N-partite correlations
$\Downarrow$
absence of $(N-2k)$-partite correlations for $k = 0,\ldots,\lfloor N/2\rfloor$

Symmetric states with no N-partite correlations

Spectral properties and antistates

Symmetric states with no N-partite correlations can always be written as

$$\rho = \sum_{i=0}^{M}\lambda_i\left(|\psi_i\rangle\langle\psi_i| + |\bar\psi_i\rangle\langle\bar\psi_i|\right),$$

where $|\psi_i\rangle$ and its antistate $|\bar\psi_i\rangle \equiv \mathfrak{N}^{\otimes N}|\psi_i\rangle$, with $\mathfrak{N}$ the one-qubit universal-not operator, are orthonormal eigenstates of ρ with eigenvalue λ_i and $\sum_i\lambda_i = 1/2$.

Entanglement criteria for symmetric states

Necessary and sufficient criterion for $N = 3$: PPT criterion

Sufficient criterion for any N: Let S^2 denote the unit sphere in $\mathbb{R}^3$, and $|n\rangle$ be the fully separable state $|\mathbf{n}\rangle^{\otimes N}$ associated with $\mathbf{n} \in S^2$. If ρ is a symmetric state such that

$$\forall\,\mathbf{n} \in S^2, \quad \langle n|\rho|n\rangle < \mathrm{tr}\rho^2, \qquad (1)$$

then ρ is genuinely entangled.

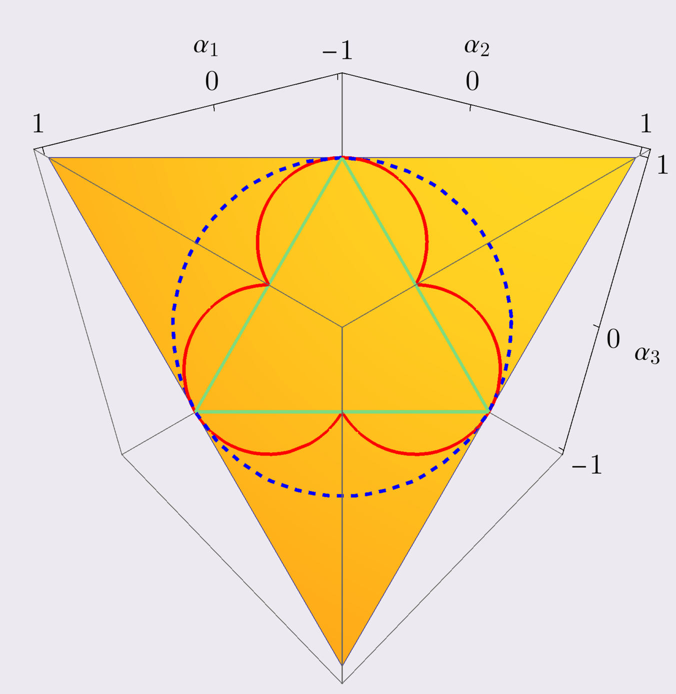
Criterion (1) is also necessary for rank-2 states ($M = 0$), and any separable rank-2 state $\rho \in \mathcal{SNC}_N$ is of the form $\frac{1}{2}|n\rangle\langle n| + \frac{1}{2}|\bar n\rangle\langle\bar n|$.

All genuinely entangled states $\in \mathcal{SNC}_3$

Any genuinely entangled three-qubit state with no 3-partite correlations can be written

$$\rho = \frac{1}{8}\mathbb{1}_4 + \frac{3}{8}\sum_{a,b=1}^{3}A_{ab}S_{ab},$$

with $\mathbb{1}_4$ the 4×4 identity matrix and A the 3×3 real symmetric matrix of two-partite correlations, $A = (\langle\sigma_{ab}\rangle_\rho)_{1\le a,b\le3}$, with $\sigma_{ab} = \sigma_a\otimes\sigma_b\otimes\mathbb{1}_2$ and $\langle\sigma_{ab}\rangle_\rho = \mathrm{tr}(\rho\,\sigma_{ab})$ and σ_a the Pauli matrices. This result follows from the PPT criterion ans our tensor representation.



Three-qubit symmetric states in the space of eigenvalues α_i of A. The orange plane (triangle down) corresponds to the condition $\mathrm{tr}A = \sum_i\alpha_i = 1$. Points inside the dashed blue circle defined by $\mathrm{tr}A^2 = \sum_i\alpha_i^2 = 1$ correspond to physical states $\rho \ge 0$. Points inside the green triangle up, defined by $\alpha_i \ge 0$, correspond to separable states. Points outside the green triangle up and inside the dashed blue circle correspond to genuinely entangled states. Points lying between the dashed circle and the solid red trilobe defined by $\max_i\alpha_i = \sum_i\alpha_i^2$ correspond to genuinely entangled states detected by the sufficient criterion (1).

Corollary: For any $\rho \in \mathcal{SNC}_3$,

$$\rho \text{ separable} \Leftrightarrow \begin{pmatrix}\langle\sigma_{11}\rangle_\rho & \langle\sigma_{12}\rangle_\rho & \langle\sigma_{13}\rangle_\rho\\ \langle\sigma_{21}\rangle_\rho & \langle\sigma_{22}\rangle_\rho & \langle\sigma_{23}\rangle_\rho\\ \langle\sigma_{31}\rangle_\rho & \langle\sigma_{32}\rangle_\rho & \langle\sigma_{33}\rangle_\rho\end{pmatrix} \ge 0 \Leftrightarrow \mathrm{tr}_1(\rho) \text{ separable}$$

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