State complexity of the multiples of the Thue-Morse set

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Basics

Definition

A DFA is ***minimal*** iff it is ***reduced*** and ***accessible***.

- Trim minimal

Theorem

For any regular language L, there exists a unique (up to isomorphism) minimal automaton accepting L.

Definition

The ***state complexity*** of a regular language is equal to the number of states of its minimal automaton.

Definition

A DFA has ***disjoint states*** if, for distinct states p and q, we have $L(p) \cap L(q) = \emptyset$.

Proposition

Any coaccessible DFA having disjoint states is reduced.

Definition

For a base b, a subset X of $\mathbb{N}$ is said to be ***b-recognizable*** if the language $0^*\mathrm{rep}_b(X)$ is regular.

Proposition

Let $b \in \mathbb{N}_{\geq 2}$ and $m \in \mathbb{N}$. If X is b-recognizable, then so is mX.

Multiplicatively independent integers:

$$(p^a = q^b) \Rightarrow (a = b = 0)$$

Theorem (Cobham, 1969)

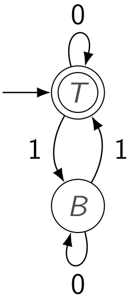
- Let b, b' be two multiplicatively independent bases. Then a subset of $\mathbb{N}$ is both b-recognizable and b'-reconnaissable if and only if it is a finite union of arithmetic progressions.
- Let b, b' be two multiplicatively dependent bases. Then a subset of $\mathbb{N}$ is b-recognizable if and only if it is b'-recognizable.

Thue-Morse set

The ***Thue-Morse set***:

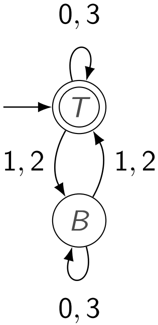
$$\mathcal{T} = \{n \in \mathbb{N} \colon |\mathrm{rep}_2(n)|_1 \in 2\mathbb{N}\}$$

Characteristic sequence: $1001011001101001\cdots$



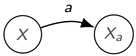
Proposition

The set $\mathcal{T}$ is 2^p-recognizable for all $p \in \mathbb{N}_{\geq 1}$ and is not b-recognizable for any other base b.



$$A_4 \cap \mathcal{T} = \{0, 3\}$$
$$A_4 \cap (\mathbb{N} \setminus \mathcal{T}) = \{1, 2\}$$

Edges:



where $X_a = \begin{cases} X & \text{if } a \in \mathcal{T} \\ \overline{X} & \text{otherwise} \end{cases}$

Main Theorem

Lemma

For any $m \in \mathbb{N}$ and $p \in \mathbb{N}_{\geq 1}$, the set $m\mathcal{T}$ is 2^p-recognizable.

Theorem

Let $m \in \mathbb{N}$ and $p \in \mathbb{N}_{\geq 1}$. Then the state complexity of the language $0^*\mathrm{rep}_{2^p}(m\mathcal{T})$ is equal to

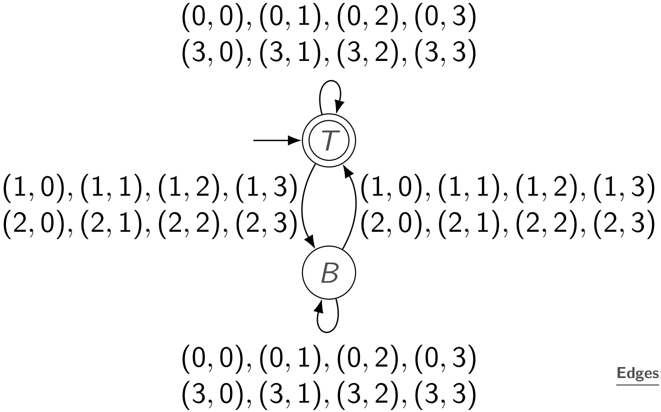
$$2k + \left\lceil \frac{z}{p} \right\rceil$$

if $m = k2^z$ with k odd.

Method

Automaton	Language
$\mathcal{A}_{\mathcal{T},2^p}$	$(0,0)^*\{\mathrm{rep}_{2^p}(t,n)\colon t \in \mathcal{T}, n \in \mathbb{N}\}$
$\mathcal{A}_{m,2^p}$	$(0,0)^*\{\mathrm{rep}_{2^p}(n,mn)\colon n \in \mathbb{N}\}$
$\mathcal{A}_{m,2^p} \times \mathcal{A}_{\mathcal{T},2^p}$	$(0,0)^*\left\{\mathrm{rep}_{2^p}\left(t,mt\right) : t \in \mathcal{T}\right\}$
$\Pi(\mathcal{A}_{m,2^p} \times \mathcal{A}_{\mathcal{T},2^p})$	$0^*\left\{\mathrm{rep}_{2^p}\left(mt\right) : t \in \mathcal{T}\right\}$

The automaton $\mathcal{A}_{\mathcal{T},2^p}$



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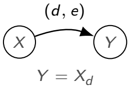
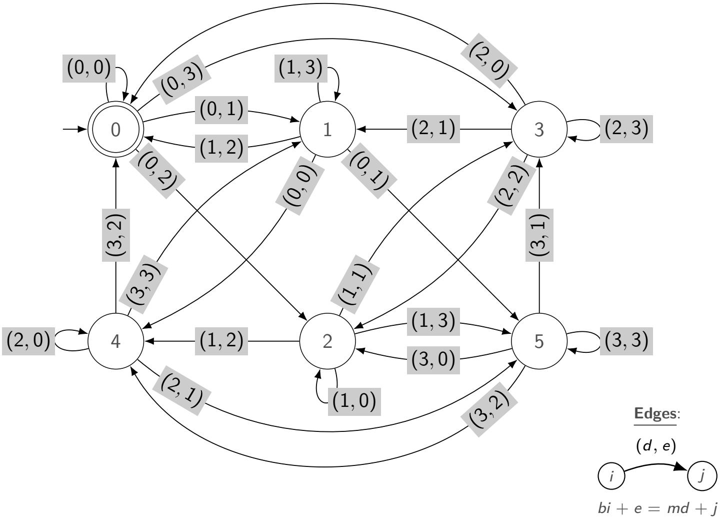


Proposition

The automaton $\mathcal{A}_{\mathcal{T},2^p}$

- accepts $(0,0)^*\{\mathrm{rep}_{2^p}(t,n) \colon t \in \mathcal{T}, n \in \mathbb{N}\}$
- is accessible
- is coaccessible
- has disjoint states
- is trim minimal
- is complete

The automaton $\mathcal{A}_{m,b}$



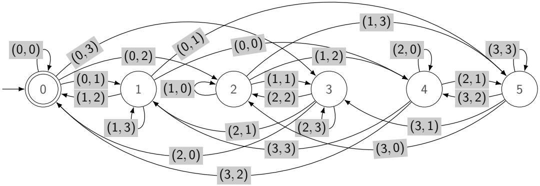
Proposition

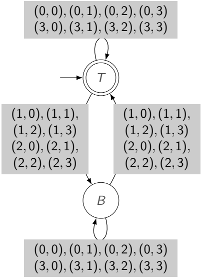
The automaton $\mathcal{A}_{m,b}$

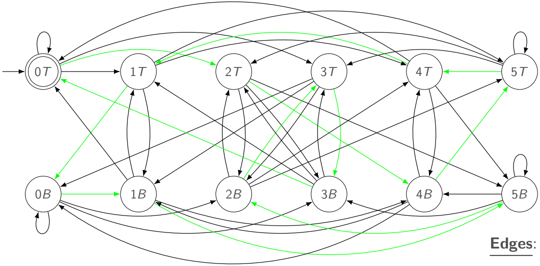
- accepts $(0,0)^*\{\mathrm{rep}_b(n, mn) \colon n \in \mathbb{N}\}$
- is accessible
- is coaccessible
- has disjoint states
- is trim minimal

Remark: The automaton $\mathcal{A}_{m,b}$ is not complete.

The product automaton $\mathcal{A}_{m,2^p} \times \mathcal{A}_{\mathcal{T},2^p}$







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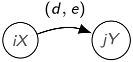


$2^p i + e = md + j$ and $Y = X_d$.

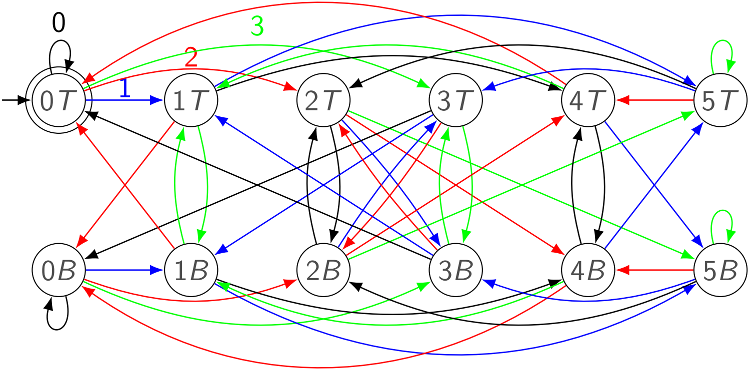
Proposition

The automaton $\mathcal{A}_{m,2^p} \times \mathcal{A}_{\mathcal{T},2^p}$

- accepts $(0,0)^* \{\mathrm{rep}_{2^p}(t, mt) : t \in \mathcal{T}\}$
- is accessible
- is coaccessible
- has disjoint states
- is trim minimal

Remark: The automaton $\mathcal{A}_{m,2^p} \times \mathcal{A}_{\mathcal{T},2^p}$ is not complete.

Projection of $\mathcal{A}_{m,2^p} \times \mathcal{A}_{\mathcal{T},2^p}$



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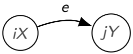


$\exists d \in A_{2p}$:
$2^p i + e = md + j$ and $Y = X_d$.

Proposition

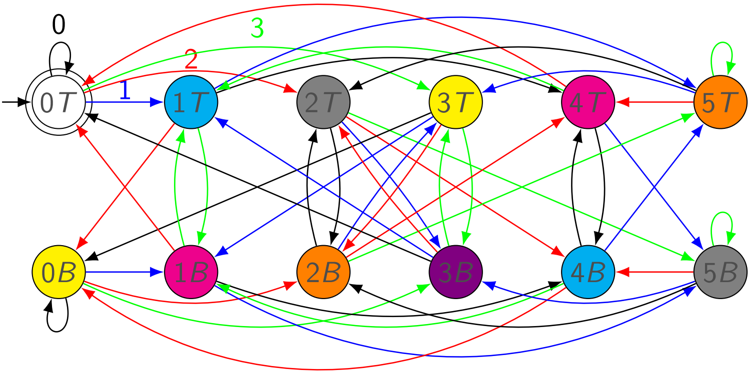
The automaton $\Pi(\mathcal{A}_{m,2^p} \times \mathcal{A}_{\mathcal{T},2^p})$

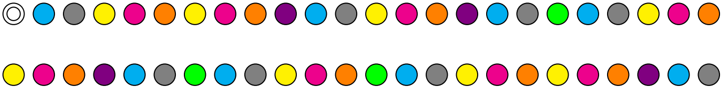
- accepts $0^* \{\mathrm{rep}_{2^p}(mt) : t \in \mathcal{T}\}$
- is deterministic
- is accessible
- is coaccessible
- has disjoint states **if** m **is odd**
- is trim minimal **if** m **is odd**

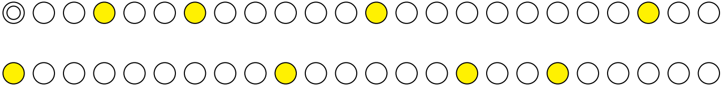
Corollary

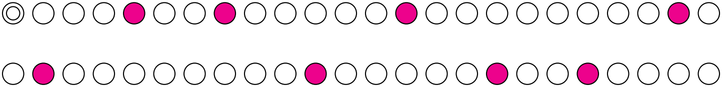
The state complexity of $m\mathcal{T}$ in base 2^p is $2m$ **if** m **is odd**.

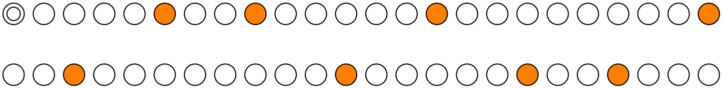
Minimisation of $\Pi(\mathcal{A}_{m,2^p} \times \mathcal{A}_{\mathcal{T},2^p})$

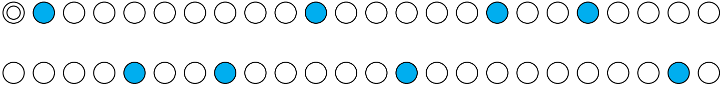


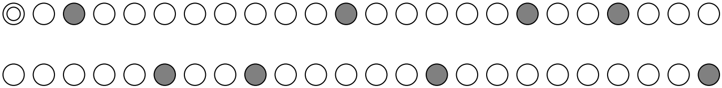












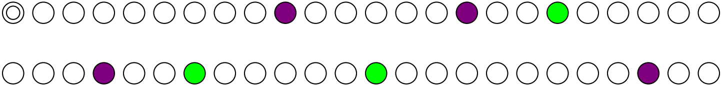
Let $m = k.2^z$ where $k, z \in \mathbb{N}$ with k odd.

For $(j, X) \in (\{1, \ldots, k-1\} \times \{T, B\}) \cup \{(0, B)\}$, the class of (j, X) is

$$[(j, X)] = \{(j + k\ell, X_\ell) : 0 \leq \ell \leq 2^z - 1\}.$$

Moreover, the class of $(0, T)$ is

$$[(0, T)] = \{(0, T)\}.$$



For $\alpha \in \{0, \ldots, z-1\}$, we define a *pre-classe* C_α of size 2^α:

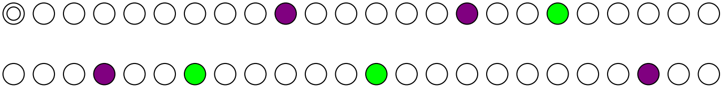
$$C_\alpha = \left\{ \left(\frac{m}{2^{\alpha+1}} + \frac{m}{2^\alpha} \ell, B_\ell \right) : 0 \leq \ell \leq 2^\alpha - 1 \right\}$$

For all $\beta \in \{0, \ldots, \left\lceil \frac{z}{p} \right\rceil - 2\}$, we define

$$\Gamma_\beta = \bigcup_{\alpha \in \{\beta p, \ldots, (\beta+1)p-1\}} C_\alpha$$

and

$$\Gamma_{\left\lceil \frac{z}{p} \right\rceil - 1} = \bigcup_{\alpha \in \left\{ \left(\left\lceil \frac{z}{p} \right\rceil - 1 \right) p, \ldots, z-1 \right\}} C_\alpha$$



In this example $m = 3.2^3$ and $b = 4$.
So, $k = 3, z = 3, p = 2,$ and $\left\lceil \frac{z}{p} \right\rceil = 2.$ We obtain

$$C_0 = \{(12, B)\}$$
$$C_1 = \{(6, B), (18, T)\}$$
$$C_2 = \{(3, B), (9, T), (15, T), (21, B)\}$$

and

$$\Gamma_1 = C_0 \cup C_1 = \{(6, B), (12, B), (18, T)\}$$
$$\Gamma_2 = C_2 = \{(3, B), (9, T), (15, T), (21, B)\}$$

Counting and Conclusion

Classes	**Number of such classes**
$[(j, X)]$ for $(j, X) \in (\{1, \ldots, k-1\} \times \{H, B\})$	$2(k-1)$
$[(0, B)]$	1
$[(0, H)]$	1
Γ_β for $\beta \in \{0, \ldots, \left\lceil \frac{z}{p} \right\rceil - 2\}$	$\left\lceil \frac{z}{p} \right\rceil - 1$
$\Gamma_{\left\lceil \frac{z}{p} \right\rceil - 1}$	1

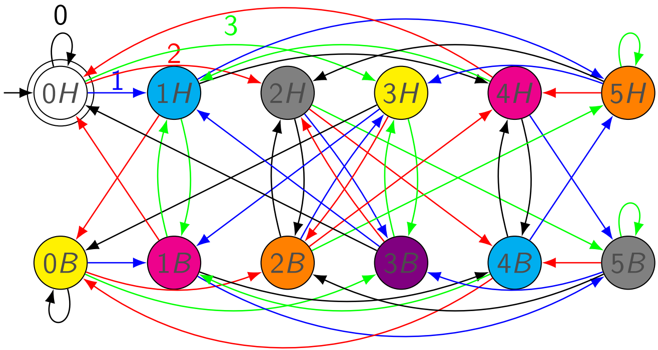
Total $= 2k + \left\lceil \frac{z}{p} \right\rceil$

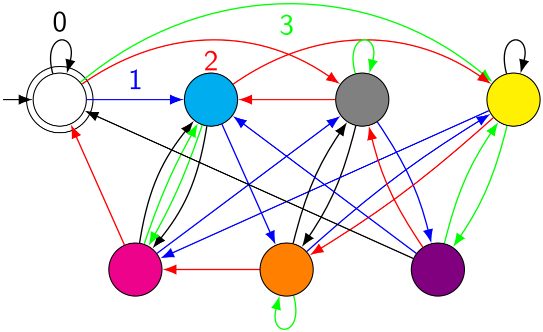
Theorem

Let $m \in \mathbb{N}$ and $p \in \mathbb{N}_{\geq 1}$. Then the state complexity of the language $0^* \mathrm{rep}_{2^p}(m\mathcal{T})$ is equal to

$$2k + \left\lceil \frac{z}{p} \right\rceil$$

if $m = k2^z$ with k odd.


0
3
2
1
0H
1H
2H
3H
4H
5H
0B
1B
2B
3B
4B
5B



The state complexity of $6\mathcal{T}$ in base 4 is equal to

$$2.3 + \left\lceil \frac{1}{2} \right\rceil .$$

Thank you!