Computing the k-binomial complexity of the Tribonacci word

June 20, 2019

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Joint work with Michel Rigo and Matthieu Rosenfeld

Plan

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Factors and subwords

Definition

Let $u = u_1 u_2 \cdots u_m$ be a finite or infinite word. A **(scattered) subword** of u is a finite subsequence of the sequence $(u_j)_{j=1}^{m}$. A **factor** of u is a contiguous subword.

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We can replace $\sim_=$ with other equivalence relations.

Other equivalence relations

Different equivalence relations from $\sim_=$ can be considered:

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- k-binomial equivalence: $u \sim_k v \Leftrightarrow \binom{u}{x} = \binom{v}{x} \ \forall x \in A^{\leq k}$

We will deal with the last one.

Plan

1. Preliminary definitions
   - Words, factors and subwords
   - Complexity functions
   - k-binomial complexity
2. State of the art
3. Next result: the Tribonacci word
   - Definition
   - The theorem
   - Introduction to templates and their parents
   - Using templates to compute $\mathbf{b}_T^{(2)}$

k-binomial equivalence

Definition (Reminder)

Let u and v be two finite words. They are **k-binomially equivalent** if

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Proposition

For all words u, v and for every nonnegative integer k,

$$u \sim_{k+1} v \Rightarrow u \sim_k v.$$

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k-binomial complexity

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If $\mathbf{w}$ is an infinite word, we can define the function

$$\mathbf{b}_{\mathbf{w}}^{(k)} : \mathbb{N} \to \mathbb{N} : n \mapsto \#(\mathsf{Fac}_{\mathbf{w}}(n)/\sim_k),$$

which is called the **k-binomial complexity** of $\mathbf{w}$.

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We have an order relation between the different complexity functions.

Proposition

$$\rho_{\mathbf{w}}^{ab}(n) \leq \mathbf{b}_{\mathbf{w}}^{(k)}(n) \leq \mathbf{b}_{\mathbf{w}}^{(k+1)}(n) \leq p_{\mathbf{w}}(n) \quad \forall n \in \mathbb{N}, k \in \mathbb{N}^{+}$$

where $\rho_{\mathbf{w}}^{ab}$ is the abelian complexity function of the word $\mathbf{w}$.

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A famous word...

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The classical Thue–Morse word, defined as the fixed point of the morphism

$$\varphi : \{0,1\}^* \to \{0,1\}^* : \left\{ \begin{array}{lcl} 0 & \mapsto & 01; \\ 1 & \mapsto & 10, \end{array} \right.$$

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has a bounded k-binomial complexity. The exact value is known.

Theorem (M. L., J. Leroy, M. Rigo, 2018)

Let k be a positive integer. For every $n \leq 2^k - 1$, we have

$$\mathbf{b}_{\mathbf{t}}^{(k)}(n) = p_{\mathbf{t}}(n),$$

while for every $n \geq 2^k$,

$$\mathbf{b}_{\mathbf{t}}^{(k)}(n) = \left\{ \begin{array}{ll} 3 \cdot 2^k - 3, & \text{if } n \equiv 0 \pmod{2^k}; \\ 3 \cdot 2^k - 4, & \text{otherwise.} \end{array} \right.$$

Another family

Definition: Sturmian words

A **Sturmian word** is an infinite word having, as factor complexity, $p(n) = n + 1$ for all $n \in \mathbb{N}$.

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Theorem (M. Rigo, P. Salimov, 2015)

Let $\mathbf{w}$ be a Sturmian word. We have

$$\mathbf{b}_{\mathbf{w}}^{(k)}(n) = p_{\mathbf{w}}(n) = n + 1,$$

for all $n \in \mathbb{N}$ and for all $k \geq 2$.

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for all $n \in \mathbb{N}$ and for all $k \geq 2$.

Since $\mathbf{b}_{\mathbf{w}}^{(k)}(n) \leq \mathbf{b}_{\mathbf{w}}^{(k+1)}(n) \leq p_{w}(n)$, it suffices to show that

$$\mathbf{b}_{\mathbf{w}}^{(2)}(n) = p_{\mathbf{w}}(n).$$

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$$\mathcal{T} = 0\underline{1}02 \cdots$$

The Tribonacci word

From now, let $A = \{0, 1, 2\}$. Let us define the Tribonacci word.

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Plan

Its k-binomial complexity

The next result was first conjectured by Michel Rigo, and then proved.

Theorem (M. L., M. Rigo, M. Rosenfeld, 2019)

For all $k \geq 2$, the k-binomial complexity of the Tribonacci word equals its factor complexity.

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For all $k \geq 2$, the k-binomial complexity of the Tribonacci word equals its factor complexity.

To show this result, it suffices to show that, for all $n \in \mathbb{N}$,

$$\left\{ \begin{array}{l} u, v \in \mathrm{Fac}_{\mathcal{T}}(n) \\ u \neq v \end{array} \right. \quad \Rightarrow \quad u \not\sim_2 v.$$

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The **Parikh vector** of a word u is classicaly defined as

$$\Psi(u) := \left(\binom{u}{0} \quad \binom{u}{1} \quad \binom{u}{2}\right)^{\intercal} \in \mathbb{N}^3 .$$

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Let us define the **extended Parikh vector** of a word u as

$$\Phi(u) := \left(\binom{u}{0} \quad \binom{u}{1} \quad \binom{u}{2} \quad \binom{u}{00} \quad \binom{u}{01} \quad \dots \quad \binom{u}{22}\right)^\intercal \in \mathbb{N}^{12} .$$

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Remark

We have $u \sim_2 v \;\Leftrightarrow\; \Phi(u) = \Phi(v) \;\Leftrightarrow\; \Phi(u) - \Phi(v) = 0.$

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Intuitive introduction to templates

We will be interested into values of $\Phi(u) - \Phi(v)$ for $u, v \in \mathrm{Fac}_{\mathcal{T}}$. We will thus express the difference using the notion of templates.

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Informally, we will associate to every pair of words several templates, which are 5-uples:

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There exists a strong link between this notion and our thesis:

$$\mathbf{b}_{\mathcal{T}}^{(2)}(n) < p_{\mathcal{T}}(n)$$
$$\Leftrightarrow$$
$$\exists (u, v) \in \mathsf{Fac}_{\mathcal{T}} \times \mathsf{Fac}_{\mathcal{T}} \leftrightsquigarrow [\mathbf{0}, \mathbf{0}, \mathbf{0}, a, b], a \neq b$$

Templates: a formal definition

Definition

A **template** is a 5-uple of the form $t = [\mathbf{d}, \mathbf{D_b}, \mathbf{D_e}, a_1, a_2]$ where $\mathbf{d} \in \mathbb{Z}^{12}$, $\mathbf{D_b}, \mathbf{D_e} \in \mathbb{Z}^3$ and $a_1, a_2 \in A$.

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where the matrix P_3 is such that, for all $\mathbf{x} \in \mathbb{Z}^9$, $P_3 \cdot \mathbf{x} = \begin{pmatrix} 0 & 0 & 0 & \mathbf{x} \end{pmatrix}^{\mathsf{T}}$, and where $\otimes$ is the usual Kronecker product: if $A \in \mathbb{Z}^{m \times n}$ and $B \in \mathbb{Z}^{p \times q}$,

$$A \otimes B := \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix} \in \mathbb{Z}^{mp \times nq}.$$

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Why are templates useful?

Theorem

There exists $n \in \mathbb{N}$ such that

$$\mathbf{b}_{\mathcal{T}}^{(2)}(n) < p_{\mathcal{T}}(n)$$

if and only if at least one template of the form $[\mathbf{0}, \mathbf{0}, \mathbf{0}, a, b]$ with $a \neq b$ is realizable by a pair of factors of $\mathcal{T}$.

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Let $i \in \{1, \ldots, n\}$ be such that $u_i \neq v_i, u_{i+1} = v_{i+1}, \ldots, u_n = v_n$.
Then $(u_1 \cdots u_i, v_1 \cdots v_i)$ realizes $[\mathbf{0}, \mathbf{0}, \mathbf{0}, u_i, v_i]$, because
$u \sim_2 v \Rightarrow u_1 \cdots u_i \sim_2 v_1 \cdots v_i$.

Using this result

We want to verify that $\forall (u, v) \in (\mathrm{Fac}_{\mathcal{T}})^2$, the pair (u, v) doesn't realize any template of the form $[\mathbf{0}, \mathbf{0}, \mathbf{0}, a, b]$ with $a \neq b$.

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Let us suppose that there exists a pair (u,v) realizing $[\mathbf{0},\mathbf{0},\mathbf{0},a,b]$ and let fix $L > 0$. Then,

- either $\min(|u|,|v|) \leq L$, or
- there exists an *ancestor template* of $[\mathbf{0},\mathbf{0},\mathbf{0},a,b]$ which is realized by a pair of words $(u',v') =$ "$((\tau^{-1})^j(u), (\tau^{-1})^j(v))$" such that $L \leq \min(|u'|,|v'|) \leq 2L$.

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Preimages

Example: intuitive definition

Recall that

$$\mathcal{T} = 01 \cdot 02 \cdot 01 \cdot 0 \cdot 01 \cdot 02 \cdot 01 \cdot 01 \cdot 02 \cdot 01 \cdot 0 \cdot 01 \cdots$$

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Definition

Let u and u' be two words. The word u' is a **preimage** of u if

- u is a factor of $\tau(u')$, and
- u' is minimal: for all factors v of u', u is not a factor of $\tau(v)$.

Preimages (continued)

A word can have several preimages.

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$$\mathcal{T} = \overbrace{01}^{\tau(0)} \cdot \overbrace{02}^{\tau(1)} \cdot 01 \cdot 0 \cdot 01 \cdot 02 \cdot 01 \cdot 01 \cdot 02 \cdot 01 \cdot 0 \cdot 01 \cdots$$

Take $u = 010$. It has 00, 01 and 02 as preimages.

Preimages (continued)

A word can have several preimages.

Example

Recall that

$$\mathcal{T} = 01 \cdot 02 \cdot \overbrace{01}^{\tau(0)} \cdot \overbrace{0}^{\tau(2)} \cdot 01 \cdot 02 \cdot 01 \cdot 01 \cdot 02 \cdot 01 \cdot 0 \cdot 01 \cdots$$

Take $u = 010$. It has 00, 01 and 02 as preimages.

Templates have parents...

We will now introduce the notion of parents of a template.

Theorem

Let t be a template and let (u, v) be a pair of factors realizing t. Let u' (resp., v') be a preimage of u (resp., v).
There always exists a template t' which is realized by (u', v'). and which is, *in some way*, related to t.

$$(u, v) \qquad \leftrightsquigarrow \qquad [\mathbf{d}, \mathbf{D_b}, \mathbf{D_e}, a_1, a_2] = t$$

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$$
\begin{tikzcd}
(u',v') & & \\
(u,v) \arrow[u, bend left, "\tau^{-1}"] & \leftrightsquigarrow & {[\mathbf{d}, \mathbf{D_b}, \mathbf{D_e}, a_1, a_2] = t}
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$$\begin{tikzcd}
(u', v') \arrow[r, leftrightsquigarrow] & {[\mathbf{d}', \mathbf{D}'_\mathbf{b}, \mathbf{D}'_\mathbf{e}, a'_1, a'_2] = t'} \\
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There always exists a template t' which is realized by (u', v') and which is, *in some way*, related to t.

The template t' is called a **parent template** of t.

$$\begin{tikzcd}
(u', v') \arrow[r, leftrightsquigarrow] & {[\mathbf{d}', \mathbf{D}'_{\mathbf{b}}, \mathbf{D}'_{\mathbf{e}}, a'_1, a'_2] = t'} \\
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Remark

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- Since a word can sometimes have several preimages, a template can also have several parents.
- There exists a formula allowing to compute all parents of a given template.

... and ancestors

Definition

Let t and t' be templates. We say that t' is an **(realizable) ancestor** of t if there exists a finite sequence of templates $t_0, \ldots, t_n$ such that

$$\begin{cases} t_0 = t', \\ t_n = t, \\ \forall i \in \{0, \ldots, n-1\}, t_i \text{ is a (realizable) parent of } t_{i+1}. \end{cases}$$

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The formal theorem

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Theorem

Let $L \geq 0$ be an integer, and let t be a template. If there exists a pair of factors (u, v) realizing t, then

- either $\min(|u|, |v|) \leq L$, or
- there exist an ancestor t' of t, and a pair (u', v') of factors realizing t', such that $L \leq \min(|u'|, |v'|) \leq 2L$.

Plan

Verifying that no template $[\mathbf{0}, \mathbf{0}, \mathbf{0}, a, b]$ is realizable

To show that $\mathbf{b}_{\mathcal{T}}^{(2)} = p_{\mathcal{T}}$, we have to show that no template from

$$T := \{[\mathbf{0}, \mathbf{0}, \mathbf{0}, a, b] \,:\, a \neq b\}$$

is realizable.

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1. We check that no pair of factors (u, v) with $\min(|u|, |v|) \leq L$ realizes a template of T.
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2. We compute all the ancestors of T and we check that none of them is realized by a pair (u', v') with $L \leq \min(|u'|, |v'|) \leq 2L$.

<u>**Problem:**</u> there exists an infinite number of ancestors.

Keeping a finite number of templates

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That will leave us with a finite number of candidates.

It is then possible to verify with a computer that, in fact, none of them is realizable by a pair (u, v) with $L \leq \min(|u|, |v|) \leq 2L$.

A matrix associated to τ

Let us consider the matrix associated to τ : $\begin{cases} 0 & \mapsto & 01; \\ 1 & \mapsto & 02; \\ 2 & \mapsto & 0. \end{cases}$

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We define its extended version M_τ, such that, for all $u \in \mathrm{Fac}_{\mathcal{T}}$, we have $M_\tau \Phi(u) = \Phi(\tau(u))$.

The extended version

We have

$$M_\tau = \left(\begin{array}{ccc|ccc|ccc|ccc}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
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0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
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\end{array}\right)$$

For example, since $\tau(0) = 01$ and 1 is present only in $\tau(0)$ while 0 occurs once in every $\tau(a)$,

$$\binom{\tau(u)}{01} = \binom{u}{0} + \binom{u}{00} + \binom{u}{10} + \binom{u}{20}.$$

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About its eigenvalues

The Perron-Frobenius eigenvalue of M'_τ is $\theta \approx 1.839$. The matrix M_τ has

- the eigenvalue θ once;
- the eigenvalue θ^2 once;
- two pairs of complex conjugate eigenvalues of modulus in $]1;\theta[$;
- a real eigenvalue of modulus less than 1, of geometric multiplicity 2;
- two pairs of complex conjugate eigenvalues of modulus less than 1.

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- the eigenvalue θ once;
- the eigenvalue θ^2 once;
- two pairs of complex conjugate eigenvalues of modulus in $]1;\theta[$;
- a real eigenvalue of modulus less than 1, of geometric multiplicity 2;
- two pairs of complex conjugate eigenvalues of modulus less than 1.

The bounds we will give on possibly realizable templates will concern projections of templates on the left eigenvectors associated to eigenvalues of modulus less than θ.

First restrictions

Theorem

Let λ be an eigenvalue of modulus less than 1. Let $\mathbf{r}$ be an associated eigenvector. If the template $t = [\mathbf{d}, \mathbf{D_b}, \mathbf{D_e}, a_1, a_2]$ is realizable, then

$$\min_{\boldsymbol{\delta} \in \Delta} |\mathbf{r} \cdot (\mathbf{d} + P_3 (\mathbf{D_b} \otimes \boldsymbol{\delta} + \boldsymbol{\delta} \otimes \mathbf{D_e}))| \leq 2C(\mathbf{r}),$$

where $C(\mathbf{r})$ is an easily computable constant such that, for all factors $w \in \mathrm{Fac}_{\mathcal{T}}$, we have

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$$\Delta = \left\{ \begin{pmatrix} \delta_0 \\ \delta_1 \\ \delta_2 \end{pmatrix} \in [-1.5; 1.5]^3 \, : \, \delta_0 + \delta_1 + \delta_2 = 0 \right\}.$$

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$$|\mathbf{r} \cdot P_3\left(\mathbf{D_b} \otimes \boldsymbol{\alpha} + \boldsymbol{\alpha} \otimes \mathbf{D_e}\right)| \leq \frac{2L - \sum_{i=1}^{3} \mathbf{d}_i}{L} C(\mathbf{r}) + \max_{\boldsymbol{\delta} \in \Delta} \frac{|\mathbf{r} \cdot (\mathbf{d} + P_3\left(\mathbf{D_b} \otimes \boldsymbol{\delta} + \boldsymbol{\delta} \otimes \mathbf{D_e}\right))|}{L},$$

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Final algorithm

Recall that we want to compute the ancestors of T and check that none of them is realized by a pair (u, v) with

$$L \leq \min(|u|, |v|) \leq 2L. \qquad (\star)$$

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- If the program stops, `seen` contains all the possibly realizable ancestors of T, which is a finite set
- Check that none of them is realized by a pair (u, v) of factors of $\mathcal{T}$ satisfying $(\star)$

Our implementation

In our implementation, we took $L = 15$.

Computing all the possibly realizable ancestors gave us a list of 241544 elements, in more or less eight hours.

The program then checks in less than three hours that none of them is realized.

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Thus, no template from $T = \{[\mathbf{0}, \mathbf{0}, \mathbf{0}, a, b] \ : \ a \neq b\}$ is realizable.

That implies that $p_{\mathcal{T}}(n) = \mathbf{b}_{\mathcal{T}}^{(2)}(n)$ for all $n \in \mathbb{N}$.

Last remark

Remark (G. Richomme, K. Saari, L. Zamboni, 2010)

The necessary conditions we found are related to the 2-balancedness property of $\mathcal{T}$, and more precisely, to the fact that, for all $w \in \mathrm{Fac}_{\mathcal{T}}$ and for all $a \in \{0, 1, 2\}$,

$$||w|_a - \alpha_a |w|| < 1.5,$$

where $\alpha_a = \lim_{n \to +\infty} \frac{|w_0 \cdots w_{n-1}|_a}{n}$ is the density of a in $\mathcal{T}$.

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To end with an open question...

Is it true that for every Arnoux-Rauzy word $\mathbf{w}$, we have

$$\mathbf{b}_{\mathbf{w}}^{(k)}(n) = p_{\mathbf{w}}(n)$$

for all $n \in \mathbb{N}$ and for all $k \geq 2$?

Thank you!