Computing the *k*-binomial complexity of the Tribonacci word





June 20, 2019 Marie Lejeune _(FNRS grantee) Joint work with Michel Rigo and Matthieu Rosenfeld

Plan

Preliminary definitions

- Words, factors and subwords
- Complexity functions
- k-binomial complexity

State of the art

3 Next result: the Tribonacci word

- Definition
- The theorem
- Introduction to templates and their parents
- Using templates to compute $\mathbf{b}_{\mathcal{T}}^{(2)}$

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Definition

Let $u = u_1 u_2 \cdots u_m$ be a finite or infinite word. A (scattered) subword of u is a finite subsequence of the sequence $(u_j)_{j=1}^m$. A factor of u is a contiguous subword.

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The factor complexity of the word w is the function

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We can replace $\sim_{=}$ with other equivalence relations.

Different equivalence relations from $\sim_{=}$ can be considered:

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We will deal with the last one.

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Definition (Reminder)

Let u and v be two finite words. They are k-binomially equivalent if

$$\binom{u}{x} = \binom{v}{x} \ \forall x \in A^{\leq k}.$$

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Some properties

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For all words u, v and for every nonnegative integer k,

 $u \sim_{k+1} v \Rightarrow u \sim_k v.$

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The words u and v are 1-abelian equivalent if

$$\begin{pmatrix} u \\ a \end{pmatrix} = |u|_a = |v|_a = \begin{pmatrix} v \\ a \end{pmatrix} \ \forall a \in A.$$

k-binomial complexity

Definition

If \mathbf{w} is an infinite word, we can define the function

$$\mathbf{b}_{\mathbf{w}}^{(k)}:\mathbb{N}\to\mathbb{N}:n\mapsto\#(\mathsf{Fac}_{\mathbf{w}}(n)/\sim_k),$$

which is called the k-binomial complexity of w.

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We have an order relation between the different complexity functions.

Proposition

$$ho_{\mathbf{w}}^{ab}(n) \leq \mathbf{b}_{\mathbf{w}}^{(k)}(n) \leq \mathbf{b}_{\mathbf{w}}^{(k+1)}(n) \leq
ho_{\mathbf{w}}(n) \quad orall n \in \mathbb{N}, k \in \mathbb{N}^+$$

where $\rho_{\mathbf{w}}^{ab}$ is the abelian complexity function of the word \mathbf{w} .

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The classical Thue-Morse word, defined as the fixed point of the morphism

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has a bounded k-binomial complexity. The exact value is known.

Theorem (M. L., J. Leroy, M. Rigo, 2018)

Let k be a positive integer. For every $n \leq 2^k - 1$, we have

$$\mathbf{b}_{\mathbf{t}}^{(k)}(n) = p_{\mathbf{t}}(n),$$

while for every $n \ge 2^k$,

$$\mathbf{b}_{\mathbf{t}}^{(k)}(n) = \begin{cases} 3 \cdot 2^k - 3, & \text{if } n \equiv 0 \pmod{2^k}; \\ 3 \cdot 2^k - 4, & \text{otherwise.} \end{cases}$$

Marie Lejeune (Liège University)

Another family

Definition: Sturmian words

A Sturmian word is an infinite word having, as factor complexity, p(n) = n + 1 for all $n \in \mathbb{N}$.

Another family

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Theorem (M. Rigo, P. Salimov, 2015)

Let w be a Sturmian word. We have

$$\mathbf{b}_{\mathbf{w}}^{(k)}(n) = p_{\mathbf{w}}(n) = n+1,$$

for all $n \in \mathbb{N}$ and for all $k \geq 2$.

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Theorem (M. Rigo, P. Salimov, 2015)

Let \mathbf{w} be a Sturmian word. We have

$$\mathbf{b}_{\mathbf{w}}^{(k)}(n) = p_{\mathbf{w}}(n) = n+1,$$

for all $n \in \mathbb{N}$ and for all $k \geq 2$.

Since $\mathbf{b}_{\mathbf{w}}^{(k)}(n) \leq \mathbf{b}_{\mathbf{w}}^{(k+1)}(n) \leq p_w(n)$, it suffices to show that

$$\mathbf{b}_{\mathbf{w}}^{(2)}(n)=p_{\mathbf{w}}(n).$$

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The Tribonacci word

From now, let $A = \{0, 1, 2\}$. Let us define the Tribonacci word.

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The Tribonacci word ${\mathcal T}$ is the fixed point of the morphism

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$$au: \{0,1,2\}^* o \{0,1,2\}^*: \left\{ egin{array}{ccc} 0 & \mapsto & 01; \ 1 & \mapsto & 02; \ 2 & \mapsto & 0. \end{array}
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Plan

Preliminary definitions

- Words, factors and subwords
- Complexity functions
- k-binomial complexity

State of the art



- The theorem
- Introduction to templates and their parents
- Using templates to compute $\mathbf{b}_{T}^{(2)}$

The next result was first conjectured by Michel Rigo, and then proved.

Theorem (M. L., M. Rigo, M. Rosenfeld, 2019)

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Theorem (M. L., M. Rigo, M. Rosenfeld, 2019)

For all $k \ge 2$, the k-binomial complexity of the Tribonacci word equals its factor complexity.

To show this result, it suffices to show that, for all $n \in \mathbb{N}$,

$$\left\{\begin{array}{ll} u, v \in \mathsf{Fac}_{\mathcal{T}}(n) \\ u \neq v \end{array}\right. \Rightarrow \quad u \not\sim_2 v.$$

The Parikh vector of a word u is classically defined as

$$\Psi(u) := egin{pmatrix} u \ 0 \end{pmatrix} egin{pmatrix} u \ 1 \end{pmatrix} egin{pmatrix} u \ 2 \end{pmatrix}^{\mathsf{T}} \in \mathbb{N}^3 \,.$$

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Let us define the extended Parikh vector of a word u as

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Remark

We have
$$u \sim_2 v \Leftrightarrow \Phi(u) = \Phi(v) \Leftrightarrow \Phi(u) - \Phi(v) = 0$$
.

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State of the art

Next result: the Tribonacci word

- Definition
- The theorem
- Introduction to templates and their parents
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Informally, we will associate to every pair of words several templates, which are 5-uples:

$$A^* \times A^* \iff \mathbb{Z}^{12} \times \mathbb{Z}^3 \times \mathbb{Z}^3 \times A \times A.$$

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There exists a strong link between this notion and our thesis:

$$egin{aligned} & egin{aligned} & egi$$

Definition

A template is a 5-uple of the form $t = [\mathbf{d}, \mathbf{D}_{\mathbf{b}}, \mathbf{D}_{\mathbf{e}}, a_1, a_2]$ where $\mathbf{d} \in \mathbb{Z}^{12}$, $\mathbf{D}_{\mathbf{b}}, \mathbf{D}_{\mathbf{e}} \in \mathbb{Z}^3$ and $a_1, a_2 \in A$.

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where the matrix P_3 is such that, for all $\mathbf{x} \in \mathbb{Z}^9$, $P_3 \cdot \mathbf{x} = \begin{pmatrix} 0 & 0 & \mathbf{x} \end{pmatrix}^{\mathsf{T}}$, and where \otimes is the usual Kronecker product: if $A \in \mathbb{Z}^{m \times n}$ and $B \in \mathbb{Z}^{p \times q}$

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There exists $n \in \mathbb{N}$ such that

$$\mathbf{b}_{\mathcal{T}}^{(2)}(n) < p_{\mathcal{T}}(n)$$

if and only if at least one template of the form [0, 0, 0, a, b] with $a \neq b$ is realizable by a pair of factors of \mathcal{T} .

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Let us suppose that there exists a pair (u, v) realizing [0, 0, 0, a, b] and let fix L > 0. Then,

- either min $(|u|, |v|) \leq L$, or
- there exists an ancestor template of [0, 0, 0, a, b] which is realized by a pair of words $(u', v') = "((\tau^{-1})^j(u), (\tau^{-1})^j(v))"$ such that $L \le \min(|u'|, |v'|) \le 2L$.

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Preimages

Example: intuitive definition

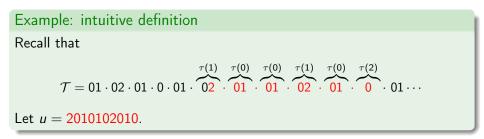
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Let u = 2010102010.

Preimages



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Let u = 2010102010. The word u' = 100102 is a *preimage* of u.

Example: intuitive definition

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Let u = 2010102010. The word u' = 100102 is a *preimage* of u.

Definition

Let u and u' be two words. The word u' is a **preimage** of u if

- u is a factor of $\tau(u')$, and
- u' is minimal: for all factors v of u', u is not a factor of $\tau(v)$.

A word can have several preimages.

Example

Recall that

Take u = 010.

A word can have several preimages.

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Take u = 010. It has 00, 01 and 02 as preimages.

A word can have several preimages.

Example Recall that $\mathcal{T} = 01 \cdot 02 \cdot 01 \cdot 0 \cdot 01 \cdot 02 \cdot \underbrace{01}^{\tau(0)} \cdot \underbrace{01}^{\tau(0)} \cdot 02 \cdot 01 \cdot 0 \cdot 01 \cdots$ Take u = 010. It has 00, 01 and 02 as preimages. A word can have several preimages.

Example

Recall that

$$\mathcal{T} = \overbrace{01}^{\tau(0)} \cdot \overbrace{02}^{\tau(1)} \cdot 01 \cdot 0 \cdot 01 \cdot 02 \cdot 01 \cdot 01 \cdot 02 \cdot 01 \cdot 0 \cdot 01 \cdots$$

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We will now introduce the notion of parents of a template.

Theorem

Let t be a template and let (u, v) be a pair of factors realizing t. Let u' (resp., v') be a preimage of u (resp., v). There always exists a template t' which is realized by (u', v'). and which is, in some way, related to t.

(u, v)

 \longleftrightarrow

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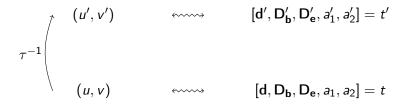
 \longrightarrow

$$[\mathsf{d},\mathsf{D}_{\mathsf{b}},\mathsf{D}_{\mathsf{e}},a_1,a_2]=t$$

We will now introduce the notion of parents of a template.

Theorem

Let t be a template and let (u, v) be a pair of factors realizing t. Let u' (resp., v') be a preimage of u (resp., v). There always exists a template t' which is realized by (u', v') and which is, *in some way*, related to t.

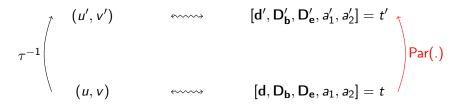


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The template t' is called a parent template of t.



Remark

• Since a word can sometimes have several preimages, a template can also have several parents.

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- There exists a formula allowing to compute all parents of a given template.

Definition

Let t and t' be templates. We say that t' is an (realizable) ancestor of t if there exists a finite sequence of templates t_0, \ldots, t_n such that

$$\begin{cases} t_0 = t', \\ t_n = t, \\ \forall i \in \{0, \dots, n-1\}, t_i \text{ is a (realizable) parent of } t_{i+1}. \end{cases}$$

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Plan

Preliminary definitions

- Words, factors and subwords
- Complexity functions
- k-binomial complexity

State of the art

Next result: the Tribonacci word

- Definition
- The theorem
- Introduction to templates and their parents
- Using templates to compute $\mathbf{b}_{\mathcal{T}}^{(2)}$

To show that $\mathbf{b}_{\mathcal{T}}^{(2)}=p_{\mathcal{T}}$, we have to show that no template from

$$T := \{ [0, 0, 0, a, b] : a \neq b \}$$

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Problem: there exists an infinite number of ancestors.

Instead of computing all the ancestors of \mathcal{T} , we will focus on the **possibly** realizable ones.

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That will leave us with a finite number of candidates.

It is then possible to verify with a computer that, in fact, none of them is realizable by a pair (u, v) with $L \leq \min(|u|, |v|) \leq 2L$.

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We define its extended version M_{τ} , such that, for all $u \in Fac_{\mathcal{T}}$, we have $M_{\tau}\Phi(u) = \Phi(\tau(u))$.

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- the eigenvalue θ once;
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- two pairs of complex conjugate eigenvalues of modulus less than 1. The bounds we will give on possibly realizable templates will concern projections of templates on the left eigenvectors associated to eigenvalues of modulus less than θ .

Let λ be an eigenvalue of modulus less than 1. Let **r** be an associated eigenvector. If the template $t = [\mathbf{d}, \mathbf{D}_{\mathbf{b}}, \mathbf{D}_{\mathbf{e}}, a_1, a_2]$ is realizable, then

$$\min_{\boldsymbol{\delta} \in \Delta} |\mathbf{r} \cdot (\mathbf{d} + P_3 (\mathbf{D}_{\mathbf{b}} \otimes \boldsymbol{\delta} + \boldsymbol{\delta} \otimes \mathbf{D}_{\mathbf{e}}))| \leq 2C(\mathbf{r}),$$

where $C(\mathbf{r})$ is an easily computable constant such that, for all factors $w \in Fac_{\mathcal{T}}$, we have

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Recall that we want to compute the ancestors of T and check that none of them is realized by a pair (u, v) with

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- Check that none of them is realized by a pair (u, v) of factors of \mathcal{T} satisfying (\star)

In our implementation, we took L = 15.

Computing all the possibly realizable ancestors gave us a list of 241544 elements, in more or less eight hours.

The program then checks in less than three hours that none of them is realized.

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Thus, no template from $T = \{[0, 0, 0, a, b] : a \neq b\}$ is realizable.

That implies that $p_{\mathcal{T}}(n) = \mathbf{b}_{\mathcal{T}}^{(2)}(n)$ for all $n \in \mathbb{N}$.

Last remark

Remark (G. Richomme, K. Saari, L. Zamboni, 2010)

The necessary conditions we found are related to the 2-balancedness property of \mathcal{T} , and more precisely, to the fact that, for all $w \in Fac_{\mathcal{T}}$ and for all $a \in \{0, 1, 2\}$,

$$|w|_{a} - \alpha_{a}|w|| < 1.5,$$

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To end with an open question...

Is it true that for every Arnoux-Rauzy word \mathbf{w} , we have

$$\mathbf{b}_{\mathbf{w}}^{(k)}(n) = p_{\mathbf{w}}(n)$$

for all $n \in \mathbb{N}$ and for all $k \geq 2$?

Thank you!