$\mathcal{H}_\infty$ optimization of positive position feedback control for mitigation of nonlinear vibrations

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Abstract
This paper investigates the potential of using positive position feedback control for the mitigation of nonlinear vibrations. Closed form expressions for the $\mathcal{H}_\infty$ optimal control parameters that minimize the maximal response of the structure are firstly derived for the linear case and then extended to the nonlinear case. The resonant vibration of a Duffing oscillator is considered to illustrate the proposed tuning of the optimal control parameters and the harmonic balance method is employed to approximate the analytical solutions. Numerical simulations are performed to demonstrate and validate the proposed method.

1 Introduction

Controlling nonlinear structural vibrations is becoming increasingly important in a number range of engineering applications such as aerospace, medicine and robotics, wherein lightweight materials are considered in the construction of systems in order to meet the increasing demand for fuel efficiency or smaller actuators [1], [2]. However, this will naturally lead to the fact that the resonances are lightly damped and to the presence of some geometric nonlinearities from large deformations. The resulting unwanted nonlinear vibrations thus become the main concern, limiting the success of these applications. One key characteristic of nonlinear vibrations is their frequency-energy dependence which in other words means that the frequency of the nonlinear oscillations depends intrinsically on the motion amplitudes [3]. As a consequence, the mature linear damping-enhanced approaches based on the superposition principle such as tuned mass dampers and piezoelectric shunting [4]–[6] (passive solutions) or direct velocity feedback, integral acceleration, positive position feedback (PPF) and force feedback controllers (active solutions) [7] are no longer effective in the presence of strong nonlinearities. In order to recover their control effectiveness for a large range of excitation levels, mechanisms that can deliver nonlinear reacting forces should be included in these linear approaches. For example, Agnes [8] suggested to integrate a positive or negative cubic spring into a linear vibration absorber for compensating the softening (hardening) nonlinear effect of the primary systems. Febbo and Machado [9] explored the potential of using a nonlinear absorber with the saturation nonlinearity for vibration mitigation of nonlinear primary oscillators. Habib et al. [3] reported that the mathematical form of the nonlinear vibration absorber should be the same as that of the primary nonlinear system such that the nonlinear vibration absorber behaves as a linear vibration absorber.

In this study, a nonlinear positive position feedback (NPPF) controller as an extension of the classical PPF controller is investigated for the mitigation of Duffing oscillations. According to the similarity principle proposed by [10], the NPPF controller introduces a cubic term on top of the original PPF controller. The proposed nonlinear PPF controller is implemented by feeding the structural position directly to the nonlinear compensator, whose output is fed through a fixed gain positively back to the structure, thus achieving damping for a particular mode. In this sense, NPPF as well as PPF controllers would be well suited for the applications where piezoelectric sensors and actuators are employed for vibration damping. This is because the voltage from the sensor is proportional to the strain of the attached structure, which can be directly measured to drive the strain-based piezoelectric actuators based on the NPPF or PPF controllers.

Although some numerical and experimental implementations of the proposed NPPF control scheme may be found in the literature, limited investigation of the optimization the NPPF controller for the Duffing vibrations exists. In this paper, an $\mathcal{H}_\infty$ optimization criterion is used to optimize the NPPF controller, and accordingly the optimal parameters are set to minimize the maximum steady state response of the structure. The objective of the study is to understand its working principle and validate the control efficiency through a harmonic balance analysis.

2 Mathematical model and $\mathcal{H}_\infty$ optimization

2.1 Modeling

The system under investigation is shown in Figure 1. The structure is modeled as an undamped single degree of freedom (dof) system, defined through a lumped mass m_1, a linear spring k_1 and a cubic spring k_3. It is excited by a harmonic force $F = F_d \cos(\omega t)$. A force actuator with its stiffness k_2 is placed in parallel to the passive mount. The control loop is implemented by feeding the displacement of the lumped mass m_1 through a nonlinear controller $h(x)$ to drive the actuator.

The governing equations of the system read:

$$m_1\ddot{x} + k_1 x + k_3 x^3 = F_d \cos(\omega t) + F_a - k_2 x \tag{1}$$

$$F_a = g_1 h(x) \tag{2}$$

where F_a is the actuating force proportional to the driving signal, g_1 represents the feedback gain and $h(x)$ is the NPPF controller.

The NPPF controller is designed based on the principle of similarity, and consequently a cubic term is included in the linear PPF controller, yielding:

$$\ddot{u}_a + 2\alpha\omega_f \dot{u}_a + \omega_f{}^2 u_a + \kappa u_a{}^3 = x \tag{3}$$

where $u_a = h(x)$, α, ω_f and κ are controller parameters.

In order to come to a more general formulation, the following parameters are introduced to normalize the system governing equations:

$$\begin{aligned} &\tau = \omega_1 t, \quad \mu = \frac{\omega_f}{\omega_1}, \quad k_t = k_1 + k_2 \quad y_1 = \frac{k_t x}{F_d}, \quad y_2 = \frac{k_t \omega_1{}^2 u_a}{F_d}, \\ &\omega_1 = \sqrt{\frac{k_t}{m}}, \quad \delta = \frac{k_3 F_d{}^2}{k_t{}^3}, \quad g = \frac{g_1}{k_t \omega_1{}^2}, \quad \beta = \frac{\kappa k_t}{k_3 \omega_1^6} \end{aligned} \tag{4}$$

The equations of motion with normalized parameters can then be written as:

$$y_1'' + y_1 + \delta y_1{}^3 - g y_2 = \cos(\Omega \tau) \tag{5}$$

$$y_2'' + 2\alpha\mu y_2' + \mu^2 y_2 + \delta\beta y_2{}^3 - y_1 = 0 \tag{6}$$

where Ω is the normalized frequency defined as $\Omega = \omega / \omega_1$.

It is shown that the forcing amplitude appears only in the expression of the nonlinear coefficients. The $\mathscr{H}_\infty$ optimization criterion is employed to optimize the controller $h(x)$ aiming to minimize the maximum magnitude of the frequency response of the system under consideration. In this context, the magnitude of the normalized driving point receptance $|y_1|$ is taken as the performance index.

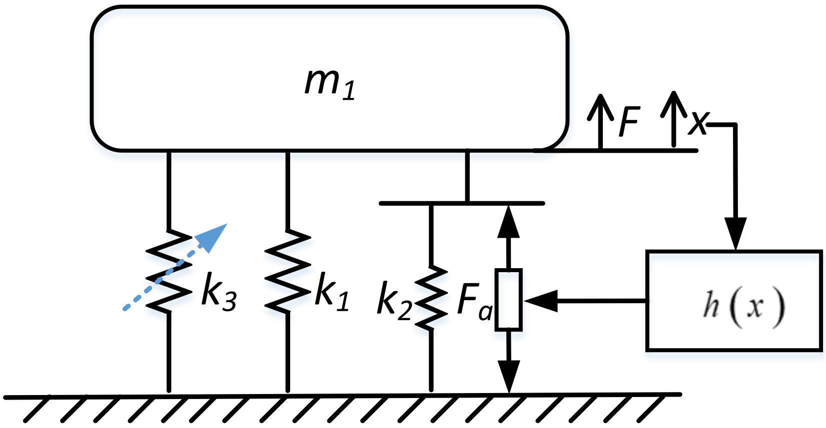


Figure 1 The scheme of the system under consideration

2.2 $\mathcal{H}_\infty$ optimization of linear PPF controller

The system with a linear PPF controller is considered first. This is done by setting the parameter δ in Eq. (5) equal to zero. The normalized driving point receptance of the primary structure is then given by:

$$y_1 = \frac{s^2 + 2\alpha\mu s + \mu^2}{s^4 + 2\alpha\mu s^3 + \mu^2 s^2 + s^2 + 2\alpha\mu s + \mu^2 - g} \tag{7}$$

where $s = j\Omega$ is the Laplace variable and the magnitude of y_1 is calculated as:

$$|y_1| = \frac{\sqrt{\Omega^4 + \left(4\alpha^2 - 2\right)\mu^2\Omega^2 + \mu^4}}{\sqrt{\left(\Omega^2 - 1\right)^2 \mu^4 + 2\left(\left(2\alpha^2 - 1\right)\Omega^4 + \left(-2\alpha^2 + 1\right)\Omega^2 + g\right)\left(\Omega^2 - 1\right)\mu^2 + \left(-\Omega^4 + \Omega^2 + g\right)^2}} \tag{8}$$

From the mathematic point of view, the control effectiveness of a PPF controller according to Eq. (7) would be similar to that of a tuned Mass damper, where an additional zero is introduced to interfere with the resonance of the primary system aiming to reduce certain vibration metrics in the frequency band of interest. Following the $\mathscr{H}_\infty$ optimization procedure proposed by Den Hartog [4], the parameters of the linear PPF controller are optimally tuned such that the responses at the fixed points are minimized. Fixed point refers to the frequency location at which the magnitude of the driving point receptance of the primary structure is invariant in terms of the damping coefficient of the TMD or the parameter α of the PPF controller.

The frequencies at which the fixed points occur can be calculated by differentiating Eq. (8) with respect to the damping coefficient, α, and equating the derivative to zero, which yields:

$$\Omega_{f1} = \frac{\sqrt{2\mu^2 + 2 - 2\sqrt{\mu^4 - 2\mu^2 + 2g + 1}}}{2} \tag{9}$$

$$\Omega_{f2} = \frac{\sqrt{2\mu^2 + 2 + 2\sqrt{\mu^4 - 2\mu^2 + 2g + 1}}}{2} \tag{10}$$

The optimal μ is set to equalize the resulting performance index as defined in Eq. (8) at the two fixed points. This can be done by substituting Eqs. (9) and (10) into Eq. (8) and equating the resulting expressions for $\alpha = 0$, yields,

$$\mu_{opt} = 1 \tag{11}$$

For the optimal α, it is sought to make the performance index horizontally pass through the fixed points. Thus, two optimal damping coefficients associated with the two fixed points are obtained:

$$\alpha_1 = \sqrt{\frac{3g}{4\sqrt{2}\left(\sqrt{2}-\sqrt{g}\right)}} \tag{12}$$

$$\alpha_2 = \sqrt{\frac{3g}{4\sqrt{2}\left(\sqrt{2}+\sqrt{g}\right)}} \tag{13}$$

The optimal α can be taken in practice by calculating the average of Eqs. (12) and (13). It should be noted that this approach is an empirical method as the resulting resonance points (the derivative of (8) with respect to Ω is equal to zero) do not necessarily coincide simultaneously with the corresponding fixed points. An exact solution for this problem was proposed in [11], with which the two resulting resonance points are equally damped. In this study this exact approach is not considered because this would result in very long and therefore rather impractical polynomial expressions.

Up to now, only the parameter g (normalized feedback gain) is left un-optimized for implementing the linear PPF controller. The function of the feedback gain g can be assessed by evaluating the magnitude of driving point receptance at the fixed points. This is done by substituting Eqs. (9) and (11) into Eq. (8) for $\alpha = 0$, yields the minimal maximum response

$$y_{1mm} = \sqrt{\frac{2}{g}} \tag{14}$$

As shown in Eq. (14), the minimal maximum response is inversely proportional to the gain g, indicating that the value of the feedback gain g should be as high as possible without compromising the stability of the active system.

The stability of an active system can be studied by applying the Routh-Hurwitz stability criterion to its closed loop characteristic equation [12]. The characteristic equation of the system can be formed as:

$$A_4 s^4 + A_3 s^3 + A_2 s^2 + A_1 s + A_0 = 0 \tag{15}$$

where A_0, A_1, A_2, A_3 and A_4 are the corresponding coefficients of Laplace variable in the denominator of Eq. (7).

The Routh-Hurwitz stability criterion states that the roots of the characteristic equation have negative real parts if and only if the following conditions are satisfied:

$$A_0, A_1, A_2, A_3, A_4 > 0 \tag{16}$$

$$A_2 A_3 - A_1 A_4 > 0 \tag{17}$$

$$A_1 A_2 A_3 - A_1^2 A_4 - A_0 A_3^2 > 0 \tag{18}$$

It can be derived that the system is stable if and only if the gain g is defined:

$$g < \mu^2 \tag{19}$$

In the following, numerical studies are performed to illustrate the control effectiveness of the linear PPF controller for the system under consideration. Figure 2 shows the performance index $|y_1|$ plotted against frequency for five different damping ratios defined as α/α_{opt} : 0, 1/4, 1, 4 and ∞, where the control parameters μ is set to its optimal value as given in Eq. (11) and the gain is set to 0.2. It can be seen that all the curves with different damping values intersect at two frequencies and only with the optimal damping the response at the two fixed frequencies become the maximum. One should also note that the system becomes dynamically softer with the application of PPF control as the control signal is positively proportional to the displacement of the system in the low frequency range whereas the PPF control effectiveness is similar to that of a negative spring. However, when the damping value approaches infinity, the stiffness softening effect disappears as the control action is lost. Figure 3 depicts the performance index $|y_1|$ plotted against frequency for four different feedback gains namely g : 0, 0.01, 0.05, and 0.5, where the control parameters μ and α are both set to their optimal values. As can be seen, the performance index indeed decreases with an increase in the gain as indicated by Eq. (14). With this respect, the feedback gain g of the PPF controller can be understood to play the same role as the mass ratio between tuned mass dampers and host primary structures, where better performance comes with a large value of this quantity. However, the approximation errors induced by estimation of the damping parameter α is more pronounced with an increase in the feedback gain. In the same fashion, the response in the low frequency range will be more amplified because of the negative stiffness effect. Therefore, the maximum feedback gain g under $\mathscr{H}_\infty$ optimization criterion is not only limited by the stability concern, but also by the amplification of the low frequency response.

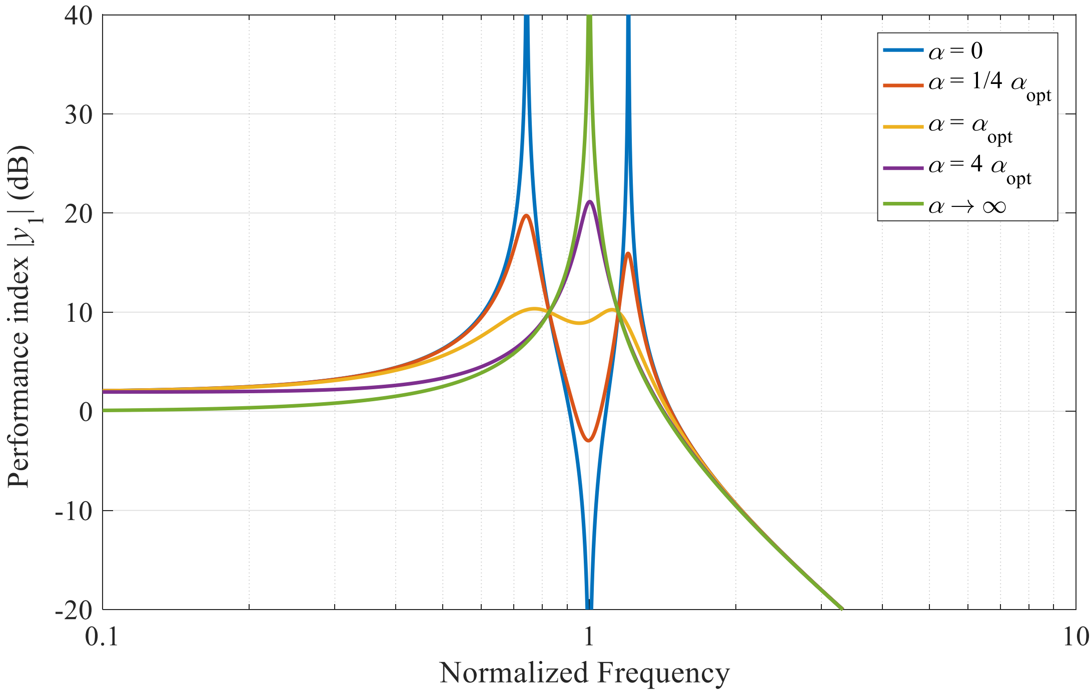


Figure 2 The driving point receptance under different active damping ratios

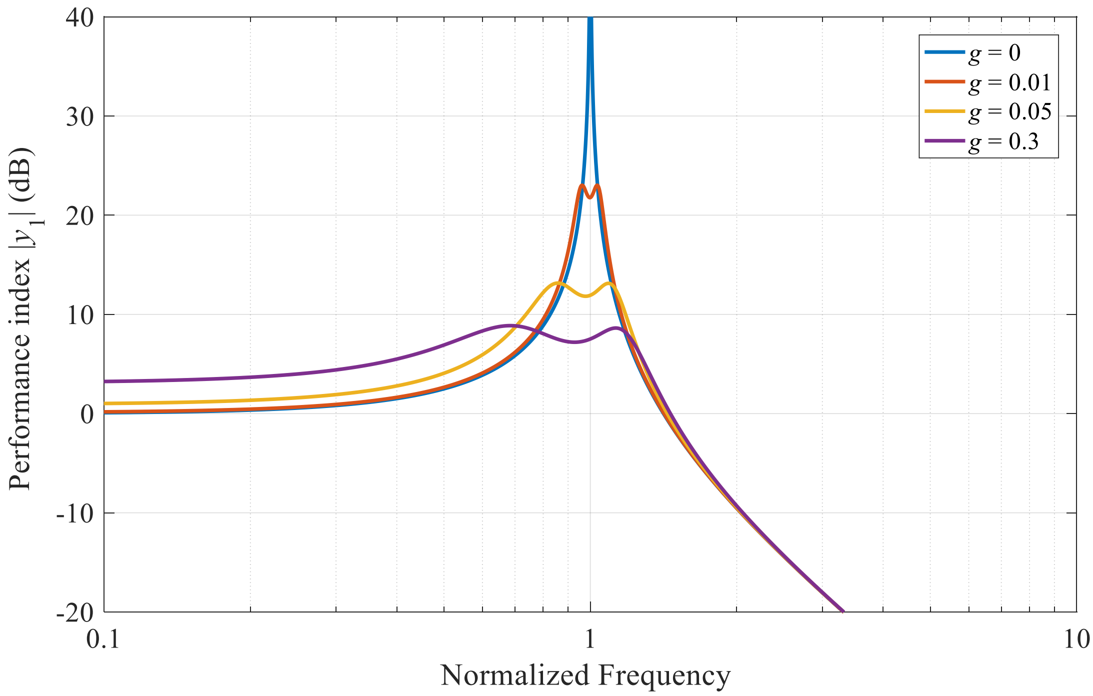


Figure 3 The driving point receptance under different feedback gains

2.3 $\mathcal{H}_\infty$ optimization of nonlinear PPF controller

In this subsection, $\mathcal{H}_\infty$ optimization of the nonlinear PPF controller is performed. Due to the cubic terms, no explicit solutions can be found for Eqs. (5) and (6). As reported in [13]–[15], harmonic solutions can be used to approximate the exact solutions with a good agreement. In the following, a pair of one-term harmonic balance approximation $y_1 = A_1\cos(\Omega\tau) + B_1\sin(\Omega\tau)$ and $y_2 = A_2\cos(\Omega\tau) + B_2\sin(\Omega\tau)$ is assumed as the solutions. Substituting the above ansatz into Eqs. (5) and (6), and applying the approximations $\cos^3(\Omega\tau) \approx 3/4\cos(\Omega\tau)$ and $\sin^3(\Omega\tau) \approx 3/4\sin(\Omega\tau)$, then balancing cosine and sine terms, yields the system of polynomial equations:

$$-A_1\Omega^2 + A_1 + g\Omega^2 A_2 + 3/4\delta A_1\left(A_1^{\,2} + B_1^{\,2}\right) = 1 \tag{20}$$

$$-B_1\Omega^2 + B_1 + g\Omega^2 B_2 + 3/4\delta B_1\left(A_1^{\,2} + B_1^{\,2}\right) = 0 \tag{21}$$

$$-A_2\Omega^2 + 2\alpha\mu\Omega B_2 - A_1 + \mu^2 A_2 + 3/4\delta\beta A_2\left(A_2^{\,2} + B_2^{\,2}\right) = 0 \tag{22}$$

$$-B_2\Omega^2 - 2\alpha\mu\Omega A_2 - B_1 + \mu^2 B_2 + 3/4\delta\beta B_2\left(A_2^{\,2} + B_2^{\,2}\right) = 0 \tag{23}$$

Up to now, it is still not possible to find explicit solutions of Eqs. (20)-(23), we thus expand the harmonic coefficients A_i and B_i into series with respect to the primary nonlinear coefficient δ, i.e. $A_1 = A_{11} + \delta A_{12}$, $B_1 = B_{11} + \delta B_{12}$, $A_2 = A_{21} + \delta A_{22}$ and $B_2 = B_{21} + \delta B_{22}$.

Substituting the above ansatz into Eqs. (20)-(23), and collecting the resulting expressions with respect to the order of the parameter δ, then after omitting the expressions whose orders are higher than δ^1, yields:

$$(gA_{21} - A_{11})\Omega^2 + A_{11} - 1 = 0 \tag{24}$$

$$1/4(4gA_{22} - 4A_{12})\Omega^2 + 3/4A_{11}^{\,3} + 3/4A_{11}B_{11}^{\,2} + A_{12} = 0 \tag{25}$$

$$\left(gB_{21}-B_{11}\right)\Omega^{2}+B_{11}=0 \tag{26}$$

$$1/4\left(4gB_{22}-4B_{12}\right)\Omega^{2}+3/4A_{11}^{2}B_{11}+3/4B_{11}^{3}+B_{12}=0 \tag{27}$$

$$1/4\left(3A_{21}^{3}+3A_{21}B_{21}^{2}\right)\beta+1/4\left(4\mu^{2}-4\Omega^{2}\right)A_{22}+2\alpha\mu\Omega B_{22}-A_{12}=0 \tag{28}$$

$$\left(\mu^{2}-\Omega^{2}\right)A_{21}+2\alpha\mu\Omega B_{21}-A_{11}=0 \tag{29}$$

$$\left(\mu^{2}-\Omega^{2}\right)B_{21}-2\alpha\mu\Omega A_{21}-B_{11}=0 \tag{30}$$

$$1/4\left(3A_{21}^{2}B_{21}+3B_{21}^{3}\right)\beta+1/4\left(4\mu^{2}-4\Omega^{2}\right)B_{22}-2\alpha\mu\Omega A_{22}-B_{12}=0 \tag{31}$$

Solving for A_{ij} and B_{ij} ($i=1,2$, $j=1,2$) from Eqs. (24)-(31), the resulting solutions are found to be in terms of the control gains α , μ, g and the normalized frequency Ω. Due to the complexity, these expressions are not given here. Nevertheless, the absolute value of the normalized frequency response $\left|y_1(\Omega)\right|$, namely the performance index, can be expressed as:

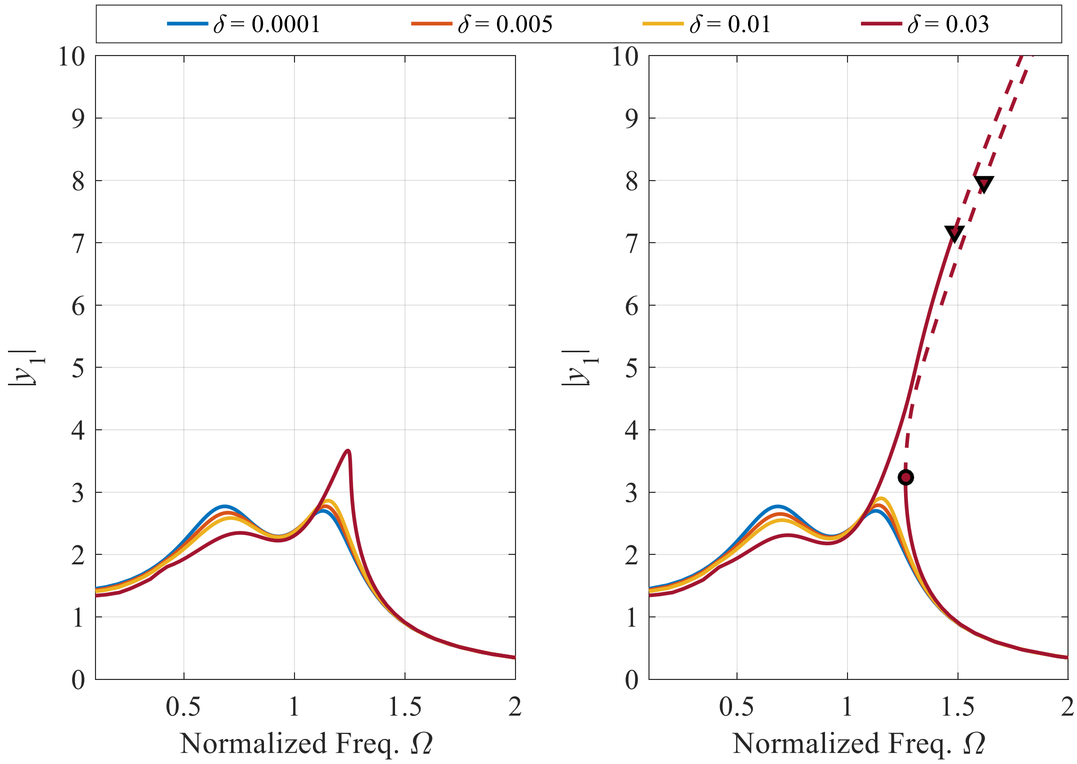
$$\left|Q(\Omega)\right|=\sqrt{A_{11}^{2}+B_{11}^{2}+2\delta\left(A_{11}A_{12}+B_{11}B_{12}\right)+\delta^{2}\left(A_{12}^{2}+B_{12}^{2}\right)} \tag{32}$$

Substituting Eqs. (9)-(13) into Eq. (32) and solving for β to maintain the equal peaks at the fixed points, yields

$$\beta=\frac{1}{4}g+\frac{2}{3}g^{2}+\mathrm{O}\left(g^{3}\right) \tag{33}$$

In fact, Eq. (33) represents a simpler and more easily interpretable relation which is the Taylor series expansion of the exact solution with respect to the feedback gain g given $g \ll 1$.

Up to now, the derivation of the explicit expressions for forming the NPPF controller $h(\cdot)$ is complete wherein the optimal control parameters are given in Eqs. (11), (12)-(13) and (33) respectively. Some numerical studies are performed to illustrate the control effectiveness of the NPPF controller for the system under consideration, where the feedback gain is set to 0.2 and the primary nonlinear coefficient δ is varied between 0.0001 and 0.03. Substituting the optimal control parameters into Eqs. (5) and (6), the resulting nonlinear equations are computed using a path-following algorithm combining shooting and pseudo-arclength continuation [12]. This algorithm provides a very accurate numerical solution to the equations of motion. Figure 4 (b) illustrates that the frequency response of the Duffing oscillator with the linear PPF controller is strongly detuned in the presence of strong nonlinearities i.e. $\delta=0.03$. Conversely, Figure 4 (a) shows that the nonlinear controller can compensate, to a certain extent, the detuned equal peaks for values of δ ranging from 0.0001 to 0.03. Although the control effectiveness degrades in terms of the differences between the two peaks when δ is large, Eq. (33) can be updated using approximated solutions with more harmonics.



(a) (b)

Figure 4 The performance of the system under consideration where the feedback gain is set to 0.2 and the primary nonlinear coefficient δ varies between 0.0001 and 0.03: (a) with NPPF controller and (b) with linear PPF controller (—:stable solution, --: unstable solution, •: fold bifurcation, ▼: Neimark-Sacker bifurcation)

3 Conclusion

A nonlinear active damping strategy based on positive position feedback is proposed. Closed-form expressions are derived using the $\mathscr{H}_\infty$ optimization criterion wherein the optimal control parameters are achieved to minimize the maximal response of the structure. The resonant vibration of a Duffing oscillator is considered to illustrate the proposed tuning of the optimal control gains and the harmonic balance method is employed to approximate the analytical solutions. With the proposed NPPF controller, it has been shown that the response around the resonance does not change substantially when the importance of the nonlinearity increases, which means that the response of the coupled system is almost proportional to the forcing amplitude, as it would be the case for a linear system.

Acknowledgements

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References

[1] F.-Y. Wang and Y. Gao, *Advanced Studies of Flexible Robotic Manipulators*. 2003.

[2] M. Darecki *et al.*, "Flightpath 2050," *Flightpath 2050 Eur. Vis. Aviat.*, p. 28, 2011.

[3] G. Habib, T. Detroux, R. Viguié, and G. Kerschen, "Nonlinear generalization of Den Hartog's equal-peak method," *Mech. Syst. Signal Process.*, vol. 52–53, no. 1, pp. 17–28, 2015.

[4] J. P. Den Hartog, *Mechanical Vibrations*. Dover Publications, 1985.

[5] G. Zhao, N. Alujević, B. Depraetere, and P. Sas, "Dynamic analysis and H_2 optimisation of a piezo-based tuned vibration absorber," *J. Intell. Mater. Syst. Struct.*, vol. 26, no. 15, pp. 1995–2010, 2015.

[6] N. W. Hagood and A. von Flotow, "Damping of structural vibrations with piezoelectric materials and passive electrical networks," *J. Sound Vib.*, vol. 146, no. 2, pp. 243–268, Apr. 1991.

[7] A. Preumont, *Vibration Control of Active Structures, an introduction*, vol. 179. Dordrecht: Springer Netherlands, 2011.

[8] G. S. Agnes and R. D. Kriz, "Performance of Nonlinear Mechanical , Resonant-Shunted Piezoelectric , and Electronic Vibration Absorbers for Multi-Degree-of-Freedom Structures by Electronic Vibration Absorbers for Multi-Degree-of-Freedom Structures," 1997.

[9] M. Febbo and S. P. Machado, "Nonlinear dynamic vibration absorbers with a saturation," vol. 332, pp. 1465–1483, 2013.

[10] G. Habib and G. Kerschen, "A principle of similarity for nonlinear vibration absorbers," *Phys. D*, vol. 332, pp. 1–8, 2016.

[11] T. Asami and O. Nishihara, "Closed-Form Exact Solution to H[sub ∞] Optimization of Dynamic Vibration Absorbers (Application to Different Transfer Functions and Damping Systems)," *J. Vib. Acoust.*, vol. 125, no. 3, p. 398, 2003.

[12] L. Meirovitch, *Dynamics and Control of Structures*. John Wiley & Sons, Inc, 1990.

[13] D. Wagg, S. Neild, C. W. S. To, and S. K. Lau, *Nonlinear Vibration with Control*, vol. 58, no. 4. 2010.

[14] T. Detroux, L. Renson, L. Masset, and G. Kerschen, "The harmonic balance method for bifurcation analysis of large-scale nonlinear mechanical systems," pp. 1–34, 2016.

[15] G. Kerschen, K. Worden, A. F. Vakakis, and J.-C. Golinval, "Past, present and future of nonlinear system identification in structural dynamics," *Mech. Syst. Signal Process.*, vol. 20, no. 3, pp. 505–592, 2006.