Simulation of the highly non linear properties of bulk superconductors: finite element approach with a backward Euler method and a single time step

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Breaking the gravity....

...with the magnetic levitation
Electrical resistivity of HTS is modelled by a non-linear power law

\[ E(J) = \rho(J)J = E_c \left( \frac{J}{J_c} \right)^n \]

- \( n \) is the critical exponent \((\text{usually large}!)\)
- \( J_c \) is the critical current density

**Asymptotic behavior**

- \( n \to \infty \): Bean model

Analytical calculations available for comparison in specific geometries
Finite-element softwares are widely used for simulating HTS-based systems

Advantages

• Many more geometries can be treated
• No extensive writing of numerical codes is required
• Treatment of non-linear problems available in most commercial packages
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Drawbacks

• **Long calculation time** on fine meshing or in 3D geometry
• **Convergence problems** when $n$ is large
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Proposed improvements

Single time step method implemented in an open-source solver, GetDP

• Better control of the algorithm parameters
• Used for simulating the penetration of an external magnetic field that varies linearly with time
Outline

• Finite-element formulation and implementation
  A- $\phi$ formulation
  Numerical resolution scheme

• Validation and comparison of the FEM model on simple geometries
The Maxwell equations are solved for two independent variables

- the **vector potential** $A$
- the **scalar potential** $\phi$

defined as

\[
B = B_{\text{react}} + B_a = \text{curl } A + \text{curl } A_a
\]
\[
E = -dA/dt - dA_a/dt - \text{grad } \phi
\]
The Maxwell equations are solved for two independent variables:

- the **vector potential** $\mathbf{A}$
- the **scalar potential** $\phi$

Defined as:

$$B = B_{\text{react}} + B_a = \text{curl } \mathbf{A} + \text{curl } \mathbf{A}_a$$

$$E = -\frac{d\mathbf{A}}{dt} - \frac{d\mathbf{A}_a}{dt} - \text{grad } \phi$$

Applied magnetic flux density (uniform)
The Maxwell equations are solved for two independent variables

- the **vector potential** \( A \)
- the **scalar potential** \( \phi \)

defined as

\[
B = B_{\text{react}} + B_a = \text{curl } A + \text{curl } A_a \\
E = - \frac{dA}{dt} - \frac{dA_a}{dt} - \text{grad } \phi
\]
A - $\phi$ formulation - Variables

The Maxwell equations are solved for two independent variables

- the **vector potential** $\mathbf{A}$
- the **scalar potential** $\phi$

approximated by

**Edge function (1st order)**

$$\mathbf{A} = \sum_i a_i \mathbf{A}_i$$

*where $\mathbf{A}_i$ and $\phi_j$ are known functions*

**Node function (1st order)**

$$\phi = \sum_j b_j \phi_j$$

*ensures continuity of the tangential component of $\mathbf{A}$*

*ensures continuity of $\phi$*
• The Maxwell equations are reduced to 2 equations

\[
\begin{align*}
\nabla \times \nabla \times \mathbf{A} &= \mu_0 \sigma(\mathbf{A}, \phi) \left( -\dot{\mathbf{A}} - \dot{\mathbf{A}}_a - \nabla \phi \right) \\
\n\nabla \cdot \{\sigma(\mathbf{A}, \phi) \left( -\dot{\mathbf{A}} - \dot{\mathbf{A}}_a - \nabla \phi \right) \} &= 0
\end{align*}
\]

Ampere’s law (rot \( H = J \))

Continuity equations (div \( J = 0 \))
The Maxwell equations are reduced to 2 equations:

\[
\begin{align*}
\nabla \times \nabla \times \mathbf{A} &= \mu_0 \sigma(\mathbf{A}, \phi) \left(-\dot{\mathbf{A}} - \dot{\mathbf{A}}_a - \nabla \phi\right) \\
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\end{align*}
\]

- **Ampere’s law** \((\text{rot } H = J)\)
- **Continuity equations** \((\text{div } J = 0)\)

A is not expressed in the Coulomb gauge, gauge condition:

\[
\mathbf{A} \cdot \mathbf{w} = 0
\]

Set of meshing edges that **connects** all the nodes **without closed contours**

**Examples**
A - $\phi$ formulation – Boundary conditions

• The Maxwell equations are reduced to 2 equations

\[
\begin{align*}
\nabla \times \nabla \times \mathbf{A} &= \mu_0 \sigma(\mathbf{A}, \phi) \left( -\dot{\mathbf{A}} - \dot{\mathbf{A}}_a - \nabla \phi \right) \\
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\textit{Ampere’s law (rot } H = J\text{)}
\textit{Continuity equations (div } J = 0\text{)}

• $\mathbf{A}$ is not expressed in the Coulomb gauge, gauge condition: $\mathbf{A} \cdot \mathbf{w} = 0$

• Boundary conditions

\[
\begin{align*}
\mathbf{A} &= 0 \\
\phi &= 0
\end{align*}
\] at infinity

\[\rightarrow\] Use of Jacobian transformation for sending the outer surface of a spherical shell to infinity
A - $\phi$ formulation – Boundary conditions

- The Maxwell equations are reduced to 2 equations
  \[
  \begin{align*}
  \nabla \times \nabla \times A &= \mu_0 \sigma(A, \phi) (-\dot{A} - \dot{A}_a - \nabla \phi) \\
  \nabla \cdot \{\sigma(A, \phi) (-\dot{A} - \dot{A}_a - \nabla \phi)\} &= 0
  \end{align*}
  \]
  \textit{Ampere's law (rot }H = J) \textit{Continuity equations (div }J = 0) \]

- $A$ is not expressed in the Coulomb gauge, gauge condition: $A \cdot w = 0$

- Boundary conditions
  \[
  \begin{align*}
  A &= 0 \\
  \phi &= 0
  \end{align*}
  \] at infinity

  \[\text{Use of Jacobian transformation for sending the outer surface of a spherical shell to infinity}\]
A - $\phi$ formulation – External field

- The Maxwell equations are reduced to 2 equations

\[
\begin{aligned}
\nabla \times \nabla \times \mathbf{A} &= \mu_0 \sigma (\mathbf{A}, \phi) (-\dot{\mathbf{A}} - \dot{\mathbf{A}}_a - \nabla \phi) \\
\nabla \cdot \{\sigma (\mathbf{A}, \phi) (-\dot{\mathbf{A}} - \dot{\mathbf{A}}_a - \nabla \phi)\} &= 0
\end{aligned}
\]

- Ampere’s law (rot $H = J$)

- Continuity equations (div $J = 0$)

- $\mathbf{A}$ is not expressed in the Coulomb gauge, gauge condition:  \( \mathbf{A} \cdot \mathbf{w} = 0 \)

- Boundary conditions

\[
\begin{aligned}
\mathbf{A} &= 0 \\
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\]

at infinity

Source field $\mathbf{A}_a$ corresponds to a uniform magnetic flux density $\mathbf{B}_a$

The source field is a temporal ramp with a constant sweep rate ($mT/s$)
Implicit time-resolution and non-linear Picard iteration

- The equations are solved with a Galerkin residual minimization method
- We use the Backward Euler method at each time step
- Non linear terms are treated with a Picard iteration loop
Implicit time-resolution and non-linear Picard iteration

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Solution \((A, \phi) @ t - \Delta t\)
Implicit time-resolution and non-linear Picard iteration

- The equations are solved with a **Galerkin residual minimization method**
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Solution \((A, \phi) \at \ t - \Delta t\)

Update of \(\sigma (A, \phi)\)
Implicit time-resolution and non-linear Picard iteration

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Solution $(A, \phi) \ @ \ t - \Delta t$

Update of $\sigma (A, \phi)$ \rightarrow Solution of implicit linear systems with GMRES
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Solution \((A, \phi) \at \Delta t\)

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Solution \( (A, \phi) @ t – \Delta t \)

Update of \( \sigma (A, \phi) \) \n
Solution of implicit linear systems with GMRES

Convergence criteria satisfied

Solution \( (A, \phi) @ t \)
Outline

- Finite-element formulation and implementation
- Validation and comparison of the FEM model on simple geometries

  Magnetic field penetration in:

  - a HTS tube of infinite extension
  - a HTS tube of finite height
Bean critical-state is solved with the E(J) model taking $n = 100$

**HTS tube of infinite height**

**Bean model**

*linear decay* of the magnetic flux density in the wall
Bean critical-state is solved with the E(J) model taking $n = 100$

**HTS tube of infinite height**

- Bean model
  - **linear decay** of the magnetic flux density in the wall

**Magnetic flux penetration**

- FEM results are consistent with the Bean model
Single time step method produces more accurate results in a smaller calculation time

Choice of the time step

*For solving the problem with* $B_a = 200 \text{ mT}$ *with a sweep rate of* $10 \text{ mT/s}*$:

1. 20 time steps of 1s

System is solved @ $t = 1\text{s, 2s, ...}, 20\text{s}$
Single time step method produces more accurate results in a smaller calculation time.

Choice of the time step

For solving the problem with $B_a = 200 \text{ mT}$ with a sweep rate of $10 \text{ mT/s}$:

1. 20 time steps of 1s
2. 1 time step of 20 s

System is solved @ $t = 20\text{s}$
Single time step method produces more accurate results in a smaller calculation time

*Magnetic field penetration of an external field increasing from 0 mT to 200 mT (10mT/s)*

Comparison of two methods

1. **in a single simulation** with 20 time steps of 1s
2. **in 20 different simulations** with time steps of 1s, 2s, ..., 20s
Single time step method produces more accurate results in a smaller calculation time

*Magnetic field penetration of an external field increasing from 0 mT to 200 mT*

Comparison of two methods

1. *in a single simulation* with 20 time steps of 1s
2. *in 20 different simulations* with time steps of 1s, 2s, ..., 20s

Analysis of the error

![Graph showing average deviation from the Bean model for different applied magnetic fields and time steps. The graph plots applied magnetic field (mT) on the x-axis and average deviation (mT) on the y-axis. There are two sets of data points: one for a single time step and another for multiple time steps. The single time step data shows a gentler slope compared to the multiple time steps, indicating more accurate results.](image-url)
Single time step method produces more accurate results in a smaller calculation time

Magnetic field penetration of an external field increasing from 0 mT to 200 mT

Comparison of two methods

1. in a single simulation with 20 time steps of 1s
2. in 20 different simulations with time steps of 1s, 2s, ..., 20s

Analysis of the error

Calculation time

for solving $B_a = 200 \text{ mT}$

- 20 time steps
- 2 days 3/4

Single time-step

20 min
3D FEM simulations on tubes with finite height are consistent with previous observations.

**HTS tube of finite height**

Simulation parameters:

- \( n = 100 \)
- \( B_a = 0 - 200 \text{ mT} \) (10 mT/s)
- Single time step method
- 3D geometry

**Magnetic flux penetration**

20 FEM simulations with single time step method

Brandt method (*semi-analytical*)

multiple step method (\( \Delta t=5.10^{-4}\text{s} \))
Single time step method is more accurate with large critical exponent

**HTS tube of finite height**

**Principle**

- Applied magnetic flux density: 200 mT
- FEM single time step compared with Brandt method with the help of $B_{center}$

$$\text{Error} = \frac{B_{center,\text{Brandt}} - B_{center,\text{FEM}}}{B_{center,\text{Brandt}}}$$

**Brandt (semi-analytical)**

*multiple time step method ($\Delta t=5.10^{-4}s$)*

**FEM**

*single time step method*
Single time step method is more accurate with large critical exponent

**HTS tube of finite height**

**Principle**

- Applied magnetic flux density: 200 mT
- FEM single time step compared with Brandt method with the help of $B_{\text{center}}$

$n \sim 20$: experimental value for the critical exponent

Single time step method is valid
Conclusion

- Implementation of a finite-element formulation in GetDP with high non linearity
- A-\( \phi \) formulation in 3D geometry for calculating the magnetic field penetration
- Single time step method in the case of linearly time varying excitation is fast and accurate
Outline

- Finite-element formulation and implementation
- Validation and comparison of the FEM model on simple geometries
- Optimization of the magnetic properties of drilled samples
  - Influence of the lattice types
Polar triangular lattice wins...

HTS tube of infinite height

- Squared
- Polar squared
- Centered rectangular
- Polar triangular

Trapped flux without holes

Magnetic flux density (mT)

Distance along the diameter (mm)
Polar triangular lattice wins ...

**HTS tube of infinite height**

- Squared
- Polar squared
- Centered rectangular
- Polar triangular

![Bar chart showing differences in Remant magnetization and Maximum trapped field between different lattices.](chart.png)