

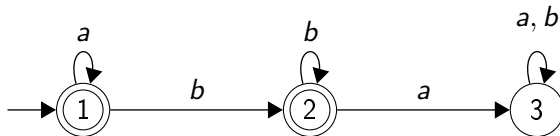
State complexity of the multiples of the Thue-Morse set

Adeline Massuir

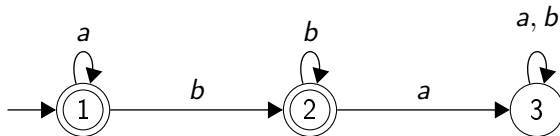
Joint work with Émilie Charlier and Célia Cisternino

April 3rd, 2019

Deterministic finite automaton (DFA) : $\mathcal{A} = (Q, q_0, F, \Sigma, \delta)$



Deterministic finite automaton (DFA) : $\mathcal{A} = (Q, q_0, F, \Sigma, \delta)$



Language – Regular language

Theorem

For any regular language L , there exists a unique (up to isomorphism) minimal automaton accepting L .

Theorem

For any regular language L , there exists a unique (up to isomorphism) minimal automaton accepting L .

Theorem

An automaton is minimal if and only if it is accessible and reduced.

Theorem

For any regular language L , there exists a unique (up to isomorphism) minimal automaton accepting L .

Theorem

An automaton is minimal if and only if it is accessible and reduced.

One algorithm :

Theorem

For any regular language L , there exists a unique (up to isomorphism) minimal automaton accepting L .

Theorem

An automaton is minimal if and only if it is accessible and reduced.

One algorithm :

- 1 Eject non accessible states

Theorem

For any regular language L , there exists a unique (up to isomorphism) minimal automaton accepting L .

Theorem

An automaton is minimal if and only if it is accessible and reduced.

One algorithm :

- 1 Eject non accessible states
- 2 Look for undistinguished states

Definition

The *state complexity* of a regular language is the number of states of its minimal automaton.

What do we want to do ?

Definition

Let $b \in \mathbb{N}_{\geq 2}$. A subset X of \mathbb{N} is *b-recognizable* if $\text{rep}_b(X)$ is regular.

What do we want to do ?

Definition

Let $b \in \mathbb{N}_{\geq 2}$. A subset X of \mathbb{N} is *b-recognizable* if $\text{rep}_b(X)$ is regular.

It is equivalent to work with $0^* \text{rep}_b(X)$.

What do we want to do ?

Definition

Let $b \in \mathbb{N}_{\geq 2}$. A subset X of \mathbb{N} is *b-recognizable* if $\text{rep}_b(X)$ is regular.

It is equivalent to work with $0^* \text{rep}_b(X)$.

Theorem

Let $b \in \mathbb{N}_{\geq 2}$ and $m \in \mathbb{N}$. If $X \subseteq \mathbb{N}$ is *b-recognizable*, so is mX .

Theorem [Alexeev, 2004]

The state complexity of the language $0^* \text{rep}_b(m\mathbb{N})$ is

$$\min_{N \geq 0} \left\{ \frac{m}{\gcd(m, b^N)} + \sum_{n=0}^{N-1} \frac{b^n}{\gcd(b^n, m)} \right\}$$

0

0

1

01

1

01

10

0110

10

0110

1001

01101001

1001

01101001

10010110

0110100110010110

10010110

0110100110010110...

0110100110010110...

Definition

The Thue-Morse set is the set

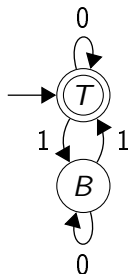
$$\mathcal{T} = \{n \in \mathbb{N} : |\text{rep}_2(n)|_1 \in 2\mathbb{N}\}.$$

0110100110010110...

Definition

The Thue-Morse set is the set

$$\mathcal{T} = \{n \in \mathbb{N} : |\text{rep}_2(n)|_1 \in 2\mathbb{N}\}.$$

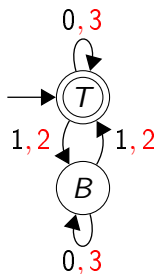


0110100110010110...

Definition

The Thue-Morse set is the set

$$\mathcal{T} = \{n \in \mathbb{N} : |\text{rep}_2(n)|_1 \in 2\mathbb{N}\}.$$



Definition

Let $p, q \in \mathbb{N}_{\geq 2}$. We say that p and q are *multiplicatively independent* if

$$p^a = q^b \Rightarrow a = b = 0.$$

They are said *multiplicatively dependent* otherwise.

Definition

Let $p, q \in \mathbb{N}_{\geq 2}$. We say that p and q are *multiplicatively independent* if

$$p^a = q^b \Rightarrow a = b = 0.$$

They are said *multiplicatively dependent* otherwise.

Theorem [Cobham, 1969]

- Let b, b' two multiplicatively independent bases. A subset of \mathbb{N} is both b -recognizable and b' -recognizable iff it is a finite union of arithmetic progressions.
- Let b, b' two multiplicatively dependent bases. A subset of \mathbb{N} is b -recognizable iff it is b' -recognizable.

$$\mathcal{T} = \{n \in \mathbb{N} : |\text{rep}_2(n)|_1 \in 2\mathbb{N}\}$$

Theorem

Let $m \in \mathbb{N}$ and $p \in \mathbb{N}_{\geq 1}$.

Then the state complexity of the language $0^* \text{rep}_{2^p}(m\mathcal{T})$ is

$$2k + \left\lceil \frac{z}{p} \right\rceil$$

if $m = k2^z$ with k odd.

Automaton	Language accepted
$\mathcal{A}_{\mathcal{I}, 2^p}$	$(0, 0)^* \text{rep}_{2^p}(\mathcal{I} \times \mathbb{N})$

Automaton	Language accepted
$\mathcal{A}_{\mathcal{I}, 2^p}$	$(0, 0)^* \text{rep}_{2^p} (\mathcal{I} \times \mathbb{N})$
$\mathcal{A}_{m, 2^p}$	$(0, 0)^* \text{rep}_{2^p} (\{(n, mn) : n \in \mathbb{N}\})$

Automaton	Language accepted
$\mathcal{A}_{\mathcal{I}, 2^p}$	$(0, 0)^* \text{rep}_{2^p} (\mathcal{I} \times \mathbb{N})$
$\mathcal{A}_{m, 2^p}$	$(0, 0)^* \text{rep}_{2^p} (\{(n, mn) : n \in \mathbb{N}\})$
$\mathcal{A}_{\mathcal{I}, 2^p} \times \mathcal{A}_{m, 2^p}$	$(0, 0)^* \text{rep}_{2^p} (\{(t, mt) : t \in \mathcal{I}\})$

Automaton	Language accepted
$\mathcal{A}_{\mathcal{I}, 2^p}$	$(0, 0)^* \text{rep}_{2^p} (\mathcal{I} \times \mathbb{N})$
$\mathcal{A}_{m, 2^p}$	$(0, 0)^* \text{rep}_{2^p} (\{(n, mn) : n \in \mathbb{N}\})$
$\mathcal{A}_{\mathcal{I}, 2^p} \times \mathcal{A}_{m, 2^p}$	$(0, 0)^* \text{rep}_{2^p} (\{(t, mt) : t \in \mathcal{I}\})$
$\pi (\mathcal{A}_{\mathcal{I}, 2^p} \times \mathcal{A}_{m, 2^p})$	$0^* \text{rep}_{2^p} (m\mathcal{I})$

The automaton $\mathcal{A}_{\mathcal{T}, 2^p}$

$$(0, 0)^* \{ \text{rep}_{2^p}(t, n) : t \in \mathcal{T}, n \in \mathbb{N} \}$$

The automaton $\mathcal{A}_{\mathcal{T}, 2^p}$

$$(0, 0)^* \{ \text{rep}_{2^p}(t, n) : t \in \mathcal{T}, n \in \mathbb{N} \}$$

States	T, B
Initial state	T
Final states	T
Alphabet	$\{0, \dots, 2^p - 1\}^2$
Transitions	$\delta_{\mathcal{A}_{\mathcal{T}, 2^p}}(X, (a, b)) = \begin{cases} X & \text{if } a \in \mathcal{T} \\ \bar{X} & \text{else.} \end{cases}$

The automaton $\mathcal{A}_{\mathcal{T}, 2^p}$

$$(0, 0)^* \{ \text{rep}_{2^p}(t, n) : t \in \mathcal{T}, n \in \mathbb{N} \}$$

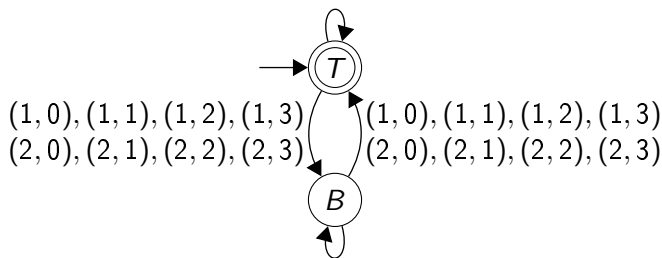
States	T, B
Initial state	T
Final states	T
Alphabet	$\{0, \dots, 2^p - 1\}^2$
Transitions	$\delta_{\mathcal{A}, 2^p}(X, (a, b)) = \begin{cases} X & \text{if } a \in \mathcal{T} \\ \bar{X} & \text{else.} \end{cases}$

For all $u, v \in \{0, \dots, 2^p - 1\}^*$,

$$\delta_{\mathcal{A}, 2^p}(X, (u, v)) = \begin{cases} X & \text{if } \text{val}_{2^p}(u) \in \mathcal{T} \\ \bar{X} & \text{else.} \end{cases}$$

The automaton $\mathcal{A}_{\mathcal{T},4}$

$(0, 0), (0, 1), (0, 2), (0, 3)$
 $(3, 0), (3, 1), (3, 2), (3, 3)$



$(0, 0), (0, 1), (0, 2), (0, 3)$
 $(3, 0), (3, 1), (3, 2), (3, 3)$

The automaton $\mathcal{A}_{m,b}$

$$(0,0)^* \{\text{rep}_b(n, mn) : n \in \mathbb{N}\}$$

The automaton $\mathcal{A}_{m,b}$

$$(0,0)^* \{ \text{rep}_b(n, mn) : n \in \mathbb{N} \}$$

States	$0, \dots, m-1$
Initial state	0
Final states	0
Alphabet	$\{0, \dots, b-1\}^2$
Transitions	$\delta_{m,b}(i, (d, e)) = j \Leftrightarrow bi + e = md + j$

The automaton $\mathcal{A}_{m,b}$

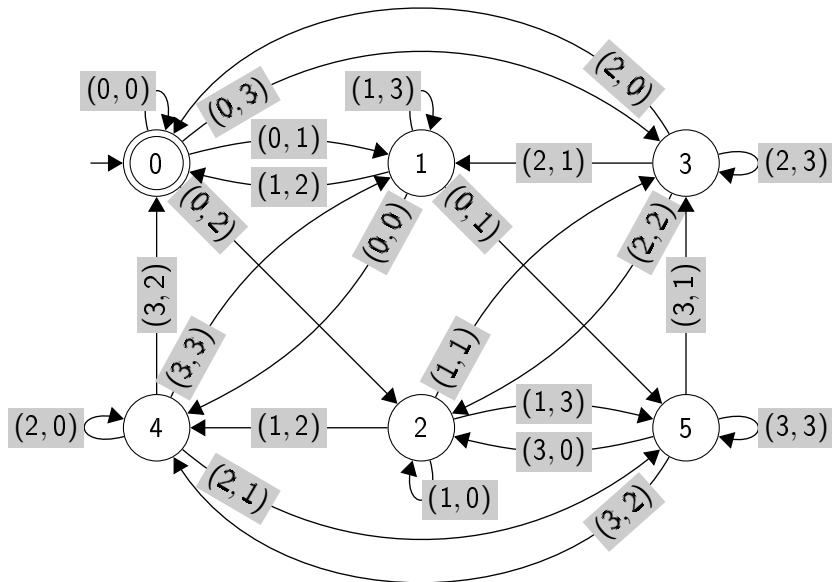
$$(0, 0)^* \{ \text{rep}_b(n, mn) : n \in \mathbb{N} \}$$

States	$0, \dots, m - 1$
Initial state	0
Final states	0
Alphabet	$\{0, \dots, b - 1\}^2$
Transitions	$\delta_{m,b}(i, (d, e)) = j \Leftrightarrow bi + e = md + j$

For all $i, j \in \{0, \dots, m - 1\}$, for all $u, v \in \{0, \dots, b - 1\}^*$,

$$\delta_{m,b}(i, (u, v)) = j \Leftrightarrow b^{|(u,v)|} i + \text{val}_b(v) = m \text{val}_b(u) + j.$$

The automaton $\mathcal{A}_{6,4}$



The product automaton $\mathcal{A}_{m,2^p} \times \mathcal{A}_{\mathcal{T},2^p}$

$$(0,0)^* \{ \text{rep}_{2^p}(t, mt) : t \in \mathcal{T} \}$$

The product automaton $\mathcal{A}_{m,2^p} \times \mathcal{A}_{\mathcal{T},2^p}$

$$(0,0)^* \{ \text{rep}_{2^p}(t, mt) : t \in \mathcal{T} \}$$

States	$(0, T), \dots, (m-1, T), (0, B), \dots, (m-1, B)$
Initial state	$(0, T)$
Final states	$(0, T)$
Alphabet	$\{0, \dots, 2^p - 1\}^2$
Transitions	$\delta_{\mathcal{A}_{\mathcal{T},2^p}}((i, X), (u, v)) = (j, Y)$ $\Leftrightarrow 2^{p (u,v)}i + \text{val}_{2^p}(v) = m \text{val}_{2^p}(u) + j$ and $Y = \begin{cases} X & \text{if } \text{val}_{2^p}(u) \in \mathcal{T} \\ \bar{X} & \text{else.} \end{cases}$

The product automaton $\mathcal{A}_{m,2^p} \times \mathcal{A}_{\mathcal{T},2^p}$

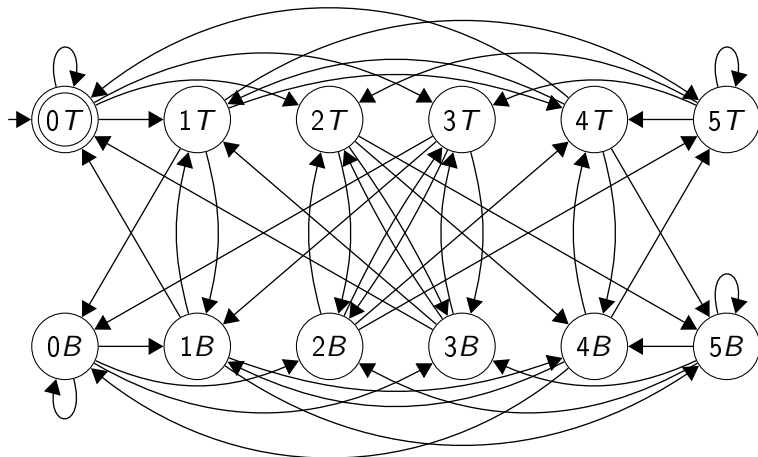
$$(0,0)^* \{ \text{rep}_{2^p}(t, mt) : t \in \mathcal{T} \}$$

States	$(0, T), \dots, (m-1, T), (0, B), \dots, (m-1, B)$
Initial state	$(0, T)$
Final states	$(0, T)$
Alphabet	$\{0, \dots, 2^p - 1\}^2$
Transitions	$\delta_{\mathcal{A}_{\mathcal{T},2^p}}((i, X), (u, v)) = (j, Y)$ $\Leftrightarrow 2^{p (u,v)} i + \text{val}_{2^p}(v) = m \text{val}_{2^p}(u) + j$ and $Y = \begin{cases} X & \text{if } \text{val}_{2^p}(u) \in \mathcal{T} \\ \bar{X} & \text{else.} \end{cases}$

Remark

If i, X, v are fixed, there exist unique j, Y, u such that we have a transition labeled by (u, v) from (i, X) to (j, Y) .

The automaton $\mathcal{A}_{6,4} \times \mathcal{A}_{\mathcal{T},4}$

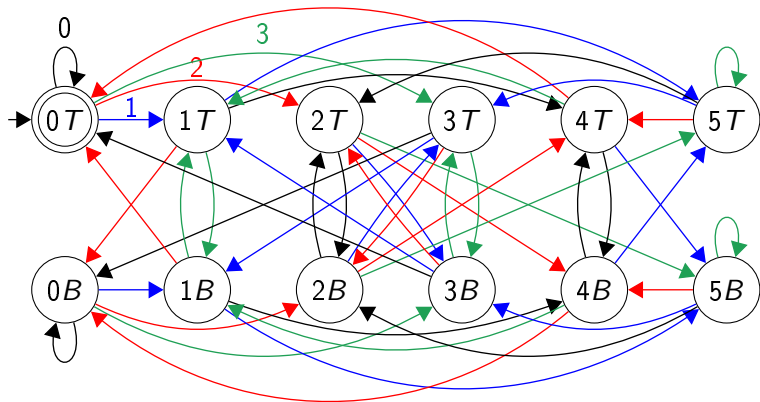


The projected automaton $\pi(\mathcal{A}_{m,2^p} \times \mathcal{A}_{\mathcal{T},2^p})$

$$0^* \text{rep}_{2^p}(m\mathcal{T}) = 0^* \{\text{rep}_{2^p}(mt) : t \in \mathcal{T}\}$$

The projected automaton $\pi(\mathcal{A}_{m,2^p} \times \mathcal{A}_{\mathcal{T},2^p})$

$$0^* \text{rep}_{2^p}(m\mathcal{T}) = 0^* \{\text{rep}_{2^p}(mt) : t \in \mathcal{T}\}$$



Proposition

The automaton $\pi (\mathcal{A}_{m,2^p} \times \mathcal{A}_{\mathcal{T},2^p})$ is

- deterministic,
- accessible,
- coaccessible.

Proposition

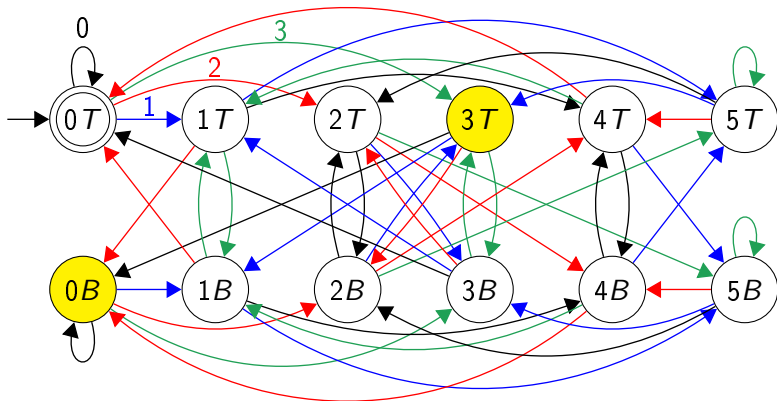
The automaton $\pi(\mathcal{A}_{m,2^p} \times \mathcal{A}_{\mathcal{T},2^p})$ is

- deterministic,
- accessible,
- coaccessible.

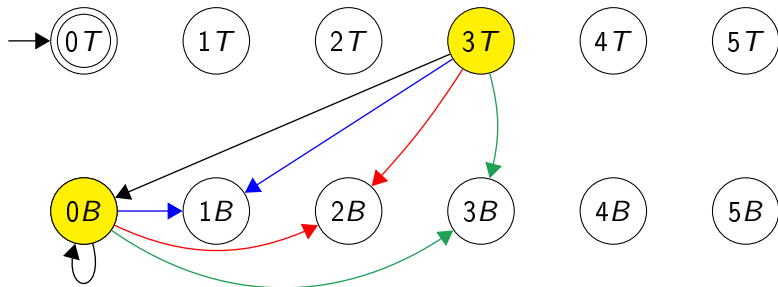
Proposition

In the automaton $\pi(\mathcal{A}_{m,2^p} \times \mathcal{A}_{\mathcal{T},2^p})$, the states (i, T) and (i, B) are disjoint for all $i \in \{0, \dots, m-1\}$.

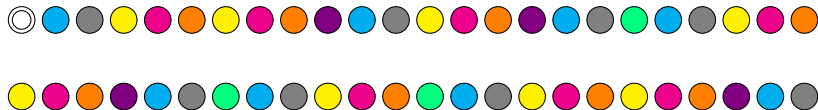
The automaton $\pi(\mathcal{A}_{6,4} \times \mathcal{A}_{\mathcal{T},4})$



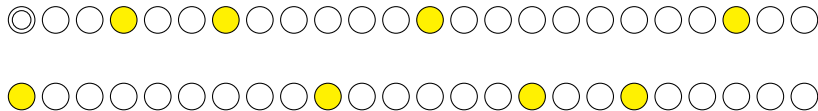
The automaton $\pi(\mathcal{A}_{6,4} \times \mathcal{A}_{\mathcal{T},4})$



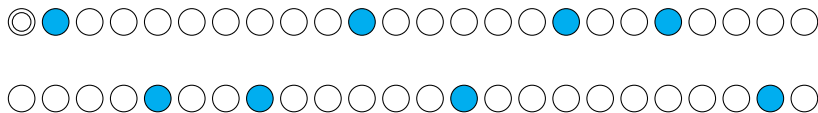
The automaton $\pi(\mathcal{A}_{24,4} \times \mathcal{A}_{\mathcal{T},4})$



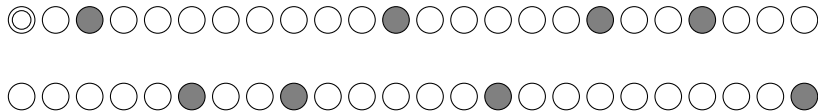
The automaton $\pi(\mathcal{A}_{24,4} \times \mathcal{A}_{\mathcal{T},4})$



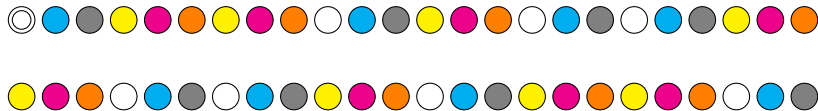
The automaton $\pi(\mathcal{A}_{24,4} \times \mathcal{A}_{\mathcal{T},4})$



The automaton $\pi(\mathcal{A}_{24,4} \times \mathcal{A}_{\mathcal{T},4})$



The automaton $\pi(\mathcal{A}_{24,4} \times \mathcal{A}_{\mathcal{T},4})$



Recall

For $m \in \mathbb{N}$, $p \in \mathbb{N}_{\geq 1}$, we write $m = k2^z$ with z odd.

Recall

For $m \in \mathbb{N}$, $p \in \mathbb{N}_{\geq 1}$, we write $m = k2^z$ with z odd.

For all $n \in \mathbb{N}$, we set

$$T_n := \begin{cases} T & \text{if } n \in \mathcal{T} \\ B & \text{else.} \end{cases}$$

Recall

For $m \in \mathbb{N}$, $p \in \mathbb{N}_{\geq 1}$, we write $m = k2^z$ with z odd.

For all $n \in \mathbb{N}$, we set

$$T_n := \begin{cases} T & \text{if } n \in \mathcal{T} \\ B & \text{else.} \end{cases}$$

Definition

For all $j \in \{1, \dots, k-1\}$, we set

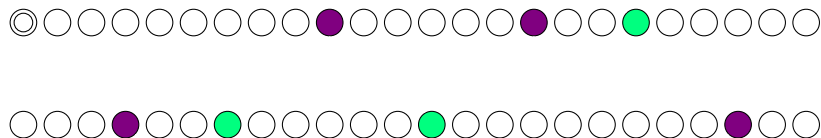
$$[(j, T)] := \{(j + k\ell, T_\ell) : 0 \leq \ell \leq 2^z - 1\}$$

$$[(j, B)] := \{(j + k\ell, \overline{T}_\ell) : 0 \leq \ell \leq 2^z - 1\}.$$

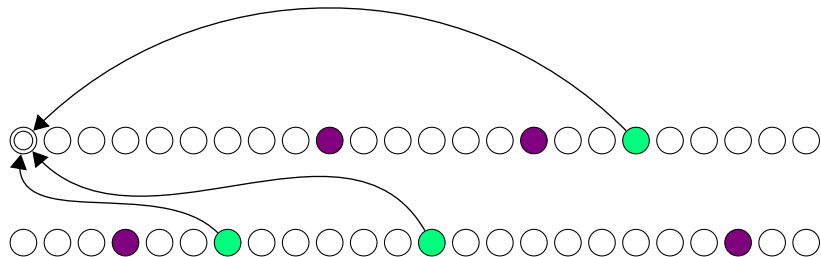
We also set

$$[(0, T)] := \{(0, T)\} \text{ and } [(0, B)] := \{(k\ell, \overline{T}_\ell) : 0 \leq \ell \leq 2^z - 1\}.$$

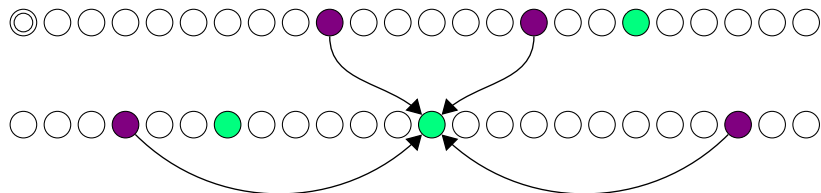
The automaton $\pi(\mathcal{A}_{24,4} \times \mathcal{A}_{\mathcal{T},4})$



The automaton $\pi(\mathcal{A}_{24,4} \times \mathcal{A}_{\mathcal{T},4})$



The automaton $\pi(\mathcal{A}_{24,4} \times \mathcal{A}_{\mathcal{T},4})$



Definition

For all $\alpha \in \{0, \dots, z-1\}$, we set

$$C_\alpha := \{(k2^{z-\alpha-1} + k2^{z-\alpha}l, \overline{T}_l) : 0 \leq l \leq 2^\alpha - 1\}.$$

For all $\beta \in \{0, \dots, \lceil \frac{z}{p} \rceil - 2\}$, we set

$$\Gamma_\beta := \bigcup_{\alpha \in \{\beta p, \dots, (\beta+1)p-1\}} C_\alpha.$$

We also set

$$\Gamma_{\lceil \frac{z}{p} \rceil - 1} := \bigcup_{\alpha \in \{(\lceil \frac{z}{p} \rceil - 1)p, \dots, z-1\}} C_\alpha.$$

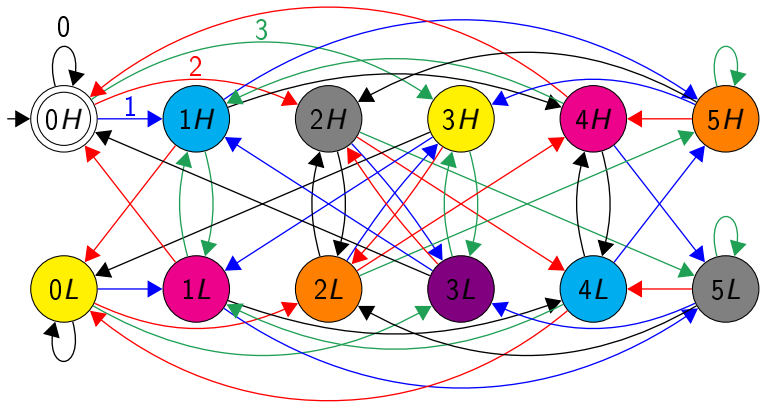
We can build a new automaton

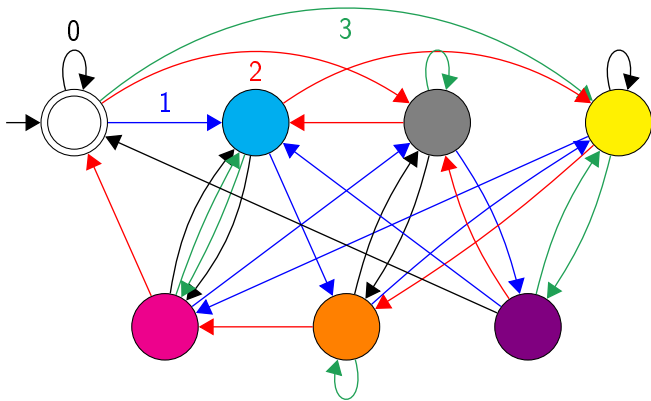
We can build a new automaton which is

- accessible

We can build a new automaton which is

- accessible
- reduced.





Theorem

Let $m \in \mathbb{N}$ and $p \in \mathbb{N}_{\geq 1}$. Then the state complexity of the language $0^* \text{rep}_{2^p}(m\mathcal{I})$ is equal to

$$2k + \left\lceil \frac{z}{p} \right\rceil$$

if $m = k2^z$ with k odd.

Theorem

Let $m \in \mathbb{N}$ and $p \in \mathbb{N}_{\geq 1}$. Then the state complexity of the language $0^* \text{rep}_{2^p}(m\mathcal{T})$ is equal to

$$2k + \left\lceil \frac{z}{p} \right\rceil$$

if $m = k2^z$ with k odd.

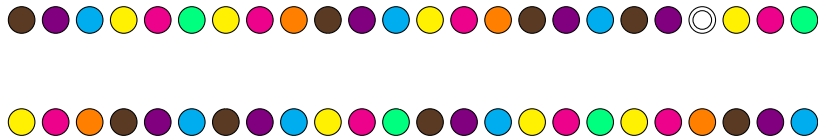
$$2 \times 3 + \left\lceil \frac{1}{2} \right\rceil = 7$$

What about the language

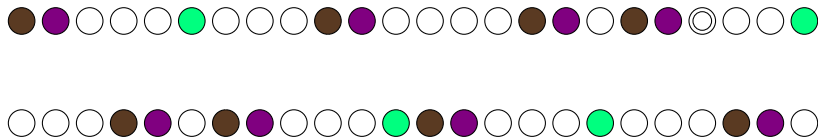
$$0^* \text{rep}_{2^p}(m\mathcal{T} + r)$$

where $r \in \{0, \dots, m-1\}$?

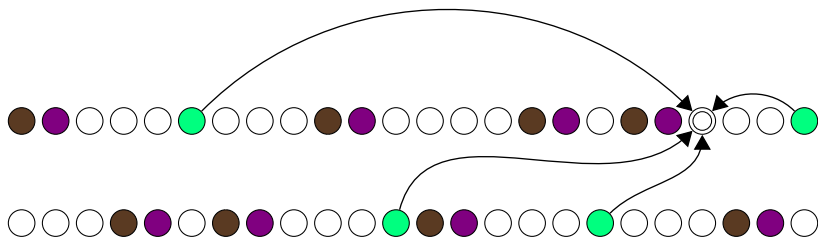
The automaton $\pi \left(\mathcal{A}_{24,4}^{20} \times \mathcal{A}_{\mathcal{T},4} \right)$



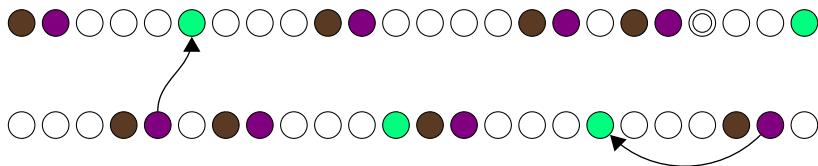
The automaton $\pi (A_{24,4}^{20} \times A_{\mathcal{T},4})$



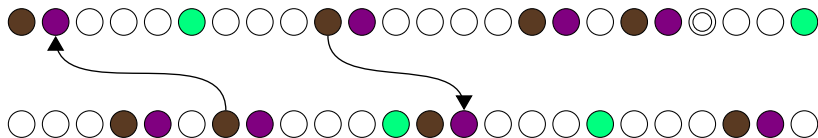
The automaton $\pi (\mathcal{A}_{24,4}^{20} \times \mathcal{A}_{\mathcal{T},4})$



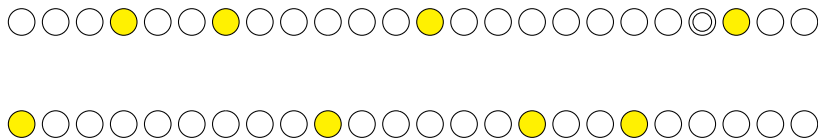
The automaton $\pi (\mathcal{A}_{24,4}^{20} \times \mathcal{A}_{\mathcal{T},4})$



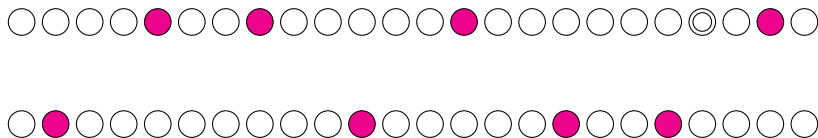
The automaton $\pi (\mathcal{A}_{24,4}^{20} \times \mathcal{A}_{\mathcal{T},4})$



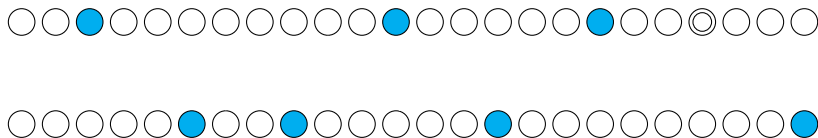
The automaton $\pi (A_{24,4}^{20} \times A_{\mathcal{T},4})$



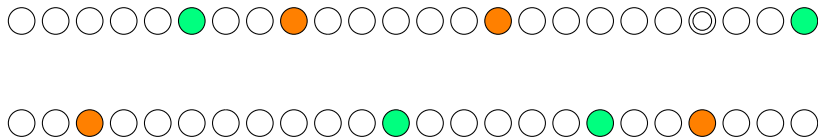
The automaton $\pi \left(\mathcal{A}_{24,4}^{20} \times \mathcal{A}_{\mathcal{T},4} \right)$



The automaton $\pi (A_{24,4}^{20} \times A_{\mathcal{T},4})$



The automaton $\pi (A_{24,4}^{20} \times A_{\mathcal{T},4})$



Definition

For all $0 \leq \alpha \leq \max \left\{ \left\lceil \frac{z}{p} \right\rceil, |\text{rep}_{2^p}(r)| \right\} =: L,$

$$R'_\alpha = \begin{cases} \left\{ \left(\left\lfloor \frac{r}{2^{\alpha p}} \right\rfloor + lk2^{z-\alpha p}, X_\ell \right) : 0 \leq \ell \leq 2^{\alpha p} - 1 \right\} & \text{if } \alpha \leq \left\lfloor \frac{z}{p} \right\rfloor \\ \left\{ \left(\left\lfloor \frac{r}{2^{\alpha p}} \right\rfloor + lk, X_\ell \right) : 0 \leq \ell \leq 2^z - 1 \right\} & \text{else.} \end{cases}$$

and $R_\alpha = R'_\alpha \setminus \bigcup_{i=0}^{\alpha-1} R'_i.$

Definition

For all $0 \leq \alpha \leq \max \left\{ \left\lceil \frac{z}{p} \right\rceil, |\text{rep}_{2^p}(r)| \right\} =: L$,

$$R'_\alpha = \begin{cases} \left\{ \left(\left\lfloor \frac{r}{2^{\alpha p}} \right\rfloor + lk2^{z-\alpha p}, X_\ell \right) : 0 \leq \ell \leq 2^{\alpha p} - 1 \right\} & \text{if } \alpha \leq \left\lfloor \frac{z}{p} \right\rfloor \\ \left\{ \left(\left\lfloor \frac{r}{2^{\alpha p}} \right\rfloor + lk, X_\ell \right) : 0 \leq \ell \leq 2^z - 1 \right\} & \text{else.} \end{cases}$$

and $R_\alpha = R'_\alpha \setminus \bigcup_{i=0}^{\alpha-1} R'_i$.

For all $1 \leq j \leq k-1$ and $Y \in \{T, B\}$ and for $j=0$ and $Y = \bar{X}$,

$$S_j^Y = [(j, Y)] \setminus \bigcup_{\alpha=0}^L R_\alpha.$$

Theorem

Let $m \in \mathbb{N}$, $r \in \{0, \dots, m-1\}$ and $p \in \mathbb{N}_{\geq 1}$. Let $X = \mathcal{I}$ or $\mathbb{N} \setminus \mathcal{I}$. Then, the state complexity of the language $0^* \text{rep}_{2^p}(mX + r)$ is equal to

$$2k + \left\lceil \frac{z}{p} \right\rceil$$

if $m = k2^z$ with k odd.