Extensions of the Pascal triangle to words

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Classical Pascal triangle

$\mathrm{P}\colon (m,k) \in \mathbb{N} \times \mathbb{N} \mapsto \binom{m}{k} \in \mathbb{N}$

$\binom{m}{k}$		k								
		0	1	2	3	4	5	6	7	$\cdots$
	0	1	0	0	0	0	0	0	0	
	1	1	1	0	0	0	0	0	0	
	2	1	2	1	0	0	0	0	0	
m	3	1	3	3	1	0	0	0	0	
	4	1	4	6	4	1	0	0	0	
	5	1	5	**10**	**10**	5	1	0	0	
	6	1	6	15	**20**	15	6	1	0	
	7	1	7	21	35	35	21	7	1	
	$\vdots$									$\ddots$

Binomial coefficients:

$$\binom{m}{k} = \frac{m!}{(m-k)!\,k!}$$

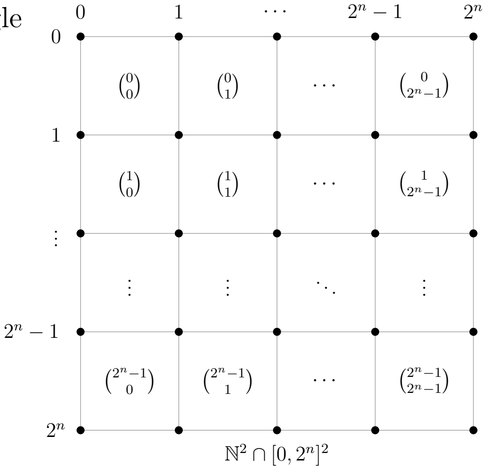
Pascal's rule:

$$\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}$$

A specific construction

- Grid: first 2^n rows and columns of the Pascal triangle

$$\left(\binom{m}{k} \bmod 2\right)_{0\le m,k<2^n}$$



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- Color each square in
  - white if $\binom{m}{k} \equiv 0 \bmod 2$
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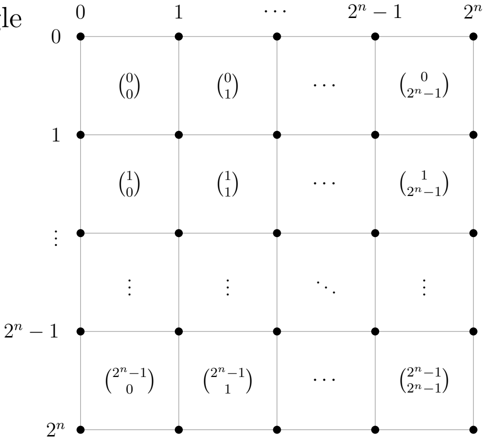

$\mathbb{N}^2 \cap [0,2^n]^2$

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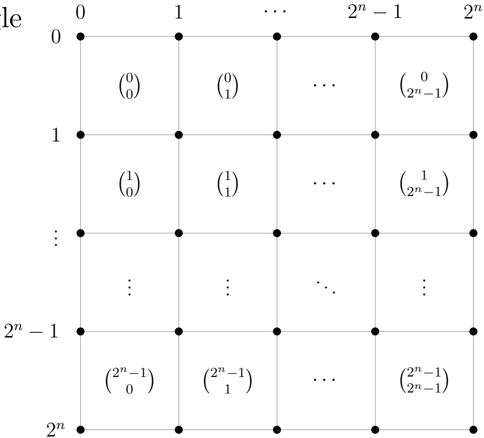
- Normalize by a homothety of ratio $1/2^n$ (bring into $[0,1]^2$)



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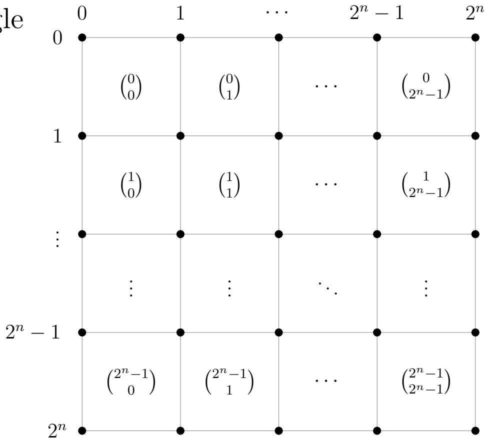
$\rightsquigarrow$ sequence of compact sets belonging to $[0,1]^2$



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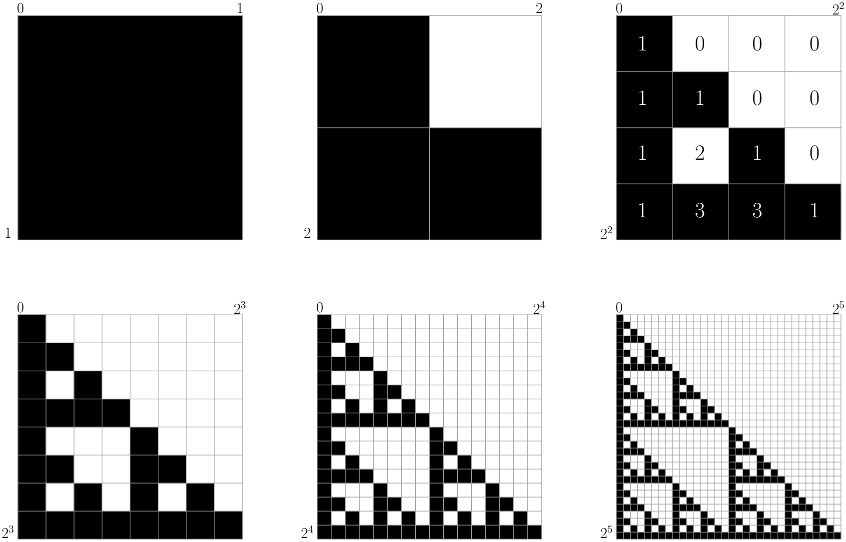
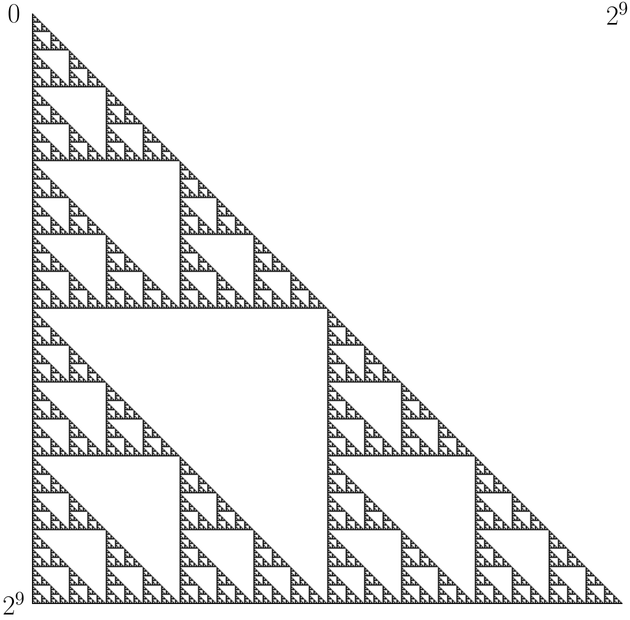
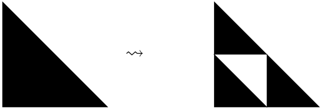

0
1
1
0
2
2
0
2^2
2^2
1 0 0 0
1 1 0 0
1 2 1 0
1 3 3 1
0
2^3
2^3
0
2^4
2^4
0
2^5
2^5

The tenth element of the sequence



The Sierpiński gasket



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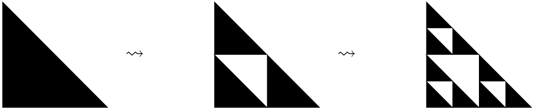

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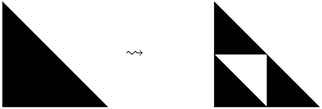
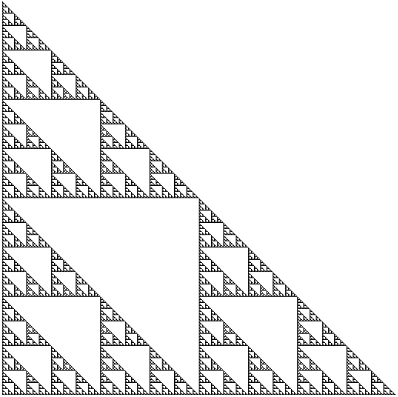
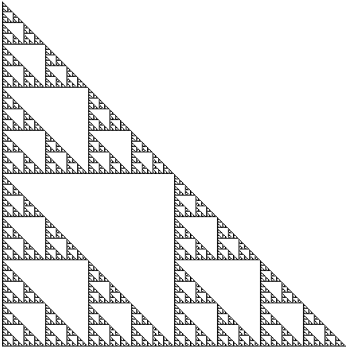




Folklore fact

The latter sequence of compact sets converges to the Sierpiński gasket (w.r.t. the Hausdorff distance).



Definitions:

- ϵ-*fattening* of a subset $S \subset \mathbb{R}^2$

$$[S]_\epsilon = \bigcup_{x \in S} B(x, \epsilon)$$

- $(\mathcal{H}(\mathbb{R}^2), d_h)$ complete space of the non-empty compact subsets of $\mathbb{R}^2$ equipped with the *Hausdorff distance* d_h

$$d_h(S, S') = \inf\{\epsilon \in \mathbb{R}_{>0} \mid S \subset [S']_\epsilon \quad \text{and} \quad S' \subset [S]_\epsilon\}$$

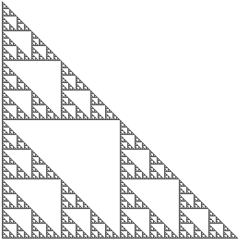
Theorem (von Haeseler, Peitgen, and Skordev, 1992)

Let p be a prime and $s > 0$.
The sequence of compact sets corresponding to

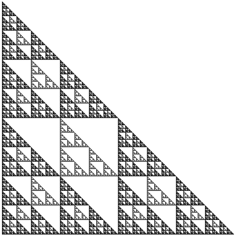
$$\left(\binom{m}{k} \bmod p^s\right)_{0 \leq m,k < p^n}$$

converges when n tends to infinity (w.r.t. the Hausdorff distance).

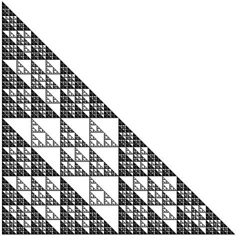
$p = 2,\ s = 1$



$p = 2,\ s = 2$



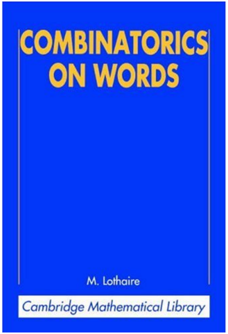
$p = 2,\ s = 3$



Replace integers by finite words.

Combinatorics on words (CoW)

- new area of discrete mathematics (± 1900)
- study sequences of symbols (called letters)
- topics include:
  - ◇ regularities and patterns in words
  - ◇ important types of words (e.g. automatic, regular, de Bruijn, Lyndon, Sturmian)
  - ◇ coding of structures (e.g. paths, trees or curves in the plane)



M. Lothaire, 1983.

Binomial coefficient of finite words

Definition: A *finite word* is a finite sequence of letters belonging to a finite set called *alphabet*.

Example: $101, 101001 \in \{0,1\}^*$

Binomial coefficient of words

Let u, v be two finite words.
The *binomial coefficient* $\binom{u}{v}$ of u and v is the number of times v occurs as a subsequence of u (meaning as a "scattered" subword).

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Example: $u = 101001$ $\qquad v = 101$

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Example: $u = \mathbf{101}001$ $\quad v = 101$ $\quad$ 1 occurrence

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Binomial coefficient of words

Let u, v be two finite words.
The *binomial coefficient* $\binom{u}{v}$ of u and v is the number of times v occurs as a subsequence of u (meaning as a "scattered" subword).

Example: $u = \mathbf{10}100\mathbf{1}$ $\quad v = 101 \quad$ 2 occurrences

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The *binomial coefficient* $\binom{u}{v}$ of u and v is the number of times v occurs as a subsequence of u (meaning as a "scattered" subword).

Example: $u = \mathbf{101001}$ $\qquad v = 101 \qquad$ 3 occurrences

Binomial coefficient of finite words

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Example: $101, 101001 \in \{0,1\}^*$

Binomial coefficient of words

Let u, v be two finite words.
The *binomial coefficient* $\binom{u}{v}$ of u and v is the number of times v occurs as a subsequence of u (meaning as a "scattered" subword).

Example: $u = \mathbf{101001}$ $\quad v = 101 \quad$ 4 occurrences

Definition: A *finite word* is a finite sequence of letters belonging to a finite set called *alphabet*.

Example: $101, 101001 \in \{0,1\}^*$

Binomial coefficient of words

Let u, v be two finite words.
The *binomial coefficient* $\binom{u}{v}$ of u and v is the number of times v occurs as a subsequence of u (meaning as a "scattered" subword).

Example: $u = 10\mathbf{1}\mathbf{0}\mathbf{0}\mathbf{1}$ $\quad v = 101 \quad$ 5 occurrences

Definition: A *finite word* is a finite sequence of letters belonging to a finite set called *alphabet*.

Example: $101, 101001 \in \{0,1\}^*$

Binomial coefficient of words

Let u, v be two finite words.
The *binomial coefficient* $\binom{u}{v}$ of u and v is the number of times v occurs as a subsequence of u (meaning as a "scattered" subword).

Example: $u = 10\mathbf{1}0\mathbf{0}\mathbf{1}$ $\quad v = 101 \quad$ 6 occurrences

Definition: A *finite word* is a finite sequence of letters belonging to a finite set called *alphabet*.

Example: $101, 101001 \in \{0,1\}^*$

Binomial coefficient of words

Let u, v be two finite words.
The *binomial coefficient* $\binom{u}{v}$ of u and v is the number of times v occurs as a subsequence of u (meaning as a "scattered" subword).

Example: $u = 101001$ $\qquad$ $v = 101$

$$\Rightarrow \binom{101001}{101} = 6$$

Remark:

Natural generalization of binomial coefficients of integers

If a is a letter,

$$\binom{a^m}{a^k} = \binom{\overbrace{a\cdots a}^{m\text{ times}}}{\underbrace{a\cdots a}_{k\text{ times}}} = \binom{m}{k} \quad \forall\, m,k \in \mathbb{N}\,.$$

Generalized Pascal triangles

Let $(A, <)$ be a totally ordered alphabet.
Let $L \subset A^*$ be an infinite language (set of words) over A.
The words in L are genealogically ordered

$$w_0 <_{\mathrm{gen}} w_1 <_{\mathrm{gen}} w_2 <_{\mathrm{gen}} \cdots .$$

The *generalized Pascal triangle* P_L associated with L is defined by

$$\mathrm{P}_L \colon (m, k) \in \mathbb{N} \times \mathbb{N} \mapsto \binom{w_m}{w_k} \in \mathbb{N} .$$

<u>Questions:</u>

- With a similar construction, can we expect the convergence to an analogue of the Sierpiński gasket?
- In particular, where should we cut to normalize a given generalized Pascal triangle?
- Could we describe this limit object?

Base 2

Definitions:

- $\text{rep}_2(n)$ greedy base-2 representation of $n \in \mathbb{N}_{>0}$ starting with 1
- $\text{rep}_2(0) = \varepsilon$ where ε is the empty word

n	$n = \sum_i c_i 2^i$ with $c_i \in \{0,1\}$	$\text{rep}_2(n)$
0		ε
1	1×2^0	1
2	$1 \times 2^1 + 0 \times 2^0$	10
3	$1 \times 2^1 + 1 \times 2^0$	11
4	$1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0$	100
5	$1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$	101
6	$1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$	110
$\vdots$	$\vdots$	$\vdots$
		$L_2 = 1\{0,1\}^* \cup \{\varepsilon\}$

Generalized Pascal triangle P_2 in base 2

	$\binom{\mathrm{rep}_2(m)}{\mathrm{rep}_2(k)}$	$\mathrm{rep}_2(k)$								
		ε	1	10	11	100	101	110	111	$\cdots$
	ε	1	0	0	0	0	0	0	0	
	1	1	1	0	0	0	0	0	0	
	10	1	1	1	0	0	0	0	0	
	11	1	**2**	0	**1**	0	0	0	0	
$\mathrm{rep}_2(m)$	100	1	1	2	0	1	0	0	0	
	101	1	2	1	1	0	1	0	0	
	110	1	2	2	1	0	0	1	0	
	111	1	3	0	**3**	0	0	0	1	
	$\vdots$									$\ddots$

Rule (not local):

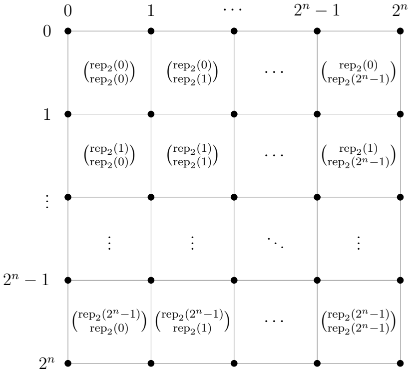
$$\binom{ua}{vb} = \binom{u}{vb} + \delta_{a,b}\binom{u}{v}$$

$\binom{\mathrm{rep}_2(m)}{\mathrm{rep}_2(k)}$		$\mathrm{rep}_2(k)$								
		ε	**1**	10	**11**	100	101	110	**111**	$\cdots$
	ε	**1**	0	0	0	0	0	0	0	
	1	**1**	**1**	0	0	0	0	0	0	
	10	1	1	1	0	0	0	0	0	
	11	**1**	**2**	0	**1**	0	0	0	0	
$\mathrm{rep}_2(m)$	100	1	1	2	0	1	0	0	0	
	101	1	2	1	1	0	1	0	0	
	110	1	2	2	1	0	0	1	0	
	111	**1**	**3**	0	**3**	0	0	0	**1**	
	$\vdots$									$\ddots$

The classical Pascal triangle

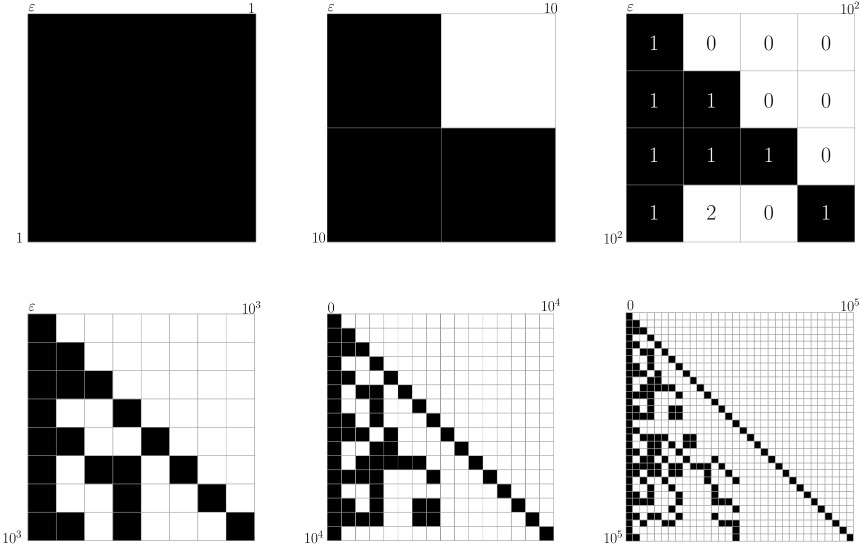
Same construction

- Grid: first 2^n rows and columns of P_2
- Color each square in
  - white if $\binom{\mathrm{rep}_2(m)}{\mathrm{rep}_2(k)} \equiv 0 \bmod 2$
  - black if $\binom{\mathrm{rep}_2(m)}{\mathrm{rep}_2(k)} \equiv 1 \bmod 2$
- Normalize by a homothety of ratio $1/2^n$ (bring into $[0,1]^2$) $\rightsquigarrow$ sequence of compact sets belonging to $[0,1]^2$

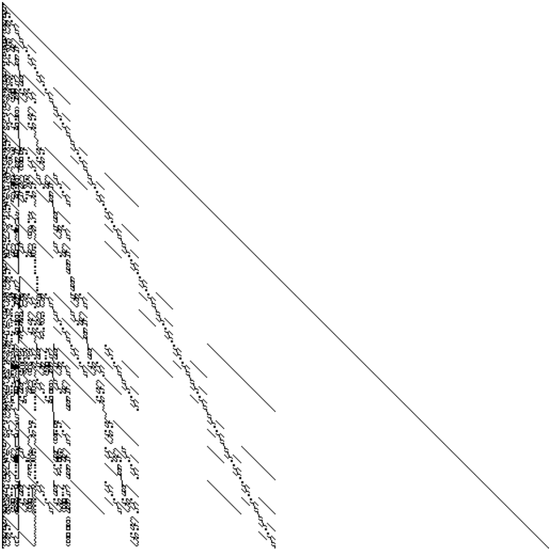


$\mathbb{N}^2 \cap [0, 2^n]^2$

$$U_n = \frac{1}{2^n} \bigcup_{\substack{u,v \in L_2,\ |u|,|v| \le n \\ \binom{u}{v} \equiv 1 \bmod 2}} \mathrm{val}_2(v,u) + [0,1]^2$$


ε
1
1
ε
10
10
ε
10^2
1 0 0 0
1 1 0 0
1 1 1 0
1 2 0 1
10^2
ε
10^3
10^3
0
10^4
10^4
0
10^5
10^5

The element U_9



Lines of different slopes: $1, 2, 4, 8, 16, \ldots$

The $(\star)$ condition

$(\star)$

$$(u,v) \text{ satisfies } (\star) \text{ iff } \begin{cases} u, v \neq \varepsilon \\ \binom{u}{v} \equiv 1 \bmod 2 \\ \binom{u}{v0} = 0 = \binom{u}{v1} \end{cases}$$

Example: $(u,v) = (101, 11)$ satisfies $(\star)$

$$\binom{101}{11} = 1 \qquad \binom{101}{110} = 0 \qquad \binom{101}{111} = 0$$

Lemma: Completion

(u, v) satisfies $(\star) \Rightarrow (u0, v0), (u1, v1)$ satisfy $(\star)$

Proof: Since (u, v) satisfies $(\star)$

$$\binom{u}{v} \equiv 1 \bmod 2, \quad \binom{u}{v0} = 0 = \binom{u}{v1}$$

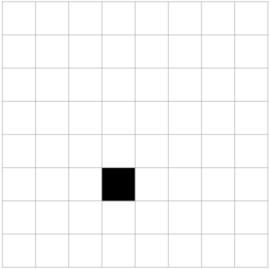
Proof for $(u0, v0)$:

$$\binom{u0}{v0} = \underbrace{\binom{u}{v0}}_{=0 \text{ since } (\star)} + \underbrace{\binom{u}{v}}_{\equiv 1 \bmod 2} \equiv 1 \bmod 2$$

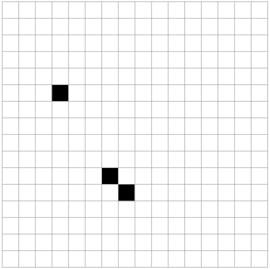
If $\binom{u0}{v00} > 0$ or $\binom{u0}{v01} > 0$, then $v0$ is a subsequence of u.
This contradicts $(\star)$.

Same proof for $(u1, v1)$. $\square$

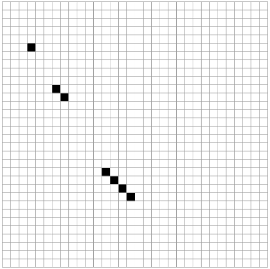
Example: $(u, v) = (101, 11)$ satisfies $(\star) \Rightarrow \binom{u}{v} \equiv 1 \bmod 2$



U_3



U_4

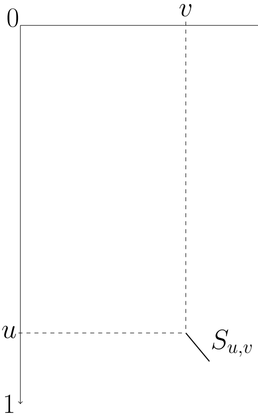


U_5

$\rightsquigarrow$ Creation of a segment of slope 1
Endpoint $(3/8, 5/8) = (\mathrm{val}_2(11)/2^3, \mathrm{val}_2(101)/2^3)$
Length $\sqrt{2} \cdot 2^{-3}$

Segments of slope 1

The $(\star)$ condition describes lines of slope 1 in $[0,1]^2$.



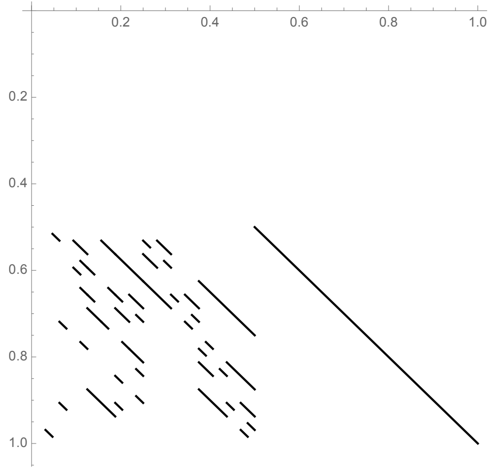
$(u,v) \in L_2 \times L_2$ satisfying $(\star)$
$\rightsquigarrow$ closed segment $S_{u,v}$

- slope 1
- length $\sqrt{2} \cdot 2^{-|u|}$
- origin

$$A_{u,v} = \mathrm{val}_2(v,u)/2^{|u|}$$
$$= (0.0^{|u|-|v|}v, 0.u)$$

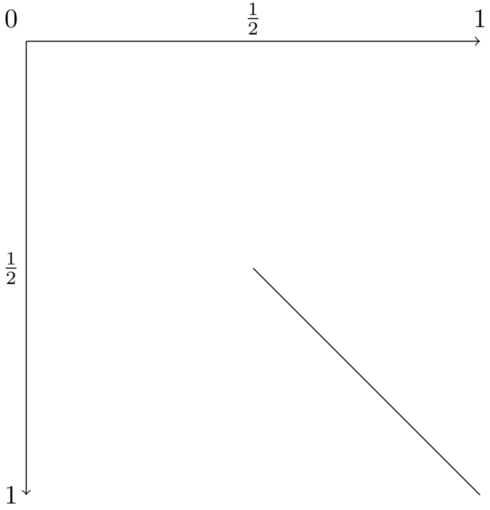
Definition: New compact set containing those lines

$$\mathcal{A}_0 = \overline{\bigcup_{\substack{(u,v)\\ \text{satisfying } (\star)}} S_{u,v}} \subset [0,1]^2$$



Modifying the slope

Two maps $c\colon (x,y) \mapsto (x/2, y/2)$ and $h\colon (x,y) \mapsto (x, 2y)$



Example: $(1,1)$ satisfies $(\star)$

Segment $S_{1,1}$
endpoint $(1/2, 1/2)$
length $\sqrt{2} \cdot 2^{-1}$

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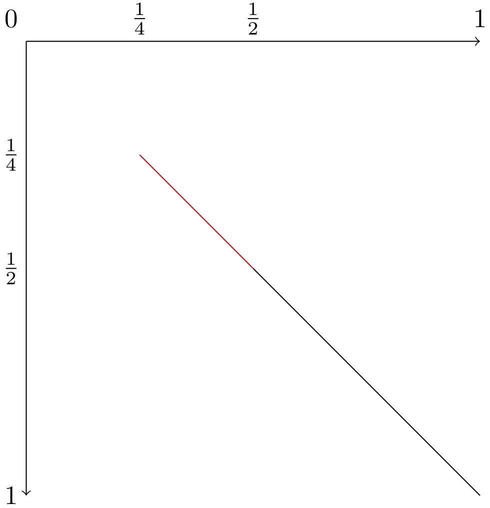

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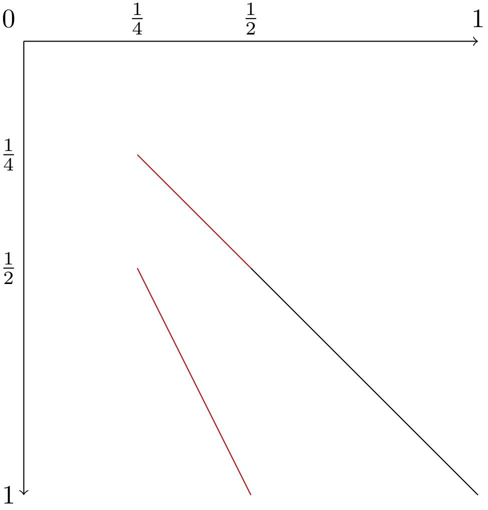

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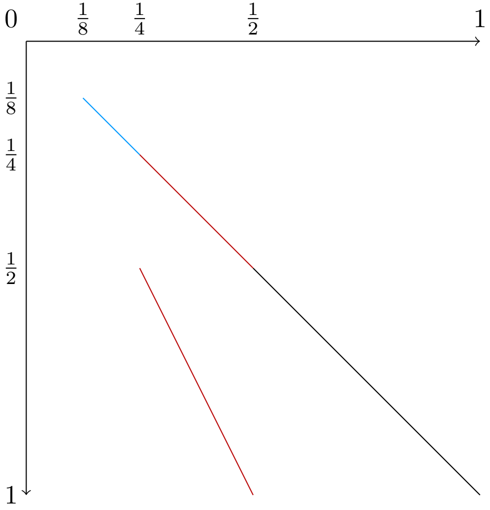
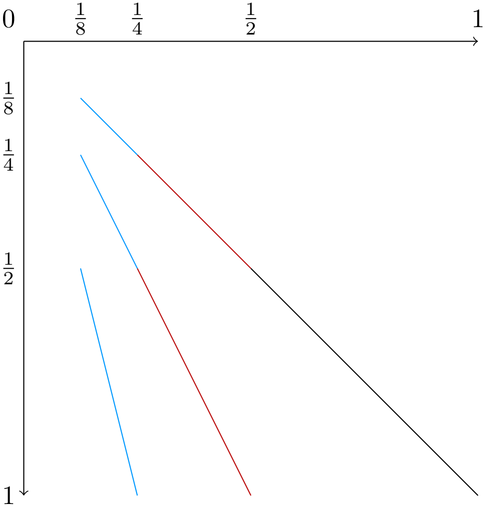


Modifying the slope

Two maps $c\colon (x,y) \mapsto (x/2, y/2)$ and $h\colon (x,y) \mapsto (x, 2y)$

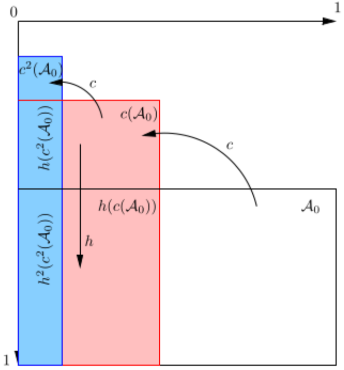


<u>Example</u>: $(1,1)$ satisfies $(\star)$

Segment $S_{1,1}$
endpoint $(1/2, 1/2)$
length $\sqrt{2} \cdot 2^{-1}$

Definition: New compact set containing lines of slopes 1, 2, 2^2, ..., 2^n
$c\colon (x,y) \mapsto (x/2, y/2)$
$h\colon (x,y) \mapsto (x, 2y)$

$$\mathcal{A}_n = \bigcup_{0 \le j \le i \le n} h^j(c^i(\mathcal{A}_0))$$



The compact sets $(\mathcal{A}_n)_{n\ge 0}$ are increasingly nested and their union is bounded.
Thus $(\mathcal{A}_n)_{n\ge 0}$ converges to

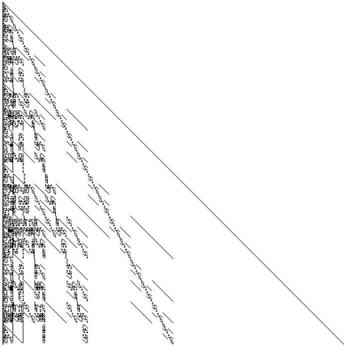
$$\mathcal{L} = \overline{\bigcup_{n\ge 0} \mathcal{A}_n}$$

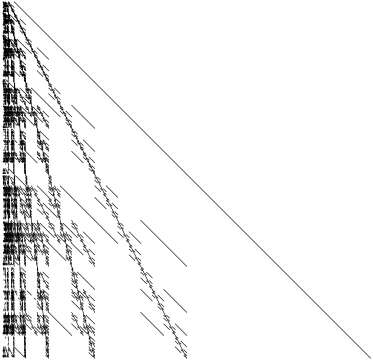
(w.r.t. the Hausdorff distance).

A key result

Theorem (Leroy, Rigo, S., 2016)

The sequence $(U_n)_{n\geq 0}$ of compact sets converges to the compact set $\mathcal{L}$ when n tends to infinity (w.r.t. the Hausdorff distance).





"Simple" characterization of $\mathcal{L}$: $(\star)$ condition

Extension modulo p

Previous result: even and odd coefficients

Theorem (Lucas, 1878)

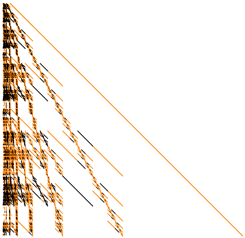
Let p be a prime number.
If $m = m_k p^k + \cdots + m_1 p + m_0$ and $n = n_k p^k + \cdots + n_1 p + n_0$ then

$$\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{m_i}{n_i} \bmod p.$$

Theorem (Leroy, Rigo, S., 2016)

Let p be a prime and $0 < r < p$.
When considering binomial coefficients congruent to $r \pmod p$, the sequence $(U_{n,p,r})_{n \geq 0}$ converges to a well-defined compact set $\mathcal{L}_{p,r}$ (w.r.t. the Hausdorff distance).

Example: $\mathcal{L}_{3,1} \cup \mathcal{L}_{3,2}$

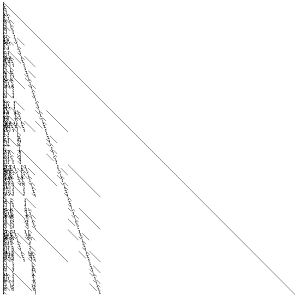


Extension to any integer base

Everything still holds for binomial coefficients $\equiv r \bmod p$ with

- integer base $b \geq 2$
- language L_b of greedy base-b representations of integers
- p a prime
- $r \in \{1, \ldots, p-1\}$

Example: base 3, $\equiv 1 \bmod 2$



Definitions:

- Fibonacci numbers $(F(n))_{n\geq 0}$
  $F(0) = 1,\ F(1) = 2,\ F(n+2) = F(n+1) + F(n) \quad \forall\, n \geq 0$
  $1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, \ldots$
- $\mathrm{rep}_F(n)$ greedy Fibonacci representation of $n \in \mathbb{N}_{>0}$ starting with 1
- $\mathrm{rep}_F(0) = \varepsilon$ where ε is the empty word

n	$n = \sum_i c_i F(i)$ with $c_i \in \{0,1\}$	$\mathrm{rep}_F(n)$
0		ε
1	$1 \times F(0)$	1
2	$1 \times F(1) + 0 \times F(0)$	10
3	$1 \times F(2) + 0 \times F(1) + 0 \times F(0)$	100
4	$1 \times F(2) + 0 \times F(1) + 1 \times F(0)$	101
5	$1 \times F(3) + 0 \times F(2) + 0 \times F(1) + 0 \times F(0)$	1000
6	$1 \times F(3) + 0 \times F(2) + 0 \times F(1) + 1 \times F(0)$	1001
$\vdots$	$\vdots$	$\vdots$
		$L_F = 1\{0, 01\}^* \cup \{\varepsilon\}$

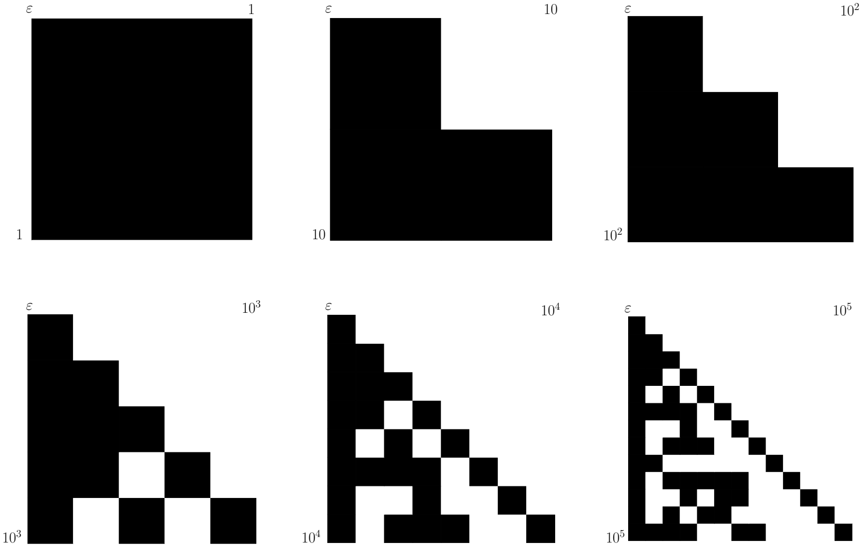
Generalized Pascal triangle P_F in Fibonacci base

$\binom{\mathrm{rep}_F(m)}{\mathrm{rep}_F(k)}$		$\mathrm{rep}_F(k)$								
		ε	1	10	100	101	1000	1001	1010	$\dots$
	ε	1	0	0	0	0	0	0	0	
	1	1	1	0	0	0	0	0	0	
	10	1	1	1	0	0	0	0	0	
	100	1	1	2	1	0	0	0	0	
$\mathrm{rep}_F(m)$	101	1	2	1	0	1	0	0	0	
	1000	1	1	3	3	0	1	0	0	
	1001	1	2	2	1	2	0	1	0	
	1010	1	2	3	1	1	0	0	1	
	$\vdots$									$\ddots$

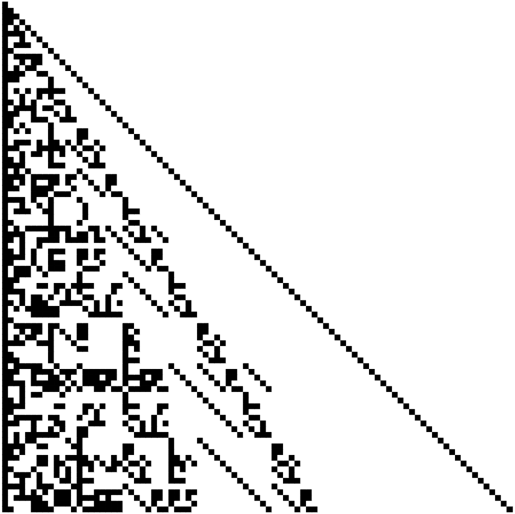
Rule (not local):

$$\binom{ua}{vb} = \binom{u}{vb} + \delta_{a,b}\binom{u}{v}$$

The first six elements of the sequence $(U'_n)_{n\geq 0}$

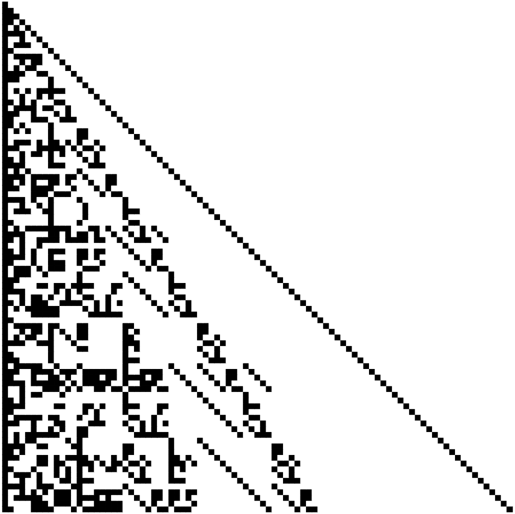


The tenth element



Lines of different slopes:

The tenth element



Lines of different slopes: φ^n, $n \geq 0$, with $\varphi = \frac{1+\sqrt{5}}{2}$ Golden Ratio

The $(\star')$ condition

Recall

- (u, v) satisfies $(\star)$ iff $\binom{u}{v} \equiv 1 \bmod 2$, and $\binom{u}{v0} = 0 = \binom{u}{v1}$.
- (u, v) satisfies $(\star) \Rightarrow (u0, v0), (u1, v1)$ satisfy $(\star)$

Problem: we cannot always add a letter 1 as a suffix in L_F.

Solution: $p(u, v) \in \mathbb{N}$ s.t. $u0^{p(u,v)}w, v0^{p(u,v)}w \in L_F$ for all $w \in 0^* L_F$

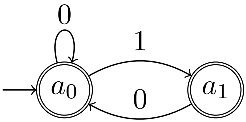
$(\star')$

$$(u, v) \text{ satisfies } (\star') \text{ iff } u = v = \varepsilon \text{ or } \begin{cases} u, v \neq \varepsilon \\ \binom{u0^{p(u,v)}}{v0^{p(u,v)}} \equiv 1 \pmod 2 \\ \binom{u0^{p(u,v)}}{v0^{p(u,v)}a} = 0 \quad \forall\, a \in \{0, 1\}. \end{cases}$$

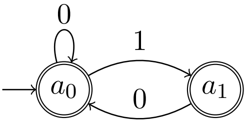
<u>Definition</u>: A deterministic finite automaton (DFA) over an alphabet A is given by a 5-tuple $\mathcal{A} = (Q, q_0, F, A, \delta)$ where

- Q is a finite set of states
- $q_0 \in Q$ is the initial state (incoming arrow)
- $F \subset Q$ is the set of final states (concentric circles)
- $\delta \colon Q \times A \mapsto Q$ is the transition function

<u>Example</u>:



$A = \{0, 1\}$
$Q = \{a_0, a_1\}$
$q_0 = a_0$
$F = \{a_0, a_1\}$
$\delta(a_0, 0) = a_0,\ \delta(a_0, 1) = a_1,\ \delta(a_1, 0) = a_0$



Reading words in the automaton:

100101	$a_0 \xrightarrow{1} a_1 \xrightarrow{0} a_0 \xrightarrow{0} a_0 \xrightarrow{1} a_1 \xrightarrow{0} a_0 \xrightarrow{1} a_1$	✓	accepted
1011	$a_0 \xrightarrow{1} a_1 \xrightarrow{0} a_0 \xrightarrow{1} a_1 \xrightarrow{1} ???$	✗	not accepted
000101	$a_0 \xrightarrow{0} a_0 \xrightarrow{0} a_0 \xrightarrow{0} a_0 \xrightarrow{1} a_1 \xrightarrow{0} a_0 \xrightarrow{1} a_1$	✓	accepted

Accepted language (set of words): 0^*L_F with $L_F = 1\{0,01\}^* \cup \{\varepsilon\}$

Definition: $p(u,v)$ is the smallest p s.t. $\delta(a_0, u0^p) = a_0 = \delta(a_0, v0^p)$

Then $u0^{p(u,v)}w, v0^{p(u,v)}w \in L_F$ for all $w \in 0^*L_F$.

Similar technique

- Completion lemma with the $(\star')$ condition
- Creation of segments of slope 1
- New compact set $\mathcal{A}'_0$ containing those lines

$$\mathcal{A}'_0 = \overline{\bigcup_{\substack{(u,v) \\ \text{satisfying } (\star')}} S_{u,v}} \subset [0,1]^2$$

- Modification of the slopes with $c\colon (x,y) \mapsto (x/\varphi, y/\varphi)$ and $h\colon (x,y) \mapsto (x, \varphi y)$
- New compact set $\mathcal{A}'_n$ containing lines of slopes $1, \varphi, \varphi^2, \ldots, \varphi^n$

$$\mathcal{A}'_n = \bigcup_{0 \le j \le i \le n} h^j(c^i(\mathcal{A}'_0))$$

- $(\mathcal{A}'_n)_{n\ge 0}$ converges to

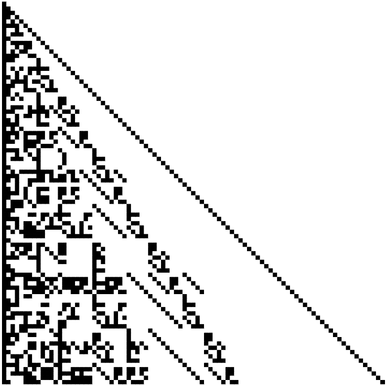
$$\mathcal{L}' = \overline{\bigcup_{n\ge 0} \mathcal{A}'_n}$$

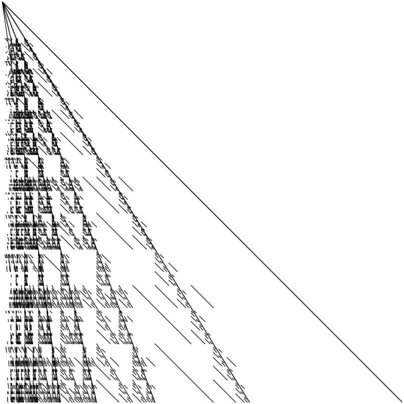
(w.r.t. the Hausdorff distance).

Similar convergence result

Theorem (S., 2018)

The sequence $(U'_n)_{n\geq 0}$ of compact sets converges to the compact set $\mathcal{L}'$ when n tends to infinity (w.r.t. the Hausdorff distance).





"Simple" characterization of $\mathcal{L}'$: $(\star')$ condition

Linear numeration system

Definition

A *numeration system* is a sequence $U = (U(n))_{n\geq 0}$ of integers s.t.

- U increasing
- $U(0) = 1$
- $\sup_{n\geq 0} \frac{U(n+1)}{U(n)}$ bounded by a constant $\rightsquigarrow$ finite alphabet.

A numeration system U is *linear* if $\exists k \geq 1, \exists a_0, \ldots, a_{k-1} \in \mathbb{Z}$ s.t.

$$U(n+k) = a_{k-1}\, U(n+k-1) + \cdots + a_0\, U(n) \quad \forall\, n \geq 0.$$

Greedy representation in $(U(n))_{n\geq 0}$:

$$n = \sum_{i=0}^{\ell} c_i U(i) \qquad \text{with} \qquad \sum_{i=0}^{j-1} c_i U(i) < U(j)$$

$$\text{rep}_U(n) = c_\ell \cdots c_0 \in \underbrace{L_U = \text{rep}_U(\mathbb{N})}_{\text{numeration language}}$$

Example: integer base $(b^n)_{n\geq 0}$ with $b \in \mathbb{N}_{>1}$

Fibonacci numeration system $(F(n))_{n\geq 0}$

Parry numbers

$\beta \in \mathbb{R}_{>1}$ $\qquad A_\beta = \{0, 1, \ldots, \lceil \beta \rceil - 1\}$

$$x \in [0,1] \rightsquigarrow x = \sum_{j=1}^{+\infty} c_j \beta^{-j}, \quad c_j \in A_\beta$$

Greedy way: $c_j \beta^{-j} + c_{j+1}\beta^{-j-1} + \cdots < \beta^{-(j-1)}$
β-expansion of x: $d_\beta(x) = c_1 c_2 c_3 \cdots$

Definition

$\beta \in \mathbb{R}_{>1}$ is a *Parry number* if $d_\beta(1)$ is ultimately periodic.

Example: $b \in \mathbb{N}_{>1}$: $d_b(1) = (b-1)^\omega$
Golden ratio φ: $d_\varphi(1) = 110^\omega$

Parry numeration system

Parry number $\beta \in \mathbb{R}_{>1} \rightsquigarrow$ linear numeration system $(U_\beta(n))_{n\geq 0}$

- $d_\beta(1) = t_1 \cdots t_m 0^\omega$

$$\begin{array}{lcll} U_\beta(0) & = & 1 & \\ U_\beta(i) & = & t_1 U_\beta(i-1) + \cdots + t_i U_\beta(0) + 1 & \forall\, 1 \leq i \leq m-1 \\ U_\beta(n) & = & t_1 U_\beta(n-1) + \cdots + t_m U_\beta(n-m) & \forall\, n \geq m \end{array}$$

- $d_\beta(1) = t_1 \cdots t_m (t_{m+1} \cdots t_{m+k})^\omega$

$$\begin{array}{lcll} U_\beta(0) & = & 1 & \\ U_\beta(i) & = & t_1 U_\beta(i-1) + \cdots + t_i U_\beta(0) + 1 & \forall\, 1 \leq i \leq m+k-1 \\ U_\beta(n) & = & t_1 U_\beta(n-1) + \cdots + t_{m+k} U_\beta(n-m-k) & \forall\, n \geq m+k \\ & & +U_\beta(n-k) & \\ & & -t_1 U_\beta(n-k-1) - \cdots - t_m U_\beta(n-m-k) & \end{array}$$

Examples:

$b \in \mathbb{N}_{>1} \rightsquigarrow (b^n)_{n\geq 0}$ base b

Golden ratio $\varphi \rightsquigarrow (F(n))_{n\geq 0}$ Fibonacci numeration system

Extension to Parry numeration systems

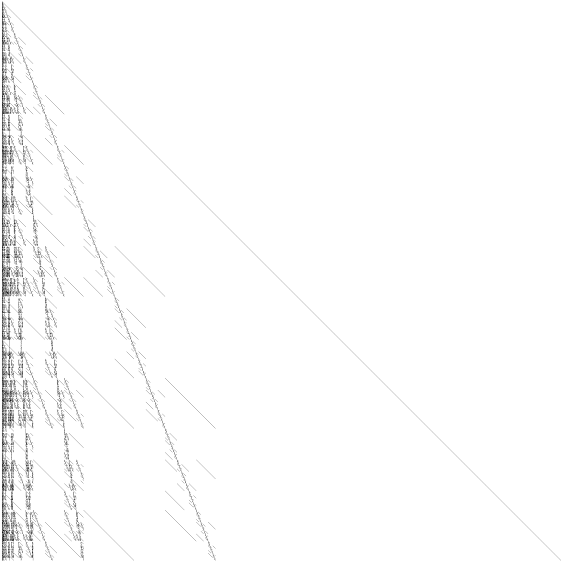
- Parry number $\beta \in \mathbb{R}_{>1}$
- Parry numeration system $(U_\beta(n))_{n\geq 0}$
- Numeration language L_{U_β}
- Generalized Pascal triangle P_β in $(U_\beta(n))_{n\geq 0}$ indexed by words of L_{U_β}
- Sequence of compact sets extracted from P_β (first $U_\beta(n)$ rows and columns of P_β)
- Convergence to a limit object (same technique)
  - Lines of different slopes: β^n, $n \geq 0$
  - $(\star')$ condition and description of segments of slope 1
  - Two maps $c\colon (x,y) \mapsto (x/\beta, y/\beta)$ and $h\colon (x,y) \mapsto (x, \beta y)$
  - Sequence of sets $\mathcal{A}_n^\beta$ containing lines of slopes 1, β, β^2, $\ldots$, β^n
  - $\mathcal{A}_n^\beta$ converges to

$$\mathcal{L}^\beta = \overline{\bigcup_{n\geq 0} \mathcal{A}_n^\beta}$$

- Works modulo any prime number

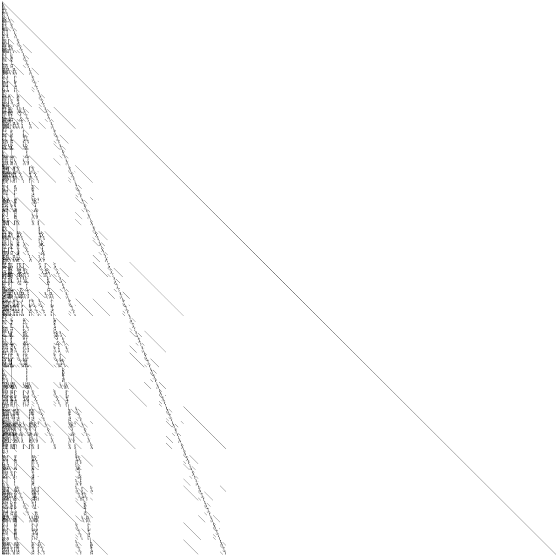
Example 1

φ^2



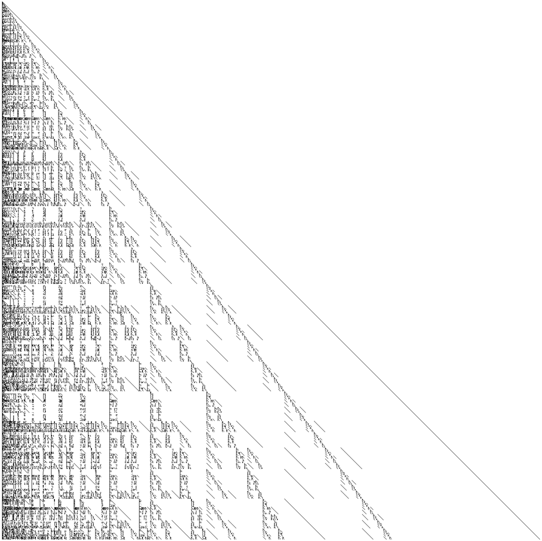
Example 2

$\beta_1 \approx 2.47098$ dominant root of $P(X) = X^4 - 2X^3 - X^2 - 1$



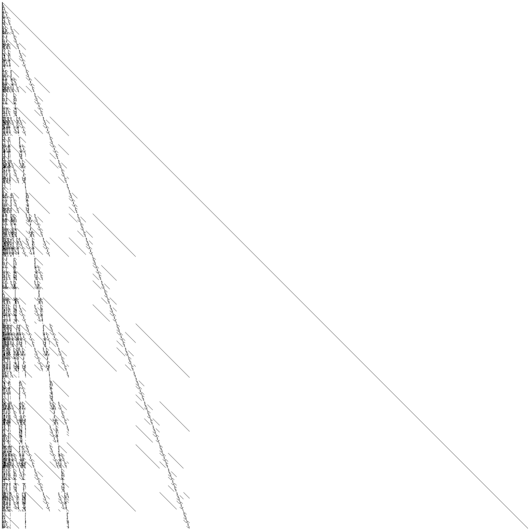
Example 3

$\beta_2 \approx 1.38028$ dominant root of $P(X) = X^4 - X^3 - 1$



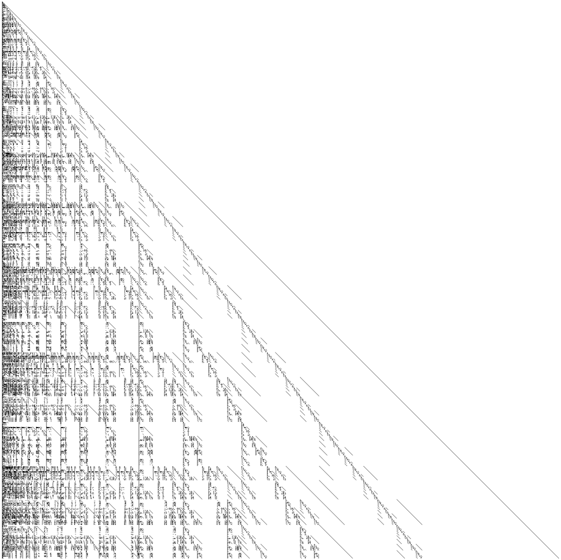
Example 4

$\beta_3 \approx 2.80399$ dominant root of $P(X) = X^4 - 2X^3 - 2X^2 - 2$



Example 5

$\beta_4 \approx 1.32472$ dominant root of polynomial $P(X) = X^5 - X^4 - 1$



In this talk:

Numeration system	Generalized Pascal triangle	Convergence mod p
Base 2	✓	✓
Integer base	✓	✓
Fibonacci	✓	✓
Parry	✓	✓

Other related work

- Regularity of the sequence counting subword occurrences: result for any integer base b and the Fibonacci numeration system
- Behavior of the summatory function: result for any integer base b (exact behavior) and the Fibonacci numeration system (asymptotics)

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