Extensions of the Pascal triangle to words
Joint work with Julien Leroy and Michel Rigo (ULiège)

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Classical Pascal triangle

\[ P : (m, k) \in \mathbb{N} \times \mathbb{N} \mapsto \binom{m}{k} \in \mathbb{N} \]

<table>
<thead>
<tr>
<th>( \binom{m}{k} )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>( \ldots )</th>
</tr>
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<tbody>
<tr>
<td>0</td>
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<td>7</td>
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<td>35</td>
<td>21</td>
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<td>1</td>
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<tr>
<td>( \vdots )</td>
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<td></td>
<td></td>
<td>( \vdots )</td>
</tr>
</tbody>
</table>

Binomial coefficients:

\[ \binom{m}{k} = \frac{m!}{(m-k)!k!} \]

Pascal’s rule:

\[ \binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1} \]

Generalized Pascal triangles

Manon Stipulanti (ULiège)
A specific construction

- Grid: first $2^n$ rows and columns of the Pascal triangle

\[
\left(\left(\begin{array}{c} m \\ k \end{array}\right) \mod 2 \right)_{0 \leq m, k < 2^n}
\]
A specific construction

- **Grid:** first $2^n$ rows and columns of the Pascal triangle
  \[
  \left( \binom{m}{k} \mod 2 \right)_{0 \leq m, k < 2^n}
  \]

- **Color each square in**
  - white if $\binom{m}{k} \equiv 0 \mod 2$
  - black if $\binom{m}{k} \equiv 1 \mod 2$

\[\begin{array}{cccc}
0 & 1 & \ldots & 2^n - 1 & 2^n \\
\hline
0 & (\binom{0}{0}) & (\binom{0}{1}) & \ldots & (\binom{0}{2^n - 1}) \\
1 & (\binom{1}{0}) & (\binom{1}{1}) & \ldots & (\binom{1}{2^n - 1}) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
2^n - 1 & (\binom{2^n - 1}{0}) & (\binom{2^n - 1}{1}) & \ldots & (\binom{2^n - 1}{2^n - 1}) \\
2^n & N^2 \cap [0, 2^n]^2
\end{array}\]
A specific construction

- Grid: first $2^n$ rows and columns of the Pascal triangle
  
  $\left( \left( \begin{array}{c} m \\ k \end{array} \right) \mod 2 \right)_{0 \leq m, k < 2^n}$

- Color each square in
  - white if $\left( \begin{array}{c} m \\ k \end{array} \right) \equiv 0 \mod 2$
  - black if $\left( \begin{array}{c} m \\ k \end{array} \right) \equiv 1 \mod 2$

- Normalize by a homothety of ratio $1/2^n$
  (bring into $[0, 1]^2$)
A specific construction

- **Grid:** first $2^n$ rows and columns of the Pascal triangle

\[
\left( \binom{m}{k} \mod 2 \right)_{0 \leq m, k < 2^n}
\]

- **Color each square in**
  - white if \( \binom{m}{k} \equiv 0 \mod 2 \)
  - black if \( \binom{m}{k} \equiv 1 \mod 2 \)

- **Normalize by a homothety** of ratio $1/2^n$
  (bring into \([0, 1]^2\))

\( \leadsto \) sequence of compact sets belonging to \([0, 1]^2\)
The first six elements of the sequence

Generalized Pascal triangles

Manon Stipulanti (ULiège)
The Sierpiński gasket
The Sierpiński gasket

Generalized Pascal triangles

Manon Stipulanti (ULiège)
Folklore fact

The latter sequence of compact sets converges to the Sierpiński gasket (w.r.t. the Hausdorff distance).

Definitions:

- \( \epsilon \)-fattening of a subset \( S \subset \mathbb{R}^2 \)

\[
[S]_\epsilon = \bigcup_{x \in S} B(x, \epsilon)
\]

- \((\mathcal{H}(\mathbb{R}^2), d_h)\) complete space of the non-empty compact subsets of \( \mathbb{R}^2 \) equipped with the Hausdorff distance \( d_h \)

\[
d_h(S, S') = \inf\{\epsilon \in \mathbb{R}_{>0} \mid S \subset [S']_\epsilon \text{ and } S' \subset [S]_\epsilon\}
\]
Theorem (von Haeseler, Peitgen, and Skordev, 1992)

Let $p$ be a prime and $s > 0$. The sequence of compact sets corresponding to

$$
\left( \binom{m}{k} \mod p^s \right)_{0 \leq m, k < p^n}
$$

converges when $n$ tends to infinity (w.r.t. the Hausdorff distance).

$p = 2, \ s = 1$

$p = 2, \ s = 2$

$p = 2, \ s = 3$
New idea: CoW

Replace integers by finite words.

Combinatorics on words (CoW)

- new area of discrete mathematics (± 1900)
- study sequences of symbols (called letters)
- topics include:
  - regularities and patterns in words
  - important types of words (e.g. automatic, regular, de Bruijn, Lyndon, Sturmian)
  - coding of structures (e.g. paths, trees or curves in the plane)

M. Lothaire, 1983.
Binomial coefficient of finite words

Definition: A finite word is a finite sequence of letters belonging to a finite set called alphabet.

Example: $101, 101001 \in \{0, 1\}^*$

Binomial coefficient of words

Let $u, v$ be two finite words.
The binomial coefficient $\binom{u}{v}$ of $u$ and $v$ is the number of times $v$ occurs as a subsequence of $u$ (meaning as a “scattered” subword).
**Binomial coefficient of finite words**

**Definition:** A *finite word* is a finite sequence of letters belonging to a finite set called *alphabet*.

**Example:** 101, 101001 ∈ \{0, 1\}*

---

**Binomial coefficient of words**

Let \( u, v \) be two finite words. The *binomial coefficient* \( ^u_v \) of \( u \) and \( v \) is the number of times \( v \) occurs as a subsequence of \( u \) (meaning as a “scattered” subword).

**Example:** \( u = 101001 \quad v = 101 \)
Definition: A finite word is a finite sequence of letters belonging to a finite set called alphabet.

Example: 101, 101001 ∈ \{0, 1\}^*

Binomial coefficient of words

Let \( u, v \) be two finite words.
The binomial coefficient \( \binom{u}{v} \) of \( u \) and \( v \) is the number of times \( v \) occurs as a subsequence of \( u \) (meaning as a “scattered” subword).

Example: \( u = 101001 \quad v = 101 \quad 1 \) occurrence
**Definition:** A *finite word* is a finite sequence of letters belonging to a finite set called *alphabet*.

**Example:** $101, 101001 \in \{0, 1\}^*$

---

**Binomial coefficient of words**

Let $u, v$ be two finite words.

The *binomial coefficient* $\binom{u}{v}$ of $u$ and $v$ is the number of times $v$ occurs as a subsequence of $u$ (meaning as a “scattered” subword).

**Example:** $u = \textbf{101001}$  
$v = 101$  
2 occurrences
Binomial coefficient of finite words

**Definition:** A *finite word* is a finite sequence of letters belonging to a finite set called *alphabet*.

**Example:** 101, 101001 ∈ {0, 1}*

---

**Binomial coefficient of words**

Let \( u, v \) be two finite words.

The *binomial coefficient* \( \binom{u}{v} \) of \( u \) and \( v \) is the number of times \( v \) occurs as a subsequence of \( u \) (meaning as a “scattered” subword).

**Example:** \( u = 101001 \) \( v = 101 \) 3 occurrences
Definition: A finite word is a finite sequence of letters belonging to a finite set called alphabet.

Example: 101, 101001 ∈ \{0, 1\}*

Binomial coefficient of words

Let \( u, v \) be two finite words.

The binomial coefficient \( \binom{u}{v} \) of \( u \) and \( v \) is the number of times \( v \) occurs as a subsequence of \( u \) (meaning as a “scattered” subword).

Example: \( u = 101001 \quad v = 101 \quad 4 \) occurrences
Binomial coefficient of finite words

**Definition:** A *finite word* is a finite sequence of letters belonging to a finite set called *alphabet*.

**Example:** 101, 101001 ∈ \{0, 1\}*

---

**Binomial coefficient of words**

Let $u, v$ be two finite words. The *binomial coefficient* $\binom{u}{v}$ of $u$ and $v$ is the number of times $v$ occurs as a subsequence of $u$ (meaning as a “scattered” subword).

**Example:** $u = 101001$ $v = 101$ 5 occurrences
Binomial coefficient of finite words

**Definition:** A finite word is a finite sequence of letters belonging to a finite set called alphabet.

**Example:** 101, 101001 ∈ \(\{0, 1\}\)^*

---

**Binomial coefficient of words**

Let \(u, v\) be two finite words. The binomial coefficient \(\binom{u}{v}\) of \(u\) and \(v\) is the number of times \(v\) occurs as a subsequence of \(u\) (meaning as a “scattered” subword).

**Example:** \(u = 10\textbf{1}001\) \(v = 101\) 6 occurrences
Definition: A finite word is a finite sequence of letters belonging to a finite set called alphabet.

Example: 101, 101001 ∈ {0, 1}*

Binomial coefficient of words

Let $u, v$ be two finite words. The binomial coefficient $(^u_v)$ of $u$ and $v$ is the number of times $v$ occurs as a subsequence of $u$ (meaning as a “scattered” subword).

Example: $u = 101001$ $v = 101$

$$\Rightarrow \binom{101001}{101} = 6$$
Remark:
Natural generalization of binomial coefficients of integers

If \( a \) is a letter,

\[
\binom{a^m}{a^k} = \binom{a \cdots a}{a \cdots a} = \binom{m}{k} \quad \forall m, k \in \mathbb{N}.
\]
Let \((A, <)\) be a totally ordered alphabet. Let \(L \subset A^*\) be an infinite language (set of words) over \(A\). The words in \(L\) are genealogically ordered

\[w_0 <_{\text{gen}} w_1 <_{\text{gen}} w_2 <_{\text{gen}} \cdots.\]

The generalized Pascal triangle \(P_L\) associated with \(L\) is defined by

\[P_L: (m, k) \in \mathbb{N} \times \mathbb{N} \mapsto \binom{w_m}{w_k} \in \mathbb{N}.\]

Questions:

- With a similar construction, can we expect the convergence to an analogue of the Sierpiński gasket?
- In particular, where should we cut to normalize a given generalized Pascal triangle?
- Could we describe this limit object?
**Base 2**

**Definitions:**
- \( \text{rep}_2(n) \) greedy base-2 representation of \( n \in \mathbb{N}_{>0} \) starting with 1
- \( \text{rep}_2(0) = \varepsilon \) where \( \varepsilon \) is the empty word

<table>
<thead>
<tr>
<th>( n )</th>
<th>( n = \sum_i c_i 2^i ) with ( c_i \in {0, 1} )</th>
<th>( \text{rep}_2(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>( \varepsilon )</td>
</tr>
<tr>
<td>1</td>
<td>( 1 \times 2^0 )</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>( 1 \times 2^1 + 0 \times 2^0 )</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>( 1 \times 2^1 + 1 \times 2^0 )</td>
<td>11</td>
</tr>
<tr>
<td>4</td>
<td>( 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0 )</td>
<td>100</td>
</tr>
<tr>
<td>5</td>
<td>( 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 )</td>
<td>101</td>
</tr>
<tr>
<td>6</td>
<td>( 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 )</td>
<td>110</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>( L_2 = 1{0, 1}^* \cup {\varepsilon} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
# Generalized Pascal triangle $P_2$ in base 2

The table below illustrates the construction of the generalized Pascal triangle $P_2$ in base 2. Each entry $P(x,y)$ is computed using the rule:

$$P(x,y) = P(x, y_0) + P(x_0, y)$$

where $x$ and $y$ are binary representations of numbers, and $P(x_0, y)$ and $P(x, y_0)$ are the entries in the row corresponding to $x$ and column corresponding to $y$, respectively.

## Rule (not local):

$$P(x_0, y_0) = P(x_0, y) + \delta_{a,b} P(x, y_0)$$

<table>
<thead>
<tr>
<th>$rep_2(m)$</th>
<th>$rep_2(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon$</td>
<td>$\varepsilon$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
</tr>
<tr>
<td>100</td>
<td>1</td>
</tr>
<tr>
<td>101</td>
<td>1</td>
</tr>
<tr>
<td>110</td>
<td>1</td>
</tr>
<tr>
<td>111</td>
<td>1</td>
</tr>
</tbody>
</table>

...
<table>
<thead>
<tr>
<th>$\text{rep}_2(m)$</th>
<th>$\text{rep}_2(k)$</th>
<th>(\varepsilon)</th>
<th>1</th>
<th>10</th>
<th>11</th>
<th>100</th>
<th>101</th>
<th>110</th>
<th>111</th>
<th>$\cdots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\cdots$</td>
</tr>
<tr>
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<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\cdots$</td>
</tr>
<tr>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>$\text{rep}_2(m)$</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>101</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>110</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\cdots$</td>
</tr>
</tbody>
</table>

$\vdots$ $\vdots$ $\vdots$

The classical Pascal triangle
• Grid: first $2^n$ rows and columns of $P_2$

• Color each square in
  • white if $\binom{\text{rep}_2(m)}{\text{rep}_2(k)} \equiv 0 \mod 2$
  • black if $\binom{\text{rep}_2(m)}{\text{rep}_2(k)} \equiv 1 \mod 2$

• Normalize by a homothety of ratio $1/2^n$
  (bring into $[0, 1]^2$)
  $\leadsto$ sequence of compact sets belonging to $[0, 1]^2$

$$U_n = \frac{1}{2^n} \bigcup_{u,v \in L_2, |u|, |v| \leq n} \text{val}_2(v, u) + [0, 1]^2$$
The elements $U_0, \ldots, U_5$

Generalized Pascal triangles

Manon Stipulanti (ULiège)
The element $U_9$

Lines of different slopes: 1, 2, 4, 8, 16, ...
The (⋆) condition

(⋆)

(u, v) satisfies (⋆) iff

\[
\begin{align*}
&u, v \neq \varepsilon \\
&\binom{u}{v} \equiv 1 \mod 2 \\
&\binom{u}{v_0} = 0 = \binom{u}{v_1}
\end{align*}
\]

Example: (u, v) = (101, 11) satisfies (⋆)

\[
\begin{align*}
\binom{101}{11} &= 1 & \binom{101}{110} &= 0 & \binom{101}{111} &= 0
\end{align*}
\]
Lemma: Completion

\((u, v)\) satisfies \((\ast)\) \(\Rightarrow\) \((u_0, v_0), (u_1, v_1)\) satisfy \((\ast)\)

Proof: Since \((u, v)\) satisfies \((\ast)\)

\[
\binom{u}{v} \equiv 1 \text{ mod } 2, \quad \binom{u}{v_0} = 0 = \binom{u}{v_1}
\]

Proof for \((u_0, v_0)\):

\[
\binom{u_0}{v_0} = \underbrace{\binom{u}{v_0}}_{=0 \text{ since } (\ast)} + \underbrace{\binom{u}{v}}_{\equiv 1 \text{ mod } 2} \equiv 1 \text{ mod } 2
\]

If \(\binom{u_0}{v_0} > 0\) or \(\binom{u_0}{v_{01}} > 0\), then \(v_0\) is a subsequence of \(u\). This contradicts \((\ast)\).

Same proof for \((u_1, v_1)\). \(\square\)
Example: \((u, v) = (101, 11)\) satisfies \((\star) \Rightarrow \binom{u}{v} \equiv 1 \mod 2\)

\[U_3\] \[U_4\] \[U_5\]

\(\leadsto\) Creation of a segment of slope 1
Endpoint \((3/8, 5/8) = (\text{val}_2(11)/2^3, \text{val}_2(101)/2^3)\)
Length \(\sqrt{2} \cdot 2^{-3}\)
The (⋆) condition describes lines of slope 1 in \([0, 1]^2\).

\[(u, v) \in L_2 \times L_2 \text{ satisfying } (\star) \implies \text{closed segment } S_{u,v} \]

- slope 1
- length \(\sqrt{2} \cdot 2^{-|u|}\)
- origin

\[A_{u,v} = \text{val}_2(v, u) / 2^{|u|} = (0.0^{|u|}-|v|, v, 0.u)\]
Definition: New compact set containing those lines

\[ A_0 = \bigcup_{(u, v)} S_{u, v} \subset [0, 1]^2 \]

satisfying \((\ast)\)
Modifying the slope

Two maps $c: (x, y) \mapsto (x/2, y/2)$ and $h: (x, y) \mapsto (x, 2y)$

Example: $(1, 1)$ satisfies $(\ast)$

Segment $S_{1,1}$
endpoint $(1/2, 1/2)$
length $\sqrt{2} \cdot 2^{-1}$
Modifying the slope

Two maps $c: (x, y) \mapsto (x/2, y/2)$ and $h: (x, y) \mapsto (x, 2y)$

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Two maps \( c: (x, y) \mapsto (x/2, y/2) \) and \( h: (x, y) \mapsto (x, 2y) \)

Example: \((1, 1)\) satisfies \((\ast)\)

Segment \( S_{1,1} \)
endpoint \((1/2, 1/2)\)
length \(\sqrt{2} \cdot 2^{-1}\)
Definition: New compact set containing lines of slopes $1$, $2$, $2^2$, $\ldots$, $2^n$
$c: (x, y) \mapsto (x/2, y/2)$
$h: (x, y) \mapsto (x, 2y)$

\[ \mathcal{A}_n = \bigcup_{0 \leq j \leq i \leq n} h^j(c^i(A_0)) \]

The compact sets $(\mathcal{A}_n)_{n \geq 0}$ are increasingly nested and their union is bounded.
Thus $(\mathcal{A}_n)_{n \geq 0}$ converges to
\[ \mathcal{L} = \bigcup_{n \geq 0} \mathcal{A}_n \]
(w.r.t. the Hausdorff distance).
A key result

Theorem (Leroy, Rigo, S., 2016)

The sequence $\left( U_n \right)_{n \geq 0}$ of compact sets converges to the compact set $\mathcal{L}$ when $n$ tends to infinity (w.r.t. the Hausdorff distance).

“Simple” characterization of $\mathcal{L}$: ($\star$) condition
Extension modulo $p$

Previous result: even and odd coefficients

**Theorem** (Lucas, 1878)

Let $p$ be a prime number.

If $m = m_k p^k + \cdots + m_1 p + m_0$ and $n = n_k p^k + \cdots + n_1 p + n_0$ then

$$\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{m_i}{n_i} \mod p.$$ 

**Theorem** (Leroy, Rigo, S., 2016)

Let $p$ be a prime and $0 < r < p$.

When considering binomial coefficients congruent to $r \pmod{p}$, the sequence $(U_{n,p,r})_{n \geq 0}$ converges to a well-defined compact set $\mathcal{L}_{p,r}$ (w.r.t. the Hausdorff distance).

Example: $\mathcal{L}_{3,1} \cup \mathcal{L}_{3,2}$
Extension to any integer base

Everything still holds for binomial coefficients \( \equiv r \mod p \) with

- integer base \( b \geq 2 \)
- language \( L_b \) of greedy base-\( b \) representations of integers
- \( p \) a prime
- \( r \in \{1, \ldots, p - 1\} \)

Example: base 3, \( \equiv 1 \mod 2 \)
**Fibonacci numeration system**

**Definitions:**

- **Fibonacci numbers** \((F(n))_{n \geq 0}\)
  
  \[ F(0) = 1, \quad F(1) = 2, \quad F(n + 2) = F(n + 1) + F(n) \quad \forall n \geq 0 \]
  
  \[
  1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, \ldots
  \]

- **rep\(_F(n)\)** greedy Fibonacci representation of \(n \in \mathbb{N}_{>0}\) starting with 1

- **rep\(_F(0) = \varepsilon\)** where \(\varepsilon\) is the empty word

<table>
<thead>
<tr>
<th>(n)</th>
<th>(n = \sum_i c_i F(i) ) with (c_i \in {0, 1})</th>
<th>rep(_F(n))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>(\varepsilon)</td>
</tr>
<tr>
<td>1</td>
<td>1 × (F(0))</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1 × (F(1)) + 0 × (F(0))</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>1 × (F(2)) + 0 × (F(1)) + 0 × (F(0))</td>
<td>100</td>
</tr>
<tr>
<td>4</td>
<td>1 × (F(2)) + 0 × (F(1)) + 1 × (F(0))</td>
<td>101</td>
</tr>
<tr>
<td>5</td>
<td>1 × (F(3)) + 0 × (F(2)) + 0 × (F(1)) + 0 × (F(0))</td>
<td>1000</td>
</tr>
<tr>
<td>6</td>
<td>1 × (F(3)) + 0 × (F(2)) + 0 × (F(1)) + 1 × (F(0))</td>
<td>1001</td>
</tr>
<tr>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
</tr>
</tbody>
</table>

\[ L_F = 1\{0, 01\}^* \cup \{\varepsilon\} \]
Generalized Pascal triangle $P_F$ in Fibonacci base

<table>
<thead>
<tr>
<th>$(\text{rep}_F(m))$</th>
<th>$(\text{rep}_F(k))$</th>
<th>$\text{rep}_F(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon$</td>
<td>$\varepsilon$</td>
<td>$\varepsilon$</td>
</tr>
<tr>
<td>1</td>
<td>1 0 0 0 0 0 0</td>
<td>0 0 0 0</td>
</tr>
<tr>
<td>10</td>
<td>1 1 1 0 0 0 0</td>
<td>0 0 0 0</td>
</tr>
<tr>
<td>100</td>
<td>1 1 1 1 0 0 0</td>
<td>0 0 0 0</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>101</td>
<td>1 2 1 0 1 0 0</td>
<td>0 0 0 0</td>
</tr>
<tr>
<td>1000</td>
<td>1 1 3 3 0 1 0</td>
<td>0 0 0 0</td>
</tr>
<tr>
<td>1001</td>
<td>1 2 2 1 2 0 1</td>
<td>0 1 0 0</td>
</tr>
<tr>
<td>1010</td>
<td>1 2 3 1 1 0 0</td>
<td>0 0 1 1</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
</tbody>
</table>

Rule (not local):

$$
\begin{pmatrix}
ua \\
vb
\end{pmatrix} =
\begin{pmatrix}
u \\
vb
\end{pmatrix} + \delta_{a,b} \begin{pmatrix}
u \\
v
\end{pmatrix}
$$
The first six elements of the sequence \((U'_n)_{n \geq 0}\)

Generalized Pascal triangles

Manon Stipulanti (ULiège)
The tenth element

Lines of different slopes:
The tenth element

Lines of different slopes: \( \varphi^n, \ n \geq 0 \), with \( \varphi = \frac{1+\sqrt{5}}{2} \) \ Golden Ratio
The (⋆′) condition

Recall

• \((u, v)\) satisfies (⋆) iff \(\binom{u}{v} \equiv 1 \mod 2\), and \(\binom{u}{v_0} = 0 = \binom{u}{v_1}\).
• \((u, v)\) satisfies (⋆) ⇒ \((u_0, v_0), (u_1, v_1)\) satisfy (⋆)

Problem: we cannot always add a letter 1 as a suffix in \(L_F\).

Solution: \(p(u, v) \in \mathbb{N}\) s.t. \(u_0^{p(u, v)} w, v_0^{p(u, v)} w \in L_F\) for all \(w \in 0^* L_F\)

\((⋆′)\)

\((u, v)\) satisfies (⋆′) iff \(u = v = \varepsilon\) or

\[
\begin{cases}
\text{\(u, v \neq \varepsilon\)} \\
\left(\binom{u_0^{p(u, v)}}{v_0^{p(u, v)}}\right) \equiv 1 \pmod{2} \\
\left(\binom{u_0^{p(u, v)}}{v_0^{p(u, v)} a}\right) = 0 \quad \forall a \in \{0, 1\}.
\end{cases}
\]
**An automaton to find** $p(u, v)$

**Definition:** A deterministic finite automaton (DFA) over an alphabet $A$ is given by a 5-tuple $\mathcal{A} = (Q, q_0, F, A, \delta)$ where

- $Q$ is a finite set of states
- $q_0 \in Q$ is the initial state (incoming arrow)
- $F \subset Q$ is the set of final states (concentric circles)
- $\delta: Q \times A \rightarrow Q$ is the transition function

**Example:**

![Automaton Diagram]

- $A = \{0, 1\}$
- $Q = \{a_0, a_1\}$
- $q_0 = a_0$
- $F = \{a_0, a_1\}$
- $\delta(a_0, 0) = a_0$, $\delta(a_0, 1) = a_1$, $\delta(a_1, 0) = a_0$
Reading words in the automaton:

100101 \quad a_0 \xrightarrow{1} a_1 \xrightarrow{0} a_0 \xrightarrow{0} a_0 \xrightarrow{1} a_1 \xrightarrow{0} a_0 \xrightarrow{1} a_1  \quad \checkmark \quad \text{accepted}

1011 \quad a_0 \xrightarrow{1} a_1 \xrightarrow{0} a_0 \xrightarrow{1} a_1 \xrightarrow{1} ???  \quad \times \quad \text{not accepted}

000101 \quad a_0 \xrightarrow{0} a_0 \xrightarrow{0} a_0 \xrightarrow{0} a_0 \xrightarrow{1} a_1 \xrightarrow{0} a_0 \xrightarrow{1} a_1  \quad \checkmark \quad \text{accepted}

Accepted language (set of words): $0^* L_F$ with $L_F = 1 \{0, 01\}^* \cup \{\varepsilon\}$

**Definition:** $p(u, v)$ is the smallest $p$ s.t. $\delta(a_0, u0^p) = a_0 = \delta(a_0, v0^p)$

Then $u0^p(u,v)w, v0^p(u,v)w \in L_F$ for all $w \in 0^* L_F$. 
Similar technique

- Completion lemma with the \((\star')\) condition
- Creation of segments of slope 1
- New compact set \(\mathcal{A}'_0\) containing those lines

\[
\mathcal{A}'_0 = \bigcup_{(u,v) \text{ satisfying } (\star')} S_{u,v} \subset [0,1]^2
\]

- Modification of the slopes with \(c: (x,y) \mapsto (x/\varphi, y/\varphi)\) and \(h: (x, y) \mapsto (x, \varphi y)\)
- New compact set \(\mathcal{A}'_n\) containing lines of slopes \(1, \varphi, \varphi^2, \ldots, \varphi^n\)

\[
\mathcal{A}'_n = \bigcup_{0 \leq j \leq i \leq n} h^j(c^i(\mathcal{A}'_0))
\]

- \((\mathcal{A}'_n)_{n \geq 0}\) converges to

\[
\mathcal{L}' = \bigcup_{n \geq 0} \mathcal{A}'_n
\]

(w.r.t. the Hausdorff distance).
Similar convergence result

**Theorem (S., 2018)**

The sequence \((U'_n)_{n \geq 0}\) of compact sets converges to the compact set \(L'\) when \(n\) tends to infinity (w.r.t. the Hausdorff distance).

“Simple” characterization of \(L'\): \((\star')\) condition
Definition

A *numeration system* is a sequence $U = (U(n))_{n \geq 0}$ of integers s.t.

- $U$ increasing
- $U(0) = 1$
- $\sup_{n \geq 0} \frac{U(n+1)}{U(n)}$ bounded by a constant $\sim$ finite alphabet.

A numeration system $U$ is *linear* if $\exists k \geq 1, \exists a_0, \ldots, a_{k-1} \in \mathbb{Z}$ s.t.

$$U(n + k) = a_{k-1} U(n + k - 1) + \cdots + a_0 U(n) \quad \forall n \geq 0.$$ 

Greedy representation in $(U(n))_{n \geq 0}$:

$$n = \sum_{i=0}^{\ell} c_i U(i) \quad \text{with} \quad \sum_{i=0}^{j-1} c_i U(i) < U(j)$$

$$\text{rep}_U(n) = c_\ell \cdots c_0 \in \mathbb{L}_U = \text{rep}_U(\mathbb{N})$$

**Example**: integer base $(b^n)_{n \geq 0}$ with $b \in \mathbb{N}_{>1}$

**Fibonacci numeration system** $(F(n))_{n > 0}$
\( \beta \in \mathbb{R}_{>1} \quad A_\beta = \{0, 1, \ldots, \lceil \beta \rceil - 1\} \)

\[ x \in [0, 1] \mapsto x = \sum_{j=1}^{+\infty} c_j \beta^{-j}, \quad c_j \in A_\beta \]

Greedy way: \( c_j \beta^{-j} + c_{j+1} \beta^{-j-1} + \cdots < \beta^{-(j-1)} \)

\( \beta \)-expansion of \( x \): \( d_\beta(x) = c_1 c_2 c_3 \cdots \)

**Definition**

\( \beta \in \mathbb{R}_{>1} \) is a *Parry number* if \( d_\beta(1) \) is ultimately periodic.

**Example:** \( b \in \mathbb{N}_{>1} \): \( d_b(1) = (b - 1)^\omega \)

Golden ratio \( \varphi \): \( d_\varphi(1) = 110^\omega \)
Parry numeration system

Parry number $\beta \in \mathbb{R}_{>1} \leadsto$ linear numeration system $(U_\beta(n))_{n \geq 0}$

- $d_\beta(1) = t_1 \cdots t_m 0^\omega$
  
  $$
  \begin{align*}
  U_\beta(0) &= 1 \\
  U_\beta(i) &= t_1 U_\beta(i-1) + \cdots + t_i U_\beta(0) + 1 \quad \forall 1 \leq i \leq m - 1 \\
  U_\beta(n) &= t_1 U_\beta(n-1) + \cdots + t_m U_\beta(n-m) \quad \forall n \geq m
  \end{align*}
  $$

- $d_\beta(1) = t_1 \cdots t_m (t_{m+1} \cdots t_{m+k})^\omega$
  
  $$
  \begin{align*}
  U_\beta(0) &= 1 \\
  U_\beta(i) &= t_1 U_\beta(i-1) + \cdots + t_i U_\beta(0) + 1 \quad \forall 1 \leq i \leq m + k - 1 \\
  U_\beta(n) &= t_1 U_\beta(n-1) + \cdots + t_{m+k} U_\beta(n-m-k) + U_\beta(n-k) \quad \forall n \geq m + k \\
  &\quad - t_1 U_\beta(n-k-1) - \cdots - t_m U_\beta(n-m-k)
  \end{align*}
  $$

Examples:

- $b \in \mathbb{N}_{>1} \leadsto (b^n)_{n \geq 0}$ base $b$
- Golden ratio $\varphi \leadsto (F(n))_{n \geq 0}$ Fibonacci numeration system
Extension to Parry numeration systems

- Parry number $\beta \in \mathbb{R}_{>1}$
- Parry numeration system $(U_\beta(n))_{n \geq 0}$
- Numeration language $L_{U_\beta}$
- Generalized Pascal triangle $P_\beta$ in $(U_\beta(n))_{n \geq 0}$ indexed by words of $L_{U_\beta}$
- Sequence of compact sets extracted from $P_\beta$ (first $U_\beta(n)$ rows and columns of $P_\beta$)
- Convergence to a limit object (same technique)
  - Lines of different slopes: $\beta^n$, $n \geq 0$
  - ($\star'$) condition and description of segments of slope 1
  - Two maps $c: (x, y) \mapsto (x/\beta, y/\beta)$ and $h: (x, y) \mapsto (x, \beta y)$
  - Sequence of sets $A^\beta_n$ containing lines of slopes 1, $\beta$, $\beta^2$, $\ldots$, $\beta^n$
  - $A^\beta_n$ converges to
    \[ L^\beta = \bigcup_{n \geq 0} A^\beta_n \]
- Works modulo any prime number
\( \varphi^2 \)
\( \beta_1 \approx 2.47098 \) dominant root of \( P(X) = X^4 - 2X^3 - X^2 - 1 \)
Example 3

$\beta_2 \approx 1.38028$ dominant root of $P(X) = X^4 - X^3 - 1$
Example 4

$\beta_3 \approx 2.80399$ dominant root of $P(X) = X^4 - 2X^3 - 2X^2 - 2$
Example 5

$\beta_4 \approx 1.32472$ dominant root of polynomial $P(X) = X^5 - X^4 - 1$
In this talk:

<table>
<thead>
<tr>
<th>Numeration system</th>
<th>Generalized Pascal triangle</th>
<th>Convergence mod $p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base 2</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Integer base</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Fibonacci</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Parry</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>
• Regularity of the sequence counting subword occurrences: result for any integer base $b$ and the Fibonacci numeration system

• Behavior of the summatory function: result for any integer base $b$ (exact behavior) and the Fibonacci numeration system (asymptotics)


