LIÈGE université Sciences

Automatic sequences based on Parry or Bertrand numeration systems

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Joint work with Jarkko Peltomäki and Michel Rigo

Discrete Mathematics Seminar

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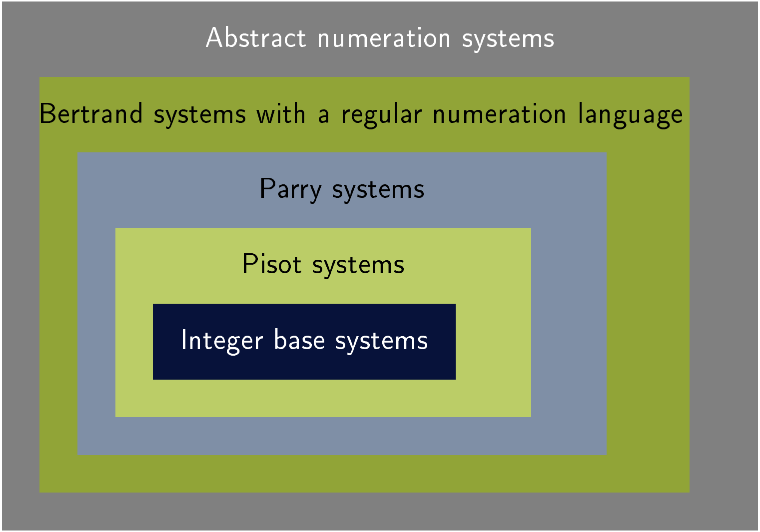
Abstract numeration systems
Bertrand systems with a regular numeration language
Parry systems
Pisot systems
Integer base systems

Numeration systems

A *numeration system* is an increasing sequence $U = (U_n)_{n \geq 0}$ of integers such that $U_0 = 1$ and $C_U := \sup_{n \geq 0} \lceil \frac{U_{n+1}}{U_n} \rceil < +\infty$.

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We set $A_U := \{0, \ldots, C_U - 1\}$.

The *greedy representation* of the positive integer n is the word $\text{rep}_U(n) = w_{l-1} \ldots w_0$ over A_U satisfying

$$\sum_{i=0}^{\ell-1} w_i U_i = n, w_{\ell-1} \neq 0 \text{ and } \forall j \in \{1, \ldots, \ell\}, \sum_{i=0}^{j-1} w_i U_i < U_j.$$

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The *numerical value* $\text{val}_U : \mathbb{Z}^* \to \mathbb{N}$ maps a word $d_{\ell-1} \dots d_0$ to the number $\sum_{i=0}^{\ell-1} d_i U_i$.

Definition

A numeration system U is a *Bertrand numeration system* if, for all $w \in A_U^*$, $w \in 0^* \operatorname{rep}_U(\mathbb{N})$ if and only if $w0 \in 0^* \operatorname{rep}_U(\mathbb{N})$.

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- Modified Fibonacci : $G_0 = 1, G_1 = 3$.
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$2 \in 0^* \operatorname{rep}_G(\mathbb{N})$ $\qquad$ $20 \notin 0^* \operatorname{rep}_G(\mathbb{N})$, because $\operatorname{rep}_G(\operatorname{val}_G(20)) = 102$.

Let $\beta > 1$ be a real number.

The β-*expansion* of a real number $x \in [0, 1]$ is the sequence $d_\beta(x) = (x_i)_{i \geq 1} \in \mathbb{N}^\omega$ that satisfies

$$x = \sum_{i=1}^{+\infty} x_i \beta^{-i}$$

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Fact (D_β) = set of finite factors occurring the the base-β expansions of real numbers in $[0, 1)$.

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If $d_\beta(1) = t_1 \dots t_m 0^\omega$ with $t_1, \dots, t_m \in A_\beta$ and $t_m \neq 0$, we say that $d_\beta(1)$ is *finite* and we set $d^*_\beta(1) = (t_1 \dots t_{m-1}(t_m - 1))^\omega$.

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Let $\beta > 1$ be a real number such that $d^*_\beta(1) = (t_i)_{i \geq 1}$. The numeration system $U_\beta = (U_n)_{n \geq 0}$ canonically associated with β is defined by

$$U_n = t_1 U_{n-1} + \dots + t_n U_0 + 1, \ \forall n \geq 0.$$

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U_3 is the classical base 3, so $0^* \operatorname{rep}_{U_3}(\mathbb{N}) = \{0,1,2\}^*$.

Theorem (Parry, 1960)

A sequence $x = (x_i)_{i \geq 1}$ over $\mathbb{N}$ is the β-expansion of a real number in $[0, 1)$ if and only if $(x_{n+i})_{i \geq 1}$ is lexicographically less than $d^*_\beta(1)$ for all $n \geq 0$.

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Deterministic finite automaton $\mathscr{A}_\beta$

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Fibonacci is Pisot, but not an integer base.

Automatic sequences

Definition

Let U be a numeration system. An infinite word $\mathbf{x} = (x_n)_{n\geq 0}$ over an alphabet B is *U-automatic* (it is an *U-automatic sequence*) if there exists a complete DFAO $(Q, q_0, A_U, \delta, \tau)$ with transition function $\delta : Q \times A_U \to Q$ and output function $\tau = Q \to B$ such that $\delta(q_0, 0) = q_0$ and

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The infinite word $\mathbf{x}$ is *k-automatic* (resp. *Parry-automatic*, resp. *Bertrand-automatic*) if $U = (k^n)_{n\geq 0}$ for an integer $k \geq 2$ (resp. U is a Parry numeration system, resp. U is a Bertrand numeration system).

Factor complexity

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Proposition (Cobham, 1972)

The factor complexity function of a k-automatic sequence is sublinear.

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Sketch of the proof : on board.

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Parry-automatic : σ quasi-uniform

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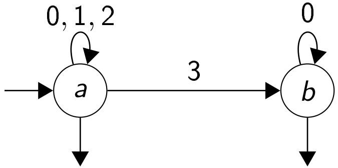

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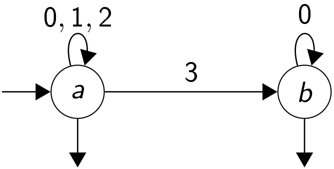

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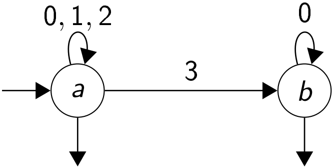

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$\mathbf{x} = \sigma^\omega(a) = aaabaaabaaabb\ldots$

Theorem (Pansiot, 1984)

Let $\mathbf{x}$ be a purely morphic word and p its complexity function. Then one of the following holds :

- $p(n) = \Theta(1)$
- $p(n) = \Theta(n)$
- $p(n) = \Theta(n \log \log n)$
- $p(n) = \Theta(n \log n)$
- $p(n) = \Theta(n^2)$.

Closure properties

Theorem

Let U be a numeration system such that $\mathrm{rep}_U(\mathbb{N})$ is regular. An infinite word $\mathbf{x} = (x_n)_{n\geq 0}$ over A is U-automatic in and only if, for all $a \in A$, the set $\{j \geq 0 | x_j = a\}$ is U-recognizable.

Closure properties

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Proposition

The image of a k-automatic sequence under a substitution of constant length is again a k-automatic sequence.

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$\mu(\mathbf{x})$ is the characteristic sequence of the set $\{tU_n | n \geq 0\}$

Proposition

Let $r \geq 2$ be an integer. If t is an integer such that $4 \leq t \leq \lfloor \beta^r \rfloor$, then the β-expansion of the number $\frac{t}{\beta^r}$ is aperiodic.

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Corollary

The set $\{tU_n | n \geq 0\}$ is not U-recognizable for $t \geq 4$. In other words, its characteristic sequence $\mu(\mathbf{x})$ is not U-automatic.

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Theorem

There exists a Parry numeration system U such that the class of U-automatic sequences is not closed under periodic deletion.

$$U_n = 3U_{n-1} + 2U_{n-2} + 3U_{n-4},\ U_0 = 1, U_1 = 4, U_2 = 15, U_3 = 54$$

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Let $\mathbf{y}$ be the characteristic sequence of the set $\{\frac{U_n}{2} | n \geq 0, U_n \in 2\,\mathbb{N}\}$:

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Proposition

The set $\{\frac{U_n}{2} | n \geq 0, U_n \in 2\,\mathbb{N}\}$ is not U-recognizable. In other words, its characteristic sequence $\mathbf{y}$ is not U-automatic.

Multidimensional sequences

Definition

Let U be a numeration system. A 2-dimensional word $\mathbf{x} = (x_{m,n})_{m,n\geq 0}$ over an alphabet B is *U-automatic* if there exists a complete DFAO $(Q, q_0, A_U \times A_U, \delta, \tau)$ with transition function $\delta : Q \times (A_U \times A_U)^* \to Q$ and output function $\tau : Q \to B$ such that $\delta(q_0, (0,0)) = q_0$ and

$$x_{m,n} = \tau\left(\delta(q_0, (0^{\ell-|\operatorname{rep}_U(m)|}\operatorname{rep}_U(m), 0^{\ell-|\operatorname{rep}_U(n)|}\operatorname{rep}_U(n)))\right)$$

$\forall m,n \geq 0$, where $\ell = \max\{|\operatorname{rep}_U(m)|, |\operatorname{rep}_U(n)|\}$.

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$\forall m, n \geq 0$, where $\ell = \max\{|\operatorname{rep}_U(m)|, |\operatorname{rep}_U(n)|\}$.
The 2-dimensional word $\mathbf{x}$ is *k-automatic* (resp. *Parry-automatic*, resp. *Bertrand-automatic*) if $U = (k^n)_{n\geq 0}$ for an integer $k \geq 2$ (resp. U is a Parry numeration system, resp. U is a Bertrand numeration system).

Let $k \geq 2$ be an integer.

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The *k-kernel* of an infinite word $\mathbf{x} = (x_n)_{n\geq 0}$ over A is the set of its subsequences of the form

$$\{(x_{k^e n+d})_{n\geq 0} \,|\, e \geq 0, 0 \leq d \leq k^e\}$$

Let U be a numeration system and $s \in A_U^*$ be a finite word. Define the ordered set of integers

$$\mathscr{I}_s := \mathrm{val}_U(0^* \, \mathrm{rep}_U(\mathbb{N}) \cap A_U^* s) = \{i(s,0), i(s,1), \ldots\}.$$

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Definition

The U-kernel of an infinite word $\mathbf{x} = (x_n)_{n \geq 0}$ over B is the set

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Definition

The U-kernel of an 2-dimensional word $\mathbf{x} = (x_{m,n})_{m,n\geq 0}$ over B is the set

$$\ker_U(\mathbf{x}) := \{\left(x_{i(s,m),i(t,n)}\right)_{m,n\geq 0} | s, t \in A_U^*, |s| = |t|\}.$$

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Let U be a numeration system such that the numeration language $\text{rep}_U(\mathbb{N})$ is regular. A 2-dimensional word $\mathbf{x} = (x_{m,n})_{m,n\geq 0}$ is U-automatic if and only if its U-kernel is finite.