An introduction to automata theory and state complexity of the multiples of the Thue-Morse set

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Young Mathematicians Symposium of the Greater Region

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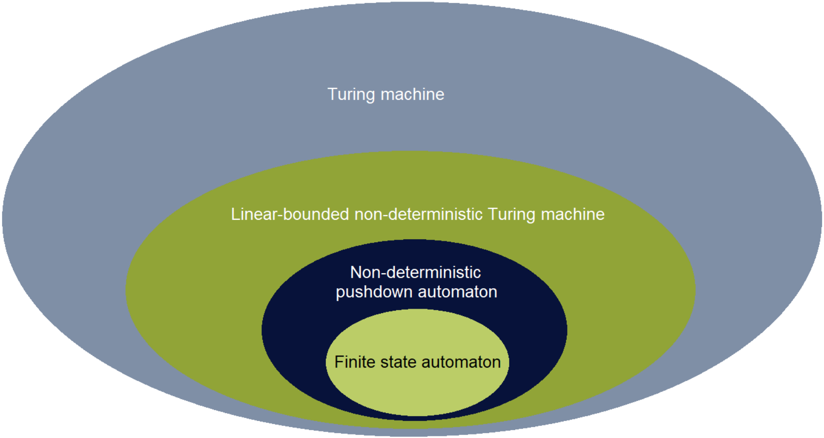
Chomsky hierarchy



Alphabet – Letter – Word

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The *empty word* ε is the only word of length 0.

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Let Σ be an alphabet, $w = w_1 \dots w_n \in \Sigma^*$ and $\sigma \in \Sigma$.

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We set

$$|w|_\sigma = \sharp \{i \in \{1, \dots, n\} : w_i = \sigma\}.$$

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$$|baabcbacaa|_c = 2$$

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Let Σ be an alphabet, $u = u_1 \dots u_n \in \Sigma^*$ and $v = v_1 \dots v_m \in \Sigma^*$.

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The *concatenation* of the words u and v is the word $w \in \Sigma^*$ defined as follows :

$$w = w_1 \dots w_{n+m} \qquad \text{where} \qquad \begin{cases} w_i = u_i & \text{if } 0 \leq i \leq n \\ w_{n+i} = v_i & \text{if } 0 \leq i \leq m. \end{cases}$$

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$$uv = 011001110$$
$$vu = 100111001$$

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A word $u \in \Sigma^*$ is a *factor* of the word w if there exists two words $x, y \in \Sigma^*$ such that

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Factors :

Length	Factors
0	ε

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1	a, b, c
2	ab, ac, ba, ca, cc
3	$aba, acc, bac, bab, cab, cca$
4	$abab, acca, bacc, caba, ccab$
5	$accab, bacca, cabab, ccaba$
6	$accaba, baccab, ccabab$
7	$accabab, baccaba$
8	$baccabab$

Combinatorics on words

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$$|w| \geq 4 \Rightarrow \exists u \in \Sigma^* : u^2 \in \text{Fac}(w).$$

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Definition

An *infinite word* over an alphabet Σ is a map

$$w : \mathbb{N} \to \Sigma.$$

We denote by Σ^ω the set of infinite words over Σ.

Thue-Morse sequence

0

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0

1

Thue-Morse sequence

01

1

Thue-Morse sequence

01

10

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0110

10

Thue-Morse sequence

0110

1001

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01101001

1001

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0110100110010110

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where Σ is an alphabet, $u \in \Sigma^*$ and $a \in \Sigma$.

Thue-Morse sequence

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The Thue-Morse sequence is an infinite word without overlap.

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Kleene star

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One more example

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2049

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Base	Decomposition	Representation
10	$2 \times 10^3 + 4 \times 10^1 + 9 \times 10^0$	$(2, 0, 4, 9)$

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Examples of languages : $\mathrm{rep}_b(\mathbb{N})$

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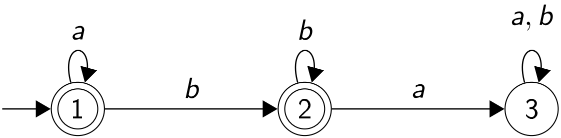
Examples of languages : $\text{rep}_b(\mathbb{N}), 0^* \text{rep}_b(\mathbb{N}), \text{rep}_b(2\,\mathbb{N}), \ldots$

Automaton

Deterministic finite automaton (DFA) : $\mathscr{A} = (Q, q_0, F, \Sigma, \delta)$



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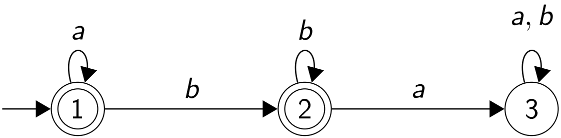

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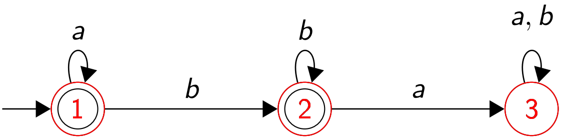

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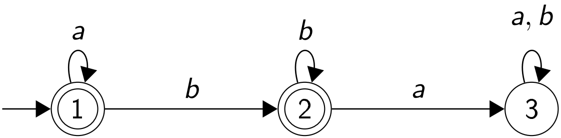

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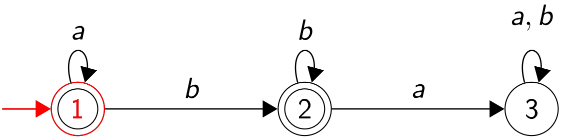

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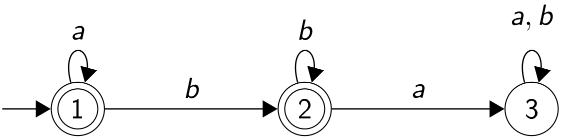

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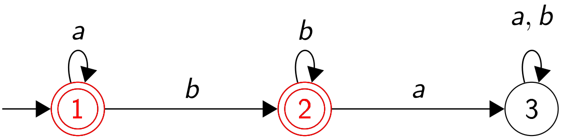

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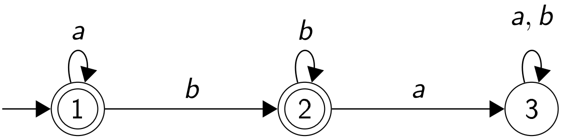

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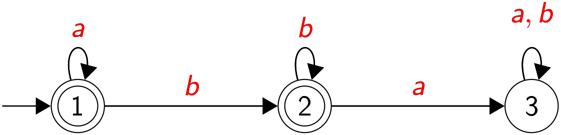

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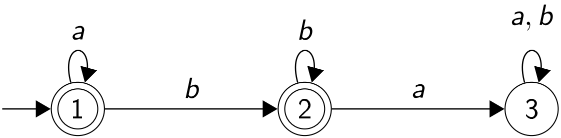

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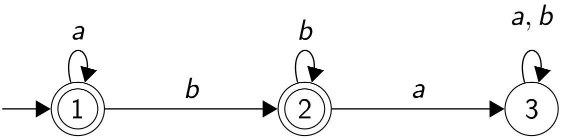

$\delta(1, a)$

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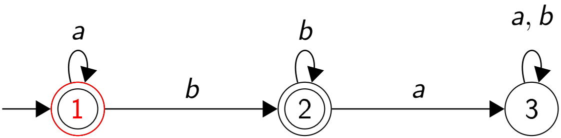

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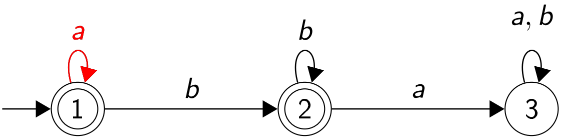

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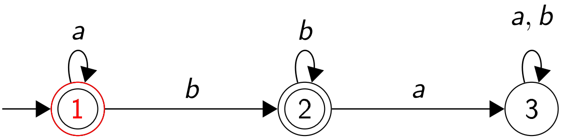

$$\delta(1, a) = 1$$

Automaton

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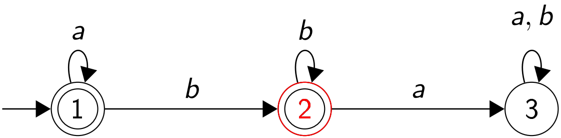

$\delta(2, a)$

Automaton

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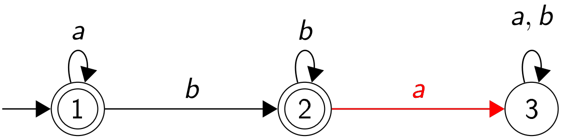

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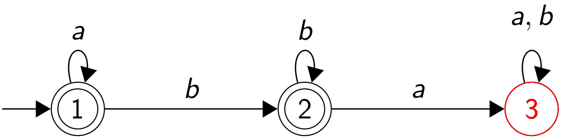

$$\delta(2, a) = 3$$

Automaton

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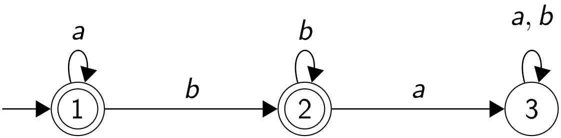

$aaab$

Automaton

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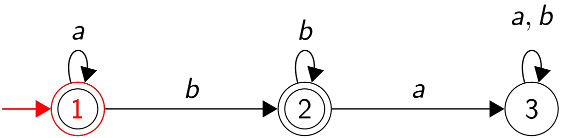

$aaab$

1

Automaton

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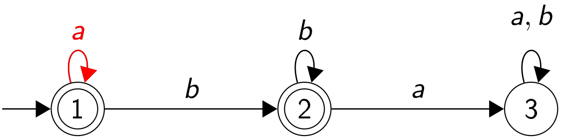

$aaab$

$1 \xrightarrow{a}$

Automaton

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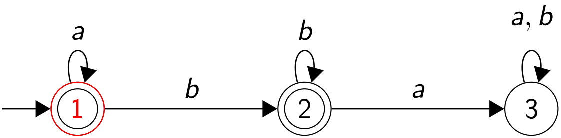

a *aab*

$1 \xrightarrow{a} 1$

Automaton

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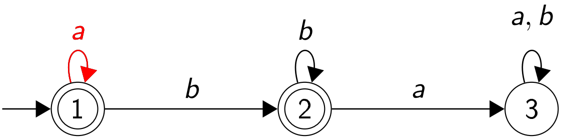

$a \quad aab$

$1 \xrightarrow{a} 1 \xrightarrow{a}$

Automaton

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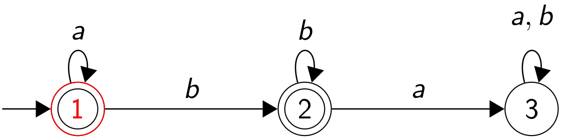

$aa \quad ab$

$1 \xrightarrow{a} 1 \xrightarrow{a} 1$

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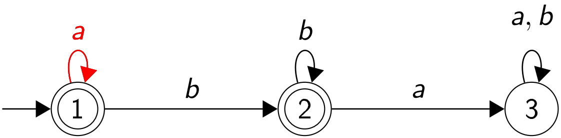

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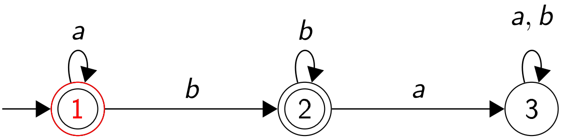

$aaa \quad b$

$1 \xrightarrow{a} 1 \xrightarrow{a} 1 \xrightarrow{a} 1$

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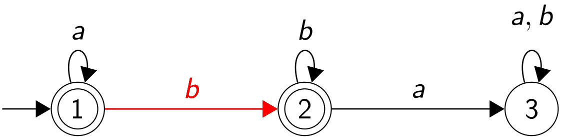

$aaa \quad b$

$1 \xrightarrow{a} 1 \xrightarrow{a} 1 \xrightarrow{a} 1 \xrightarrow{b}$

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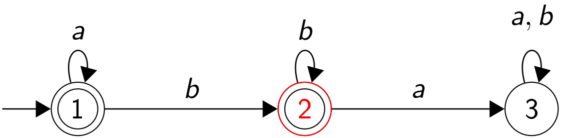

$aaab$

$1 \xrightarrow{a} 1 \xrightarrow{a} 1 \xrightarrow{a} 1 \xrightarrow{b} 2$

Automaton

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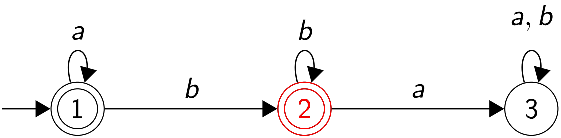


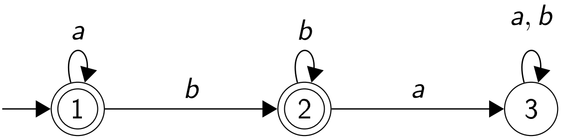
aaab

$1 \xrightarrow{a} 1 \xrightarrow{a} 1 \xrightarrow{a} 1 \xrightarrow{b} 2 \qquad \delta(1, aaab) = 2$

Final $\rightsquigarrow$ Accepted word

Automaton

DFA : $\mathscr{A} = (\{1,2,3\}, 1, \{1,2\}, \{a,b\}, \delta)$



ba

Automaton

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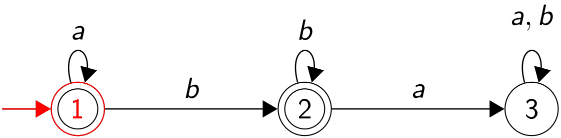

ba

1

Automaton

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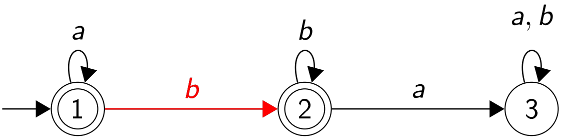

ba

$1 \xrightarrow{b}$

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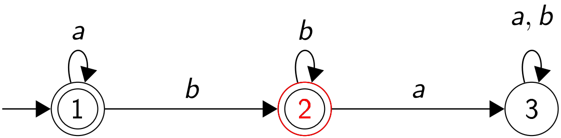

$b \quad a$

$1 \xrightarrow{b} 2$

Automaton

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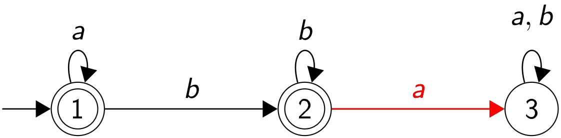

$b \quad a$

$1 \xrightarrow{b} 2 \xrightarrow{a}$

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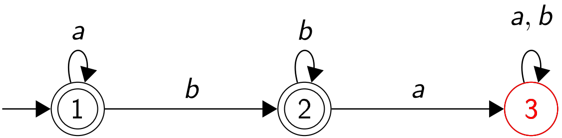

ba

$1 \xrightarrow{b} 2 \xrightarrow{a} 3$

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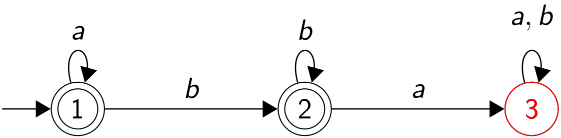

ba

$1 \xrightarrow{b} 2 \xrightarrow{a} 3 \qquad \delta(1, ba) = 3$

Not final $\rightsquigarrow$ Non-accepted word

Automata and languages I

Definition

For each automaton $\mathscr{A} = (Q, q_0, F, \Sigma, \delta)$, the *language accepted by* $\mathscr{A}$ is the set

$$L(\mathscr{A}) := \{w \in \Sigma^* : \delta(q_0, w) \in F\}.$$

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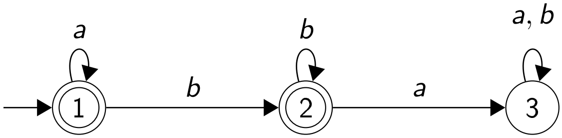

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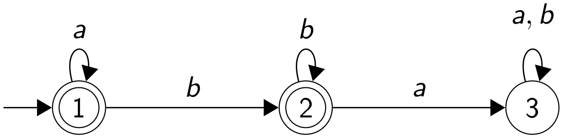

$L(\mathscr{A}) = a^* b^*$

Automata and languages II

Let Σ be an alphabet.

Recall

A language over Σ is a subset of Σ^*.

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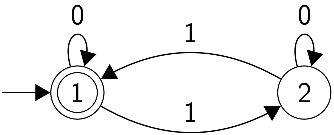

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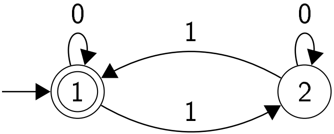

$\{a^n b^n : n \in \mathbb{N}\}$

DFA vs NDFA

Deterministic Finite Automaton – Non-Deterministic Finite Automaton

DFA vs NDFA

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Transitions	Function on Σ	Relation on Σ^*

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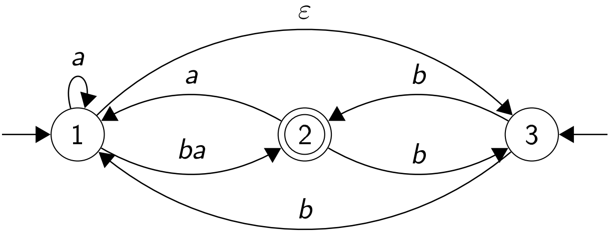

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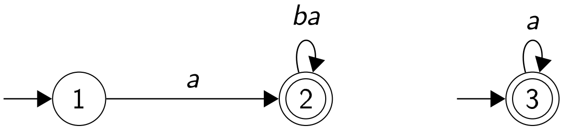

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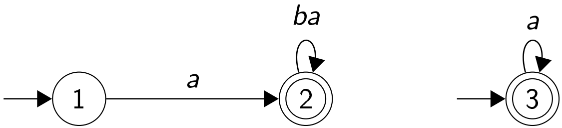

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Every language accepted by a NDFA is accepted by a DFA.

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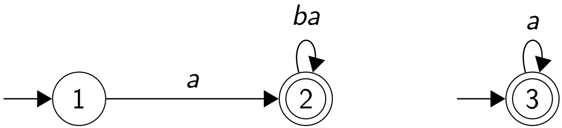

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Existence of an algorithm

Product of automata

Given two regular languages L and M, is the language $L \cap M$ also regular ?

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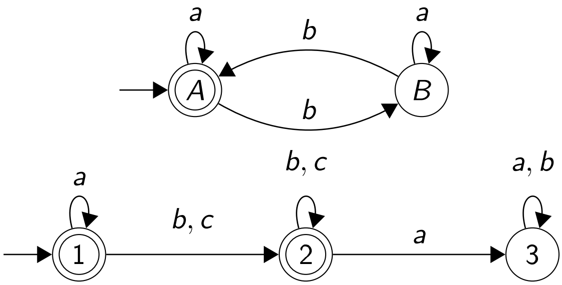

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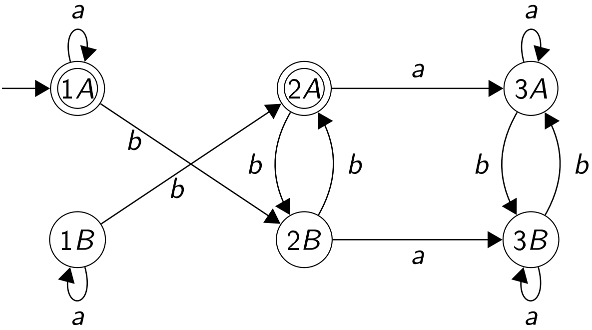

Complete automaton

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An automaton $\mathscr{A} = (Q, q_0, F, \Sigma, \delta)$ is *complete* if $\forall q \in Q, \forall \sigma \in \Sigma$,

$$\delta(q, \sigma)$$

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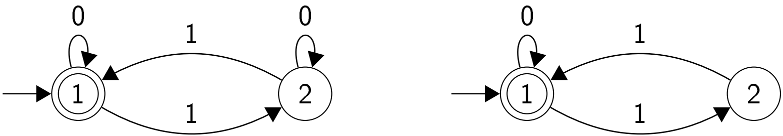

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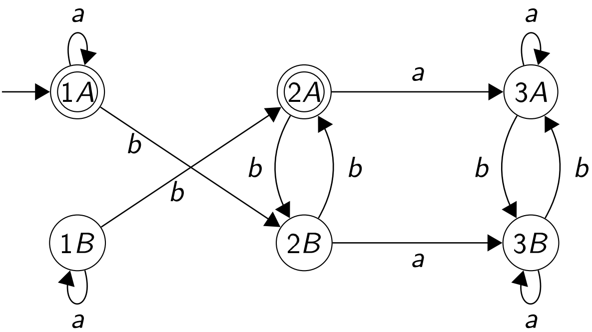

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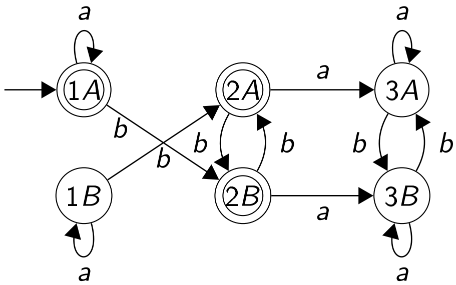

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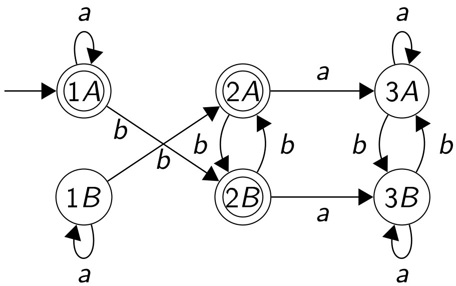

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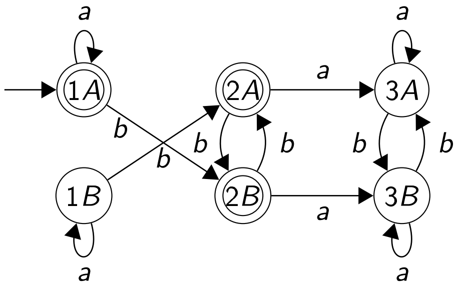
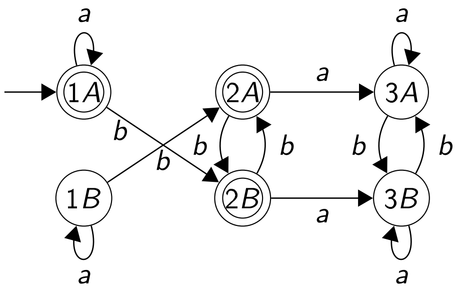
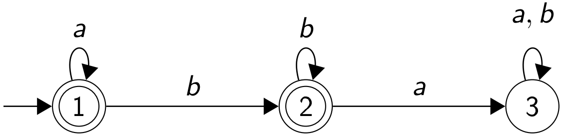


Coaccessible + with disjoint states $\Rightarrow$ reduced


a
a
1A
2A
3A
b
b
b
b
b
b
1B
2B
3B
a
a
a


a
b
a, b
b
a
1
2
3

Minimal automaton II

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For any regular language L, there exists a unique (up to isomorphism) minimal automaton accepting L.

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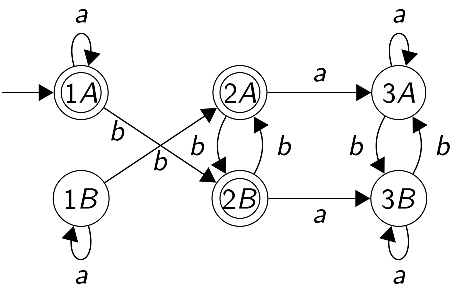

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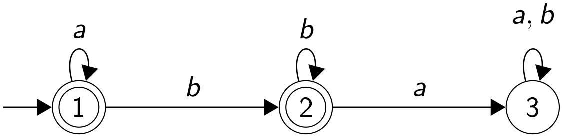

State complexity

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The *state complexity* of a regular language is the number of states of its minimal automaton.

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The state complexity of a^*b^* is 3.

Theorem [Alexeev, 2004]

The state complexity of the language $0^* \operatorname{rep}_b(m\mathbb{N})$ is

$$\min_{N \geq 0} \left\{ \frac{m}{\gcd(m, b^N)} + \sum_{n=0}^{N-1} \frac{b^n}{\gcd(b^n, m)} \right\}$$

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$$\mathscr{T} = \{n \in \mathbb{N} : |\operatorname{rep}_2(n)|_1 \in 2\,\mathbb{N}\}$$

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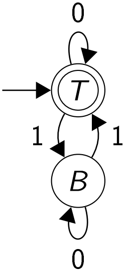

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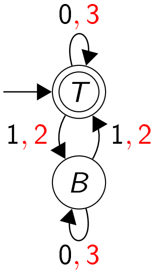

The result

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Automaton	Language accepted
$\mathscr{A}_{\mathscr{T},2^p}$	$(0,0)^* \operatorname{rep}_{2^p}(\mathscr{T} \times \mathbb{N})$

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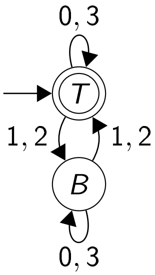
The automaton $\mathscr{A}_{\mathscr{T},2^p}$: $(0,0)^* \operatorname{rep}_{2^p}(\mathscr{T} \times \mathbb{N})$



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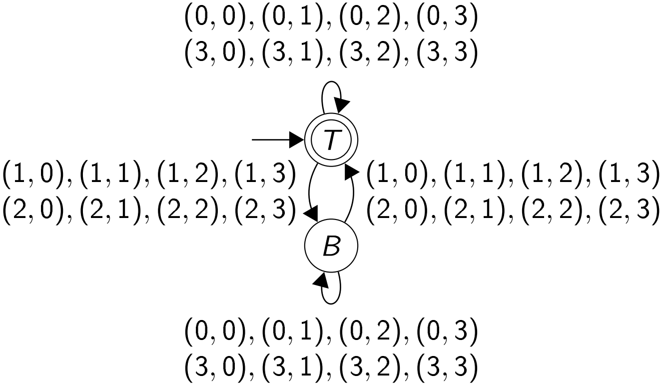
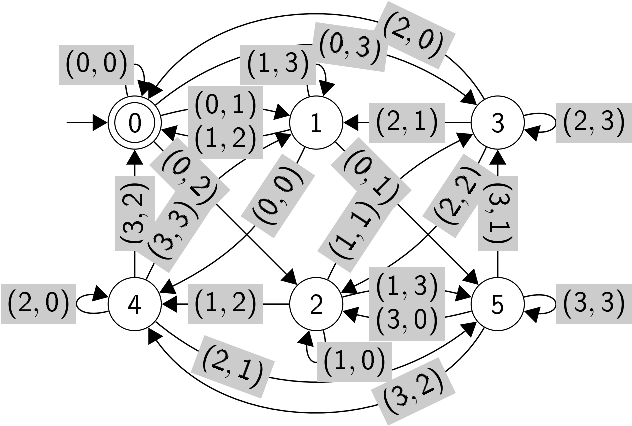


$$\delta_{\mathscr{T},2^p}(X,(d,e)) = \begin{cases} X \text{ if } d \in \mathscr{T} \\ \overline{X} \text{ otherwise} \end{cases}$$

The automaton $\mathscr{A}_{m,b}$: $(0,0)^* \operatorname{rep}_b(\{(n, mn) : n \in \mathbb{N}\})$



$$\delta_{m,b}(i, (d, e)) = j \qquad \Leftrightarrow \qquad bi + e = md + j$$

$\mathscr{A}_{m,2^p} \times \mathscr{A}_{\mathscr{T},2^p}$: $(0,0)^* \operatorname{rep}_{2^p}(\{(t, mt) : t \in \mathscr{T}\})$

$$(0, T), \ldots, (m-1, T) \qquad (0, B), \ldots (m-1, B)$$

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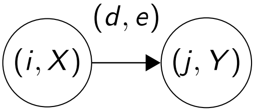
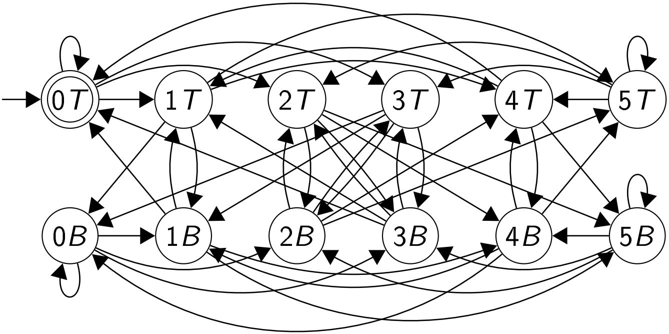
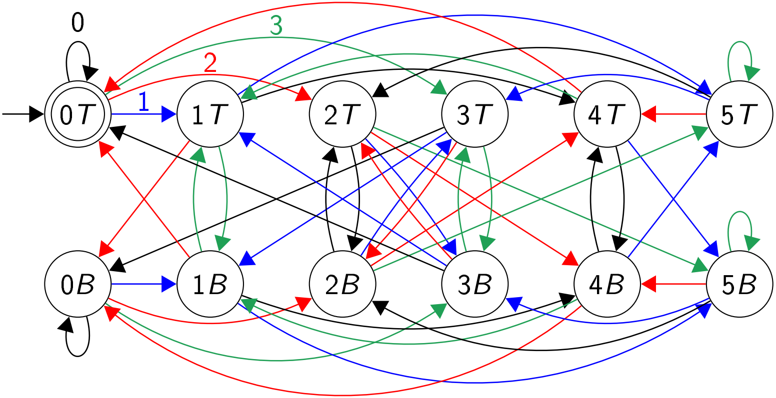


$$2^p i + e = md + j$$

$$Y = \begin{cases} X \text{ if } d \in \mathscr{T} \\ \overline{X} \text{ otherwise} \end{cases}$$



The automaton $\pi(\mathscr{A}_{m,2^p} \times \mathscr{A}_{\mathscr{T},2^p})$: $0^* \operatorname{rep}_{2^p}(m\mathscr{T})$



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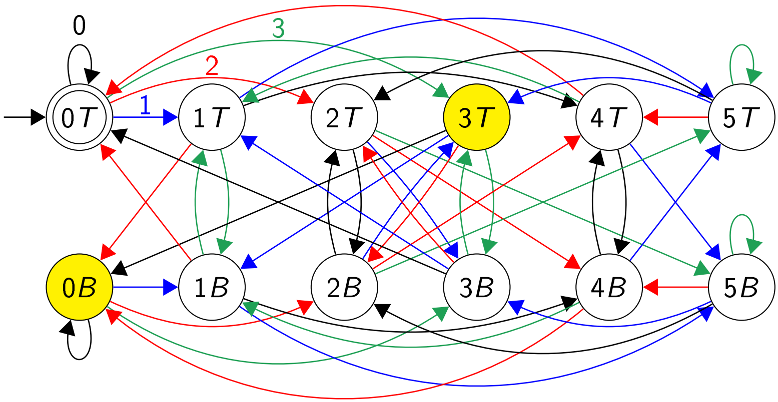
In the automaton $\pi\left(\mathscr{A}_{m,2^p} \times \mathscr{A}_{\mathscr{T},2^p}\right)$, the states (i, T) and (i, B) are disjoint for all $i \in \{0, \ldots, m-1\}$.

The automaton $\pi(\mathscr{A}_{6,4} \times \mathscr{A}_{\mathscr{T},4})$



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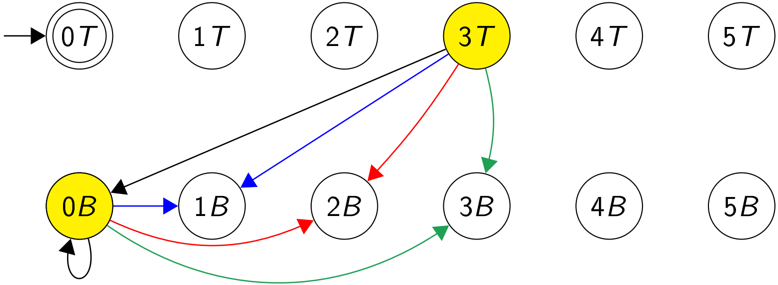

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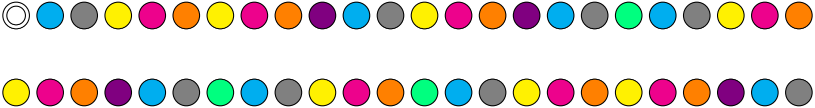
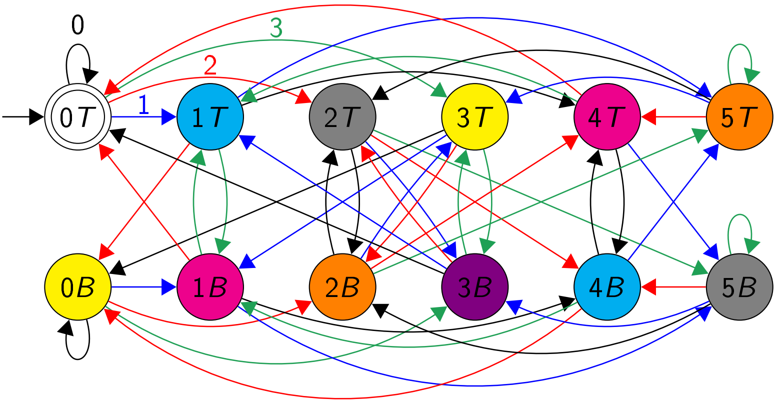


Automaton recognizing $6\mathscr{T}$ in base 4



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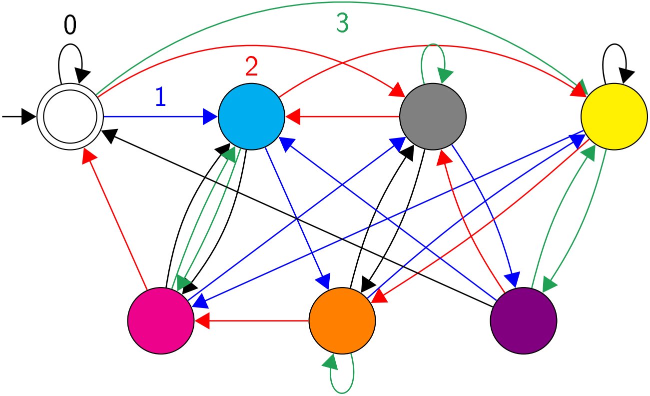

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$2 \times 3 + \left\lceil \frac{1}{2} \right\rceil = 7$

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