Computing the k-binomial complexity of the Tribonacci word

April 03, 2019

Marie Lejeune (FNRS grantee)

Plan

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Factors and subwords

Definition

Let $u = u_1 u_2 \cdots u_m$ be a finite or infinite word. A **(scattered) subword** of u is a finite subsequence of the sequence $(u_j)_{j=1}^m$. A **factor** of u is a contiguous subword.

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where $u \sim_{=} v \Leftrightarrow u = v$.

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We can replace $\sim_=$ with other equivalence relations.

Other equivalence relations

Different equivalence relations from $\sim_=$ can be considered:

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We will deal with the last one.

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k-binomial equivalence

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Let u and v be two finite words. They are **k-binomially equivalent** if

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For all words u, v and for every nonnegative integer k,

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If $\mathbf{w}$ is an infinite word, we can define the function

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We have an order relation between the different complexity functions.

Proposition

$$\rho_{\mathbf{w}}^{ab}(n) \leq \mathbf{b}_{\mathbf{w}}^{(k)}(n) \leq \mathbf{b}_{\mathbf{w}}^{(k+1)}(n) \leq p_{\mathbf{w}}(n) \quad \forall n \in \mathbb{N}, k \in \mathbb{N}^+$$

where $\rho_{\mathbf{w}}^{ab}$ is the abelian complexity function of the word **w**.

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The classical Thue–Morse word, defined as the fixed point of the morphism

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has a bounded k-binomial complexity. The exact value is known.

Theorem (M. L., J. Leroy, M. Rigo, 2018)

Let k be a positive integer. For every $n \leq 2^k - 1$, we have

$$\mathbf{b}_{\mathbf{t}}^{(k)}(n) = p_{\mathbf{t}}(n),$$

while for every $n \geq 2^k$,

$$\mathbf{b}_{\mathbf{t}}^{(k)}(n) = \left\{ \begin{array}{ll} 3 \cdot 2^k - 3, & \text{if } n \equiv 0 \pmod{2^k}; \\ 3 \cdot 2^k - 4, & \text{otherwise.} \end{array} \right.$$

Another family

Definition: Sturmian words

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Theorem (M. Rigo, P. Salimov, 2015)

Let $\mathbf{w}$ be a Sturmian word. We have

$$\mathbf{b}_{\mathbf{w}}^{(k)}(n) = p_{\mathbf{w}}(n) = n + 1,$$

for all $n \in \mathbb{N}$ and for all $k \geq 2$.

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$$\mathbf{b}_{\mathbf{w}}^{(k)}(n) = p_{\mathbf{w}}(n) = n + 1,$$

for all $n \in \mathbb{N}$ and for all $k \geq 2$.

Since $\mathbf{b}_{\mathbf{w}}^{(k)}(n) \leq \mathbf{b}_{\mathbf{w}}^{(k+1)}(n) \leq p_w(n)$, it suffices to show that

$$\mathbf{b}_{\mathbf{w}}^{(2)}(n) = p_{\mathbf{w}}(n).$$

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We have

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The **Tribonacci word** $\mathcal{T}$ is the fixed point of the morphism

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Remark

We have $u \sim_2 v \iff \Psi(u) = \Psi(v) \iff \Psi(u) - \Psi(v) = 0.$

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and we will finally desubstitute. That means that we will take preimages of factors (by applying, in some sense, τ^{-1}) and parents of templates.

Templates: a formal definition

Definition

A **template** is a 5-uple of the form $t = [\mathbf{d}, \mathbf{D_b}, \mathbf{D_e}, a_1, a_2]$ where $\mathbf{d} \in \mathbb{Z}^{12}$, $\mathbf{D_b}, \mathbf{D_e} \in \mathbb{Z}^3$ and $a_1, a_2 \in A$.

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$$\mathbf{D_b} \otimes \Psi'(u) = \begin{pmatrix} 0 & 0 & 0 & -3 & -1 & -1 & 0 & 0 & 0 \end{pmatrix}^{\intercal} \text{ and}$$
$$\Psi'(u) \otimes \mathbf{D_e} = \begin{pmatrix} 3 & 3 & -3 & 1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}^{\intercal}.$$

Then,

$$\mathbf{D_b} \otimes \Psi'(u) + \Psi'(u) \otimes \mathbf{D_e}$$
$$= \begin{pmatrix} 3 & 3 & -3 & -2 & 0 & -2 & 1 & 1 & -1 \end{pmatrix}^{\intercal}.$$

Example: building a realizable template

Take $u = 01020$, $v = 20102 \in \text{Fac}_{\mathcal{T}}$. We have

$$\Psi(u) = \begin{pmatrix} 3 & 1 & 1 & 3 & 1 & 2 & 2 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}^{\intercal}$$
$$\Psi(v) = \begin{pmatrix} 2 & 1 & 2 & 1 & 1 & 2 & 1 & 0 & 1 & 2 & 1 & 1 \end{pmatrix}^{\intercal}.$$

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$$P_3(\mathbf{D_b} \otimes \Psi'(u) + \Psi'(u) \otimes \mathbf{D_e})$$
$$= \begin{pmatrix} 0 & 0 & 0 & 3 & 3 & -3 & -2 & 0 & -2 & 1 & 1 & -1 \end{pmatrix}^{\intercal}.$$

Example: building a realizable template (continued)

Then,

$$P_3\left(\mathbf{D_b}\otimes\Psi'(u)+\Psi'(u)\otimes\mathbf{D_e}\right)$$
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Existence of an infinite number of realizable templates

Remark

For a given pair (u, v) of words ending with letters a_1 and a_2 respectively, and given $\mathbf{D_b}, \mathbf{D_e}$, there always exists $\mathbf{d} \in \mathbb{Z}^{12}$ such that the pair (u, v) realizes the template $[\mathbf{d}, \mathbf{D_b}, \mathbf{D_e}, a_1, a_2]$.
Indeed, it suffices to take

$$\mathbf{d} = \Psi(u) - \Psi(v) - P_3\left(\mathbf{D_b} \otimes \Psi'(u) + \Psi'(u) \otimes \mathbf{D_e}\right).$$

Preimages

Example: intuitive definition

Recall that

$$\mathcal{T} = 01 \cdot 02 \cdot 01 \cdot 0 \cdot 01 \cdot 02 \cdot 01 \cdot 01 \cdot 02 \cdot 01 \cdot 0 \cdot 01 \cdots$$

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Recall that

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Let $u = 2010102010$. The word $u' = 100102$ is a *preimage* of u.

Preimages

Example: intuitive definition

Recall that

$$\mathcal{T} = 01 \cdot 02 \cdot 01 \cdot 0 \cdot 01 \cdot \overbrace{0\textcolor{red}{2}}^{\tau(\textcolor{blue}{1})} \textcolor{red}{\cdot} \overbrace{\textcolor{red}{01}}^{\tau(\textcolor{blue}{0})} \textcolor{red}{\cdot} \overbrace{\textcolor{red}{01}}^{\tau(\textcolor{blue}{0})} \textcolor{red}{\cdot} \overbrace{\textcolor{red}{02}}^{\tau(\textcolor{blue}{1})} \textcolor{red}{\cdot} \overbrace{\textcolor{red}{01}}^{\tau(\textcolor{blue}{0})} \textcolor{red}{\cdot} \overbrace{\textcolor{red}{0}}^{\tau(\textcolor{blue}{2})} \cdot 01 \cdots$$

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Definition

Let u and u' be two words. The word u' is a **preimage** of u if

- u is a factor of $\tau(u')$, and
- u' is minimal: for all factors v of u', u is not a factor of $\tau(v)$.

Preimages (continued)

A word can have several preimages.

Example

Recall that

$$\mathcal{T} = 01 \cdot 02 \cdot 01 \cdot 0 \cdot 01 \cdot 02 \cdot 01 \cdot 01 \cdot 02 \cdot 01 \cdot 0 \cdot 01 \cdots$$

Take $u = 010$.

Preimages (continued)

A word can have several preimages.

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Take $u = 010$. It has 00, 01 and 02 as preimages.

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Templates have parents

We will now introduce the notion of parents of a template.

Theorem

Let t be a template and let (u, v) be a pair of factors realizing t. Let u' (resp., v') be a preimage of u (resp., v).
There always exists a template t' which is realized by (u', v'). and which is, *in some way*, related to t.

$$(u, v) \quad \leftrightsquigarrow \quad [\mathbf{d}, \mathbf{D_b}, \mathbf{D_e}, a_1, a_2] = t$$

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The template t' is called a **parent template** of t.

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Remark

- Since a word can sometimes have several preimages, a template can also have several parents.

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- Since a word can sometimes have several preimages, a template can also have several parents.
- There exists a formula allowing to compute all the parents of a given template.

Plan

Why are templates useful?

These templates are useful to see if $\mathbf{b}_{\mathcal{T}}^{(2)} = p_{\mathcal{T}}$.

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Theorem

There exists $n \in \mathbb{N}$ such that

$$\mathbf{b}_{\mathcal{T}}^{(2)}(n) < p_{\mathcal{T}}(n)$$

if and only if at least one template of the form $[\mathbf{0}, \mathbf{0}, \mathbf{0}, a, b]$ with $a \neq b$ is realizable by a pair of factors of $\mathcal{T}$.

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$\Leftarrow$ $\exists (u, v) \in (\mathrm{Fac}_{\mathcal{T}})^2$ realizing $[\mathbf{0}, \mathbf{0}, \mathbf{0}, a, b]$.

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Proof

$\Leftarrow$ $\exists (u, v) \in (\mathrm{Fac}_{\mathcal{T}})^2$ realizing $[\mathbf{0}, \mathbf{0}, \mathbf{0}, a, b]$. So, $\Psi(u) - \Psi(v) = 0$ and $u \neq v$.

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There exists $n \in \mathbb{N}$ such that

$$\mathbf{b}_{\mathcal{T}}^{(2)}(n) < p_{\mathcal{T}}(n)$$

if and only if at least one template of the form $[\mathbf{0}, \mathbf{0}, \mathbf{0}, a, b]$ with $a \neq b$ is realizable by a pair of factors of $\mathcal{T}$.

Proof

$\Leftarrow$ $\exists (u, v) \in (\mathrm{Fac}_{\mathcal{T}})^2$ realizing $[\mathbf{0}, \mathbf{0}, \mathbf{0}, a, b]$. So, $\Psi(u) - \Psi(v) = 0$ and $u \neq v$.

$\Rightarrow$ $\exists u = u_1 \cdots u_n,\ v = v_1 \cdots v_n \in \mathrm{Fac}_{\mathcal{T}}$ such that $u \neq v$ and $u \sim_2 v$.

Why are templates useful?

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Let $i \in \{1, \ldots, n\}$ be such that $u_i \neq v_i, u_{i+1} = v_{i+1}, \ldots, u_n = v_n$.
Then $(u_1 \cdots u_i, v_1 \cdots v_i)$ realizes $[\mathbf{0}, \mathbf{0}, \mathbf{0}, u_i, v_i]$, because
$u \sim_2 v \Rightarrow u_1 \cdots u_i \sim_2 v_1 \cdots v_i$.

From parents to ancestors

Definition

Let t and t' be templates. We say that t' is an **(realizable) ancestor** of t if there exists a finite sequence of templates $t_0, \dots, t_n$ such that

$$\begin{cases} t_0 = t', \\ t_n = t, \\ \forall i \in \{0, \dots, n-1\}, t_i \text{ is a (realizable) parent of } t_{i+1}. \end{cases}$$

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- there exist an ancestor t' of t, and a pair (u', v') of factors realizing t', such that $L \leq \min(|u'|, |v'|) \leq 2L$.

Verifying that no template $[\mathbf{0}, \mathbf{0}, \mathbf{0}, a, b]$ is realizable

To show that $\mathbf{b}_{\mathcal{T}}^{(2)} = p_{\mathcal{T}}$, we have to show that no template from

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Problem: there exists an infinite number of ancestors.

Plan

Keeping a finite number of templates

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Remark (G. Richomme, K. Saari, L. Zamboni, 2010)

The necessary conditions we found are related to the 2-balancedness property of $\mathcal{T}$, and more precisely, to the fact that, for all $w \in \mathrm{Fac}_{\mathcal{T}}$ and for all $a \in \{0, 1, 2\}$,

$$||w|_a - \alpha_a|w|| < 1.5,$$

where $\alpha_a = \lim_{n \to +\infty} \frac{|w_0 \cdots w_{n-1}|_a}{n}$ is the density of a in $\mathcal{T}$.

A matrix associated to τ

Let us consider the matrix associated to $\tau : \begin{cases} 0 & \mapsto & 01; \\ 1 & \mapsto & 02; \\ 2 & \mapsto & 0. \end{cases}$

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We define its extended version M_τ, such that, for all $u \in \mathrm{Fac}_{\mathcal{T}}$, we have $M_\tau \Psi(u) = \Psi(\tau(u))$.

The extended version

We have

$$M_\tau = \left(\begin{array}{ccc|ccc|ccc|ccc}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
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For example, since $\tau(0) = 01$ and 1 is present only in $\tau(0)$ while 0 occurs once in every $\tau(a)$,

$$\binom{\tau(u)}{01} = \binom{u}{0} + \binom{u}{00} + \binom{u}{10} + \binom{u}{20}.$$

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The extended version

We have

$$M_\tau \cdot \Psi(u) = \left(\begin{array}{ccc|ccc|ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right) \begin{pmatrix} \binom{u}{0} \\ \binom{u}{1} \\ \binom{u}{2} \\ \binom{u}{00} \\ \binom{u}{01} \\ \binom{u}{02} \\ \binom{u}{10} \\ \binom{u}{11} \\ \binom{u}{12} \\ \binom{u}{20} \\ \binom{u}{21} \\ \binom{u}{22} \end{pmatrix}$$

For example, since $\tau(0) = 01$ and 1 is present only in $\tau(0)$ while 0 occurs once in every $\tau(a)$,

$$\binom{\tau(u)}{01} = \binom{u}{0} + \binom{u}{00} + \binom{u}{10} + \binom{u}{20}.$$

About its eigenvalues

The Perron-Frobenius eigenvalue of M'_τ is $\theta \approx 1.839$. The matrix M_τ has

- the eigenvalue θ once;
- the eigenvalue θ^2 once;
- two pairs of complex conjugate eigenvalues of modulus in $]1;\theta[$;
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The bounds we will give on possibly realizable templates will concern projections of templates on the left eigenvectors associated to eigenvalues of modulus less than θ.

First restrictions

Theorem

Let λ be an eigenvalue of modulus less than 1. Let $\mathbf{r}$ be an associated eigenvector. If the template $t = [\mathbf{d}, \mathbf{D_b}, \mathbf{D_e}, a_1, a_2]$ is realizable, then

$$\min_{\boldsymbol{\delta} \in \boldsymbol{\Delta}} |\mathbf{r} \cdot (\mathbf{d} + P_3 (\mathbf{D_b} \otimes \boldsymbol{\delta} + \boldsymbol{\delta} \otimes \mathbf{D_e}))| \leq 2C(\mathbf{r}),$$

where $C(\mathbf{r})$ is an efficiently computable constant such that, for all factors $w \in \mathrm{Fac}_{\mathcal{T}}$, we have

$$|\mathbf{r} \cdot \Psi(w)| \leq C(\mathbf{r}).$$

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$$\Delta = \left\{ \begin{pmatrix} \delta_0 \\ \delta_1 \\ \delta_2 \end{pmatrix} \in [-1.5; 1.5]^3 \ : \ \delta_0 + \delta_1 + \delta_2 = 0 \right\}.$$

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Let λ be an eigenvalue of modulus in $]1, \theta[$. Let $\mathbf{r}$ be an associated eigenvector. If the template $t = [\mathbf{d}, \mathbf{D_b}, \mathbf{D_e}, a_1, a_2]$ is realizable by a pair (u, v) with $|u| \geq L$, then

$$|\mathbf{r} \cdot P_3 (\mathbf{D_b} \otimes \boldsymbol{\alpha} + \boldsymbol{\alpha} \otimes \mathbf{D_e})| \leq \frac{2L - \sum_{i=1}^{3} \mathbf{d}_i}{L} C(\mathbf{r}) + \max_{\boldsymbol{\delta} \in \Delta} \frac{|\mathbf{r} \cdot (\mathbf{d} + P_3 (\mathbf{D_b} \otimes \boldsymbol{\delta} + \boldsymbol{\delta} \otimes \mathbf{D_e}))|}{L},$$

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Recall that we want to compute the ancestors of T and check that none of them is realized by a pair (u, v) with

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- Initialize `toSee = T` and `seen = {}`
- While $\texttt{toSee} \neq \{\}$, take a template $\texttt{t} \in \texttt{toSee}$
  - Compute its parents
  - Keep, among them, only the ones that verify the previous theorems for all eigenvalues of modulus less than θ
  - Add them in `toSee`, if they are neither in `toSee`, nor in `seen`
  - $\texttt{toSee = toSee}\setminus\{\texttt{t}\}$ and $\texttt{seen = seen} \cup \{\texttt{t}\}$
- If the program stops, `seen` contains all the possibly realizable ancestors of T, which is a finite set
- Check that none of them is realized by a pair (u, v) of factors of $\mathcal{T}$ satisfying (**??**)

In our implementation, we took $L = 15$.

Computing all the possibly realizable ancestors gave us a list of 241544 elements, in more or less eight hours.

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To end with an open question...

Is it true that for every Arnoux-Rauzy word $\mathbf{w}$, we have

$$\mathbf{b}^{(k)}_{\mathbf{w}}(n) = p_{\mathbf{w}}(n)$$

for all $n \in \mathbb{N}$ and for all $k \geq 2$?

Thank you!