

Palindromic Ziv–Lempel and Crochemore Factorizations of m -Bonacci Infinite Words

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Abstract

We introduce a variation of the Ziv–Lempel and Crochemore factorizations of words by requiring each factor to be a palindrome. We compute these factorizations for the Fibonacci word, and more generally, for all m -bonacci words.

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1 Introduction

The Ziv–Lempel [9] and Crochemore [4] factorizations are two well-known factorizations of words used in text compression and other text algorithms. Here we apply them to infinite words. Let $|u|$ denote the length of a finite word u . In this paper, we start indexing words at 0, i.e., if u is a finite word over the alphabet A , then we write $u = u_0 \cdots u_{|u|-1}$ where $u_i \in A$ for all $0 \leq i < |u|$. If \mathbf{w} is an infinite word and u is a finite word, we say there is an *occurrence of u at position j in \mathbf{w}* if $\mathbf{w} = pu\mathbf{w}'$ for some word p of length j and some infinite word \mathbf{w}' . Given an infinite word \mathbf{w} , the *Ziv–Lempel* or *z -factorization* of \mathbf{w} is the factorization

$$z(\mathbf{w}) = (z_1, z_2, z_3, \dots)$$

where z_i is the shortest prefix of $z_i z_{i+1} z_{i+2} \cdots$ such that there is no occurrence of z_i in \mathbf{w} at any position $j < |z_1 z_2 \cdots z_{i-1}|$. The *Crochemore* or *c -factorization* of \mathbf{w} is the factorization

$$c(\mathbf{w}) = (c_1, c_2, c_3, \dots)$$

where c_i is the longest prefix of $c_i c_{i+1} c_{i+2} \cdots$ such that there is an occurrence of c_i in \mathbf{w} at some position $j < |c_1 c_2 \cdots c_{i-1}|$, or, if this prefix does not exist, the factor c_i is just a single letter.

For instance, if \mathbf{f} is the Fibonacci word, we have

$$z(\mathbf{f}) = (0, 1, 00, 101, 00100, 10100101, \dots)$$

and

$$c(\mathbf{f}) = (0, 1, 0, 010, 10010, 01010010, \dots).$$

Note that if \mathbf{w} is ultimately periodic the z -factorization is not well-defined, since eventually there will be no factors that do not occur previously in \mathbf{w} . Similarly, if \mathbf{w} is ultimately periodic the definition of the c -factorization will result in some factor c_i being an infinite word. We are not interested in ultimately periodic words in this paper and will therefore ignore this possibility and assume that any infinite word considered in this paper is aperiodic.

In the context of combinatorics on words, these factorizations have been computed for certain important families of words. Berstel and Savelli [2] computed the c -factorizations of all standard Sturmian words. They also observed that the z -factorization of the Fibonacci word coincides with the *singular factorization* of the Fibonacci word introduced by Wen and Wen [13]. Fici [5] has given an excellent survey of these and other factorizations of the Fibonacci word. Ghareghani, Mohammad–noori, and Sharifani [6] determined the z - and c -factorizations of standard episturmian words. Constantinescu and Ilie [3] used the z -factorization to define the *Lempel–Ziv complexity* of an infinite word.

We introduce the *palindromic z -factorization* $pz(\mathbf{w})$ and *palindromic c -factorization* $pc(\mathbf{w})$ by requiring that each of the factors in the previous definitions be palindromes. That is, the *palindromic z -factorization* of \mathbf{w} is the factorization

$$pz(\mathbf{w}) = (z_1, z_2, z_3, \dots)$$

where z_i is the shortest *palindromic* prefix of $z_i z_{i+1} z_{i+2} \cdots$ such that there is no occurrence of z_i in \mathbf{w} at any position $j < |z_1 z_2 \cdots z_{i-1}|$. The palindromic z -factorization may not exist for

certain infinite words \mathbf{w} . For instance, if \mathbf{w} only contains palindromes of bounded length, then the palindromic z -factorization will not exist. This type of factorization is therefore only interesting when applied to infinite words with arbitrarily long palindromic factors. The *palindromic c -factorization* of \mathbf{w} is the factorization

$$pc(\mathbf{w}) = (c_1, c_2, c_3, \dots)$$

where c_i is the longest *palindromic* prefix of $c_i c_{i+1} c_{i+2} \dots$ such that there is an occurrence of c_i in \mathbf{w} at some position $j < |c_1 c_2 \dots c_{i-1}|$, or, if this prefix does not exist, the factor c_i is just a single letter.

For instance, if \mathbf{f} is the Fibonacci word, we have

$$pz(\mathbf{f}) = (0, 1, 00, 101, 00100, 10100101, \dots)$$

and

$$pc(\mathbf{f}) = (0, 1, 0, 010, 1001, 0010100, \dots).$$

It turns out that $pz(\mathbf{f})$ and $z(\mathbf{f})$ are the same, and in fact are equal to the *singular factorization* of \mathbf{f} (which we define later). However, the factorizations $pc(\mathbf{f})$ and $c(\mathbf{f})$ are not the same. We show that the factors of $pc(\mathbf{f})$ can also be written in terms of the singular words and the factorization $pc(\mathbf{f})$ (except for the first few factors) coincides with a nice factorization of \mathbf{f} that appears in [5].

We believe that it could be of interest to compare the ordinary z - and c -factorizations of certain infinite words with their palindromic z - and c -factorizations, in the same way that one can compare the ordinary complexity function of an infinite word with its palindromic complexity function (see [1]).

The main results of this paper give a description of the palindromic z - and c -factorizations of the Fibonacci word and, more generally, the m -bonacci word for $m \geq 2$.

2 Basics from combinatorics on words

Let A be a finite *alphabet*, i.e., a finite set made of *letters*. A (*finite*) *word* w over A is a finite sequence of letters belonging to A . If $w = w_0 w_1 \dots w_n \in A^*$ with $n \geq 0$ and $w_i \in A$ for all $i \in \{0, \dots, n\}$, then the *length* $|w|$ of w is $n + 1$, i.e., it is the number of letters that w contains. We let ε denote the empty word. This special word is the neutral element for concatenation of words, and its length is set to be 0. The set of all finite words over A is denoted by A^* , and we let $A^+ = A^* \setminus \{\varepsilon\}$ denote the set of non-empty finite words over A . An *infinite word* \mathbf{w} over A is any infinite sequence over A . The set of all infinite words over A is denoted by A^ω . Note that in this paper infinite words are written in bold.

A finite word $w \in A^*$ is a *prefix* (resp., *suffix*) of another finite word $z \in A^*$ if there exists $u \in A^*$ such that $z = wu$ (resp., $z = uw$). The word $w \in A^*$ is said to be a *factor* of $z \in A^*$ if there exist $u, v \in A^*$ such that $z = uvw$. If $z = xy$ is a finite word over A , we write $x^{-1}z = y$ and $zy^{-1} = x$. Observe that if $z = xyt$ with $t, x, y, z \in A^*$, then $(xy)^{-1}z = y^{-1}(x^{-1}z)$ and $z(yt)^{-1} = (zt^{-1})y^{-1}$. In particular, for any words $u, v \in A^*$, we have $(uv)^{-1} = v^{-1}u^{-1}$.

In the same way, a finite word $w \in A^*$ is a *prefix* of an infinite word $\mathbf{z} \in A^\omega$ if there exist $\mathbf{u} \in A^\omega$ such that $\mathbf{z} = w\mathbf{u}$. The word $w \in A^*$ is said to be a *factor* of $\mathbf{z} \in A^\omega$ if there exist $u \in A^*$ and $\mathbf{v} \in A^\omega$ such that $\mathbf{z} = uw\mathbf{v}$.

Let $w = w_0w_1\cdots w_n \in A^*$ with $n \geq 0$ and $w_i \in A$ for all $i \in \{0, \dots, n\}$. The *mirror image*, or *reversal*, of w is the word $w^R = w_nw_{n-1}\cdots w_0$ over A , i.e., the word obtained by reading w from right to left. We say that a word w over A is a *palindrome* if $w^R = w$.

A *factorization* of a finite word $w \in A^*$ is a finite sequence $(x_n)_{0 \leq n \leq m}$ of finite words over A such that

$$w = \prod_{n=0}^m x_n.$$

Similarly, a *factorization* of an infinite word $\mathbf{w} \in A^\omega$ is a sequence $(x_n)_{n \geq 0}$ of finite words over A such that

$$\mathbf{w} = \prod_{n \geq 0} x_n.$$

A *morphism* on A is a map $\sigma : A^* \rightarrow A^*$ such that for all $u, v \in A^*$, we have $\sigma(uv) = \sigma(u)\sigma(v)$. In order to define a morphism, it suffices to provide the image of letters belonging to A . A morphism is said to be *prolongable* on a letter $a \in A$ if $\sigma(a) = au$ with $u \in A^+$ and σ is non-erasing, i.e., the image of no letter is the empty word. If σ is prolongable on a , then $\sigma^n(a)$ is a proper prefix of $\sigma^{n+1}(a)$ for all $n \geq 0$. Therefore, the sequence $(\sigma^n(a))_{n \geq 0}$ of finite words defines an infinite word \mathbf{w} that is a fixed point of σ .

In combinatorics on words, given an alphabet A , a set $X \subset A^+$ of non-empty words is a *code* on A if any word $w \in A^*$ has at most one factorization using words of X . For more on this topic, see, for instance, [10, Chapter 6]. The following result can be found in [10, Chapter 6].

Proposition 1. *Let A, B be two finite alphabets, and let $\sigma : A^* \rightarrow B^*$ be an injective morphism. If $X \subset A^+$ is a code on A , then $\sigma(X)$ is a code on B .*

In the following definition, we introduce two new factorizations of interest.

Definition 2. Let \mathbf{w} be an infinite word over A . The *palindromic Ziv–Lempel* or *palindromic z -factorization* of \mathbf{w} is the factorization

$$pz(\mathbf{w}) = (z_1, z_2, z_3, \dots)$$

where z_i is the shortest *palindromic* prefix of $z_i z_{i+1} z_{i+2} \cdots$ such that there is no occurrence of z_i in \mathbf{w} at any position $j < |z_1 z_2 \cdots z_{i-1}|$. The *palindromic Crochemore* or *palindromic c -factorization* of \mathbf{w} is the factorization

$$pc(\mathbf{w}) = (c_1, c_2, c_3, \dots)$$

where c_i is the longest *palindromic* prefix of $c_i c_{i+1} c_{i+2} \cdots$ such that there is an occurrence of c_i in \mathbf{w} at some position $j < |c_1 c_2 \cdots c_{i-1}|$, or, if this prefix does not exist, the factor c_i is just a single letter.

3 The Fibonacci case

3.1 Some known results and preliminaries

Before establishing the two palindromic factorizations of the Fibonacci word, we gather some definitions and necessary results. Some of them are well known and can be found in [5, 13]. In the following definition, we follow the lines of [5].

Definition 3. Let \mathbf{f} be the (infinite) Fibonacci word, i.e., the fixed point of the morphism $\varphi : 0 \mapsto 01, 1 \mapsto 0$, starting with 0. For all $n \geq 0$, define the finite word $h_n = \varphi^n(0)$ to be the n th iteration of φ on 0. The first few words of the sequence $(h_n)_{n \geq 0}$ are 0, 01, 010, 01001. It is well known that the Fibonacci word \mathbf{f} is the limit of $(h_n)_{n \geq 0}$. Let $(p_n)_{n \geq 3}$ be the sequence of the palindromic prefixes of \mathbf{f} , which are also called *central words*. The first few terms of this sequence are $\varepsilon, 0, 010, 010010, \dots$. The *singular words* $(\hat{f}_n)_{n \geq 1}$ satisfy $\hat{f}_1 = 0$, $\hat{f}_2 = 1$ and, for all $n \geq 1$, $\hat{f}_{2n+1} = 0p_{2n+1}0$ and $\hat{f}_{2n+2} = 1p_{2n+2}1$. The first few singular words are 0, 1, 00, 101, 00100.

The following properties of the singular words can be found in [13].

Proposition 4. Let $(F_n)_{n \geq 0}$ be the sequence of Fibonacci numbers with initial conditions $F_0 = 0$ and $F_1 = 1$.

- (1) For all $n \geq 1$, \hat{f}_n is a palindrome.
- (2) For all $n \geq 1$, $|\hat{f}_n| = F_n$.
- (3) For all $n \geq 4$, $\hat{f}_n = \hat{f}_{n-2}\hat{f}_{n-3}\hat{f}_{n-2}$.
- (4) For all $n \geq 1$, \hat{f}_n is not a factor of \hat{f}_{n+1} .
- (5) For all $n \geq 1$, \hat{f}_n is not a factor of $\prod_{m=1}^{n-1} \hat{f}_m$.
- (6) Let $n \geq 1$ and let $\hat{f}_{n+1} = wa$ where $w \in \{0, 1\}^*$ and $a \in \{0, 1\}$. If $\hat{f}'_{n+1} = w\bar{a}$ with $\bar{a} = 1 - a$, then $\hat{f}_{n+2} = \hat{f}_n \hat{f}'_{n+1}$.
- (7) Let $n \geq 3$ and define α_n to be 0 if n is odd, or 1 if n is even. Then $\hat{f}_n = \alpha_n \prod_{m=1}^{n-2} \hat{f}_m$.

The following result can be found in [5]. Note that the first factorization of the Fibonacci word \mathbf{f} also appears in [13].

Proposition 5. We have the following two factorizations of the Fibonacci word

$$\mathbf{f} = \prod_{n \geq 1} \hat{f}_n \tag{1}$$

$$= 0 \cdot 1 \cdot 00 \cdot 101 \cdot 00100 \cdot 10100101 \dots$$

$$= 010 \prod_{n \geq 2} \hat{f}_{n-1} \hat{f}_n \hat{f}_{n-1} \tag{2}$$

$$= 010 \cdot (0 \cdot 1 \cdot 0) \cdot (1 \cdot 00 \cdot 1) \cdot (00 \cdot 101 \cdot 00) \cdot (101 \cdot 00100 \cdot 101) \dots$$

Moreover, the Ziv–Lempel factorization of the Fibonacci word is given by the sequence of singular words, i.e.,

$$z(\mathbf{f}) = (\hat{f}_1, \hat{f}_2, \hat{f}_3, \dots).$$

As a matter of fact, the palindromic z -factorization of \mathbf{f} is easily deduced from the previous result, as shown in the next section. However, the palindromic c -factorization of \mathbf{f} cannot be obtained from already known results, and, to that aim, we define a sequence of specific prefixes of \mathbf{f} .

Definition 6. For all $n \geq 2$, define

$$g_n := 010 \prod_{2 \leq m \leq n-1} \hat{f}_{m-1} \hat{f}_m \hat{f}_{m-1}.$$

From (2), observe that, for all $n \geq 2$, we have

$$\mathbf{f} = g_n \cdot (\hat{f}_{n-1} \hat{f}_n \hat{f}_{n-1}) \cdot \prod_{m \geq n+1} \hat{f}_{m-1} \hat{f}_m \hat{f}_{m-1}.$$

Interestingly, the prefix g_n of \mathbf{f} can be factorized as a particular product of singular words.

Proposition 7. For all $n \geq 2$, we have

$$g_n = \hat{f}_1 \hat{f}_2 \cdots \hat{f}_{n-1} \hat{f}_n \hat{f}_{n-1} \hat{f}_{n-2}. \quad (3)$$

Proof. Proceed by induction on $n \geq 2$. The result holds for $n = 2$ because $g_2 = 010 = \hat{f}_1 \hat{f}_2 \hat{f}_1$. For $n = 3$, we get $g_3 = 010 \cdot (0 \cdot 1 \cdot 0)$ by Definition 6 and therefore

$$g_3 = 0 \cdot 1 \cdot 00 \cdot 1 \cdot 0 = \hat{f}_1 \hat{f}_2 \hat{f}_3 \hat{f}_2 \hat{f}_1,$$

as desired. Assume that $n \geq 3$. Now we suppose the result holds up to n and we show it still holds for $n + 1$. Using Definition 6, we have

$$g_{n+1} = g_n (\hat{f}_{n-1} \hat{f}_n \hat{f}_{n-1}).$$

By the induction hypothesis, we get

$$\begin{aligned} g_{n+1} &= (\hat{f}_1 \hat{f}_2 \cdots \hat{f}_{n-1} \hat{f}_n \hat{f}_{n-1} \hat{f}_{n-2}) (\hat{f}_{n-1} \hat{f}_n \hat{f}_{n-1}) \\ &= \hat{f}_1 \hat{f}_2 \cdots \hat{f}_{n-1} \hat{f}_n (\hat{f}_{n-1} \hat{f}_{n-2} \hat{f}_{n-1}) \hat{f}_n \hat{f}_{n-1}. \end{aligned}$$

Since $n + 1 \geq 4$, Proposition 4 implies that $\hat{f}_{n+1} = \hat{f}_{n-1} \hat{f}_{n-2} \hat{f}_{n-1}$, and we deduce that

$$g_{n+1} = \hat{f}_1 \hat{f}_2 \cdots \hat{f}_{n-1} \hat{f}_n \hat{f}_{n+1} \hat{f}_n \hat{f}_{n-1},$$

which ends the proof. □

3.2 The palindromic z -factorization of the Fibonacci word

In this (very) short section, we obtain the palindromic z -factorization of the Fibonacci word, which easily follows from already known results.

Theorem 8. The palindromic z -factorization of the Fibonacci word \mathbf{f} is

$$pz(\mathbf{f}) = (\hat{f}_1, \hat{f}_2, \hat{f}_3, \dots).$$

Proof. From Proposition 5, $z(\mathbf{f}) = (\hat{f}_1, \hat{f}_2, \hat{f}_3, \dots)$. Since the factors \hat{f}_n are all palindromes by Proposition 4, this factorization is also $pz(\mathbf{f})$. □

3.3 The palindromic c -factorization of the Fibonacci word

In this section, we show that, after the prefix of length 3, the factorization (2) coincides with the factorization $pc(\mathbf{f})$. Note that in this case $pc(\mathbf{f})$ and $c(\mathbf{f})$ are not the same, since the factors in $c(\mathbf{f})$ are not palindromes.

Lemma 9. *For all $n \geq 1$, the only suffix of \hat{f}_n that is also a prefix of \hat{f}_{n+1} is the empty word.*

Proof. We proceed by induction on $n \geq 1$. From Definition 3, the first two singular words are $\hat{f}_1 = 0$ and $\hat{f}_2 = 1$, so the result can be checked by hand for $n = 1$.

Now suppose that $n \geq 2$, and that the only suffix of \hat{f}_k that is also a prefix of \hat{f}_{k+1} is the empty word, for all $k \in \{1, \dots, n-1\}$. We show that the result still holds for $k = n$. Proceed by contradiction and suppose there exists a word $x \in \{0, 1, \dots, m-1\}^*$ which is a non-empty suffix of \hat{f}_n and a non-empty prefix of \hat{f}_{n+1} . We have $1 \leq |x| \leq |\hat{f}_n|$. Using Proposition 4(6), \hat{f}_{n+1} starts and ends with \hat{f}_{n-1} .

If $1 \leq |x| \leq |\hat{f}_{n-1}|$, then x is a prefix of \hat{f}_{n-1} (recall that x is a prefix of \hat{f}_{n+1}). Consequently, x^R is a non-empty suffix of \hat{f}_{n-1} and a non-empty prefix of \hat{f}_n . This contradicts the inductive assumption.

If $|\hat{f}_{n-1}| \leq |x| \leq |\hat{f}_n|$, then \hat{f}_{n-1} is a prefix of x (recall that x is a prefix of \hat{f}_{n+1}). In particular, \hat{f}_{n-1} is a factor of x , and also a factor of \hat{f}_n (recall that x is a suffix of \hat{f}_n). This contradicts Proposition 4(4). \square

In the following lemma, recall that we start indexing words at 0.

Lemma 10. *Let $n \geq 1$. There are exactly two occurrences of the factor \hat{f}_n inside the word g_{n+1} : one at position $\sum_{m=1}^{n-1} |\hat{f}_m|$, the other at position $\sum_{m=1}^{n+1} |\hat{f}_m|$.*

Proof. If $n = 1$, then $g_2 = 010$ and the factor $\hat{f}_1 = 0$ occurs in g_2 at positions 0 and $2 = |\hat{f}_1| + |\hat{f}_2|$. If $n = 2$, then $g_3 = 010010 = g_{3,0} \cdots g_{3,5}$ with $g_{3,i} \in \{0, 1\}$ for all $1 \leq i \leq 5$. There are exactly two occurrences of $\hat{f}_2 = 1$ in g_3 starting either at position $1 = |\hat{f}_1|$ or $4 = |\hat{f}_1| + |\hat{f}_2| + |\hat{f}_3|$.

Suppose that $n \geq 3$. Using (3), let us write $g_{n+1} = p\hat{f}_n\hat{f}_{n+1}\hat{f}_n\hat{f}_{n-1}$ with $p = \hat{f}_1\hat{f}_2 \cdots \hat{f}_{n-1}$. Thanks to this factorization, we immediately see that \hat{f}_n occurs at least twice as a factor of g_{n+1} : one starting at position $|p| = \sum_{m=1}^{n-1} |\hat{f}_m|$, the other beginning at position $|p\hat{f}_n\hat{f}_{n+1}| = \sum_{m=1}^{n+1} |\hat{f}_m|$. We now show that there are no other occurrences of \hat{f}_n as a factor of g_{n+1} . There are several cases to consider.

Case 1. The word \hat{f}_n cannot be a factor of p , otherwise it contradicts Proposition 4(5).

Case 2. The word \hat{f}_n cannot be a factor of \hat{f}_{n+1} , otherwise it contradicts Proposition 4(4).

Case 3. The word \hat{f}_n cannot be a factor of \hat{f}_{n-1} since $|\hat{f}_{n-1}| = F_{n-1} < F_n = |\hat{f}_n|$ by Proposition 4(2) (note that $n-1 \geq 2$).

Case 4. Suppose that \hat{f}_n is a factor of $p\hat{f}_n$, overlapping p and \hat{f}_n . Using Proposition 4(2) ($n-2 \geq 1$), we know that

$$|\hat{f}_n| = F_n = F_{n-1} + F_{n-2} = |\hat{f}_{n-1}| + |\hat{f}_{n-2}|.$$

Consequently, \hat{f}_n is a factor of $\hat{f}_{n-2}\hat{f}_{n-1}\hat{f}_n$. If \hat{f}_n starts somewhere within \hat{f}_{n-2} , or if \hat{f}_n starts with the first letter of \hat{f}_{n-1} , then \hat{f}_{n-1} is a factor of \hat{f}_n , which contradicts Proposition 4(4). Therefore \hat{f}_n must be a factor of $\hat{f}_{n-1}\hat{f}_n$, i.e., there exist a non-empty suffix x of \hat{f}_{n-1} and a

non-empty prefix y of \hat{f}_n such that $\hat{f}_n = xy$. Then x is also a non-empty prefix of \hat{f}_n , which contradicts Lemma 9.

Case 5. Suppose that \hat{f}_n is a factor of $\hat{f}_n\hat{f}_{n+1}$, overlapping \hat{f}_n and \hat{f}_{n+1} . This case is similar to the fourth case above. Indeed, observe that, since $n+1 \geq 4$, Proposition 4(3) gives

$$\hat{f}_{n+1} = \hat{f}_{n-1}\hat{f}_{n-2}\hat{f}_{n-1}.$$

Using Proposition 4 again, we know that $|\hat{f}_n| = F_n = F_{n-1} + F_{n-2} = |\hat{f}_{n-1}| + |\hat{f}_{n-2}|$. Consequently, \hat{f}_n is a factor of $\hat{f}_n\hat{f}_{n-1}\hat{f}_{n-2}$, so $(\hat{f}_n)^R = \hat{f}_n$ is a factor of $(\hat{f}_n\hat{f}_{n-1}\hat{f}_{n-2})^R = \hat{f}_{n-2}\hat{f}_{n-1}\hat{f}_n$, which is impossible due to the fourth case.

Case 6. Suppose that \hat{f}_n is a factor of $\hat{f}_{n+1}\hat{f}_n$, overlapping \hat{f}_{n+1} and \hat{f}_n . In this case, $(\hat{f}_n)^R = \hat{f}_n$ is a factor of $(\hat{f}_{n+1}\hat{f}_n)^R = \hat{f}_n\hat{f}_{n+1}$ since the singular words are palindromes. As in the fifth case, we raise a contradiction.

Case 7. Suppose that \hat{f}_n is a factor of $\hat{f}_n\hat{f}_{n-1}$, overlapping \hat{f}_n and \hat{f}_{n-1} . In this case, $(\hat{f}_n)^R = \hat{f}_n$ is a factor of $(\hat{f}_n\hat{f}_{n-1})^R = \hat{f}_{n-1}\hat{f}_n$ since the singular words are palindromes. As in the fourth case, we reach a contradiction. \square

We prove a technical result before getting the palindromic c -factorization of \mathbf{f} .

Proposition 11. *Let $n \geq 2$. Let w be a non-empty common finite prefix of the infinite words*

$$\hat{f}_{n-1} \cdot (\hat{f}_n\hat{f}_{n+1}\hat{f}_n) \cdot (\hat{f}_{n+1}\hat{f}_{n+2}\hat{f}_{n+1}) \cdots$$

and

$$(\hat{f}_{n+1}\hat{f}_{n+2}\hat{f}_{n+1}) \cdot (\hat{f}_{n+2}\hat{f}_{n+3}\hat{f}_{n+2}) \cdots$$

Then $\hat{f}_n\hat{f}_{n+1}\hat{f}_nw$ is not a palindrome.

Proof. Let us define

$$u_{n+1} := \hat{f}_n\hat{f}_{n+1}\hat{f}_nw$$

where w is taken as in the statement. Using Proposition 4, since $\hat{f}_{n+1} = \hat{f}_{n-1}\hat{f}'_n$ and $|\hat{f}_{n-1}| + |\hat{f}'_n| = |\hat{f}_{n+1}|$, we know that $0 < |w| < |\hat{f}_{n+1}|$. Now proceed by contradiction and suppose that u_{n+1} is a palindrome. Then we have

$$\hat{f}_n\hat{f}_{n+1}\hat{f}_nw = u_{n+1} = u_{n+1}^R = w^R\hat{f}_n\hat{f}_{n+1}\hat{f}_n. \quad (4)$$

The bounds on the length of w lead to an overlap between the occurrence of \hat{f}_{n+1} at position $|\hat{f}'_n|$ (in the leftmost word in (4)), and the occurrence \hat{f}_n at position $|w^R| = |w|$ (in the rightmost word in (4)). This is impossible due to either Proposition 4(4), or Lemma 9. \square

Theorem 12. *Let $pc(\mathbf{f}) = (c_{-1}, c_0, c_1, c_2, \dots)$ denote the palindromic c -factorization of the Fibonacci word \mathbf{f} . Then, we have $c_{-1} = 0$, $c_0 = 1$, $c_1 = 0$ and, for all $n \geq 2$,*

$$c_n = \hat{f}_{n-1}\hat{f}_n\hat{f}_{n-1}.$$

Proof. By definition of the palindromic c -factorization of the Fibonacci word \mathbf{f} , we clearly have $c_{-1} = 0$, $c_0 = 1$ and $c_1 = 0$. For the second part of the result, proceed by induction on

$n \geq 2$. Suppose $n = 2$. Let us find the factor c_2 of the palindromic c -factorization of \mathbf{f} . We have

$$\mathbf{f} = 0 \cdot 1 \cdot 0 \cdot 01010010010100101 \cdots$$

and the longest palindrome starting with 0 and occurring before is

$$c_2 = 010 = 0 \cdot 1 \cdot 0 = \hat{f}_1 \hat{f}_2 \hat{f}_1,$$

as expected.

For the induction step, suppose $n \geq 2$ and assume that, for all $2 \leq m \leq n$, we have $c_m = \hat{f}_{m-1} \hat{f}_m \hat{f}_{m-1}$. We show it is still true for $m = n + 1$. On the one hand, by the induction hypothesis, we have

$$\begin{aligned} \mathbf{f} &= c_{-1} c_0 c_1 c_2 \cdots c_n c_{n+1} c_{n+2} \cdots \\ &= 010 \left(\prod_{2 \leq m \leq n} c_m \right) c_{n+1} c_{n+2} \cdots \\ &= 010 \left(\prod_{2 \leq m \leq n} \hat{f}_{m-1} \hat{f}_m \hat{f}_{m-1} \right) c_{n+1} c_{n+2} \cdots \end{aligned} \quad (5)$$

and the goal is to find the next factor of the palindromic c -factorization of \mathbf{f} , i.e., the word c_{n+1} . On the other hand, using (2) first and then (3) since n is large enough, we get

$$\begin{aligned} \mathbf{f} &= 010 \left(\prod_{2 \leq m \leq n} \hat{f}_{m-1} \hat{f}_m \hat{f}_{m-1} \right) (\hat{f}_n \hat{f}_{n+1} \hat{f}_n) (\hat{f}_{n+1} \hat{f}_{n+2} \hat{f}_{n+1}) \cdots \\ &= g_{n+1} (\hat{f}_n \hat{f}_{n+1} \hat{f}_n) (\hat{f}_{n+1} \hat{f}_{n+2} \hat{f}_{n+1}) \cdots \\ &= (\hat{f}_1 \hat{f}_2 \cdots \hat{f}_{n-1} \hat{f}_n \hat{f}_{n+1} \hat{f}_n \hat{f}_{n-1}) (\hat{f}_n \hat{f}_{n+1} \hat{f}_n) (\hat{f}_{n+1} \hat{f}_{n+2} \hat{f}_{n+1}) \cdots \end{aligned} \quad (6)$$

Using (6), it is clear that $|c_{n+1}| \geq |\hat{f}_n \hat{f}_{n+1} \hat{f}_n|$ since $\hat{f}_n \hat{f}_{n+1} \hat{f}_n$ is a palindrome occurring before. Therefore, there exists a word $w \in \{0, 1\}^*$ such that $c_{n+1} = \hat{f}_n \hat{f}_{n+1} \hat{f}_n w$. We claim that w is in fact the empty word and proceed by contradiction.

By Lemma 10, we know that there are exactly two occurrences of \hat{f}_n in g_{n+1} : one starts at position $\sum_{m=1}^{n-1} |\hat{f}_m|$, and the other at position $\sum_{m=1}^{n+1} |\hat{f}_m|$.

Case 1. Let us deal with the occurrence of \hat{f}_n in g_{n+1} at position $\sum_{m=1}^{n-1} |\hat{f}_m|$. In this case, w must be a common prefix of the infinite words

$$\hat{f}_{n-1} \cdot (\hat{f}_n \hat{f}_{n+1} \hat{f}_n) \cdot (\hat{f}_{n+1} \hat{f}_{n+2} \hat{f}_{n+1}) \cdots$$

and

$$(\hat{f}_{n+1} \hat{f}_{n+2} \hat{f}_{n+1}) \cdot (\hat{f}_{n+2} \hat{f}_{n+3} \hat{f}_{n+2}) \cdots$$

By Proposition 11, we know that $\hat{f}_n \hat{f}_{n+1} \hat{f}_n w$ is not a palindrome if w is non-empty, a contradiction.

Case 2. Let us consider the occurrence of \hat{f}_n in g_{n+1} at position $\sum_{m=1}^{n+1} |\hat{f}_m|$. In this case, $\hat{f}_n \hat{f}_{n+1} \hat{f}_n w$ must be a common prefix of the infinite words

$$\hat{f}_n \hat{f}_{n-1} \cdot (\hat{f}_n \hat{f}_{n+1} \hat{f}_n) \cdot (\hat{f}_{n+1} \hat{f}_{n+2} \hat{f}_{n+1}) \cdots$$

and

$$(\hat{f}_n \hat{f}_{n+1} \hat{f}_n) \cdot (\hat{f}_{n+1} \hat{f}_{n+2} \hat{f}_{n+1}) \cdot (\hat{f}_{n+2} \hat{f}_{n+3} \hat{f}_{n+2}) \cdots$$

Using Proposition 4, we know that

$$|\hat{f}_{n+1}| = F_{n+1} = F_n + F_{n-1} = |\hat{f}_n| + |\hat{f}_{n-1}|.$$

Consequently, $\hat{f}_{n+1} = \hat{f}_{n-1} \hat{f}_n$, which violates Proposition 4 (items (4) or (6)).

As a conclusion, the longest palindrome starting with the first letter of \hat{f}_n and occurring before is

$$c_{n+1} = \hat{f}_n \hat{f}_{n+1} \hat{f}_n,$$

as required. □

4 The m -bonacci case

In this section, we extend the results obtained for the Fibonacci word to any m -bonacci word, namely we get the palindromic z - and c -factorizations of any m -bonacci word. The strategy is similar to the one adopted in the previous case: we define a particular sequence $(z_n^{(m)})_{n \geq -1}$ of finite words that we will call *p -singular words*, and we write the palindromic z - and c -factorizations of any m -bonacci word in terms of this sequence. In the case $m = 2$, the words $(z_n^{(2)})_{n \geq 0}$ turn out to be the singular words $(\hat{f}_n)_{n \geq 1}$ (see Proposition 22).

4.1 Preliminaries

Definition 13. Let $m \geq 2$. We define the morphism ϕ_m on $\{0, 1, \dots, m-1\}$ by

$$\phi_m : 0 \mapsto 01, 1 \mapsto 02, \dots, (m-2) \mapsto 0(m-1), (m-1) \mapsto 0.$$

When $m = 2$, then $\phi_m = \varphi$, and we fall into the Fibonacci case above.

Let \mathbf{w}_m be the (infinite) m -bonacci word, i.e., the fixed point of the morphism ϕ_m , starting with 0. For all $n \geq 0$, define $h_n^{(m)} = \phi_m^n(0)$ to be the n th iteration of ϕ_m on 0. It is well known that the m -bonacci word \mathbf{w}_m is the limit of $(h_n^{(m)})_{n \geq 0}$. For the sake of simplicity, when the context is clear, we write h_n instead of $h_n^{(m)}$.

From now on, m is a fixed integer greater than 1 unless otherwise specified.

Example 14. If $m = 2$, then $\mathbf{w}_2 = \mathbf{f}$ is the Fibonacci word. See also Definition 3. If $m = 3$, then $\phi_3 : 0 \mapsto 01, 1 \mapsto 02, 2 \mapsto 0$, and the infinite word \mathbf{w}_3 is called the Tribonacci word. If $m = 4$, then $\phi_4 : 0 \mapsto 01, 1 \mapsto 02, 2 \mapsto 03, 3 \mapsto 0$, and the infinite word \mathbf{w}_4 is called the Quadribonacci word. In Table 1, the first few words of the sequences $(h_n^{(m)})_{n \geq 0}$ are given for $m \in \{2, 3, 4\}$.

Lemma 15. *The set of non-empty words $\{01, 02, \dots, 0(m-1), 0\}$ is a code on the finite alphabet $\{0, 1, \dots, m-1\}$.*

Proof. It directly follows from Proposition 1, the fact that ϕ_m is an injective morphism, and $\{0, 1, \dots, m-1\}$ is a code on $\{0, 1, \dots, m-1\}$. □

m	$(h_n^{(m)})_{n \geq 0}$
2	0,01,010,01001,...
3	0,01,0102,0102010,0102010010201,...
4	0,01,0102,01020103,010201030102010,...

Table 1: The first few words of the sequences $(h_n^{(m)})_{n \geq 0}$ for $m \in \{2, 3, 4\}$.

Remark 16. Observe that not all words over $\{0, 1, \dots, m-1\}$ have a factorization in terms of blocks of $\{01, 02, \dots, 0(m-1), 0\}$. For instance, the word 1 does not have any such factorization. However if a word has such a factorization, then it is unique.

The following lemma will be useful to prove properties similar to those given in Proposition 4.

Lemma 17. *Let $x, y \in \{0, 1, \dots, m-1\}^*$ be two finite words.*

- (1) *If $\phi_m(x)0$ is a factor of $\phi_m(y)0$, then x is a factor of y .*
- (2) *If $\phi_m(x)$ is a factor of $\phi_m(y)$ and x does not end with the letter $m-1$, then x is a factor of y .*

Proof. If x is the empty word, then both items are true. Now assume that x is non-empty, so is y . From Lemma 15, the set of words $C = \{01, 02, \dots, 0(m-1), 0\}$ is a code, and thus the words $\phi_m(x)$ and $\phi_m(y)$ respectively admit a unique factorization in terms of blocks belonging to C . There exist positive integers ℓ, k and words $x_1, \dots, x_\ell, y_1, \dots, y_k \in C$ such that $\phi_m(x) = x_1 \cdots x_\ell$ and $\phi_m(y) = y_1 \cdots y_k$.

Let us prove item (1). Note that $\phi_m(x(m-1)) = \phi_m(x)0 = x_1 \cdots x_\ell 0$ and $\phi_m(y(m-1)) = \phi_m(y)0 = y_1 \cdots y_k 0$. By assumption, there exist words $w, t \in \{0, 1, \dots, m-1\}^*$ such that $\phi_m(y)0 = w\phi_m(x)0t$. Using the form of the blocks in C , there exist $1 \leq i \leq j \leq k$ such that $y_i = x_1$ and $y_j = x_\ell$ (since no words in C start with a letter different from 0 and since 00 implies that the first letter 0 is a block in C). By uniqueness of the factorization, we also have $y_{i+r-1} = x_r$ for all $1 \leq r \leq \ell$ and $j = i + \ell - 1$. Consequently, $w = y_0 y_1 \cdots y_{i-1}$ and $0t = y_{i+\ell} \cdots y_k 0$. From the form of the words w and $0t$, we deduce that there exist words $w', t' \in \{0, 1, \dots, m-1\}^*$ such that $\phi_m(w') = w$ and $\phi_m(t') = 0t$. Thus,

$$\phi_m(y)0 = w \cdot \phi_m(x) \cdot (0t) = \phi_m(w')\phi_m(x)\phi_m(t') = \phi_m(w'xt').$$

By injectivity of ϕ_m , $y = w'xt'$, and x is a factor of y , as desired.

Let us show that item (2) also holds. By hypothesis, $\phi_m(x)$ is a factor of $\phi_m(y)$ and the last letter of x is not $m-1$ (i.e., the block x_ℓ is of length 2 and ends with a letter different from 0). By an analogous reasoning, there exists $1 \leq i \leq k$ such that $y_{i+r-1} = x_r$ for all $1 \leq r \leq \ell$. Now let $w = y_0 y_1 \cdots y_{i-1}$ and $t = y_{i+\ell} \cdots y_k$. We have

$$\phi_m(y) = (y_0 \cdots y_{i-1}) \cdot (y_i \cdots y_{i+\ell-1}) \cdot (y_{i+\ell} \cdots y_k) = w\phi_m(x)t.$$

As before, there exist $w', t' \in \{0, 1, \dots, m-1\}^*$ such that $\phi_m(w') = w$ and $\phi_m(t') = t$. By injectivity of ϕ_m , x is again a factor of y . \square

Example 18. Let $x = 1012$ and $y = 1010$ be words in $\{0, 1, 2\}^*$ ($m = 3$ here). We see that $\phi_m(x) = 0201020$ is a factor of $\phi_m(y) = 02010201$ while x is not a factor of y . This is due to the fact that x ends with $m-1 = 2$.

4.2 Properties of p-singular words

In [5, 13], the Fibonacci word \mathbf{f} is factorized into singular words (see Proposition 5). In [11], this notion of singular words is extended to cover the case of characteristic Sturmian words. In particular, any characteristic Sturmian word \mathbf{c}_α has a singular decomposition and those singular words are useful to find the palindromic factors of \mathbf{c}_α . Leaving the framework of a two-letter alphabet, it is shown in [12] that there are two kinds of singular words in the Tribonacci case ($m = 3$), and that the Tribonacci word possesses a decomposition into singular words. Afterwards, the study of singular words has been extended to include standard episturmian words. More particularly, a standard episturmian \mathbf{s} word over $\{a_1, \dots, a_k\}$ is k -strict if every letter a_i , $1 \leq i \leq k$, occurs infinitely many times in its directive word. In fact, the k -strict standard episturmian words are exactly the k -letter Arnoux–Rauzy sequences. To learn more about the subject, we refer the reader to [7]. In [7, Chapter 7], it is shown that any word \mathbf{s} in a class of specific k -strict standard episturmian words has several kinds of generalized singular words. Roughly, those singular words turn out to be notably useful to study factors of \mathbf{s} (e.g., squares, cubes and other powers), and can also be used to factorize \mathbf{s} (this particular factorization is referred to as a *partition* in [7, Chapter 7]).

Following the same lead, we define the p-singular words in the general case of the m -bonacci word \mathbf{w}_m . Those particular words are useful to obtain the palindromic z - and c -factorizations of \mathbf{w}_m . In this section, we study some of their properties.

Definition 19. Define the sequence $(z_n^{(m)})_{n \geq -1}$ of finite words over the alphabet $\{0, 1, \dots, m-1\}$ by $z_{-1}^{(m)} = \varepsilon$, $z_0^{(m)} = 0$, and

- (1) For all $1 \leq n \leq m-1$, $z_n^{(m)} = z_{n-2}^{(m)} z_{n-3}^{(m)} \cdots z_1^{(m)} z_0^{(m)} n z_0^{(m)} z_1^{(m)} \cdots z_{n-3}^{(m)} z_{n-2}^{(m)}$;
- (2) For all $n \geq m$, $z_n^{(m)} = z_{n-2}^{(m)} z_{n-3}^{(m)} \cdots z_{n-(m-1)}^{(m)} z_{n-m}^{(m)} z_{n-(m+1)}^{(m)} z_{n-m}^{(m)} z_{n-(m-1)}^{(m)} \cdots z_{n-3}^{(m)} z_{n-2}^{(m)}$.

Note that for $0 \leq n \leq m-1$ (resp. $n \geq m$), $z_n^{(m)}$ is centered at n (resp., $z_{n-(m+1)}^{(m)}$).

In the Fibonacci case when $m = 2$, we will show in Proposition 22 that the corresponding words are the singular words $(\hat{f}_n)_{n \geq 1}$. For that reason, the sequence $(z_n^{(m)})_{n \geq -1}$ is the sequence of words called *p-singular words*. The p-singular words satisfy a number of identities related to the standard and central words; for instance, the following result gives another way we could have chosen to define the p-singular words; see [6], where the ordinary z - and c -factorizations of episturmian words are best described in terms of the words h_n^R .

Lemma 20. *For all $n \geq 0$, we have*

$$z_n^{(m)} = \begin{cases} (h_1^{(m)R} h_3^{(m)R} \cdots h_n^{(m)R} \left((h_0^{(m)R} h_2^{(m)R} \cdots h_{n-1}^{(m)R} \right)^{-1}, & \text{if } n \text{ is odd;} \\ (h_0^{(m)R} h_2^{(m)R} \cdots h_n^{(m)R} \left((h_1^{(m)R} h_3^{(m)R} \cdots h_{n-1}^{(m)R} \right)^{-1}, & \text{if } n \text{ is even.} \end{cases}$$

The lemma can be proved directly from the definitions, but we are able to give a much more elegant proof after first proving some preliminary results. We therefore postpone the proof until after Lemma 29.

m	$(z_n^{(m)})_{n \geq -1}$
2	$\varepsilon, 0, 1, 00, 101, 00100, \dots$
3	$\varepsilon, 0, 1, 020, 1001, 020101020, 100102010201001, \dots$
4	$\varepsilon, 0, 1, 020, 10301, 0201001020, 1030102010102010301, \dots$
5	$\varepsilon, 0, 1, 020, 10301, 02010401020, 10301020100102010301, \dots$

Table 2: The first few words of the sequences $(z_n^{(m)})_{n \geq -1}$ for $m \in \{2, 3, 4, 5\}$.

Example 21. In Table 2, the first few p-singular words are displayed for $m \in \{2, 3, 4, 5\}$.

In fact, in the context of the Fibonacci word ($m = 2$), the word $z_n^{(2)}$ is the $(n + 1)$ st singular word, as shown below. As a consequence, the palindromic z - and c -factorizations of the Fibonacci word can be rewritten in terms of the sequence $(z_n^{(2)})_{n \geq 0}$ of p-singular words; see Theorems 8 and 12. In the same way, we will show that the palindromic z - and c -factorizations of any m -bonacci word involve the p-singular words.

Proposition 22. For all $n \geq 0$, we have $\hat{f}_{n+1} = z_n^{(2)}$.

Proof. We proceed by induction on $n \geq 0$. The result is true for $n \in \{0, 1, 2\}$. Now suppose that $n \geq 3$ and the result holds up to $n - 1$. We show it is still true for n . Using Proposition 4, then the induction hypothesis and finally Definition 19, we get the result

$$\hat{f}_{n+1} = \hat{f}_{n-1}\hat{f}_{n-2}\hat{f}_{n-1} = z_{n-2}^{(2)}z_{n-3}^{(2)}z_{n-2}^{(2)} = z_n^{(2)}. \quad \square$$

Again, for the sake of simplicity, when the context is clear, we write z_n instead of $z_n^{(m)}$. By induction and Definition 19, it is clear that the p-singular words are palindromes.

Proposition 23. For all $n \geq -1$, z_n is a palindrome.

Also from Definition 19, we know the prefixes and suffixes of length at most 3 of the p-singular words.

Proposition 24. For all $n \geq 0$, z_n starts and ends with the letter 0 (resp., 1) if n is even (resp., odd). Moreover, for all even $n \geq 2$, z_n starts and ends with 00 if $m = 2$, or 020 if $m \geq 3$; for all odd $n \geq 2$, z_n starts with 101 if $m = 2$, or 100 if $m = 3$, or 103 if $m \geq 4$, and ends with 101 if $m = 2$, or 001 if $m = 3$, or 301 if $m \geq 4$.

Proof. For the first part of the result, we proceed by induction on $n \geq 0$. From Definition 19, $z_0 = 0$ and $z_1 = 1$, so the result is true for $n \in \{0, 1\}$. Now suppose that $n \geq 2$, and that the result holds for values less than n . If $1 \leq n \leq m - 1$ (resp., $n \geq m$), then Definition 19(1) (resp., Definition 19(2)) shows that z_n ends and starts with z_{n-2} . Using the induction hypothesis since $n - 2 \geq 0$, we know that z_{n-2} starts and ends with 0 (resp., 1) if $n - 2$ is even (resp., odd). Consequently, z_n starts and ends with the letter 0 (resp., 1) if n is even (resp., odd).

The proof of the second part of the statement is obtained in the same manner by first observing that Definition 19 (or Table 2) gives $z_2 = 00$ if $m = 2$, or $z_2 = 020$ if $m \geq 3$, and

$$z_3 = \begin{cases} 101, & \text{if } m = 2; \\ 1001, & \text{if } m = 3; \\ 10301, & \text{if } m \geq 4. \end{cases}$$

□

In the following two corollaries, resulting from Definition 19, we study the length of p-singular words.

Corollary 25. *We have $|z_0| = 1$, and for all $1 \leq n \leq m - 1$, $|z_n| = 2 \sum_{k=0}^{n-2} |z_k| + 1$. In particular, for all $1 \leq n \leq m - 1$, $|z_n| = |z_{n-1}| + 2|z_{n-2}|$.*

Proof. From Definition 19, we have $|z_0| = |0| = 1$. Now let $1 \leq n \leq m - 1$. From Definition 19(1), we get

$$|z_n| = 2 \sum_{k=0}^{n-2} |z_k| + 1,$$

which proves the first part of the statement. Let us show the second part of the statement. The case $n = 1$ is easily handled. Suppose that $2 \leq n \leq m - 1$. From the first part of the result with $n - 1 \in \{1, \dots, m - 1\}$, we know that

$$|z_n| = 2 \sum_{k=0}^{n-2} |z_k| + 1 = \left(2 \sum_{k=0}^{n-3} |z_k| + 1 \right) + 2|z_{n-2}| = |z_{n-1}| + 2|z_{n-2}|. \quad \square$$

In the following corollary, when n is big enough, the length of the p-singular word z_n is expressed in terms of the length of the previous m p-singular words z_{n-1}, \dots, z_{n-m} . Note that, when $m = 2$, then the following result is implied by Propositions 4(2) and 22. Also observe that, when m is even, the sequence $(|z_n|)_{n \geq 0}$ of positive integers satisfies a m -bonacci type recurrence relation. However, that is not the case when m is odd.

Corollary 26. *If m is even, then, for all $n \geq m - 1$, we have*

$$|z_n| = |z_{n-1}| + |z_{n-2}| + \dots + |z_{n-m}|.$$

If m is odd, then, for all $n \geq m - 1$, we have

$$|z_n| = |z_{n-1}| + |z_{n-2}| + \dots + |z_{n-m}| + (-1)^n.$$

Proof. If $m = 2$, the result follows from Propositions 4(2) and 22. Now suppose that $m \geq 3$, and, as a first case, suppose that m is even. Proceed by induction on $n \geq m - 1$. If $n = m - 1$, then using Corollary 25 several times, we have

$$\begin{aligned} |z_{m-1}| &= |z_{m-2}| + 2|z_{m-3}| = |z_{m-2}| + |z_{m-3}| + |z_{m-4}| + 2|z_{m-5}| = |z_{m-2}| + |z_{m-3}| + \dots + |z_2| + 2|z_1| \\ &= |z_{m-2}| + |z_{m-3}| + \dots + |z_2| + |z_1| + |z_0| + |z_{-1}| \end{aligned}$$

since $|z_1| = |z_0|$, and $|z_{-1}| = 0$. Now suppose that $n \geq m$ and the result holds for values less than n . From Definition 19, we obtain

$$\begin{aligned} |z_n| &= 2|z_{n-2}| + 2|z_{n-3}| + \dots + 2|z_{n-m}| + |z_{n-(m+1)}| \\ &= (|z_{n-2}| + |z_{n-3}| + \dots + |z_{n-1-(m-1)}| + |z_{n-1-m}|) + |z_{n-2}| + |z_{n-3}| + \dots + |z_{n-m}|, \end{aligned}$$

and using the induction hypothesis, we find

$$|z_n| = |z_{n-1}| + |z_{n-2}| + |z_{n-3}| + \dots + |z_{n-m}|.$$

Secondly assume that m is odd, and as is the previous case, proceed by induction on $n \geq m - 1$. If $n = m - 1$, then using Corollary 25 several times, we have

$$\begin{aligned} |z_{m-1}| &= |z_{m-2}| + 2|z_{m-3}| = |z_{m-2}| + |z_{m-3}| + |z_{m-4}| + 2|z_{m-5}| = |z_{m-2}| + |z_{m-3}| + \cdots + |z_1| + 2|z_0| \\ &= |z_{m-2}| + |z_{m-3}| + \cdots + |z_1| + |z_0| + |z_{-1}| + 1 \end{aligned}$$

since $|z_0| = 1$ and $|z_{-1}| = 0$. Now suppose that $n \geq m$, and assume that result holds for all values less than n . From Definition 19, we have

$$\begin{aligned} |z_n| &= 2|z_{n-2}| + 2|z_{n-3}| + \cdots + 2|z_{n-m}| + |z_{n-(m+1)}| \\ &= (|z_{n-2}| + |z_{n-3}| + \cdots + |z_{n-m}| + |z_{n-(m+1)}|) - (-1)^n + |z_{n-2}| + |z_{n-3}| + \cdots + |z_{n-m}| + (-1)^n \\ &= (|z_{n-2}| + |z_{n-3}| + \cdots + |z_{n-m}| + |z_{n-(m+1)}|) + (-1)^{n-1} + |z_{n-2}| + |z_{n-3}| + \cdots + |z_{n-m}| + (-1)^n. \end{aligned}$$

The induction hypothesis allows us to conclude that

$$|z_n| = |z_{n-1}| + |z_{n-2}| + |z_{n-3}| + \cdots + |z_{n-m}| + (-1)^n,$$

as desired. □

The following inequalities on the lengths of p -singular words will be useful later on.

Proposition 27. *We have the following inequalities.*

(1) For all $0 \leq n \leq m - 1$,

$$|z_n| \geq \sum_{k=0}^{n-1} |z_k| = \sum_{k=-1}^{n-1} |z_k|.$$

(2) For all $n \geq 1$, $|z_n| \geq |z_{n-1}| + |z_{n-2}|$.

(3) For all $n \geq 1$, $|z_{n+1}| > |z_n|$.

Proof. Let us prove (1) by induction on $0 \leq n \leq m - 1$. If $n = 0$, then $|z_{-1}| = 0 \leq 1 = |z_0|$. If $n = 1$, then $|z_{-1}| + |z_0| = 1 \leq 1 = |z_1|$. Now suppose that $2 \leq n \leq m - 1$, and that the result is true for values less than n . By the induction hypothesis, we have

$$\sum_{k=0}^{n-1} |z_k| = \sum_{k=0}^{n-3} |z_k| + |z_{n-2}| + |z_{n-1}| \leq 2|z_{n-2}| + |z_{n-1}|,$$

and by Corollary 25, we have

$$\sum_{k=0}^{n-1} |z_k| \leq |z_n|.$$

Let us prove (2). First, suppose that $1 \leq n \leq m - 1$. Then Corollary 25 implies that $|z_n| = |z_{n-1}| + 2|z_{n-2}| \geq |z_{n-1}| + |z_{n-2}|$ since $|z_{n-2}| \geq |z_{-1}| = 0$. Suppose that $n \geq m$. If m is even, then by Corollary 26, we know that

$$|z_n| = |z_{n-1}| + |z_{n-2}| + |z_{n-3}| + \cdots + |z_{n-m}| \geq |z_{n-1}| + |z_{n-2}|$$

since $|z_{n-3}|, \dots, |z_{n-m}| \geq |z_{-1}| = 0$ (when $m = 2$, the inequality above is an equality). If m is odd, then by Corollary 26, we have

$$|z_n| = |z_{n-1}| + |z_{n-2}| + \dots + |z_{n-m}| + (-1)^n.$$

When n is even, then clearly $|z_n| \geq |z_{n-1}| + |z_{n-2}|$. When n is odd, then $|z_{n-m}| - 1 \geq 0$, so we have $|z_n| \geq |z_{n-1}| + |z_{n-2}|$.

Let us show that (3) holds. Suppose that $n \geq 1$. Since $|z_{n-1}| > 0$, the result can easily be deduced as a corollary of (2). \square

From Table 2, one can observe that the first few words in two consecutive sequences of p-singular words are the same. In the following proposition, we compare the first $m + 1$ terms of the sequences $(z_n^{(m)})_{n \geq -1}$ and $(z_n^{(m+1)})_{n \geq -1}$ by showing that they are equal. Also notice the words differ after that.

Proposition 28. *For all $-1 \leq n \leq m - 1$, we have $z_n^{(m)} = z_n^{(m+1)}$.*

Proof. We proceed by induction on n , with $-1 \leq n \leq m - 1$. It is clear that for all $m \geq 2$, we have $z_{-1}^{(m)} = \varepsilon$ and $z_0^{(m)} = 0$, so the base case is true. Now suppose that $1 \leq n \leq m - 1$, and that $z_k^{(m)} = z_k^{(m+1)}$ for $-1 \leq k < n$. From Definition 19(1), we have

$$z_n^{(m)} = z_{n-2}^{(m)} z_{n-3}^{(m)} \dots z_1^{(m)} z_0^{(m)} n z_0^{(m)} z_1^{(m)} \dots z_{n-3}^{(m)} z_{n-2}^{(m)}.$$

Now using the induction hypothesis, we have

$$z_n^{(m)} = z_{n-2}^{(m+1)} z_{n-3}^{(m+1)} \dots z_1^{(m+1)} z_0^{(m+1)} n z_0^{(m+1)} z_1^{(m+1)} \dots z_{n-3}^{(m+1)} z_{n-2}^{(m+1)},$$

and using Definition 19(1) again, we get $z_n^{(m)} = z_n^{(m+1)}$. \square

The idea to obtain the palindromic z - and c -factorizations of the m -bonacci word is to mimic the reasoning in the previous case. Namely, we establish results similar to Propositions 4, 5 and 7. Before getting those properties in the more general m -bonacci case, a few preliminaries are necessary. In the following lemma, we get a formula for the p-singular word z_n in terms of the morphism ϕ_m and the p-singular word z_{n-1} .

Lemma 29. *For all $n \geq 0$,*

$$z_n = \begin{cases} 0^{-1} \phi_m(z_{n-1}), & \text{if } n \text{ is odd;} \\ \phi_m(z_{n-1}) 0, & \text{if } n \text{ is even.} \end{cases}$$

Proof. We proceed by induction on $n \geq 0$. If $n = 0$, then $z_0 = 0 = \phi_m(\varepsilon) 0 = \phi_m(z_{-1}) 0$. If $n = 1$, then

$$z_1 = 1 = 0^{-1} 0 1 = 0^{-1} \phi_m(0) = 0^{-1} \phi_m(z_0).$$

Now suppose that $n \geq 2$ and that the result is true for values less than n . As a first case, suppose that $2 \leq n \leq m - 1$. In particular, $1 \leq n - 1 \leq m - 2$, and we deduce from Definition 19(1) that

$$z_{n-1} = z_{n-3} z_{n-4} \dots z_1 z_0 (n - 1) z_0 z_1 \dots z_{n-4} z_{n-3}.$$

If n is even, then

$$\begin{aligned}\phi_m(z_{n-1})0 &= \phi_m(z_{n-3})\phi_m(z_{n-4})\cdots\phi_m(z_1)\phi_m(z_0)0n\phi_m(z_0)\phi_m(z_1)\cdots\phi_m(z_{n-4})\phi_m(z_{n-3})0 \\ &= \phi_m(z_{n-3})00^{-1}\phi_m(z_{n-4})\cdots\phi_m(z_1)00^{-1}\phi_m(z_0)0n00^{-1}\phi_m(z_0)\phi_m(z_1)0\cdots0^{-1}\phi_m(z_{n-4}) \\ &\quad \phi_m(z_{n-3})0.\end{aligned}$$

By the induction hypothesis, we obtain

$$\begin{aligned}\phi_m(z_{n-1})0 &= z_{n-2}z_{n-3}\cdots z_2z_10n0z_1z_2\cdots z_{n-3}z_{n-2} \\ &= z_{n-2}z_{n-3}\cdots z_2z_1z_0nz_0z_1z_2\cdots z_{n-3}z_{n-2},\end{aligned}$$

and the last equality holds because $z_0 = 0$. From Definition 19(1), we have $\phi_m(z_{n-1})0 = z_n$.

If n is odd, then

$$\begin{aligned}0^{-1}\phi_m(z_{n-1}) &= 0^{-1}\phi_m(z_{n-3})\phi_m(z_{n-4})\cdots\phi_m(z_1)\phi_m(z_0)0n\phi_m(z_0)\phi_m(z_1)\cdots\phi_m(z_{n-4})\phi_m(z_{n-3}) \\ &= 0^{-1}\phi_m(z_{n-3})\phi_m(z_{n-4})0\cdots\phi_m(z_1)00^{-1}\phi_m(z_0)0n00^{-1}\phi_m(z_0)\phi_m(z_1)0\cdots\phi_m(z_{n-4})0 \\ &\quad 0^{-1}\phi_m(z_{n-3}).\end{aligned}$$

By the induction hypothesis, we obtain

$$\begin{aligned}0^{-1}\phi_m(z_{n-1}) &= z_{n-2}z_{n-3}\cdots z_2z_10n0z_1z_2\cdots z_{n-3}z_{n-2} \\ &= z_{n-2}z_{n-3}\cdots z_2z_1z_0nz_0z_1z_2\cdots z_{n-3}z_{n-2},\end{aligned}$$

and the last equality is true because $z_0 = 0$. From Definition 19(1), we have $0^{-1}\phi_m(z_{n-1}) = z_n$.

Now suppose that $n = m$ (this implies that $m = n \geq 2$). By Definition 19(1), we have

$$z_{m-1} = z_{m-3}z_{m-4}\cdots z_1z_0(m-1)z_0z_1\cdots z_{m-4}z_{m-3}.$$

If $n = m$ is even, then

$$\begin{aligned}\phi_m(z_{m-1})0 &= \phi_m(z_{m-3})\phi_m(z_{m-4})\cdots\phi_m(z_1)\phi_m(z_0)0\phi_m(z_0)\phi_m(z_1)\cdots\phi_m(z_{m-4})\phi_m(z_{m-3})0 \\ &= \phi_m(z_{m-3})00^{-1}\phi_m(z_{m-4})\cdots\phi_m(z_1)00^{-1}\phi_m(z_0)000^{-1}\phi_m(z_0)\phi_m(z_1)0\cdots0^{-1}\phi_m(z_{m-4}) \\ &\quad \phi_m(z_{m-3})0.\end{aligned}$$

By the induction hypothesis and since $z_0 = 0$, we obtain

$$\begin{aligned}\phi_m(z_{m-1})0 &= z_{m-2}z_{m-3}\cdots z_2z_100z_1z_2\cdots z_{m-3}z_{m-2} \\ &= z_{m-2}z_{m-3}\cdots z_2z_1z_0z_0z_1z_2\cdots z_{m-3}z_{m-2}.\end{aligned}$$

From Definition 19(2), we have $\phi_m(z_{m-1})0 = z_m$. If $n = m$ is odd, then

$$\begin{aligned}0^{-1}\phi_m(z_{m-1}) &= 0^{-1}\phi_m(z_{m-3})\phi_m(z_{m-4})\cdots\phi_m(z_1)\phi_m(z_0)0\phi_m(z_0)\phi_m(z_1)\cdots\phi_m(z_{m-4})\phi_m(z_{m-3}) \\ &= 0^{-1}\phi_m(z_{m-3})\phi_m(z_{m-4})0\cdots\phi_m(z_1)00^{-1}\phi_m(z_0)000^{-1}\phi_m(z_0)\phi_m(z_1)0\cdots\phi_m(z_{m-4})0 \\ &\quad 0^{-1}\phi_m(z_{m-3}).\end{aligned}$$

By the induction hypothesis and since $z_0 = 0$, we obtain

$$\begin{aligned} 0^{-1}\phi_m(z_{m-1}) &= z_{m-2}z_{m-3}\cdots z_2z_100z_1z_2\cdots z_{m-3}z_{m-2} \\ &= z_{m-2}z_{m-3}\cdots z_2z_1z_0z_0z_1z_2\cdots z_{m-3}z_{m-2}. \end{aligned}$$

From Definition 19(2), we have $0^{-1}\phi_m(z_{m-1}) = z_m$.

Finally, assume that $n \geq m + 1$. By Definition 19(2), we have

$$z_{n-1} = z_{n-3}z_{n-4}\cdots z_{n-1-(m-1)}z_{n-1-m}z_{n-1-(m+1)}z_{n-1-m}z_{n-1-(m-1)}\cdots z_{n-4}z_{n-3}.$$

If n is even, then

$$\begin{aligned} \phi_m(z_{n-1})0 &= \phi_m(z_{n-3})\phi_m(z_{n-4})\cdots\phi_m(z_{n-1-(m-1)})\phi_m(z_{n-1-m}) \\ &\quad \phi_m(z_{n-1-(m+1)})\phi_m(z_{n-1-m})\phi_m(z_{n-1-(m-1)})\cdots\phi_m(z_{n-4})\phi_m(z_{n-3})0. \end{aligned}$$

Inserting 00^{-1} where needed (places where to insert it differ when m is even or odd) and using the induction hypothesis, we obtain

$$\phi_m(z_{n-1})0 = z_{n-2}z_{n-3}\cdots z_{n-(m-1)}z_{n-m}z_{n-(m+1)}z_{n-m}z_{n-(m-1)}\cdots z_{n-3}z_{n-2}.$$

From Definition 19(2), we have $\phi_m(z_{n-1})0 = z_n$ as desired. If n is odd, then

$$\begin{aligned} 0^{-1}\phi_m(z_{m-1}) &= 0^{-1}\phi_m(z_{n-3})\phi_m(z_{n-4})\cdots\phi_m(z_{n-1-(m-1)})\phi_m(z_{n-1-m}) \\ &\quad \phi_m(z_{n-1-(m+1)})\phi_m(z_{n-1-m})\phi_m(z_{n-1-(m-1)})\cdots\phi_m(z_{n-4})\phi_m(z_{n-3}). \end{aligned}$$

As before, inserting 00^{-1} where needed and making use of the induction hypothesis, we get

$$0^{-1}\phi_m(z_{n-1}) = z_{n-2}z_{n-3}\cdots z_{n-(m-1)}z_{n-m}z_{n-(m+1)}z_{n-m}z_{n-(m-1)}\cdots z_{n-3}z_{n-2}.$$

From Definition 19(2), we have $0^{-1}\phi_m(z_{n-1}) = z_n$. This ends the proof. \square

Using the previous lemma, we are able to prove Lemma 20.

Proof of Lemma 20. For the sake of simplicity, let us drop the exponent (m) in this proof. We equivalently show that, if $n \geq 0$ is odd,

$$h_{n-1}h_{n-3}\cdots h_2h_0z_n = h_nh_{n-2}\cdots h_3h_1$$

and if $n \geq 0$ is even,

$$h_{n-1}h_{n-3}\cdots h_3h_1z_n = h_nh_{n-2}\cdots h_2h_0.$$

We proceed by induction on $n \geq 0$. If $n = 0$, then $\varepsilon \cdot z_0 = 0 = h_0$ holds. If $n = 1$, then $h_0z_1 = 01 = h_1$. Now suppose that $n \geq 2$ and that the result is true for values less than n .

If n is odd, then $n - 1$ is even and the induction hypothesis yields

$$h_{n-2}h_{n-4}\cdots h_3h_1z_{n-1} = h_{n-1}h_{n-3}\cdots h_2h_0.$$

Applying ϕ_m on both sides, we get

$$h_{n-1}h_{n-3}\cdots h_4h_2\phi_m(z_{n-1}) = h_nh_{n-2}\cdots h_3h_1.$$

We may now insert 00^{-1} before $\phi_m(z_{n-1})$ in the left-hand side of the previous equality to obtain

$$h_{n-1}h_{n-3}\cdots h_4h_200^{-1}\phi_m(z_{n-1}) = h_n h_{n-2}\cdots h_3 h_1.$$

We conclude by using the fact that $h_0 = 0$ and $0^{-1}\phi_m(z_{n-1}) = z_n$ thanks to Lemma 29.

If n is even, then $n - 1$ is odd and the induction assumption gives

$$h_{n-2}h_{n-4}\cdots h_2h_0z_{n-1} = h_{n-1}h_{n-3}\cdots h_3h_1.$$

Applying ϕ_m on both sides and appending a letter 0, we obtain

$$h_{n-1}h_{n-3}\cdots h_3h_1\phi_m(z_{n-1})0 = h_n h_{n-2}\cdots h_4h_20.$$

We end this case by using the fact that $h_0 = 0$ and $\phi_m(z_{n-1})0 = z_n$ thanks to Lemma 29. \square

The following result matches Proposition 4(4) in the Fibonacci case.

Proposition 30. *For all $n \geq 0$, z_n is not a factor of z_{n+1} .*

Proof. Observe first that the case $m = 2$ is covered using Propositions 4(4) and 22. So we can suppose that $m \geq 3$, and we proceed by induction on $n \geq 0$. The result can be checked by hand for $0 \leq n \leq 2$ since $z_0 = 0$, $z_1 = 1$, and $z_2 = 020$ (see Definition 19). Suppose that $n \geq 3$ and assume that z_k is not a factor of z_{k+1} for $0 \leq k \leq n - 1$. We show it still holds for $k = n$, i.e., z_n is not a factor of z_{n+1} . Proceed by contradiction and suppose that z_n is a factor of z_{n+1} . We divide the proof into two cases according to the parity of n .

Case 1. Suppose that n is odd. From Lemma 29, we know that $z_n = 0^{-1}\phi_m(z_{n-1})$ and $z_{n+1} = \phi_m(z_n)0$. We prove that $0z_n = 00^{-1}\phi_m(z_{n-1}) = \phi_m(z_{n-1})$ is a factor of $z_{n+1} = \phi_m(z_n)0$. By hypothesis, there exist words $x, y \in \{0, 1, \dots, m - 1\}^*$ such that $z_{n+1} = 0xz_ny$. If $x = \varepsilon$, then the statement is true. If $|x| = \ell > 0$, write $x = x_1 \cdots x_\ell$ with $x_i \in \{0, 1, \dots, m - 1\}$ for all $1 \leq i \leq \ell$. By definition of the morphism ϕ_m , we find $x_\ell = 0$ since z_n starts with a positive letter, which ends the intermediate result. Now, we claim that $\phi_m(z_{n-1})$ is in fact a factor of $\phi_m(z_n)$. First, using Proposition 24,

$$z_{n-1} = 0u0 \quad \text{and} \quad z_n = 1v1$$

for non-empty words $u, v \in \{0, 1, \dots, m - 1\}^+$ (recall that $n - 1 \geq 2$). Thus $\phi_m(z_{n-1})$ (resp., $\phi_m(z_n)$) starts and ends with 01 (resp., 02). Consequently, $\phi_m(z_{n-1})$ is a factor of $\phi_m(z_n)$, as claimed. From Lemma 17, z_{n-1} is a factor of z_n , which contradicts the induction hypothesis.

Case 2. Assume that n is even. From Lemma 29, we know that $z_n = \phi_m(z_{n-1})0$ and $z_{n+1} = 0^{-1}\phi_m(z_n)$. By hypothesis, $z_n = \phi_m(z_{n-1})0$ is a factor of $0z_{n+1} = 00^{-1}\phi_m(z_n) = \phi_m(z_n)$. Thus, $\phi_m(z_{n-1})$ is in fact a factor of $\phi_m(z_n)$. Using Proposition 24,

$$z_{n-1} = 1u1 \quad \text{and} \quad z_n = 0v0$$

for non-empty words $u, v \in \{0, 1, \dots, m - 1\}^+$ (recall that $n - 1 \geq 2$). From Lemma 17, z_{n-1} is a factor of z_n , which again contradicts the induction hypothesis. \square

The following result is the counterpart to Proposition 4(5) in the Fibonacci case.

Proposition 31. *For all $n \geq 1$, z_n is not a factor of the product $\prod_{k=0}^{n-1} z_k$.*

Proof. For the sake of simplicity, we define, for all $n \geq 1$,

$$P(n) = \prod_{k=0}^{n-1} z_k.$$

Observe first that the case $m = 2$ follows from Propositions 4(5) and 22. So we can suppose that $m \geq 3$. To prove the result, we proceed by induction on $n \geq 1$.

If $1 \leq n \leq m - 1$, then by Proposition 27, we have $|z_n| \geq \sum_{k=0}^{n-1} |z_k|$. If the inequality is strict, then we are done. If we actually have an equality, then z_{n-1} would be a factor of z_n , which contradicts Proposition 30.

Suppose that $n \geq m$ and assume that z_i is not a factor of $P(i)$ for $1 \leq i \leq n - 1$. We show it still holds for $i = n$, i.e., z_n is not a factor of $P(n)$. Proceed by contradiction and suppose that z_n is a factor of $P(n)$. We divide the proof into two cases according to the parity of n .

Case 1. Suppose that n is odd. From Lemma 29, we get

$$\begin{aligned} P(n) &= z_0 z_1 \cdots z_{n-2} z_{n-1} \\ &= (\phi_m(z_{-1})0)(0^{-1}\phi_m(z_0)) \cdots (0^{-1}\phi_m(z_{n-3}))(\phi_m(z_{n-2})0) \\ &= \phi_m(z_{-1}z_0 \cdots z_{n-3}z_{n-2})0 \\ &= \phi_m(P(n-1))0. \end{aligned} \tag{7}$$

By hypothesis, $z_n = 0^{-1}\phi_m(z_{n-1})$ is a factor of $P(n) = \phi_m(P(n-1))0$, so $0z_n = 00^{-1}\phi_m(z_{n-1}) = \phi_m(z_{n-1})$ is also a factor of $0P(n) = 0\phi_m(P(n-1))0$ (the reasoning is similar to the one developed in the previous proof). We claim that $\phi_m(z_{n-1})$ is in fact a factor of $\phi_m(P(n-1))$. First, using Proposition 24, we have

$$z_{n-1} = 0u0 \quad \text{and} \quad P(n-1) = 0v1$$

for two non-empty words $u, v \in \{0, 1, \dots, m-1\}^+$. Consequently, $\phi_m(z_{n-1})$ starts and ends with 01 , and $\phi_m(P(n-1))$ starts with 01 and ends with 02 . Thus, $\phi_m(z_{n-1})$ is a factor of $\phi_m(P(n-1))$, as expected. From Lemma 17, z_{n-1} is a factor of $P(n-1)$, which contradicts the induction hypothesis.

Case 2. Assume that n is even. From Lemma 29, we get

$$\begin{aligned} P(n) &= z_0 z_1 \cdots z_{n-2} z_{n-1} \\ &= (\phi_m(z_{-1})0)(0^{-1}\phi_m(z_0)) \cdots (\phi_m(z_{n-3})0)(0^{-1}\phi_m(z_{n-2})) \\ &= \phi_m(z_{-1}z_0 \cdots z_{n-3}z_{n-2}) \\ &= \phi_m(P(n-1)). \end{aligned} \tag{8}$$

By hypothesis, $z_n = \phi_m(z_{n-1})0$ is a factor of $P(n) = \phi_m(P(n-1))$, so $\phi_m(z_{n-1})$ is also a factor of $\phi_m(P(n-1))$. Using Proposition 24, we have

$$z_{n-1} = 1u1 \quad \text{and} \quad P(n-1) = 0v0$$

for two non-empty words $u, v \in \{0, 1, \dots, m-1\}^+$. From Lemma 17, z_{n-1} is a factor of $P(n-1)$, which also contradicts the induction hypothesis. \square

Proposition 32. *For all $0 \leq n \leq m - 1$, the words $z_{-1}, z_0, z_1, \dots, z_{n-1}$ do not contain the letter n .*

Proof. If $n = 0$, then z_{-1} is the empty word, and we are done. If $n = 1$, each of the words $z_{-1} = \varepsilon$ and $z_0 = 0$ does not contain the letter 1. Let $2 \leq n \leq m - 1$. Then the words $(z_i^{(n)})_{i \geq -1}$ are well defined (see Definition 19). Iteratively applying Proposition 28, we obtain $z_i^{(n)} = z_i^{(m)}$ for all $-1 \leq i \leq n - 1$. Since $z_i^{(n)}$ is a word defined over the alphabet $\{0, 1, \dots, n - 1\}$ for any $i \geq -1$, the conclusion follows. \square

The following result compares the prefixes of z_{n-1} to suffixes of z_n . Its proof follows the same lines as the proof of Lemma 9 in the Fibonacci case.

Lemma 33. *For all $n \geq 0$, the only suffix of z_{n-1} that is also a prefix of z_n is the empty word.*

Proof. We proceed by induction on $n \geq 0$. From Definition 19, we have $z_{-1} = \varepsilon$, $z_0 = 0$ and $z_1 = 1$, so the result can be checked by hand for $n \in \{0, 1\}$.

Now suppose that $n \geq 2$, and that the only suffix of z_{k-1} that is also a prefix of z_k is the empty word, for all $k \in \{0, 1, \dots, n - 1\}$. We show that the result still holds for $k = n$. Proceed by contradiction and suppose there exists a word $x \in \{0, 1, \dots, m - 1\}^*$ which is a non-empty suffix of z_{n-1} and a non-empty prefix of z_n . We have $1 \leq |x| \leq |z_{n-1}|$. Using Definition 19, z_n starts and ends with z_{n-2} .

If $1 \leq |x| \leq |z_{n-2}|$, then x is a prefix of z_{n-2} (recall that x is a prefix of z_n). Consequently, x^R is a non-empty suffix of z_{n-2} and a non-empty prefix of z_{n-1} . This contradicts the inductive assumption.

If $|z_{n-2}| \leq |x| \leq |z_{n-1}|$, then z_{n-2} is a prefix of x (recall that x is a prefix of z_n). In particular, z_{n-2} is a factor of x , and also a factor of z_{n-1} (recall that x is a suffix of z_{n-1}). This contradicts Proposition 30. \square

4.3 Two particular factorizations of the m -bonacci word

In this section, we study two different factorizations of the m -bonacci word in terms of p -singular words (see Propositions 34 and 37), extending Proposition 5. The first one is similar to the factorization (1) of the Fibonacci word given in Proposition 5. To see this, simply put (1) and Proposition 22 altogether.

Proposition 34. *We have the following factorization of the m -bonacci word*

$$\mathbf{w}_m = \prod_{n \geq 0} z_n.$$

Proof. For all $n \geq 0$, set $P(n) = \prod_{k=0}^{n-1} z_k$ (when $n = 0$, $P(0)$ is the empty word). To prove the statement, we show two things:

- (1) For all $n \geq 1$, $|P(n)| > |P(n - 1)|$,
- (2) $(P(n))_{n \geq 0}$ is a sequence of prefixes of \mathbf{w}_m .

Then, the mentioned factorization easily follows. For all $n \geq 1$, we trivially have

$$|P(n)| = |P(n-1)| + |z_{n-1}| > |P(n-1)|,$$

since $n \geq 1$ implies $|z_{n-1}| > 0$. Thus (1) is proved. For (2), we proceed by induction on $n \geq 0$. The m -bonacci word \mathbf{w}_m starts with 01, so it is clear that $P(n)$ is a prefix of \mathbf{w}_m for $n \in \{0, 1, 2\}$. Suppose that $n \geq 3$ and that $P(n-1)$ is a prefix of \mathbf{w}_m . The proof is again divided into two parts, according to the parity of n .

Case 1. Suppose that n is odd. From (7) (which is valid for any odd $n \geq 0$), we know that $P(n) = \phi_m(P(n-1))0$, and using Proposition 24, $P(n-1)$ ends with 1. By the induction hypothesis, $P(n-1)$ is a prefix of \mathbf{w}_m ending with 1. Thus, there exists an infinite word \mathbf{z} over $\{0, 1, \dots, m-1\}$ such that $\mathbf{w}_m = P(n-1)0\mathbf{z}$. Since \mathbf{w}_m is a fixed point of ϕ_m , we get

$$\mathbf{w}_m = \phi_m(\mathbf{w}_m) = \phi_m(P(n-1)01\phi_m(\mathbf{z})) = P(n)1\phi_m(\mathbf{z}),$$

showing that $P(n)$ is also a prefix of \mathbf{w}_m .

Case 2. Assume that n is even. From (8) (which is valid for any even $n \geq 0$), we already have $P(n) = \phi_m(P(n-1))$. By the induction hypothesis, there exists an infinite word \mathbf{z} over $\{0, 1, \dots, m-1\}$ such that $\mathbf{w}_m = P(n-1)\mathbf{z}$. Since \mathbf{w}_m is a fixed point of ϕ_m , we get

$$\mathbf{w}_m = \phi_m(\mathbf{w}_m) = \phi_m(P(n-1))\phi_m(\mathbf{z}) = P(n)\phi_m(\mathbf{z}),$$

as desired. □

The factorization of the m -bonacci word in Proposition 37 is similar to the factorization (2) of the Fibonacci word given in Proposition 5. We first need some notations.

Definition 35. Let p_m be the finite word over $\{0, 1, \dots, m-1\}$ defined by

$$z_0 z_1 \cdots z_{m-4} z_{m-3} z_{m-2} z_{m-3} z_{m-4} \cdots z_1 z_0 (m-1).$$

For all $n \geq m-2$, define the finite word $Q(n)$ over $\{0, 1, \dots, m-1\}$ by

$$Q(n) = z_{n-(m-1)} z_{n-(m-2)} \cdots z_{n-2} z_{n-1} z_n z_{n-1} z_{n-2} \cdots z_{n-(m-2)} z_{n-(m-1)}.$$

Note that the word $Q(n)$ is centered at z_n .

Example 36. If $m = 2$, $p_2 = z_0(2-1) = 01$ and $Q(n) = z_{n-1}z_n z_{n-1}$ for all $n \geq 0$. When $m = 3$, we find $p_3 = z_0 z_1 z_0(3-1) = 0102$, and for all $n \geq 1$, we have $Q(n) = z_{n-2} z_{n-1} z_n z_{n-1} z_{n-2}$.

Proposition 37. *We have the following factorization of the m -bonacci word*

$$\mathbf{w}_m = p_m \cdot \prod_{n \geq m-2} Q(n).$$

Proof. From Proposition 34, we get

$$\mathbf{w}_m = \prod_{n \geq 0} z_n = z_0 \cdot z_1 \cdots z_{m-2} \cdot z_{m-1} \cdot \prod_{n \geq m} z_n. \quad (9)$$

Using Definition 19(1), we have

$$z_{m-1} = z_{m-3}z_{m-4} \cdots z_1z_0(m-1)z_0z_1 \cdots z_{m-4}z_{m-3},$$

and for all $n \geq m$, Definition 19(2) shows that

$$z_n = z_{n-2}z_{n-3} \cdots z_{n-(m-1)}z_{n-m}z_{n-(m+1)}z_{n-m}z_{n-(m-1)} \cdots z_{n-3}z_{n-2}.$$

Plugging these equalities into (9), we find

$$\begin{aligned} \mathbf{w}_m &= z_0 \cdot z_1 \cdots z_{m-2} \cdot (z_{m-3}z_{m-4} \cdots z_1z_0(m-1)z_0z_1 \cdots z_{m-4}z_{m-3}) \\ &\quad \cdot \prod_{n \geq m} z_{n-2}z_{n-3} \cdots z_{n-(m-1)}z_{n-m}z_{n-(m+1)}z_{n-m}z_{n-(m-1)} \cdots z_{n-3}z_{n-2} \\ &= p_m \cdot z_0z_1 \cdots z_{m-4}z_{m-3} \\ &\quad \cdot \prod_{n \geq m} z_{n-2}z_{n-3} \cdots z_{n-(m-1)}z_{n-m}z_{n-(m+1)}z_{n-m}z_{n-(m-1)} \cdots z_{n-3}z_{n-2} \end{aligned}$$

using Definition 35. Since $z_{-1} = \varepsilon$, we finally get

$$\mathbf{w}_m = p_m \prod_{n \geq m-2} z_{n-(m-1)}z_{n-(m-2)} \cdots z_{n-2}z_{n-1}z_nz_{n-1}z_{n-2} \cdots z_{n-(m-2)}z_{n-(m-1)} = p_m \cdot \prod_{n \geq m-2} Q(n). \quad \square$$

In the following proposition, we get a particular factorization of the prefix p_m of the m -bonacci word \mathbf{w}_m . This factorization is a step forward to obtain the palindromic c -factorization of the m -bonacci word \mathbf{w}_m .

Proposition 38. *The word p_m can be factorized as*

$$p_m = z_0 \cdot 1 \cdot z_0 \cdot 2 \cdot (z_0z_1z_0) \cdot 3 \cdots (z_0z_1 \cdots z_{m-4}z_{m-3}z_{m-4} \cdots z_1z_0) \cdot (m-1).$$

In particular, this factorization contains $2m-2$ factors and all of them are palindromes. Moreover, if this factorization is written as

$$p_m = q_1q_2 \cdots q_{2m-2},$$

then, for all $1 \leq k \leq 2m-2$ and for any infinite word \mathbf{w} , q_k is the longest palindromic prefix of $q_kq_{k+1} \cdots q_{2m-2}\mathbf{w}$ with a previous occurrence in $p_m\mathbf{w} = q_1q_2 \cdots q_{2m-2}\mathbf{w}$, or if this prefix does not exist, the factor q_k is a single letter.

Proof. To prove this result, we proceed by induction on $m \geq 2$. To avoid any confusion, from Definition 35, write

$$p_m = z_0^{(m)}z_1^{(m)} \cdots z_{m-4}^{(m)}z_{m-3}^{(m)}z_{m-2}^{(m)}z_{m-3}^{(m)}z_{m-4}^{(m)} \cdots z_1^{(m)}z_0^{(m)}(m-1).$$

The case $m = 2$ is easily checked for we have $p_2 = 01 = z_0^{(2)} \cdot 1$. Now suppose that $m \geq 3$ and assume that the result holds for values less than m . Let us prove the first part of the statement. Using Definition 19(1) to rewrite $z_{m-2}^{(m)}$, we first have

$$\begin{aligned} p_m &= z_0^{(m)}z_1^{(m)} \cdots z_{m-4}^{(m)}z_{m-3}^{(m)}(z_{m-4}^{(m)}z_{m-5}^{(m)} \cdots z_1^{(m)}z_0^{(m)}(m-2)z_0^{(m)}z_1^{(m)} \cdots z_{m-5}^{(m)}z_{m-4}^{(m)}) \\ &\quad z_{m-3}^{(m)}z_{m-4}^{(m)} \cdots z_1^{(m)}z_0^{(m)}(m-1) \\ &= z_0^{(m)}z_1^{(m)} \cdots z_{m-5}^{(m)}z_{m-4}^{(m)}z_{m-3}^{(m)}z_{m-4}^{(m)}z_{m-5}^{(m)} \cdots z_1^{(m)}z_0^{(m)}(m-2) \\ &\quad z_0^{(m)}z_1^{(m)} \cdots z_{m-5}^{(m)}z_{m-4}^{(m)}z_{m-3}^{(m)}z_{m-4}^{(m)}z_{m-5}^{(m)} \cdots z_1^{(m)}z_0^{(m)}(m-1). \end{aligned} \quad (10)$$

From Proposition 28, the two finite words

$$z_0^{(m)} z_1^{(m)} \cdots z_{m-5}^{(m)} z_{m-4}^{(m)} z_{m-3}^{(m)} z_{m-4}^{(m)} z_{m-5}^{(m)} \cdots z_1^{(m)} z_0^{(m)} (m-2)$$

and

$$z_0^{(m-1)} z_1^{(m-1)} \cdots z_{m-5}^{(m-1)} z_{m-4}^{(m-1)} z_{m-3}^{(m-1)} z_{m-4}^{(m-1)} z_{m-5}^{(m-1)} \cdots z_1^{(m-1)} z_0^{(m-1)} (m-2)$$

are equal since $z_k^{(m)} = z_k^{(m-1)}$ for all $-1 \leq k \leq m-2$. Since the latest word is p_{m-1} by Definition 35, we deduce that

$$p_m = p_{m-1} z_0^{(m)} z_1^{(m)} \cdots z_{m-5}^{(m)} z_{m-4}^{(m)} z_{m-3}^{(m)} z_{m-4}^{(m)} z_{m-5}^{(m)} \cdots z_1^{(m)} z_0^{(m)} (m-1).$$

By the induction hypothesis, we know that

$$\begin{aligned} p_{m-1} = & z_0^{(m-1)} \cdot 1 \cdot z_0^{(m-1)} \cdot 2 \cdot (z_0^{(m-1)} z_1^{(m-1)} z_0^{(m-1)}) \cdot 3 \cdots \\ & (z_0^{(m-1)} z_1^{(m-1)} \cdots z_{m-5}^{(m-1)} z_{m-4}^{(m-1)} z_{m-5}^{(m-1)} \cdots z_1^{(m-1)} z_0^{(m-1)}) \cdot (m-2). \end{aligned}$$

Proposition 28 finally gives

$$\begin{aligned} p_m = & z_0^{(m)} \cdot 1 \cdot z_0^{(m)} \cdot 2 \cdot (z_0^{(m)} z_1^{(m)} z_0^{(m)}) \cdot 3 \cdots (z_0^{(m)} z_1^{(m)} \cdots z_{m-5}^{(m)} z_{m-4}^{(m)} z_{m-5}^{(m)} \cdots z_1^{(m)} z_0^{(m)}) \cdot (m-2) \\ & (z_0^{(m)} z_1^{(m)} \cdots z_{m-5}^{(m)} z_{m-4}^{(m)} z_{m-3}^{(m)} z_{m-4}^{(m)} z_{m-5}^{(m)} \cdots z_1^{(m)} z_0^{(m)}) \cdot (m-1), \end{aligned}$$

as expected. Moreover, using the induction hypothesis, this factorization contains $(2 \cdot (m-1) - 2) + 2 = 2m - 2$ factors, which are all palindromes. Note that we have

$$q_{2m-3} = z_0^{(m)} z_1^{(m)} \cdots z_{m-5}^{(m)} z_{m-4}^{(m)} z_{m-3}^{(m)} z_{m-4}^{(m)} z_{m-5}^{(m)} \cdots z_1^{(m)} z_0^{(m)}$$

and $q_{2m-2} = m-1$. This ends the proof of the first part of the statement.

Let us show that the second part of the statement also holds. The proof is divided into three cases according to the value of the index of the considered factor q_k .

Case 1. Suppose that $1 \leq k \leq 2m-4$. For all infinite word \mathbf{w} , q_k is the longest palindromic prefix of $q_k q_{k+1} \cdots q_{2m-2} \mathbf{w}$ with a previous occurrence in $p_m \mathbf{w} = q_1 q_2 \cdots q_{2m-2} \mathbf{w}$, or if this prefix does not exist, q_k is limited to a single letter. Indeed, by the induction hypothesis and since $q_{2m-3} q_{2m-2} \mathbf{w}$ is a particular infinite word, q_k is the longest palindromic prefix of $q_k q_{k+1} \cdots q_{2m-4} (q_{2m-3} q_{2m-2} \mathbf{w})$ with a previous occurrence in $p_{m-1} (q_{2m-3} q_{2m-2} \mathbf{w}) = p_m \mathbf{w} = q_1 q_2 \cdots q_{2m-2} \mathbf{w}$, or if this prefix does not exist, q_k is limited to a single letter.

Case 2. Assume that $k = 2m-3$, and let \mathbf{w} be any infinite word. Looking at (10), we get

$$p_m \mathbf{w} = q_1 q_2 \cdots q_{2m-4} q_{2m-3} q_{2m-2} \mathbf{w} = q_{2m-3} (m-2) q_{2m-3} (m-1) \mathbf{w}.$$

Using Proposition 32, we see that q_{2m-3} is the longest palindromic prefix of $q_{2m-3} q_{2m-2} \mathbf{w}$ that has already occurred in $p_m \mathbf{w}$.

Case 3. Suppose that $k = 2m-2$, and let \mathbf{w} be any infinite word. Proposition 32 shows that $m-1 = q_{2m-2}$ does not appear previously in $p_m \mathbf{w}$. Hence, $q_{2m-2} = m-1$ also satisfies the second part of the statement. \square

Since the idea is to adopt the same strategy as in the previous case, we define a sequence of specific prefixes of the m -bonacci word \mathbf{w}_m . This definition gives the sequence of prefixes of Definition 6 in the Fibonacci case as proved in Remark 40.

Definition 39. For all $n \geq m - 2$, define

$$g_n := p_m \prod_{m-2 \leq k \leq n-1} Q(k),$$

where the words p_m and $(Q(k))_{k \geq m-2}$ are given in Definition 35. Notice that $g_{m-2} = p_m \cdot \varepsilon = p_m$. From Proposition 37, also observe that, for all $n \geq m - 2$, we have

$$w_m = g_n \cdot Q(n) \cdot \prod_{k \geq n+1} Q(k).$$

Remark 40. Let $(g'_n)_{n \geq 2}$ denote the sequence of words of Definition 6. For all $n \geq 1$, Proposition 22 gives

$$g'_{n+1} = 010 \prod_{2 \leq k \leq n} \hat{f}_{k-1} \hat{f}_k \hat{f}_{k-1} = 010 \prod_{2 \leq k \leq n} z_{k-2} z_{k-1} z_{k-2} = 01 \prod_{0 \leq k \leq n-1} z_{k-1} z_k z_{k-1} = g_n.$$

This shows that Definition 39 agrees with Definition 6 when $m = 2$.

As in the Fibonacci case (see Proposition 7), any word g_n can be written using p-singular words.

Proposition 41. For all $n \geq m - 1$, we have

$$g_n = z_0 z_1 \cdots z_{n-m} \cdot Q(n) \cdot z_{n-m}.$$

Proof. We proceed by induction on $n \geq m - 1$. For the base case $n = m - 1$, Definitions 35 and 39 give

$$\begin{aligned} g_{m-1} &= p_m Q(m-2) \\ &= z_0 z_1 \cdots z_{m-4} z_{m-3} z_{m-2} z_{m-3} z_{m-4} \cdots z_1 z_0 (m-1) \\ &\quad (z_{-1} z_0 \cdots z_{m-4} z_{m-3} z_{m-2} z_{m-3} z_{m-4} \cdots z_0 z_{-1}). \end{aligned}$$

By Definition 19(1) ($1 \leq m - 1$), we have

$$z_{m-1} = z_{m-3} z_{m-4} \cdots z_1 z_0 (m-1) z_0 z_1 \cdots z_{m-4} z_{m-3}.$$

Using Definition 35, we thus have

$$\begin{aligned} g_{m-1} &= z_0 z_1 \cdots z_{m-4} z_{m-3} z_{m-2} z_{m-1} z_{m-2} z_{m-3} z_{m-4} \cdots z_0 z_{-1} \\ &= z_{-1} (z_0 z_1 \cdots z_{m-4} z_{m-3} z_{m-2} z_{m-1} z_{m-2} z_{m-3} z_{m-4} \cdots z_1 z_0) z_{-1} \\ &= z_{-1} \cdot Q(m-1) \cdot z_{-1}, \end{aligned}$$

as expected. Assume that $n \geq m - 1$, and suppose the result holds up to n and we show it still holds for $n + 1$. Using Definition 39, we have $g_{n+1} = g_n Q(n)$. By the induction hypothesis, we get

$$g_{n+1} = z_0 z_1 \cdots z_{n-m} \cdot Q(n) \cdot z_{n-m} \cdot Q(n).$$

Rewriting $Q(n)$ using Definition 35, we find

$$\begin{aligned} g_{n+1} &= z_0 z_1 \cdots z_{n-m} \cdot (z_{n-(m-1)} z_{n-(m-2)} \cdots z_{n-2} z_{n-1} z_n z_{n-1} z_{n-2} \cdots z_{n-(m-2)} z_{n-(m-1)}) \cdot z_{n-m} \\ &\quad \cdot (z_{n-(m-1)} z_{n-(m-2)} \cdots z_{n-2} z_{n-1} z_n z_{n-1} z_{n-2} \cdots z_{n-(m-2)} z_{n-(m-1)}). \end{aligned}$$

Since $n + 1 \geq m$, from Definition 19(2), we have

$$z_{n+1} = z_{n-1}z_{n-2} \cdots z_{n+1-(m-1)}z_{n+1-m}z_{n+1-(m+1)}z_{n+1-m}z_{n+1-(m-1)} \cdots z_{n-2}z_{n-1},$$

and we deduce that

$$\begin{aligned} g_{n+1} &= z_0z_1 \cdots z_{n-m}z_{n+1-m}z_{n+1-(m-1)} \cdots z_{n-2}z_{n-1}z_nz_{n+1}z_nz_{n-1}z_{n-2} \cdots z_{n+1-(m-1)}z_{n+1-m} \\ &= z_0z_1 \cdots z_{n-m}z_{n+1-m}(z_{n+1-(m-1)}z_{n+1-(m-2)} \cdots z_{n-1}z_nz_{n+1}z_nz_{n-1} \cdots z_{n+1-(m-1)})z_{n+1-m}. \end{aligned}$$

Consequently, from Definition 35, we obtain

$$g_{n+1} = z_0z_1 \cdots z_{n-m}z_{n+1-m} \cdot Q(n+1) \cdot z_{n+1-m},$$

which ends the proof. \square

4.4 The palindromic z -factorization of the m -bonacci word

In this section, we obtain the palindromic z -factorization of the m -bonacci word.

Lemma 42. *Let p be a non-empty palindromic factor of \mathbf{w}_m .*

- *If p begins with the letter 0, then $p = \phi_m(p')0$, where p' is a palindromic factor of \mathbf{w}_m .*
- *If p begins with the letter $a \neq 0$, then $p = 0^{-1}\phi_m(p')$, where p' is a palindromic factor of \mathbf{w}_m .*

Proof. First, let us write $\mathbf{w}_m = upv$ where u (resp., v) is a finite (resp., infinite) word over $\{0, 1, \dots, m-1\}$. By definition, we have

$$\mathbf{w}_m = \phi_m(\mathbf{w}_m) \quad \text{and} \quad |\phi_m(up)| > |up|. \quad (11)$$

The proof is by induction on $|p|$. The result is certainly true when p is a single letter (for instance, combine Propositions 37 and 38), so suppose $|p| > 1$.

Case 1a. Suppose p begins with 00. If $p = 00$, then $p = \phi_m(m-1)0$, as required (observe that $m-1$ is indeed a palindromic factor of \mathbf{w}_m : for instance, make use of Propositions 37 and 38). Suppose $p = 00q00$. By the induction hypothesis, we have $0q0 = \phi_m(q')0$, where q' is a palindromic factor of \mathbf{w}_m . We get $p = 0\phi_m(q')00 = \phi_m((m-1)q'(m-1))0$. It is clear that $p' = (m-1)q'(m-1)$ is a palindrome. Let us show that p' is also a factor of \mathbf{w}_m , then we are done. From (11), $p = \phi_m(p')0$ being a factor of up implies that it is also a factor of $\phi_m(up)0$. By Lemma 17, p' is a factor of up , so of \mathbf{w}_m .

Case 1b. Suppose p begins with $0a$, where $a \in \{1, 2, \dots, m-1\}$. Then $p = 0aqa0$. By the induction hypothesis, we have $aqa = 0^{-1}\phi_m(p')$, where p' is a palindromic factor of \mathbf{w}_m . We get $p = 00^{-1}\phi_m(p')0 = \phi_m(p')0$, as required.

Case 2. Suppose p begins with a , where $a \in \{1, 2, \dots, m-1\}$. Then $p = aqa$, where q is a palindromic factor of \mathbf{w}_m that begins with 0. By the induction hypothesis, we have $q = \phi_m(q')0$ for a palindromic factor q' of \mathbf{w}_m . We get $p = 0^{-1}0a\phi_m(q')0a = 0^{-1}\phi_m((a-1)q'(a-1))$. Again, it is easy to see that $p' = (a-1)q'(a-1)$ is a palindrome. It remains to prove that p' is a factor of \mathbf{w}_m . From (11), we deduce that $p = 0^{-1}\phi_m(p')$ being a factor of up implies that it is a factor of $\phi_m(up)$ too. Thus, $\phi_m(p')$ is a factor of $\phi_m(up)$. By Lemma 17, p' is a factor of up , so also of \mathbf{w}_m . \square

Lemma 43. *Let $n \geq -1$. The set of palindromic prefixes $\mathcal{P}(z_n)$ of z_n is*

$$\mathcal{P}(z_n) = \begin{cases} \{z_{-1}, z_1, z_3, \dots, z_{n-2}, z_n\}, & \text{if } n \text{ is odd;} \\ \{z_{-1}, z_0, z_2, \dots, z_{n-2}, z_n\}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. The proof is by induction on n . The result is clearly true for $n = -1, 0, 1$. Suppose $n > 1$ is even, the case where n is odd is similar. First, let p be a palindromic prefix of z_n . Since n is even, the word z_n , and hence p , begins with 0 by Proposition 24. By Lemma 42 (and also Proposition 34), we have $p = \phi_m(p')0$ where p' is a palindrome. By Lemma 29, the word p' is a palindromic prefix of z_{n-1} . By the induction hypothesis, we have $p' \in \mathcal{P}(z_{n-1})$; i.e., $p' = z_i$ for some $i \in \{-1, 1, 3, \dots, n-3, n-1\}$. By Lemma 29 again, we have $p = \phi_m(z_i)0 = z_{i+1} \in \mathcal{P}(z_n)$. We have just showed that $\mathcal{P}(z_n) \subset \{z_{-1}, z_0, z_2, \dots, z_{n-2}, z_n\}$.

Let us prove that the other inclusion holds too. We clearly have $z_{-1}, z_n \in \mathcal{P}(z_n)$. Now let $i \in \{0, 2, \dots, n-2\}$. By Lemma 29, $z_i = \phi_m(z_{i-1})0$. By the induction hypothesis, we know that $z_{i-1} \in \mathcal{P}(z_{n-1})$, i.e., there exists a non-empty word $w \in \{0, 1, \dots, m-1\}^*$ such that $z_{n-1} = z_{i-1}w$. Using Lemma 29 again, we get $z_n = \phi_m(z_{n-1})0 = \phi_m(z_{i-1})\phi_m(w)0$. By definition of ϕ_m , we have $\phi_m(w)0 = 0w'$ with $w' \in \{0, 1, \dots, m-1\}^*$. As a consequence, we find $z_n = \phi_m(z_{i-1})0w' = z_iw'$ and $z_i \in \mathcal{P}(z_n)$, as expected. \square

The last result of this section establishes the z -factorization of the m -bonacci word \mathbf{w}_m in terms of p -singular words.

Theorem 44. *The palindromic z -factorization of the m -bonacci word \mathbf{w}_m is*

$$pz(\mathbf{w}_m) = (z_0, z_1, z_2, \dots).$$

Proof. If $m = 2$, one simply has to combine Theorem 8 and Proposition 22. Now assume that $m \geq 3$. Let $pz(\mathbf{w}_m) = (t_0, t_1, t_2, \dots)$ be the palindromic z -factorization of the m -bonacci word \mathbf{w}_m . We proceed by induction on $n \geq 0$ to show that $t_n = z_n$. From Proposition 34, we have

$$\mathbf{w}_m = \prod_{n \geq 0} z_n = 0 \cdot 1 \cdot 020 \cdot z_3 \cdot z_4 \cdots, \quad (12)$$

and we see that the first two factors of the z -factorization of the m -bonacci word \mathbf{w}_m are $t_0 = 0 = z_0$ and $t_1 = 1 = z_1$. Now suppose that $n \geq 2$. From (12), we deduce that

$$\mathbf{w}_m = \prod_{n \geq 0} z_n = \left(\prod_{0 \leq k \leq n-1} z_k \right) z_n z_{n+1} \cdots.$$

To prove that $t_n = z_n$, we need to show that every palindromic prefix of z_n which is different from z_n is a factor of

$$P(n) = \prod_{k=0}^{n-1} z_k.$$

This is clear from Lemma 43. \square

4.5 The palindromic c -factorization of the m -bonacci word

In the following lemma, which is similar to Lemma 10, recall that we start indexing words at 0.

Lemma 45. *Let $n \geq m - 1$. There are exactly two occurrences of the factor z_n inside the word g_{n+1} : one at position $\sum_{k=0}^{n-1} |z_k|$, the other at position $\sum_{k=0}^{n+1} |z_k|$.*

Proof. First consider the case $m = 2$, and let $n \geq 1$. Using the notation introduced in Remark 40, Lemma 10 shows that there are exactly two occurrences of the factor $\hat{f}_{n+1} = z_n$ inside the word $g'_{n+2} = g_{n+1}$, one occurring at position $\sum_{k=1}^n |\hat{f}_k| = \sum_{k=0}^{n-1} |z_k|$, the other at position $\sum_{k=1}^{n+2} |\hat{f}_k| = \sum_{k=0}^{n+1} |z_k|$, as desired.

Assume that $m \geq 3$. Let $n \geq m - 1$. Using Definition 35 and Proposition 41, let us write

$$\begin{aligned} g_{n+1} &= z_0 z_1 \cdots z_{n+1-m} Q(n+1) z_{n+1-m} \\ &= z_0 z_1 \cdots z_{n+1-m} (z_{n+2-m} z_{n+3-m} \cdots z_{n-1} z_n z_{n+1} z_n z_{n-1} \cdots z_{n+3-m} z_{n+2-m}) z_{n+1-m} \\ &= u z_n z_{n+1} z_n v \end{aligned}$$

with $u = z_0 z_1 \cdots z_{n-1}$ and $v = z_{n-1} z_{n-2} \cdots z_{n+1-m}$. Thanks to this factorization, we immediately see that z_n occurs at least twice as a factor of g_{n+1} : one starting at position $|u| = \sum_{k=0}^{n-1} |z_k|$, the other beginning at position $|u z_n z_{n+1}| = \sum_{k=0}^{n+1} |z_k|$. We now show that there are no other occurrences of z_n as a factor of g_{n+1} . There are several cases to consider.

Case 1. The word z_n cannot be a factor of u , otherwise it contradicts Proposition 31.

Case 2. The word z_n cannot be a factor of z_{n+1} , otherwise it contradicts Proposition 30.

Case 3. If the word z_n were a factor of v , then $(z_n)^R = z_n$ would be a factor of $v^R = z_{n+1-m} \cdots z_{n-2} z_{n-1}$, since the p -singular words are palindromes. This is impossible due to Proposition 31.

Case 4. Suppose that z_n is a factor of $u z_n$, overlapping u and z_n . Using Corollary 26 ($n \geq m - 1$), we know that

$$|z_n| - 1 \leq |z_{n-1}| + |z_{n-2}| + \cdots + |z_{n-m}|.$$

Consequently, z_n is a factor of $z_{n-m} z_{n-(m-1)} \cdots z_{n-2} z_{n-1} z_n$, overlapping $z_{n-m} z_{n-(m-1)} \cdots z_{n-2} z_{n-1}$ and z_n .

If z_n starts somewhere within z_{n-k} , with $k \in \{2, 3, \dots, m\}$, or if z_n starts with the first letter of z_{n-1} , then, in each case, z_{n-1} is a factor of z_n , which contradicts Proposition 30. Therefore the occurrence of z_n must start after the first letter of z_{n-1} in $z_{n-m} z_{n-(m-1)} \cdots z_{n-2} z_{n-1} z_n$, i.e., there exist a non-empty suffix x of z_{n-1} and a non-empty prefix y of z_n such that

$$z_n = xy.$$

Observe that, in this case, x is also a non-empty prefix of z_n . This contradicts Lemma 33.

Case 5. Suppose that z_n is a factor of $z_n z_{n+1}$, overlapping z_n and z_{n+1} . There exist a non-empty suffix x' of z_n and a non-empty prefix y' of z_{n+1} such that

$$z_n = x' y'.$$

In this case, notice that y' is also a non-empty suffix of z_n . This violates Lemma 33.

Case 6. Suppose that z_n is a factor of $z_{n+1}z_n$, overlapping z_{n+1} and z_n . Since the p-singular words are palindromes, we obtain that $(z_n)^R = z_n$ is a factor of $(z_{n+1}z_n)^R = z_nz_{n+1}$, overlapping z_n and z_{n+1} . As in the fifth case, we raise a contradiction.

Case 7. Suppose that z_n is a factor of z_nv , overlapping z_n and v . Then $(z_n)^R = z_n$ is a factor of $(z_nv)^R = z_{n+1-m}z_{n-m}\cdots z_{n-1}z_n$, overlapping (at least) z_{n-1} and z_n . In the view of the fourth case, we also reach a contradiction. \square

The following result is the counterpart to Proposition 11.

Proposition 46. *Let $n \geq m - 1$. Let w be a non-empty common finite prefix of the infinite words*

$$\mathbf{u}_{n+1,1} = z_{n-(m-1)}Q(n+1)Q(n+2)\cdots$$

and

$$\mathbf{u}_{n+1,2} = Q(n+2)Q(n+3)\cdots.$$

Then $Q(n+1)w$ is not a palindrome.

Proof. If $m = 2$, the result directly follows from Propositions 11 and 22. Suppose that $m \geq 3$. We proceed by induction on n . First, suppose $n = m - 1$. Then $\mathbf{u}_{m,1} = 0Q(m)Q(m+1)\cdots$ and $\mathbf{u}_{m,2} = Q(m+1)Q(m+2)\cdots$. Note that $Q(m)$ begins with $z_1 = 1$ and $Q(m+1)$ begins with $z_2 = 020$. The only possibility for w is $w = 0$ and in this case $Q(m)0$ is not a palindrome, as required.

We now suppose that the result holds for $\mathbf{u}_{n,1}$ and $\mathbf{u}_{n,2}$. We proceed by contradiction and suppose that $Q(n+1)w$ is a palindrome.

We first observe that by Definition 35 and Lemma 29 we have $Q(i) = \phi_m(Q(i-1))0$ if $Q(i)$ begins with 0 and $Q(i) = 0^{-1}\phi_m(Q(i-1))$ if $Q(i)$ begins with 1. Thus either

$$\mathbf{u}_{n+1,1} = \phi_m(z_{n-m})00^{-1}\phi_m(Q(n))\phi_m(Q(n+1))0\cdots = \phi_m(\mathbf{u}_{n,1})$$

and

$$\mathbf{u}_{n+1,2} = \phi_m(Q(n+1))00^{-1}\phi_m(Q(n+2))\cdots = \phi_m(\mathbf{u}_{n,2});$$

or

$$\mathbf{u}_{n+1,1} = 0^{-1}\phi_m(z_{n-m})\phi_m(Q(n))00^{-1}\phi_m(Q(n+1))\cdots = 0^{-1}\phi_m(\mathbf{u}_{n,1})$$

and

$$\mathbf{u}_{n+1,2} = 0^{-1}\phi_m(Q(n+1))\phi_m(Q(n+2))0\cdots = 0^{-1}\phi_m(\mathbf{u}_{n,2}).$$

By Definition 39, observe that $Q(n+1)w$ is a factor of \mathbf{w}_m . Now using Lemma 42, we have either $Q(n+1)w = \phi_m(Q(n)w')0 = Q(n+1)0^{-1}\phi_m(w')0$ or $Q(n+1)w = 0^{-1}\phi_m(Q(n)w') = Q(n+1)\phi_m(w')$, where $Q(n)w'$ is a palindrome and w' is a common prefix of $\mathbf{u}_{n,1}$ and $\mathbf{u}_{n,2}$. This contradicts the induction hypothesis. We conclude that $Q(n+1)w$ is not a palindrome, as required. \square

In this last result, we obtain the c -factorization of the m -bonacci word \mathbf{w}_m in terms of p-singular words.

Theorem 47. *Let $pc(\mathbf{w}_m) = (c_{-m}, c_{-(m-1)}, c_{-(m-2)}, \dots)$ denote the palindromic c -factorization of the m -bonacci word \mathbf{w}_m . Then, the first $2m - 2$ factors are given by the factorization of p_m emphasized in Proposition 38, and, for all $n \geq m - 2$, $c_n = Q(n)$.*

Proof. Consider first the case where $m = 2$. From Theorem 12, we know that $c_{-2} = 0$, $c_{-1} = 1$, $c_0 = 0$ and, for all $n \geq 1$,

$$c_n = \hat{f}_n \hat{f}_{n+1} \hat{f}_n.$$

We clearly have $c_0 = \varepsilon \cdot 0 \cdot \varepsilon = z_{-1} z_0 z_{-1} = Q(0)$, and for all $n \geq 1$, we also get $c_n = z_{n-1} z_n z_{n-1} = Q(n)$ using Proposition 22. Assume now that $m \geq 3$. From Definition 35, Propositions 37 and 38, we have

$$\begin{aligned} \mathbf{w}_m &= p_m \prod_{n \geq m-2} Q(n) \\ &= z_0 z_1 \cdots z_{m-4} z_{m-3} z_{m-2} z_{m-3} z_{m-4} \cdots z_1 z_0 (m-1) \prod_{n \geq m-2} Q(n) \end{aligned} \quad (13)$$

$$= z_0 \cdot 1 \cdot z_0 \cdot 2 \cdot (z_0 z_1 z_0) \cdot 3 \cdots (z_0 z_1 \cdots z_{m-4} z_{m-3} z_{m-4} \cdots z_1 z_0) \cdot (m-1) \cdot \prod_{n \geq m-2} Q(n). \quad (14)$$

Using the definition of the c -palindromic factorization and looking at (14), the first $(2m-2)$ factors $c_{-m}, c_{-(m-1)}, \dots, c_{m-3}$ of $pc(\mathbf{w}_m)$ are given by the factorization of p_m emphasized in Proposition 38 (recall the second part of Proposition 38: each $q_k = c_{k-m-1}$, for $1 \leq k \leq 2m-2$, is the longest palindromic prefix of $q_k q_{k+1} \cdots q_{2m-2} \cdot \prod_{n \geq m-2} Q(n)$ that has already occurred in $\mathbf{w}_m = p_m \prod_{n \geq m-2} Q(n)$, or if this prefix does not exist, q_k is a single letter).

For the second part of the result, proceed by induction on $n \geq m-2$. If $n = m-2$, we must find the factor c_{m-2} of the palindromic c -factorization of \mathbf{w}_m . Using Definition 35, we have

$$Q(m-2) = z_{-1} z_0 \cdots z_{m-4} z_{m-3} z_{m-2} z_{m-3} z_{m-4} \cdots z_0 z_{-1},$$

and from (13), we get

$$\begin{aligned} \mathbf{w}_m &= z_0 z_1 \cdots z_{m-4} z_{m-3} z_{m-2} z_{m-3} z_{m-4} \cdots z_1 z_0 (m-1) (z_{-1} z_0 \cdots z_{m-4} z_{m-3} z_{m-2} z_{m-3} z_{m-4} \cdots z_0 z_{-1}) \\ &\quad \prod_{n \geq m-1} Q(n) \\ &= Q(m-2) \cdot (m-1) \cdot Q(m-2) \cdot \prod_{n \geq m-1} Q(n). \end{aligned} \quad (15)$$

On the one hand, observe that $Q(m-1)$ starts with z_0 by Definition 35, and $z_0 \neq m-1$. On the other hand, $Q(m-2)$ does not contain the letter $m-1$ by Proposition 32. Thus, the longest palindrome occurring before is $c_{m-2} = Q(m-2)$.

If $n = m-1$, we must find the factor c_{m-1} of the palindromic c -factorization of \mathbf{w}_m . Using Definition 35, we have

$$Q(m-1) = z_0 z_1 \cdots z_{m-3} z_{m-2} z_{m-1} z_{m-2} z_{m-3} \cdots z_1 z_0.$$

Starting from (15) and using Definition 19(1), we have

$$\begin{aligned} \mathbf{w}_m &= z_0 z_1 \cdots z_{m-4} z_{m-3} z_{m-2} (z_{m-3} z_{m-4} \cdots z_1 z_0 (m-1) z_0 z_1 \cdots z_{m-4} z_{m-3}) z_{m-2} z_{m-3} z_{m-4} \cdots z_0 \\ &\quad \prod_{n \geq m-1} Q(n) \\ &= z_0 z_1 \cdots z_{m-4} z_{m-3} z_{m-2} \cdot z_{m-1} \cdot z_{m-2} z_{m-3} z_{m-4} \cdots z_0 \cdot Q(m-1) \cdot \prod_{n \geq m} Q(n). \end{aligned}$$

On the one hand, $Q(m)$ starts with z_1 by Definition 35, and $z_1 \neq z_0$. On the other hand, Proposition 32 shows that $m-1$ never occurs in $z_{-1}, z_0, z_1, \dots, z_{m-2}$. So the longest palindrome occurring before is $c_{m-1} = Q(m-1)$.

For the induction step, suppose $n \geq m-1$ and assume that, for all $1 \leq k \leq n$, we have $c_k = Q(k)$. We show it is still true for $k = n+1$. On the one hand, using the induction hypothesis, Proposition 38 and finally Definition 39, we have

$$\begin{aligned} \mathbf{w}_m &= c_{-m}c_{-(m-1)} \cdots c_{m-3} \left(\prod_{m-2 \leq k \leq n} c_k \right) c_{n+1}c_{n+2} \cdots \\ &= p_m \left(\prod_{m-2 \leq k \leq n} Q(k) \right) c_{n+1}c_{n+2} \cdots \\ &= g_{n+1}c_{n+1}c_{n+2} \cdots, \end{aligned} \tag{16}$$

and the goal is to find the next factor of the palindromic c -factorization of \mathbf{w}_m , i.e., the word c_{n+1} . On the other hand, using Proposition 37 first, then Definition 39 and finally Proposition 41, we get

$$\begin{aligned} \mathbf{w}_m &= p_m \left(\prod_{m-2 \leq k \leq n} Q(k) \right) Q(n+1)Q(n+2) \cdots \\ &= g_{n+1}Q(n+1)Q(n+2) \cdots \\ &= (z_0z_1 \cdots z_{n+1-m} \cdot Q(n+1) \cdot z_{n+1-m})Q(n+1)Q(n+2) \cdots. \end{aligned} \tag{17}$$

Comparing (16) and (17), we see that $|c_{n+1}| \geq |Q(n+1)|$ since $Q(n+1)$ is a palindrome occurring before in g_{n+1} . Therefore, there exists a word $w \in \{0, 1, \dots, m-1\}^*$ such that $c_{n+1} = Q(n+1)w$. We claim that w is in fact the empty word. For the following argument, recall that Definition 35 gives

$$Q(n+1) = z_{n+2-m}z_{n+3-m} \cdots z_{n-1}z_nz_{n+1}z_nz_{n-1} \cdots z_{n+3-m}z_{n+2-m}.$$

By Lemma 45, we know that there are exactly two occurrences of z_n in g_{n+1} : one starting at position $\sum_{k=0}^{n-1} |z_k|$, the other at position $\sum_{k=0}^{n+1} |z_k|$.

Case 1. Let us deal with the first occurrence of z_n in g_{n+1} . In this case, w must be a common prefix of the infinite words

$$\mathbf{u}_{n+1,1} = z_{n+1-m}Q(n+1)Q(n+2) \cdots$$

and

$$\mathbf{u}_{n+1,2} = Q(n+2)Q(n+3) \cdots.$$

But by Proposition 46, we know that $Q(n+1)w$ is not a palindrome unless w is the empty word.

Case 2. Let us examine the second occurrence of z_n in g_{n+1} . In this case, the suffix $z_nz_{n+1}z_nz_{n-1} \cdots z_{n+3-m}z_{n+2-m}$ of $Q(n+1)$ starts at position $\sum_{k=0}^{n+1} |z_k|$ in g_{n+1} . In particular, z_{n+1} is a prefix of the infinite word

$$\begin{aligned} & z_{n-1}z_{n-2} \cdots z_{n+3-m}z_{n+2-m}z_{n+1-m}Q(n+1)Q(n+2) \cdots \\ &= z_{n-1}z_{n-2} \cdots z_{n+3-m}z_{n+2-m}z_{n+1-m}(z_{n+2-m}z_{n+3-m} \cdots z_{n-1}z_nz_{n+1}z_nz_{n-1} \cdots z_{n+3-m}z_{n+2-m}) \\ & \quad Q(n+2)Q(n+3) \cdots. \end{aligned}$$

Since $n + 1 \geq m$, Definition 19 gives

$$z_{n+1} = z_{n-1}z_{n-2} \cdots z_{n+2-m}z_{n+1-m}z_{n-m}z_{n+1-m}z_{n+2-m} \cdots z_{n-2}z_{n-1}.$$

We now show that

$$|z_{n-1}z_{n-2} \cdots z_{n+3-m}z_{n+2-m}z_{n+1-m}z_{n+2-m}z_{n+3-m} \cdots z_{n-2}z_{n-1}| < |z_{n+1}|, \quad (18)$$

and

$$|z_{n+1}| \leq |z_{n-1}z_{n-2} \cdots z_{n+3-m}z_{n+2-m}z_{n+1-m}z_{n+2-m}z_{n+3-m} \cdots z_{n-2}z_{n-1}z_n|. \quad (19)$$

Let us show (18). We have

$$\begin{aligned} & |z_{n+1}| - |z_{n-1}z_{n-2} \cdots z_{n+3-m}z_{n+2-m}z_{n+1-m}z_{n+2-m}z_{n+3-m} \cdots z_{n-2}z_{n-1}| \\ &= 2 \sum_{k=n+1-m}^{n-1} |z_k| + |z_{n-m}| - \left(2 \sum_{k=n+2-m}^{n-1} |z_k| + |z_{n+1-m}| \right) \\ &= |z_{n+1-m}| + |z_{n-m}| > 0 \end{aligned}$$

since $n + 1 \geq m$ implies that $|z_{n+1-m}| > 0$. Let us prove (19). We get

$$\begin{aligned} & |z_{n-1}z_{n-2} \cdots z_{n+3-m}z_{n+2-m}z_{n+1-m}z_{n+2-m}z_{n+3-m} \cdots z_{n-2}z_{n-1}z_n| - |z_{n+1}| \\ &= 2 \sum_{k=n+2-m}^{n-1} |z_k| + |z_{n+1-m}| + |z_n| - \left(2 \sum_{k=n+1-m}^{n-1} |z_k| + |z_{n-m}| \right) \\ &= |z_n| - |z_{n+1-m}| - |z_{n-m}| \geq 0. \end{aligned}$$

Indeed, to see this, we make use of Corollary 26. If m is even, then the result is clear. If m is odd, then $|z_n| \geq |z_{n+1-m}| + |z_{n-m}|$ since $|z_{n-1}| + |z_{n-2}| + \cdots + |z_{n+2-m}| + (-1)^n \geq 0$. As a consequence of (18) and (19), z_{n+1} must end with a non-empty prefix of z_n , which contradicts Lemma 33.

As a conclusion to both cases, the longest palindrome occurring before is $c_{n+1} = Q(n + 1)$ as required. \square

5 Open Problems

Problem 48. Let $A = \{0, 1, \dots, m - 1\}$ be a finite alphabet of size m , $m \geq 1$. Define the family \mathcal{F}_m of infinite words \mathbf{w} over A such that $z(\mathbf{w}) = pz(\mathbf{w})$. For instance, when $m = 2$, observe that $\mathbf{f} \in \mathcal{F}_2$ (Proposition 5 and Theorem 8) and

$$010^21^20^31^3 \cdots \in \mathcal{F}_2,$$

but the Thue–Morse word

$$\mathbf{t} = 0110100110010110 \cdots$$

which is the fixed point of the morphism $\tau : 0 \rightarrow 01, 1 \rightarrow 10$ does not belong to \mathcal{F}_2 . Give a characterization of this family. Also give a characterization of the set of automatic words among this family.

The m -bonacci words belong to the family of episturmian words. When studying episturmian words, it is standard to introduce a particular sequence $(h_n)_{n \geq 0}$ of finite words related to their directive word and palindromic prefixes. Justin and Pirillo [8] showed that the sequence $(u_n)_{n \geq 1}$ of palindromic prefixes of a standard episturmian word \mathbf{s} verifies

$$u_{n+2} = h_n h_{n-1} \cdots h_1 h_0 = h_0^R h_1^R \cdots h_{n-1}^R h_n^R \quad \forall n \geq 0.$$

When it comes to m -bonacci words, the sequence $(h_n)_{n \geq 0}$ coincides with the one from Definition 13. From Lemma 20, we can also show that for all $n \geq 0$, we have

$$u_{n+2} = z_0 z_1 \cdots z_{n-2} z_{n-1} z_n z_{n-1} z_{n-2} \cdots z_1 z_0,$$

which in particular gives another way of showing that the words $(z_n)_{n \geq -1}$ are all palindromes. We observe that the sequences $(h_n)_{n \geq 0}$ and $(z_n)_{n \geq -1}$ are intimately bonded, so a natural question is the following open problem.

Problem 49. Find the palindromic z - and c -factorizations of other infinite words such as the Thue–Morse word, or more specifically episturmian words, billiard words, or rich words.

As we observed in Section 4.2, Lemma 20 gives another definition of the p -singular words, which may possibly be the more useful one when trying to extend the results of this paper to episturmian words (see the ordinary z - and c -factorizations of episturmian words given by Ghareghani, Mohammad-Noori, and Sharifani [6]).

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