Enhanced Laplace transform and holomorphic Paley-Wiener-type theorems

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Polya's theorem

Theorem (Polya, 1929 and Martineau, 1963)

Let K be a non-empty compact convex subset of $\mathbb{C}$ and h_K its support function. Then

$$\mathcal{O}^0(\mathbb{C}\setminus K) = \{f \in \mathcal{O}(\mathbb{C}\setminus K) : \lim_{z\to\infty} f(z) = 0\}$$

and

$$\mathrm{Exp}(K) = \{g \in \mathcal{O}(\mathbb{C}) : \forall \varepsilon > 0, \sup_{w\in\mathbb{C}} |g(w)|e^{-h_K(w)-\varepsilon|w|} < \infty\}$$

are topologically isomorphic through

$$\mathcal{P} : \mathcal{O}^0(\mathbb{C}\setminus K) \ni f \mapsto \left(w \mapsto \frac{1}{2i\pi}\int_{C(0,R)^+} e^{zw} f(z)dz\right) \in \mathrm{Exp}(K).$$

Méril's theorem

Let $S \subset \mathbb{C}$ be a non-compact closed convex subset which contains no lines. For such a convex S, its asymptotic cone is defined by

$$S_\infty = \{z \in \mathbb{C} : z + S \subset S\}.$$

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Its polar cone is defined by4

$$S_\infty^* = \{w \in \mathbb{C} : \forall z \in S_\infty,\ \Re(zw) \leq 0\}.$$

It is a closed convex cone with a non-empty interior. Let us fix ξ_0 a point in the interior of S_∞^*.

Theorem (Méril, 1983)

There is a topological isomorphism between

$$\varprojlim_{\varepsilon'\to 0} \frac{\{f \in \mathcal{O}(\mathbb{C}\setminus S) : \forall r > \varepsilon > 0, \sup_{z\in S_r\setminus S_\varepsilon^\circ} |e^{\varepsilon'\xi_0 z} f(z)| < \infty)\}}{\{f \in \mathcal{O}(\mathbb{C}) : \forall r > 0, \sup_{z\in S_r} |e^{\varepsilon'\xi_0 z} f(z)| < \infty)\}}$$

and

$$\{g \in \mathcal{O}((S_\infty^*)^\circ) : \forall \varepsilon, \varepsilon' > 0, \sup_{w\in S_\infty^* + \varepsilon'\xi_0} |g(w)| e^{-h_S(w)-\varepsilon|w|} < \infty\}$$

given by

$$\mathcal{P} : \mathscr{H}_S(\mathbb{C}) \ni ([u_{\varepsilon'}])_{\varepsilon'} \mapsto \frac{1}{2i\pi} \int_{\partial S_\varepsilon^+} e^{zw} u_{\varepsilon'}(z)\, dz \in \mathrm{Exp}(S),$$

where ∂S_ε^+ is the boundary (positively oriented) of a thickening S_ε.

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$\rightsquigarrow$What is the good setting for a cohomological Laplace transform ?

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$$\mathrm{D}^{\mathrm{b}}(\mathbb{C}_M) \rightsquigarrow \mathrm{D}^{\mathrm{b}}(\mathbb{C}_M^{\mathrm{sub}}) \rightsquigarrow \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{M_\infty}^{\mathrm{sub}}) \rightsquigarrow \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{M_\infty \times \mathbb{R}_\infty}^{\mathrm{sub}}),$$

where $\mathbb{R}_\infty = (\mathbb{R}, \mathbb{R} \cup \{-\infty, +\infty\})$.

Let $M_\infty = (M, \widehat{M})$ be a subanalytic bordered space and let

$$\mu, q_1, q_2 : M_\infty \times \mathbb{R}_\infty \times \mathbb{R}_\infty \to M_\infty \times \mathbb{R}_\infty$$

be the morphisms defined by

$$\mu(x, t_1, t_2) = (x, t_1 + t_2)$$

and

$$q_1(x, t_1, t_2) = (x, t_1), \qquad q_2(x, t_1, t_2) = (x, t_2).$$

Definition

One defines the two *convolution functors*

$$\overset{+}{\otimes} : \mathrm{D^b}(\mathbb{C}^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty}) \times \mathrm{D^b}(\mathbb{C}^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty}) \to \mathrm{D^b}(\mathbb{C}^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty}),$$
$$\mathscr{I}hom^{+} : \mathrm{D^-}(\mathbb{C}^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty})^{\mathrm{op}} \times \mathrm{D^+}(\mathbb{C}^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty}) \to \mathrm{D^+}(\mathbb{C}^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty})$$

by

$$F_1 \overset{+}{\otimes} F_2 = \mathrm{R}\mu_{!!}(q_1^{-1}F_1 \otimes q_2^{-1}F_2),$$
$$\mathscr{I}hom^{+}(F_1, F_2) = \mathrm{R}q_{1*}\mathrm{R}\mathscr{I}hom(q_2^{-1}F_1, \mu^! F_2).$$

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Let $\varphi : M \to \mathbb{R}$ be a continuous function. Let us denote by $\mu_\varphi : M_\infty \times \mathbb{R}_\infty \to M_\infty \times \mathbb{R}_\infty$ the map defined by $\mu_\varphi(x,t) = (x, t+\varphi(x))$. Then,

$$\mathbb{C}_{\{t=\varphi(x)\}} \overset{+}{\otimes} F \simeq R\mu_{\varphi_*}F \simeq \mathscr{I}hom^{+}(\mathbb{C}_{\{t=-\varphi(x)\}}, F).$$

Definition

On a subanalytic bordered space $M_\infty = (M, \widehat{M})$, one defines the category of (bounded) *enhanced subanalytic sheaves* by setting

$$\begin{aligned}\mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty}) &= \mathrm{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty})/\{F : (\mathbb{C}_{\{t \geq 0\}} \oplus \mathbb{C}_{\{t \leq 0\}}) \overset{+}{\otimes} F \simeq 0\} \\ &\simeq \mathrm{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty})/\{F : \exists L \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty}), F \simeq \pi^{-1} L\},\end{aligned}$$

where $\pi : M_\infty \times \mathbb{R}_\infty \to M_\infty$ is the projection.

We denote by

$$Q_{M_\infty} : \mathrm{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$$

the quotient functor.

The quotient functor Q_{M_∞} admits a left adjoint L^{E} and a right adjoint R^{E} defined by

$$\mathsf{L}^{\mathsf{E}}(F) = (\mathbb{C}_{\{t\geq 0\}} \oplus \mathbb{C}_{\{t\leq 0\}}) \overset{+}{\otimes} F,$$
$$\mathsf{R}^{\mathsf{E}}(F) = \mathscr{I}hom^{+}(\mathbb{C}_{\{t\geq 0\}} \oplus \mathbb{C}_{\{t\leq 0\}}, F),$$

for all $F \in \mathsf{E}^{\mathsf{b}}(\mathbb{C}^{\mathsf{sub}}_{M_\infty})$. Moreover, these functors are fully faithful and hence, through L^{E}, one can identify $\mathsf{E}^{\mathsf{b}}(\mathbb{C}^{\mathsf{sub}}_{M_\infty})$ to a full subcategory of $\mathsf{D}^{\mathsf{b}}(\mathbb{C}^{\mathsf{sub}}_{M_\infty\times\mathbb{R}_\infty})$.

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Remark that $\overset{+}{\otimes}$ and $\mathscr{I}hom^{+}$ factor through the quotient but not $\overset{\mathsf{L}}{\otimes}$ and $\mathsf{R}\mathscr{I}hom$.

To define the four other Grothendieck operations, let us consider a morphism $f : M_\infty \to N_\infty$ of subanalytic bordered spaces. We set

$$f_\mathbb{R} := f \times \mathrm{id}_\mathbb{R} : M_\infty \times \mathbb{R}_\infty \to N_\infty \times \mathbb{R}_\infty \, .$$

There are four functors

$$\begin{aligned} \mathrm{R}f_{\mathbb{R}*}, \mathrm{R}f_{\mathbb{R}!!} &: \mathrm{D}^\mathrm{b}(\mathbb{C}^\mathrm{sub}_{M_\infty \times \mathbb{R}_\infty}) \to \mathrm{D}^\mathrm{b}(\mathbb{C}^\mathrm{sub}_{N_\infty \times \mathbb{R}_\infty}), \\ f_\mathbb{R}^{-1}, f_\mathbb{R}^! &: \mathrm{D}^\mathrm{b}(\mathbb{C}^\mathrm{sub}_{N_\infty \times \mathbb{R}_\infty}) \to \mathrm{D}^\mathrm{b}(\mathbb{C}^\mathrm{sub}_{M_\infty \times \mathbb{R}_\infty}). \end{aligned}$$

These functors factor through the quotients and we note

$$\begin{aligned} \mathrm{E}f_*, \mathrm{E}f_{!!} &: \mathrm{E}^\mathrm{b}(\mathbb{C}^\mathrm{sub}_{M_\infty}) \to \mathrm{E}^\mathrm{b}(\mathbb{C}^\mathrm{sub}_{N_\infty}), \\ \mathrm{E}f^{-1}, \mathrm{E}f^! &: \mathrm{E}^\mathrm{b}(\mathbb{C}^\mathrm{sub}_{N_\infty}) \to \mathrm{E}^\mathrm{b}(\mathbb{C}^\mathrm{sub}_{M_\infty}) \end{aligned}$$

their factorisation.

Definition

Let $\pi : M_\infty \times \mathbb{R}_\infty \to M_\infty$. One defines the hom functor

$$\mathrm{R}\mathscr{I}hom^{\mathrm{E}} : \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})^{\mathrm{op}} \times \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty}) \to \mathrm{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$$

by

$$\mathrm{R}\mathscr{I}hom^{\mathrm{E}}(F_1, F_2) = \mathrm{R}\pi_* \mathrm{R}\mathscr{I}hom(\mathrm{R}^{\mathrm{E}} F_1, \mathrm{R}^{\mathrm{E}} F_2).$$

One also defines

$$\mathrm{R}\mathcal{H}om^{\mathrm{E}} : \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})^{\mathrm{op}} \times \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty}) \to \mathrm{D}^{\mathrm{b}}(\mathbb{C}_M)$$

by $\mathrm{R}\mathcal{H}om^{\mathrm{E}} = \rho^{-1} \circ \mathrm{R}\mathscr{I}hom^{\mathrm{E}}$. Finally, one sets

$$\mathrm{RHom}^{\mathrm{E}} = \mathrm{R}\Gamma(M, \mathrm{R}\mathcal{H}om^{\mathrm{E}}).$$

Remark that

$$\mathrm{Hom}_{\mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})}(F_1, F_2) \simeq H^0 \mathrm{RHom}^{\mathrm{E}}(F_1, F_2)$$

for all $F_1, F_2 \in \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$.

Enhanced Fourier-Sato functors

Let $\mathbb{V}$ be a n-dimensional complex vector space and $\mathbb{V}^*$ its complex dual. We consider the complex bordered spaces $\mathbb{V}_\infty = (\mathbb{V}, \overline{\mathbb{V}})$ and $\mathbb{V}^*_\infty = (\mathbb{V}^*, \overline{\mathbb{V}}^*)$ where $\overline{\mathbb{V}}$ (resp. $\overline{\mathbb{V}}^*$) is the proj. compactification of $\mathbb{V}$ (resp $\mathbb{V}^*$). We note $\langle -, - \rangle : \mathbb{V} \times \mathbb{V}^* \to \mathbb{C}$ the duality bracket and p, q the two projections on $\mathbb{V}_\infty \times \mathbb{V}^*_\infty$.

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Definition

The enhanced Fourier-Sato functors

$$^{\mathsf{E}}\mathcal{F}_{\mathbb{V}}, {}^{\mathsf{E}}\mathcal{F}^a_{\mathbb{V}} : \mathsf{E}^{\mathrm{b}}(\mathbb{C}^{\mathsf{sub}}_{\mathbb{V}_\infty}) \to \mathsf{E}^{\mathrm{b}}(\mathbb{C}^{\mathsf{sub}}_{\mathbb{V}^*_\infty})$$

are defined by

$$^{\mathsf{E}}\mathcal{F}_{\mathbb{V}}(F) = \mathsf{E}q_{!!}(\mathbb{C}_{\{t=\Re\langle z,w\rangle\}} \overset{+}{\otimes} \mathsf{E}p^{-1}F),$$

$$^{\mathsf{E}}\mathcal{F}^a_{\mathbb{V}}(F) = \mathsf{E}q_{!!}(\mathbb{C}_{\{t=-\Re\langle z,w\rangle\}} \overset{+}{\otimes} \mathsf{E}p^{-1}F).$$

The first main theorem

Theorem (M. Kashiwara, P. Schapira, 2016)

The functor ${}^{\mathsf{E}}\mathcal{F}^{a}_{\mathbb{V}}$ *is an equivalence of categories whose inverse is given by* ${}^{\mathsf{E}}\mathcal{F}_{\mathbb{V}^*}[2n]$*. Moreover, one has an isomorphism*

$$\mathrm{RHom}^{\mathsf{E}}(F_1, F_2) \simeq \mathrm{RHom}^{\mathsf{E}}({}^{\mathsf{E}}\mathcal{F}^{a}_{\mathbb{V}}(F_1), {}^{\mathsf{E}}\mathcal{F}^{a}_{\mathbb{V}}(F_2)), \tag{1}$$

functorial with respect to $F_1, F_2 \in \mathsf{E}^{\mathsf{b}}(\mathbb{C}^{\mathsf{sub}}_{\mathbb{V}_\infty})$.

Tempered distributions

Definition

Let M be a real analytic manifold and U an open subset of M. One sets

$$\mathcal{D}b^{\mathrm{t}}_M(U) = \{u \in \mathcal{D}b_M(U) : \exists v \in \mathcal{D}b_M(M), v|_U = u\}.$$

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- The functor $\mathcal{D}b^{\mathrm{t}}_M : U \mapsto \mathcal{D}b^{\mathrm{t}}_M(U)$ is a subanalytic sheaf thanks to the Lojasiewicz inequality. It is obviously quasi-injective, thus acyclic pour $f_*, f_{!!}, \Gamma_S$, etc ...

Enhanced distributions

Let $\mathsf{P} = \mathbb{R} \cup \{\infty\}$ and $i : M_\infty \times \mathbb{R}_\infty \to \widehat{M} \times \mathsf{P}$.

Definition

Let M_∞ be a real analytic bordered space. One sets

$$\mathcal{D}b^{\mathsf{T}}_{M_\infty} = i^{-1}(\ker(\mathcal{D}b^{\mathsf{t}}_{\widehat{M}\times\mathsf{P}} \xrightarrow{\partial_t - 1} \mathcal{D}b^{\mathsf{t}}_{\widehat{M}\times\mathsf{P}})).$$

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On a complex bordered space X_∞ of complex dimension d_X, one defines *the complex of enhanced holomorphic p-forms* $\Omega^{\mathsf{E},p}_{X_\infty} \in \mathsf{E}^{\mathsf{b}}(\mathbb{C}^{\mathsf{sub}}_{X_\infty})$ by the Dolbeault complex

$$Q_{X_\infty}\left(0 \to \mathcal{D}b^{\mathsf{T},p,0}_{X_\infty} \xrightarrow{\bar\partial} \mathcal{D}b^{\mathsf{T},p,1}_{X_\infty} \to \cdots \to \mathcal{D}b^{\mathsf{T},p,d_X}_{X_\infty} \to 0\right).$$

One sets $\mathcal{O}^{\mathsf{E}}_{X_\infty} = \Omega^{\mathsf{E},0}_{X_\infty}$ et $\Omega^{\mathsf{E}}_{X_\infty} = \Omega^{\mathsf{E},d_X}_{X_\infty}$.

The second main theorem

Theorem (C. D., 2018)

One has an isomorphism

$$ {}^{\mathsf{E}}\mathcal{F}^{a}_{\mathbb{V}}(\Omega^{\mathsf{E}}_{\mathbb{V}_\infty})[n] \xrightarrow{\sim} \mathcal{O}^{\mathsf{E}}_{\mathbb{V}^*_\infty} \tag{2} $$

in $\mathsf{E}^{\mathsf{b}}(\mathbb{C}^{\mathsf{sub}}_{\mathbb{V}^*_\infty})$*, derived from the classical positive Laplace transform.*

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Sketch of the proof : First, we consider the following sequence of morphisms

$$ \begin{aligned} q_{\mathbb{R}!!}({\mu_{-\langle z,w\rangle}}_* p_{\mathbb{R}}^{-1}\mathcal{D}b^{\mathrm{T},n,\bullet+n}_{\mathbb{V}_\infty}) &\to q_{\mathbb{R}!!}({\mu_{-\langle z,w\rangle}}_*\mathcal{D}b^{\mathrm{T},n,\bullet+n}_{\mathbb{V}_\infty\times\mathbb{V}^*_\infty}) \\ &\to q_{\mathbb{R}!!}(\mathcal{D}b^{\mathrm{T},n,\bullet+n}_{\mathbb{V}_\infty\times\mathbb{V}^*_\infty}) \\ &\to \mathcal{D}b^{\mathrm{T},0,\bullet}_{\mathbb{V}^*_\infty} \end{aligned} $$

which clearly encodes the classical positive Laplace transform.

1) The pullback of enhanced distributions by $p_{\mathbb{R}}$, namely the map

$$p_{\mathbb{R}}^{-1}\,\mathcal{D}b_{\mathbb{V}_\infty}^{\mathsf{T},n,\bullet+n} \to \mathcal{D}b_{\mathbb{V}_\infty\times\mathbb{V}_\infty^*}^{\mathsf{T},n,\bullet+n}$$

is well-defined since

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2) The integration of enhanced distributions along the fibers of $q_{\mathbb{R}}$, namely the map

$$q_{\mathbb{R}!!}\,\mathcal{D}b_{\mathbb{V}_\infty\times\mathbb{V}_\infty^*}^{\mathsf{T},n,\bullet+n} \to \mathcal{D}b_{\mathbb{V}_\infty^*}^{\mathsf{T},0,\bullet}$$

is well-defined since

$$(\partial_t u = u) \Longrightarrow \left(\partial_t \int_{q_{\mathbb{R}}} u = \int_{q_{\mathbb{R}}} \partial_t u = \int_{q_{\mathbb{R}}} u\right).$$

3) The key point is the translation map

$$\mu_{-\langle z,w\rangle *}\mathcal{D}b^{\mathsf{T},n,\bullet+n}_{\mathbb{V}_\infty\times\mathbb{V}^*_\infty}\to\mathcal{D}b^{\mathsf{T},n,\bullet+n}_{\mathbb{V}_\infty\times\mathbb{V}^*_\infty}.$$

If u is enhanced, then $u(z,w,t)=e^t\rho(z,w)$ for a certain distribution ρ. Hence,

$$u(z,w,t+\Re\langle z,w\rangle)=e^{t+\Re\langle z,w\rangle}\rho(z,w)=e^{\Re\langle z,w\rangle}u(z,w,t).$$

This operation preserves both the tempered and the enhanced condition. It is then enough to compose it with the multiplication by $e^{i\Im\langle z,w\rangle}$.

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$$\mu_{-\langle z,w\rangle_*}\,\mathcal{D}b^{\mathsf{T},n,\bullet+n}_{\mathbb{V}_\infty\times\mathbb{V}^*_\infty}\to\mathcal{D}b^{\mathsf{T},n,\bullet+n}_{\mathbb{V}_\infty\times\mathbb{V}^*_\infty}\,.$$

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This operation preserves both the tempered and the enhanced condition. It is then enough to compose it with the multiplication by $e^{i\Im\langle z,w\rangle}$.

4) Our map therefore corresponds to the transformation

$$u\mapsto\int_{q_{\mathbb{R}}}e^{\langle z,w\rangle}p^*_{\mathbb{R}}u.$$

Secondly, since $\mathcal{D}b^{\mathsf{T},p,q}_{\mathbb{V}_\infty \times \mathbb{V}^*_\infty}$ is acyclic for $q_{\mathbb{R}!!}$, one obtains a derived morphism

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Finally, one can show that this map corresponds to the isomorphism

$$^{\mathsf{E}}\mathcal{F}^{a}_{\mathbb{V}}(\Omega^{\mathsf{E}}_{\mathbb{V}_\infty})[n] \simeq \mathcal{O}^{\mathsf{E}}_{\mathbb{V}^*_\infty}$$

proved par M. Kashiwara et P. Schapira in 2016.

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This last isomorphism relies on the enhanced Riemann-Hilbert correspondence and a classical isomorphism of Katz and Laumon (1985) for $\mathcal{D}$-modules.

Consequence de (1) et (2)

Theorem

Let $F \in \mathsf{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{\mathbb{V}_\infty})$, *there is an isomorphism*

$$\begin{aligned} \mathrm{RHom}^{\mathsf{E}}(F, \Omega^{\mathsf{E}}_{\mathbb{V}_\infty})[n] &\simeq \mathrm{RHom}^{\mathsf{E}}({}^{\mathsf{E}}\mathcal{F}^{a}_{\mathbb{V}}(F), {}^{\mathsf{E}}\mathcal{F}^{a}_{\mathbb{V}}(\Omega^{\mathsf{E}}_{\mathbb{V}_\infty}))[n] \\ &\xrightarrow{\sim} \mathrm{RHom}^{\mathsf{E}}({}^{\mathsf{E}}\mathcal{F}^{a}_{\mathbb{V}}(F), \mathcal{O}^{\mathsf{E}}_{\mathbb{V}^*_\infty}) \end{aligned}$$

given by the positive Laplace transform.

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It is now enough to choose a judicious F to obtain a Paley-Wiener-type theorem.

Fourier-Sato functor and Legendre transform

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• For any $f \in \mathrm{Conv}(\mathbb{V})$, one defines a function $f^* : \mathbb{V}^* \to \mathbb{R} \cup \{+\infty\}$ by setting

$$f^*(w) = \sup_{z \in \mathrm{dom}(f)} (\Re\langle z, w\rangle - f(z)).$$

It is called *the Legendre transform of f*. It is an element of $\mathrm{Conv}(\mathbb{V}^*)$.

Proposition (M. Kashiwara, P. Schapira, 2016)

Let $f \in \mathrm{Conv}(\mathbb{V})$ *and let* $d(f)$ *be the real dimension of* $H(f^*)^\perp$*, where* $H(f^*)$ *is the affine space generated by* $\mathrm{dom}(f^*)$*. One has an isomorphism*

$$^{\mathrm{E}}\mathcal{F}^{a}_{\mathbb{V}}(\mathbb{C}_{\{t \geq f(z)\}}) \simeq \mathbb{C}_{\{t \geq -f^*(w), w \in \mathrm{dom}^\circ(f^*)\}} \otimes or_{H(f^*)^\perp}[-d(f)].$$

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Assume that $f \in \mathrm{Conv}(\mathbb{V})$ is such that $d(f) = 0$. One gets an isomorphism

$$ \mathrm{RHom}^{\mathsf{E}}(\mathbb{C}_{\{t \geq f(z)\}}, \Omega^{\mathsf{E}}_{\mathbb{V}_{\infty}})[n] \xrightarrow{\sim} \mathrm{RHom}^{\mathsf{E}}(\mathbb{C}_{\{t \geq -f^*(w), w \in \mathrm{dom}^{\circ}(f^*)\}}, \mathcal{O}^{\mathsf{E}}_{\mathbb{V}^*_{\infty}}), $$

given by the positive Laplace transform.

Definition

Let M be a real analytic manifold and U a subanalytic open subset of M. A function $f : U \to \mathbb{R}$ is *globally subanalytic* on M if its graph $\Gamma_f \subset U \times \mathbb{R}$ is subanalytic in $M \times \overline{\mathbb{R}}$. A continuous function $f : U \to \mathbb{R}$ is *almost $\mathcal{C}_\infty$-subanalytic* on M if there is a $\mathcal{C}_\infty$-function $g : U \to \mathbb{R}$, globally subanalytic on M, such that

$$\exists\, C > 0, \forall x \in U : |f(x) - g(x)| < C.$$

In this case, we say that g is in the (ASA)-class of f.

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Conjecture (M. Kashiwara, P. Schapira, 2016)

Let M be a real analytic manifold and U a subanalytic open subset of M. Then any continuous globally subanalytic function $f : U \to \mathbb{R}$ on M is almost $\mathcal{C}_\infty$-subanalytic on M.

Definition

Let $M_\infty = (M, \widehat{M})$ be a real analytic bordered space and let U be an subanalytic open subset of M. Let $f : U \to \mathbb{R}$ be a continuous almost $\mathcal{C}_\infty$-subanalytic function on $\widehat{M}$. For any V and any $r \in \mathbb{Z}$, we set

$$e^{-f}\,\mathcal{D}b^{\mathrm{t},r}_{M_\infty}(V) = \{u \in \mathcal{D}b^{r}_{M}(U \cap V) : e^{g}u \in \mathcal{D}b^{\mathrm{t},r}_{\widehat{M}}(U \cap V)\},$$

where g is in the (ASA)-class of f. This definition does not depend on g and the correspondence $V \mapsto e^{-f}\,\mathcal{D}b^{\mathrm{t},r}_{M_\infty}(V)$ clearly defines a quasi-injective subanalytic sheaf on M_∞.

Proposition (D'Agnolo, 2014 and M. Kashiwara, P. Schapira, 2016)

Let $M_\infty = (M, \widehat{M})$ be a real analytic bordered space and U be an subanalytic open subset of M. Let $f : U \to \mathbb{R}$ be a continuous almost $\mathcal{C}_\infty$-subanalytic function on $\widehat{M}$. There is an isomorphism

$$e^{-f}\,\mathcal{D}b^{\mathrm{t},r}_{M_\infty} \simeq \mathrm{R}\mathscr{I}hom^{\mathsf{E}}(\mathbb{C}_{\{t\geq f(x), x\in U\}}, Q_{M_\infty}(\mathcal{D}b^{\mathsf{T},r}_{M_\infty}))$$

for each $r \in \mathbb{Z}$, which is given on sections by $u \mapsto e^t u$. In particular, the right hand side is concentrated in degree 0.

Corollary

Let $M_\infty = (M, \widehat{M})$ *be a real analytic bordered space and let also* $f : M \to \mathbb{R}$ *be a continuous almost* $\mathcal{C}_\infty$*-subanalytic function on* $\widehat{M}$*. Let* S *be a subanalytic closed subset of* M*. There is an isomorphism*

$$\Gamma_S(e^{-f}\,\mathcal{D}b^{\mathrm{t},r}_{M_\infty}) \simeq \mathrm{R}\mathscr{I}hom^{\mathsf{E}}(\mathbb{C}_{\{t \geq f(x), x \in S\}}, Q_{M_\infty}(\mathcal{D}b^{\mathsf{T},r}_{M_\infty}))$$

for each $r \in \mathbb{Z}$*, which is given on sections by* $u \mapsto e^t u$*. In particular, the right hand side is concentrated in degree* 0*.*

Let $f : \mathbb{V} \to \mathbb{R}$ be a continuous almost $\mathcal{C}_\infty$-subanalytic function on $\overline{\mathbb{V}}$ and S be a non-empty subanalytic closed subset of $\mathbb{V}$. Let us denote by f_S the function equal to f on S and to $+\infty$ on $\mathbb{V} \setminus S$.

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(i) $f_S \in \mathrm{Conv}(\mathbb{V})$,

(ii) $H(f_S^*)^\perp = \{0\}$,

(iii) the convex set $\mathrm{dom}^\circ(f_S^*)$ is subanalytic,

(iv) the function $f_S^* : \mathrm{dom}^\circ(f_S^*) \to \mathbb{R}$ is continuous and almost $\mathcal{C}_\infty$-subanalytic on $\overline{\mathbb{V}}^*$.

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Then

$$\begin{aligned} H^n_S(\mathbb{V}, e^{-f}\Omega^{\mathrm{t}}_{\overline{\mathbb{V}}}) &\xrightarrow{\sim} H^0(\mathbb{V}^*, e^{f_S^*}\,\mathcal{O}^{\mathrm{t}}_{\overline{\mathbb{V}}^*}) \\ &\simeq e^{f_S^*}\,\mathcal{D}b^{\mathrm{t}}_{\overline{\mathbb{V}}^*}(\mathrm{dom}^\circ(f_S^*)) \cap \mathcal{O}_{\overline{\mathbb{V}}^*}(\mathrm{dom}^\circ(f_S^*)). \end{aligned}$$

Theorem (C.D., 2018)

This last isomorphism can be explicitly computed by

$$\frac{\Gamma_S(\mathbb{V}, e^{-f}\,\mathcal{D}b^{\mathrm{t},n,n}_{\overline{\mathbb{V}}})}{\bar{\partial}\Gamma_S(\mathbb{V}, e^{-f}\,\mathcal{D}b^{\mathrm{t},n,n-1}_{\overline{\mathbb{V}}})} \ni [u] \mapsto \mathcal{L}^+\,u \in H^0(\mathbb{V}^*, e^{f_S^*}\,\mathcal{O}^{\mathrm{t}}_{\overline{\mathbb{V}}^*}),$$

where $u \mapsto \mathcal{L}^+\,u := \int_q e^{\langle z,w\rangle} p^* u.$

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where $u \mapsto \mathcal{L}^+ u := \int_q e^{\langle z,w\rangle} p^* u.$

Proof.

$$e^{-t} \int_{q_{\mathbb{R}}} e^{\langle z,w\rangle} p^*_{\mathbb{R}}(e^t u) = \mathcal{L}^+ u.$$

□

Link with Polya's theorem

Let $\mathbb{V} = \mathbb{C}$. We identify $\mathbb{V}^*$ with $\mathbb{C}$ in such a way that $\langle z, w \rangle = zw$. Let us fix a non-empty convex compact subset K of $\mathbb{C}$ and let us consider the null function $f = 0$ on $\mathbb{C}$. For all $\varepsilon > 0$, we thus get a function f_{K_ε} defined by

$$f_{K_\varepsilon}(z) = \begin{cases} 0 & \text{if } z \in K_\varepsilon, \\ +\infty & \text{else.} \end{cases}$$

Clearly, this function is convex of domain K_ε.

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Clearly, this function is convex of domain K_ε. Its Legendre transform is given by

$$f^*_{K_\varepsilon}(w) = \sup_{z \in K_\varepsilon} \Re(zw) = h_{K_\varepsilon}(w) = h_K(w) + h_{\overline{D}(0,\varepsilon)}(w) = h_K(w) + \varepsilon|w|$$

for all $w \in \mathbb{C}$. In particular $\mathrm{dom}^\circ(f^*_{K_\varepsilon}) = \mathbb{C}$.

Under some subanalytic conditions, one obtains an isomorphism

$$\mathcal{L}^+ : \frac{\Gamma_{K_\varepsilon}(\mathbb{C}, \mathcal{D}b_{\mathbb{P}}^{\mathsf{t},1,1})}{\bar{\partial}\Gamma_{K_\varepsilon}(\mathbb{C}, \mathcal{D}b_{\mathbb{P}}^{\mathsf{t},1,0})} \simeq H^1_{K_\varepsilon}(\mathbb{C}, \Omega^{\mathsf{t}}_{\mathbb{P}}) \xrightarrow{\sim} e^{h_{K_\varepsilon}} \mathcal{O}^{\mathsf{t}}_{\mathbb{P}}(\mathbb{C}).$$

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Moreover, there is a distinguished triangle :

$$\mathrm{R}\Gamma_{K_\varepsilon}(\mathbb{C}, \Omega^{\mathrm{t}}_{\mathbb{P}}) \to \mathrm{R}\Gamma(\mathbb{C}, \Omega^{\mathrm{t}}_{\mathbb{P}}) \to \mathrm{R}\Gamma(\mathbb{C} \setminus K_\varepsilon, \Omega^{\mathrm{t}}_{\mathbb{P}}) \xrightarrow{+1},$$

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which leads to the exact sequence

$$0 \to \Omega^{\mathsf{t}}_{\mathbb{P}}(\mathbb{C}) \to \Omega^{\mathsf{t}}_{\mathbb{P}}(\mathbb{C} \setminus K_\varepsilon) \to H^1_{K_\varepsilon}(\mathbb{C}, \Omega^{\mathsf{t}}_{\mathbb{P}}) \to 0.$$

Proposition

Let $\varepsilon > 0$. *One has a isomorphism given by*

$$\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{C} \setminus K_\varepsilon)/\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{C}) \ni [u] \mapsto [\bar{\partial}\underline{u}] \in \frac{\Gamma_{K_\varepsilon}(\mathbb{C}, \mathcal{D}b^{\mathrm{t},1,1}_{\mathbb{P}})}{\bar{\partial}\Gamma_{K_\varepsilon}(\mathbb{C}, \mathcal{D}b^{\mathrm{t},1,0}_{\mathbb{P}})},$$

where $\underline{u}$ *is a distributional extension of* u *to* $\mathbb{C}$.

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where $\underline{u}$ is a distributional extension of u to $\mathbb{C}$.

Corollary

One has a canonical isomorphism

$$\Omega^{\mathrm{t}\infty}_{\mathbb{P}}(\mathbb{C}\setminus K)/\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{C}) \xrightarrow{\sim} \varprojlim_{\varepsilon\to 0} H^1_{K_\varepsilon}(\mathbb{C}, \Omega^{\mathrm{t}}_{\mathbb{P}}).$$

Let $\varepsilon > 0$ and let ψ_ε be a $\mathcal{C}_\infty$-cutoff function which is equal to 1 on $\mathbb{C}\setminus K_\varepsilon$ and to 0 on $K_{\varepsilon/2}$. Then the image of $[u]$ through the canonical map $\Omega^{\mathrm{t}\infty}_{\mathbb{P}}(\mathbb{C}\setminus K)/\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{C}) \to H^1_{K_\varepsilon}(\mathbb{C}, \Omega^{\mathrm{t}}_{\mathbb{P}})$ is given by $[\bar\partial(\psi_\varepsilon u)]$.

Theorem

There is a canonical isomorphism of $\mathbb{C}$-vector spaces

$$\Omega_{\mathbb{P}}^{\mathrm{t}\infty}(\mathbb{C}\setminus K)/\Omega_{\mathbb{P}}^{\mathrm{t}}(\mathbb{C}) \xrightarrow{\sim} \varprojlim_{\varepsilon\to 0} e^{h_{K_\varepsilon}}\,\mathcal{O}_{\mathbb{P}}^{\mathrm{t}}(\mathbb{C}).$$

Given by $[f(z)dz] \mapsto g$ *with*

$$g(w) = \int_{C(0,r)^+} e^{zw} f(z)dz,$$

where $C(0,r)^+$ *is a positively oriented circle, which encloses* K.

<u>**Sketch of the proof**</u> : Let $f(z)dz$ be in $\Omega_{\mathbb{P}}^{\mathrm{t}\infty}(\mathbb{C}\setminus K)$ and let us fix $r > 0$ such that $K \subsetneq \overline{D}(0,r)$. Let us consider $\varepsilon > 0$ small enough such that $K \subsetneq K_\varepsilon \subsetneq \overline{D}(0,r)$.

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This last theorem is nothing more but Polya's theorem. First, the canonical map

$$\mathcal{O}^0(\mathbb{C} \setminus K) \ni f \mapsto [fdz] \in \Omega_{\mathbb{P}}^{\mathsf{t}_\infty}(\mathbb{C} \setminus K)/\Omega_{\mathbb{P}}^{\mathsf{t}}(\mathbb{C})$$

is clearly bijective.

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is clearly bijective. Secondly, the inclusion

$$\mathrm{Exp}(K) \subset \{g \in \mathcal{O}(\mathbb{C}) : \forall \varepsilon > 0, g \in e^{h_{K_\varepsilon}}\, \mathcal{D}b_{\mathbb{P}}^{\mathrm{t}}(\mathbb{C})\}$$

is an equality. Indeed, if $e^{-h_{K_\varepsilon}}g$ is tempered at infinity, then $e^{-h_{K_{2\varepsilon}}}g$ is bounded.

Link with Méril's theorem

Let S be a proper non-compact closed convex subset of $\mathbb{C}$ which contains no lines and $\xi_0 \in (S^*_\infty)^\circ$. For all $\varepsilon' > 0$, let $f_{\varepsilon'} : \mathbb{C} \to \mathbb{R}$ be defined by $f_{\varepsilon'}(z) = \Re(\varepsilon' \xi_0 z)$. It is globally subanalytic on $\mathbb{P}$.

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$$f_{\varepsilon,\varepsilon'}(z) = \begin{cases} \Re(\varepsilon' \xi_0 z) & \text{if } z \in S_\varepsilon, \\ +\infty & \text{else.} \end{cases}$$

Clearly, this function is convex of domain S_ε.

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Clearly, this function is convex of domain S_ε. Its Legendre transform is given by

$$f^*_{\varepsilon,\varepsilon'}(w) = \sup_{z \in S_\varepsilon} \Re(z(w - \varepsilon'\xi_0)) = h_{S_\varepsilon}(w - \varepsilon'\xi_0),$$

for all $w \in \mathbb{C}$. One has $\mathrm{dom}^\circ(f^*_{\varepsilon,\varepsilon'}) = (S^*_\infty)^\circ + \varepsilon'\xi_0$.

Under some subanalytic conditions, one obtains an isomorphism

$$\mathcal{L}^{+} : \frac{\Gamma_{S_\varepsilon}(\mathbb{C}, e^{-\varepsilon' \xi_0 z}\, \mathcal{D}b_{\mathbb{P}}^{\mathsf{t},1,1})}{\bar{\partial}\Gamma_{S_\varepsilon}(\mathbb{C}, e^{-\varepsilon' \xi_0 z}\, \mathcal{D}b_{\mathbb{P}}^{\mathsf{t},1,0})} \simeq H^1_{S_\varepsilon}(\mathbb{C}, e^{-\varepsilon' \xi_0 z}\Omega^{\mathsf{t}}_{\mathbb{P}})$$

$$\xrightarrow{\sim} e^{h_{S_\varepsilon}(w - \varepsilon' \xi_0)}\, \mathcal{O}^{\mathsf{t}}_{\mathbb{P}^*}((S^*_\infty)^\circ + \varepsilon' \xi_0).$$

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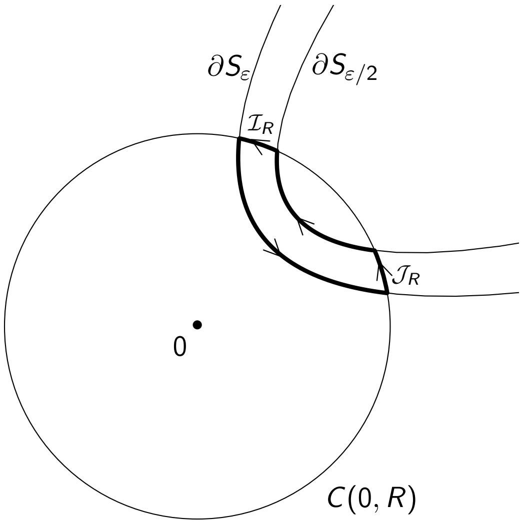
$$\xrightarrow{\sim} e^{h_{S_\varepsilon}(w-\varepsilon'\xi_0)}\, \mathcal{O}^t_{\mathbb{P}^*}((S^*_\infty)^\circ + \varepsilon'\xi_0).$$

The strategy is then globally the same that in the compact case. However, concerning the application of Green's theorem, one has to consider the following picture :



∂S_ε
$\partial S_{\varepsilon/2}$
$\mathcal{I}_R$
$\mathcal{J}_R$
0
$C(0, R)$

By taking the projective limit on $\varepsilon \to 0^+$ one gets the isomorphism

$$\mathcal{P} : \mathscr{H}_S^{\mathsf{t}}(\mathbb{C}, \varepsilon') \xrightarrow{\sim} \mathsf{Exp}_{\varepsilon'}^{\mathsf{t}}(S). \tag{3}$$

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By taking the projective limit on $\varepsilon' \to 0^+$ one gets Méril's theorem

$$\mathcal{P} : \mathscr{H}_S(\mathbb{C}) \xrightarrow{\sim} \mathrm{Exp}(S). \tag{4}$$

By taking the projective limit on $\varepsilon \to 0^+$ one gets the isomorphism

$$\mathcal{P} : \mathscr{H}_S^{\mathsf{t}}(\mathbb{C}, \varepsilon') \xrightarrow{\sim} \mathsf{Exp}_{\varepsilon'}^{\mathsf{t}}(S). \quad (3)$$

By taking the projective limit on $\varepsilon' \to 0^+$ one gets Méril's theorem

$$\mathcal{P} : \mathscr{H}_S(\mathbb{C}) \xrightarrow{\sim} \mathsf{Exp}(S). \quad (4)$$

The isomorphism (3) involves spaces which are bigger than in isomorphism (4). However the inverse map is not explicit.