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Holomorphic Cohomological Convolution, Enhanced Laplace Transform and Applications

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“*Ce n’est pas quand il a découvert l’Amérique, mais quand il a été sur le point de la découvrir, que Colomb a été heureux.*”

Fiodor Dostoïevski

Abstract

The Hadamard product of power series has been studied for more than one hundred years and has become a classical tool in complex analysis. Nonetheless, this product only concerns functions which are holomorphic near the origin. In 2009, T. Pohlen studied an extension of this Hadamard product on functions defined on open subsets of the Riemann sphere, which do not necessarily contain the origin. Using ad-hoc and explicit constructions, he could define this product thanks to a contour integration formula. However, his construction is non-symmetric with respect to 0 and ∞.

The first part of this thesis consists in the study of a generalization of Pohlen's extended Hadamard product. Using singular homology theory, we introduce more symmetric cycles and define a generalized Hadamard product which is equivalent to Pohlen's product when the functions vanish at infinity. Then, we show that this generalized Hadamard product is a particular case of a more general phenomenon called "holomorphic cohomological convolution". We study this convolution in detail on the multiplicative complex Lie group $\mathbb{C}^*$ and provide a contour integration formula to compute it.

The second part of the thesis is devoted to the study of holomorphic Paley-Wiener type theorems due to Polya (in the compact case) and to Méril (in the non-compact case). These theorems use a contour integration version of the Laplace transform. Thanks to the theory of enhanced subanalytic sheaves developed by A. D'Agnolo and M. Kashiwara as well as the enhanced Laplace transform introduced by M. Kashiwara and P. Schapira, we show that such theorems can be understood from a cohomological point of view. Under some convex subanalytic conditions, we are even able to provide stronger Laplace isomorphisms between spaces which are described by tempered growth conditions.

It appears that these spaces can be linked to certain spaces of analytic functionals. In the non-compact case, we define a convolution product between analytic functionals and conjecture that it is compatible with the additive version of the previously studied holomorphic cohomological convolution. Thanks to our results on the enhanced Laplace transform, we prove the conjecture in the subanalytic case.

Résumé

Le produit d'Hadamard entre séries de puissances entières a été étudié depuis plus de cent ans et est devenu un outil classique de l'analyse complexe. Néanmoins, ce produit concerne uniquement les fonctions holomorphes au voisinage de l'origine. En 2009, T. Pohlen a étudié une extension de ce produit d'Hadamard pour des fonctions définies sur des ouverts de la sphère de Riemann, qui ne contiennent pas nécessairement l'origine. En utilisant des constructions ad-hoc et explicites, il a pu définir ce produit via une intégrale de contour. Cependant, cette construction n'est pas symétrique par rapport à 0 et ∞.

La première partie de cette thèse consiste en l'étude d'une généralisation du produit d'Hadamard étendu par Pohlen. Au moyen de la théorie de l'homologie singulière, nous introduisons des cycles plus symétriques et définissons un produit d'Hadamard généralisé, équivalent à celui de Pohlen quand les fonctions s'annulent à l'infini. Nous montrons ensuite que ce produit d'Hadamard généralisé est un cas particulier d'un phénomène plus général appelé "convolution cohomologique holomorphe". Nous étudions en détail cette convolution dans le cas du groupe de Lie complexe multiplicatif $\mathbb{C}^*$ et fournissons une formule à base d'intégrales de contour pour la calculer.

La deuxième partie de la thèse est consacrée à l'étude de théorèmes de type Paley-Wiener holomorphes dus à Polya (dans le cas compact) et à Méril (dans le cas non compact). Ces théorèmes utilisent une version de la transformation de Laplace à base d'intégrales de contour. Grâce à la théorie des faisceaux sous-analytiques enrichis développée par A. D'Agnolo et M. Kashiwara, ainsi qu'à la transformation de Laplace enrichie introduite par M. Kashiwara et P. Schapira, nous montrons que ces théorèmes peuvent être compris d'un point de vue cohomologique. Sous certaines hypothèses de convexité et de sous-analyticité, il est même possible de prouver de plus forts isomorphismes de Laplace entre des espaces décrits par des conditions de croissance tempérée.

Ces espaces peuvent être liés à certains espaces de fonctionnelles analytiques. Dans le cas non compact, nous définissons un produit de convolution entre fonctionnelles analytiques et conjecturons que ce produit est compatible avec la version additive de la convolution cohomologique holomorphe précédemment étudiée. Grâce à nos résultats sur la transformation de Laplace enrichie, nous prouvons cette conjecture dans le cas sous-analytique.

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The second half of my thesis was done in the University of Padova, under the supervision of Andrea D'Agnolo. I would like to thank him for his hospitality and his infinite patience. Indeed, month after month, he answered to dozens of my questions concerning his famous theory of "enhanced subanalytic sheaves". This tool, which was incredibly complex to me at the beginning, has now become a familiar Swiss knife, helping me to understand a lot of analytical problems. Furthermore, Andrea read my article on the enhanced Laplace transform multiple times and helped me to correct many mistakes. Besides, Luca Prelli allowed me to live my research trip in Padova in the best conditions by providing me a residence and all the informations I needed. He also kindly invited me to present my work in a conference that he organised. I am very pleased to have Andrea and Luca in my jury and I want to thank them again.

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Finally, I would like to dedicate this thesis to my grandmother, who left us when I was in Italy. You have always been a model of intelligence and perseverance to me and I hope you are proud of this little brick that I shaped. Thank you.

Contents

Introduction

The easiest possible way one can imagine to define the product of two complex power series $A(z) = \sum_{n=0}^{+\infty} a_n z^n$ and $B(z) = \sum_{n=0}^{+\infty} b_n z^n$ is by setting

$$(A \star B)(z) = \sum_{n=0}^{+\infty} a_n b_n z^n.$$

This operation is called the Hadamard product of A and B (see [42]). Using Taylor representations, it is possible to extend this operation to holomorphic functions defined in a neighbourhood of the origin. One then has

$$(f \star g)(z) = \frac{1}{2i\pi} \int_{C(0,r)^+} f(\zeta) g\left(\frac{z}{\zeta}\right) \frac{d\zeta}{\zeta},$$

where $C(0,r)^+$ is a certain positively oriented circle around 0. Highly studied during the twentieth century, this formula led to interesting developments (see e.g. [29], [85], [94], [95] and [93]). In 2009, in order to study several problems of universality, T. Pohlen extended this notion to holomorphic functions defined on open subsets of the Riemann sphere, which do not necessarily contain the origin (see [86], [87] and [88]). In this new context, the circle which appears in the above formula is replaced by a Hadamard cycle, i.e. a curve which verifies specific winding number conditions related to the holomorphic domain of f and g and which are non-symmetric with respect to 0 and ∞. Moreover, T. Pohlen assumes that f and g vanish at infinity. Using singular homology theory and orientation classes, we propose a notion of *generalised Hadamard cycles* which itself allows to define a *generalised Hadamard product* between functions which do not necessarily vanish at infinity. Using the functoriality of the construction, we easily prove the classical properties which were already observed by T. Pohlen. Moreover, our construction is more symmetric and equivalent to his extended Hadamard product if one adds the vanishing condition at infinity. However, without this assumption our product is not commutative. As we have already suspected in our master thesis (see [26]), the good objects to consider are not holomorphic functions, but equivalence classes of holomorphic functions in a suitable quotient. Since the Hadamard product is nothing more but a contour-integration-multiplicative-convolution-formula, it seems natural to relate it to the usual convolution product of functions/distributions.

For that purpose and aware of the importance of functoriality, we introduce the general notion of holomorphic cohomological convolution on any complex Lie group

(G, μ). Like any convolution, it is defined as the combination of an exterior tensor product and a push-forward (integration over the fibers of μ). We then study in detail the case of the multiplicative group $\mathbb{C}^*$. Let S_1 and S_2 be two proper convolvable closed subsets of $\mathbb{C}^*$ such that $S_1 S_2 \neq \mathbb{C}^*$. In this setting, the holomorphic cohomological convolution gives a morphism

$$\star : H^1_{S_1}(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \otimes H^1_{S_2}(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \to H^1_{S_1 S_2}(\mathbb{C}^*, \Omega_{\mathbb{C}^*}),$$

which can be seen as a bilinear map

$$\star : \Omega(\mathbb{C}^* \setminus S_1)/\Omega(\mathbb{C}^*) \times \Omega(\mathbb{C}^* \setminus S_2)/\Omega(\mathbb{C}^*) \to \Omega(\mathbb{C}^* \setminus S_1 S_2)/\Omega(\mathbb{C}^*).$$

In the first part of this thesis, the main objective is to present a complete method to compute this morphism by the mean of contour integration formulas. These results are summarized in Theorem 2.2.12. Furthermore, if one adds an extra-condition on S_1 and S_2 (called *strong convolvability*), the statement can be simplified and we shall prove that this bilinear map is given by our generalized Hadamard product (see Proposition 2.2.17). In particular, this shows how the tools developed by T. Pohlen naturally appear thanks to a suitable cohomological framework.

The additive group $\mathbb{C}$ can be treated in a similar way and it is therefore a natural question to ask whether this notion could be linked with a contour integration Laplace transform. In [25], [69] and [70], the authors point out that convolution operators can be related to certain spaces of analytic functionals, which are themselves isomorphic to other interesting spaces, described by subexponential growth conditions. For example, if K is a proper convex compact subset of $\mathbb{C}$, the Polya-Ehrenpreis-Martineau theorem, or simply Polya's theorem (see [75] and [89]) states that

$$\begin{tikzcd} \mathcal{O}^0(\mathbb{C} \setminus K) \arrow[rr, "\mathcal{P}"] & & \mathrm{Exp}(K) \\ & \mathcal{O}'(K) \arrow[ul, "\mathcal{C}"] \arrow[ur, "\mathcal{F}"'] & \end{tikzcd}$$

is a commutative diagram of topological isomorphisms, where $\mathrm{Exp}(K)$ is the space of entire functions of exponential h_K-type (h_K being the support function of the convex K), $\mathcal{O}^0(\mathbb{C} \setminus K) \simeq \mathcal{O}(\mathbb{C} \setminus K)/\mathcal{O}(\mathbb{C})$ is the space of holomorphic functions defined on the complementary of K which vanish at infinity and $\mathcal{O}'(K)$ is the space of analytic functionals carried by K. The isomorphisms can be made explicit thanks to the Fourier-Borel transform $\mathcal{F}$, the Cauchy transform $\mathcal{C}$ and the Polya transform $\mathcal{P}$. This last application is of particular interest for us because by definition,

$$\mathcal{P}(f)(w) = \int_{C(0,r)^+} e^{zw} f(z) dz,$$

where $C(0, r)^+$ is a positively oriented circle which encloses K. By elementary computations, we can show that the convolution of analytic functionals (defined as for distributions) is compatible through the Cauchy transform $\mathcal{C}$ with the additive holomorphic cohomological convolution, which itself verifies the formula

$$\mathcal{P}(f \star g) = \mathcal{P}(f)\mathcal{P}(g). \tag{1}$$

Hence, we get a contour integration version of a classical real analysis theorem.

However, difficulties dramatically increase if one wants to deal with non-compact closed subsets of $\mathbb{C}$. The adaptation of Polya's theorem to the non-compact setting was first done by M. Morimoto in the particular case of half-strips (see [80], [81] and [82]). In this version, the Polya transform $\mathcal{P}$ is computed over the infinite boundary of a thickening of the half-strip and the integrability is assured by specific subexponential growth conditions. This result had plenty of consequences (see e.g. [83], [84], [113], [114], [115], [116] and [117]) in classical complex analysis. In 1978, J.W. De Roever solved the general case by using much more technical tools (see [99]). According to him, the functional spaces which appear to be isomorphic with the space of analytic functionals carried by a non-compact proper convex closed subset S of $\mathbb{C}$ are useful in quantum field mechanics. However, his method does not use a contour integration over the boundary of S_ε for some $\varepsilon > 0$. This issue was definitively solved by A. Méril in 1983, by adapting the proof of M. Morimoto for general convex subsets (see [77]). One should nonetheless note that these ideas were not new and were already present in [71], where A. J. Macintyre studied the holomorphic Laplace transform on convex cones.

The convolution of non-compactly carried analytic functionals was only studied in a particular case (see [78]). This is the reason why we take time to develop a general definition of such a convolution, by mimicking the distributions' one. By doing so, we remark that one has to impose a specific geometric condition on the non-compact closed subsets, that we call *compatibility*. Using some properties relative to convex geometry and asymptotic cones, we can actually prove that the compatibility and the convolvability conditions are the same. This allows to formulate Conjecture 3.2.30, which asserts that the convolution of analytic functionals is compatible through the Cauchy transform with the additive holomorphic cohomological convolution morphism. Unfortunately, we were not able to obtain a proof in the general case.

While trying to solve this conjecture, we felt that we needed a deeper understanding of Méril's theorem and that the cohomological tool could again be the key point. Furthermore, it seemed that lots of Paley-Wiener-type theorems (see e.g. [31], [63], [79], [110] and [111]) were similar to Polya's and Méril's theorems and we were willing to believe that all these results could be derived from a unique cohomological phenomenon.

The Laplace transform had already been studied from a sheaf-theoretic point of view in [57]. However, the results were only valid for conic sheaves. This work was extended to the non-conic setting by A. D'Agnolo in [18]. In particular, he explained how this abstract transformation allows to get some links with classical real Paley-Wiener-type theorems (see e.g. [30]). More recently, in [60], M. Kashiwara and P. Schapira made a full rewriting of the theory of integral transforms with irregular kernel, using the notion of enhanced ind-sheaves introduced in [19]. In particular, they treated the case of the Laplace transform. More precisely, let $\mathbb{V}$ be a n-dimensional complex vector space and $\mathbb{V}^*$ its complex dual. Let us consider the bordered spaces $\mathbb{V}_\infty = (\mathbb{V}, \overline{\mathbb{V}})$ and $\mathbb{V}^*_\infty = (\mathbb{V}^*, \overline{\mathbb{V}}^*)$ where $\overline{\mathbb{V}}$ (resp. $\overline{\mathbb{V}}^*$) is the projective compactification of $\mathbb{V}$ (resp $\mathbb{V}^*$). In [60], the authors proved that there is a canonical abstract isomorphism

$$ {}^{\mathrm{E}}\mathcal{F}^a_{\mathbb{V}}(\Omega^{\mathrm{E}}_{\mathbb{V}_\infty})[n] \simeq \mathcal{O}^{\mathrm{E}}_{\mathbb{V}^*_\infty} \tag{2}$$

in $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathbb{V}^*_\infty})$, where ${}^{\mathrm{E}}\mathcal{F}^a_{\mathbb{V}}$ is the enhanced Fourier-Sato functor and $\Omega^{\mathrm{E}}_{\mathbb{V}_\infty}$ (resp. $\mathcal{O}^{\mathrm{E}}_{\mathbb{V}^*_\infty}$) is the complex of enhanced holomorphic top-forms on $\mathbb{V}_\infty$ (resp. enhanced holomorphic functions on $\mathbb{V}^*_\infty$). In the second part of this thesis, we remark that (2) can be derived from a very explicit morphism. Using the Dolbeault complex $\mathcal{D}b^{T,\bullet,\bullet}$ of enhanced distributions, we show that there is a canonical morphism

$$ q_{!!}(\mu_{-\langle z,w\rangle *} p^{-1}\,\mathcal{D}b^{\mathrm{T},n,\bullet+n}_{\mathbb{V}_\infty}) \to \mathcal{D}b^{\mathrm{T},0,\bullet}_{\mathbb{V}^*_\infty}, $$

where $p : \mathbb{V}_\infty \times \mathbb{V}^*_\infty \to \mathbb{V}_\infty$ and $q : \mathbb{V}_\infty \times \mathbb{V}^*_\infty \to \mathbb{V}^*_\infty$ are the two projections and $\mu_{-\langle z,w\rangle}$ is the translation by $-\langle z, w\rangle$. This morphism encodes the usual positive Laplace transform of distributions and is equivalent to (2) in $\mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{\mathbb{V}^*_\infty})$ (a more concrete category that can replace $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathbb{V}^*_\infty})$). In order to prove this result, we have to trace back all the steps in the construction of (2), which leads to several morphisms defined in [52], [56] and [58]. The sketch of this historical compilation is synthesized in Theorem 5.1.10.

This remark has an immediate application. Let $f : \mathbb{V} \to \mathbb{R}$ be a continuous function and S be a subanalytic closed subset of $\mathbb{V}$. Let us denote by f_S the function which is equal to f on S and to $+\infty$ on $\mathbb{V}\setminus S$ and assume that f_S is convex. Under suitable conditions, we shall show that there is a commutative diagram

$$\begin{tikzcd} H^n_S(\mathbb{V}, e^{-f}\Omega^{\mathrm{t}}_{\overline{\mathbb{V}}}) \arrow[r, "\sim"] & H^0(\mathbb{V}^*, e^{f^*_S}\,\mathcal{O}^{\mathrm{t}}_{\overline{\mathbb{V}}^*}) \arrow[d] \\ \Gamma_S(\mathbb{V}, e^{-f}\,\mathcal{D}b^{\mathrm{t},n,n}_{\overline{\mathbb{V}}}) \arrow[u] \arrow[r] & \Gamma(\mathbb{V}^*, e^{f^*_S}\,\mathcal{D}b^{\mathrm{t}}_{\overline{\mathbb{V}}^*}) \end{tikzcd}$$

where f^*_S is the Legendre transform of f_S and $\mathcal{D}b^{t,\bullet,\bullet}$ (resp. $\Omega^{\mathrm{t}}, \mathcal{O}^{\mathrm{t}}$) is the Dolbeault complex of tempered distributions (resp. complex of tempered holomorphic forms, functions). Here, the top isomorphism comes from [60] and the bottom one is given by the positive Laplace transform of distributions.

The second main objective of this thesis consists in explaining how this diagram allows to obtain holomorphic Paley-Wiener-type theorems. As examples, we show how the contour integration formulas and the bijectivity of $\mathcal{P}$ in Polya's and Méril's theorems can naturally be obtained through a projective limit of tempered Laplace isomorphisms. We even get a stronger result than Méril (see Theorem 5.2.20). Then, introducing the notion of tempered holomorphic cohomological convolution and applying our previous results, we solve Conjecture 3.2.30 in the subanalytic case. In particular, (1) is valid in the non-compact subanalytic setting. We hope that this will convince the reader that the cohomological framework is well-fit to study the holomorphic convolution and the holomorphic Laplace transform as well as the link between them.

Let us now briefly resume the content of each chapter.

In chapter 1, we recall the basic mathematical facts that are needed to understand the rest of the thesis. We particularly highlight the Mittag-Leffler theorem for projective systems, some remarks about singular homology and winding numbers, usual constructions on distributional forms, which are highly used in all the next chapters, and finally some basic facts of convex geometry, especially concerning asymptotic cones.

In chapter 2, we essentially present the results of [28]. We first start by recalling the usual definition of the Hadamard product and the extension of T. Pohlen. We then introduce our generalized Hadamard product and prove the link with Pohlen's product. Secondly, we give the general definition of the holomorphic cohomological convolution and we fully treat the case of $\mathbb{C}^*$ in order to obtain a computable formula. Finally, we explain how this formula can be simplified in the case of strong convolvability and how it is linked with our generalized Hadamard product. We also remark that all these considerations can be adapted in the additive setting.

In chapter 3, we introduce the concept of analytic functionals carried by a convex closed subset of $\mathbb{C}$ and state Polya's theorem (in the compact case) and Méril's theorem (in the non-compact case). We easily make the link with the additive holomorphic cohomological convolution in the compact case. In the non-compact case, we completely define the notion of convolution of compatible analytic functionals and then prove that compatibility and convolvability are the same notions. We finish the chapter by conjecturing that this convolution is compatible with the additive holomorphic cohomological convolution morphism.

In chapter 4, we set all the tools needed for chapter 5. In particular, we recall in detail the construction of the category of enhanced subanalytic sheaves on a bordered space as well as the sheaf-theoretic definition of tempered distributions and tempered holomorphic forms. We also introduce the key notion of enhanced distributions and prove important facts related to integration and pullback of such distributions.

In chapter 5, we notably present the results of [27]. We define the enhanced Laplace transform morphism, explain how it can be derived from the usual Laplace transform for distributions and remark that it is equivalent to the isomorphism obtained by M. Kashiwara and P. Schapira in [60]. Then, we apply this result to the

Legendre transform in order to get holomorphic Paley-Wiener-type theorems. Modulo some subanalytic hypothesis, we explain how to obtain back Polya's theorem as well as a stronger version of Méril's theorem. Finally, we introduce the tempered holomorphic cohomological convolution and put all the pieces together to solve the main conjecture in the subanalytic case.

We conclude our thesis by proposing some lines of thought for the future.

Chapter 1

Preliminaries

1.1 Categories and sheaves

For basic category theory, we refer to [14] and [72]. For abelian, triangulated and derived categories, we refer to [15], [55] and [59]. In this thesis, we follow all the conventions about Grothendieck universes of [59] and do not write them explicitly.

For sheaf theory, we refer to [16], [35], [49] and [55]. Let us recall some classical notations that we shall use throughout this text.

Let X be a topological space and $\mathscr{R}$ a sheaf of rings with finite global homological dimension. The category of sheaves of $\mathscr{R}$-modules will be noted $\mathrm{Mod}(\mathscr{R})$. The associated derived category (resp. bounded, bounded below and bounded above derived category) will be noted $\mathrm{D}(\mathscr{R})$ (resp. $\mathrm{D^b}(\mathscr{R}), \mathrm{D}^+(\mathscr{R})$ and $\mathrm{D}^-(\mathscr{R})$).

If U is an open subset of X, we denote by $\Gamma(U, -)$ the functor of sections on U. If F is a sheaf, we sometimes write $F(U)$ instead of $\Gamma(U, F)$. We also write for short $H^k(U, F)$ instead of $H^k\mathrm{R}\Gamma(U, F)$.

Recall that there are five traditional "Grothendieck operations" on sheaves. Two internals : $- \otimes_{\mathscr{R}} -, \mathcal{H}om_{\mathscr{R}}(-, -)$ and three externals : $f_*, f^{-1}, f_!$, if $f : X \to Y$ is a continuous map between topological locally compact spaces. One can as well consider their derived version : $- \overset{\mathrm{L}}{\otimes}_{\mathscr{R}} -, \mathrm{R}\mathcal{H}om_{\mathscr{R}}(-, -), \mathrm{R}f_*, f^{-1}$ and $\mathrm{R}f_!$. If $\mathscr{R} = A_X$ with A a commutative ring (most of the time $\mathbb{Z}$ or $\mathbb{C}$), the Poincaré-Verdier duality states that $\mathrm{R}f_!$ has a right adjoint that we shall denote by $f^!$.

Let Z be a locally closed subset of X and $F \in \mathrm{Mod}(\mathscr{R})$. Let us also write $j : Z \to X$ the inclusion map. We set

$$F_Z = j_! j^{-1} F \qquad \text{and} \qquad \Gamma_Z(F) = \mathcal{H}om_{\mathscr{R}}(\mathscr{R}_Z, F).$$

If U is an open subset of X, remark that $\Gamma_Z(U, F) := \Gamma(U, \Gamma_Z(F))$ is the submodule of sections of F on U which are supported by Z. We denote by $\Gamma_c(U, F)$ the submodule of sections on U which are compactly supported. Finally, one can define the sections of F on Z by setting

$$\Gamma(Z, F) = \Gamma(Z, j^{-1}F).$$

1.2 The Mittag-Leffler theorem

The aim of this section is to recall the Mittag-Leffler theorem for projective systems and present an important cohomological application. Our main references are [22], [38] and [55].

Definition 1.2.1. Let $G = \{G_n, \varphi_{n,p}\}$ be a projective system of abelian groups indexed by $\mathbb{N}$. We say that G verifies *the Mittag-Leffler condition* if, for any $n \in \mathbb{N}$, the decreasing sequence $\{\varphi_{n,p}(G_p)\}_{p\geq n}$ stabilizes at some point.

The category of projective systems of abelian groups indexed by $\mathbb{N}$ is an abelian category with the obvious definition of morphisms. Hence, one can talk about exact sequences of such projective systems. In general the projective limit functor $\varprojlim$ is left exact but not exact. However, thanks to the Mittag-Leffler condition, we get the following result :

Theorem 1.2.2. *Let $0 \to G \to G' \to G'' \to 0$ be an exact sequence of projective systems of abelian groups indexed by $\mathbb{N}$. Assume that G verifies the Mittag-Leffler condition, then the sequence*

$$0 \to \varprojlim G \to \varprojlim G' \to \varprojlim G'' \to 0$$

is exact.

Now, we consider complexes of such projective systems, that is to say, objects of the form $G^\bullet = \{G^k, d^k\}$ where $G^k = \{G^k_n, \varphi^k_{n,p}\}$ is a projective system of abelian groups for each $k \in \mathbb{Z}$ and where the morphisms d^k and $\varphi^k_{n,p}$ verify the obvious compatibility conditions. To $G^\bullet$, one can associate the complex

$$G^\bullet_\infty = \varprojlim G^\bullet = \{\varprojlim G^k, d^k\}.$$

Hence, for each $k \in \mathbb{Z}$ one gets a canonical morphism

$$\phi_k : H^k(G^\bullet_\infty) \to \varprojlim_n H^k(G^\bullet_n).$$

In order to get isomorphisms (i.e. switch the projective limit and the cohomologies), we need again the Mittag-Leffler condition.

Proposition 1.2.3. *Assume that G^k verifies the Mittag-Leffler condition for each $k \in \mathbb{Z}$, then ϕ_k is surjective for each $k \in \mathbb{Z}$.*

If moreover the projective system $H^{k-1}(G^\bullet)$ satisfies the Mittag-Leffler condition for a given $k \in \mathbb{Z}$, then ϕ_k is bijective.

From this proposition, we can derive an important corollary :

Corollary 1.2.4 ([55], Proposition 2.7.1). *Let X be a topological space and let F be an object of $\mathrm{D}^+(\mathbb{Z}_X)$. Let $\{U_n\}_{n\in\mathbb{N}}$ be an increasing sequence of open subsets of X and $\{Z_n\}_{n\in\mathbb{N}}$ a decreasing sequence of closed subsets of X. Set $U = \cup_n U_n$ and $Z = \cap_n Z_n$. Then, for any $k \in \mathbb{Z}$, the natural map*

$$\phi_k : H^k_Z(U, F) \to \varprojlim_n H^k_{Z_n}(U_n, F)$$

is surjective.

If moreover the projective system $\{H^{k-1}_{Z_n}(U_n, F)\}_n$ satisfies the Mittag-Leffler condition for a given $k \in \mathbb{Z}$, then ϕ_k is bijective.

1.3 Algebraic topology

Singular homology theorey will be highly used in chapter 2. For classical facts about this field, we refer to [36], [43] and [76]. Let us nonetheless recall some key points.

1.3.1 Borel-Moore homology and orientation

Definition 1.3.1. Let X be a topological locally compact space and let us write $a_X : X \to \{\mathrm{pt}\}$ the canonical map which sends every element of X to a unique point. We set

$$\omega_X = a_X^! \, \mathbb{Z}_{\{\mathrm{pt}\}}$$

and call it *the orientation complex* of X.

Proposition 1.3.2 ([55], Proposition 3.3.6). *If X is a topological manifold of pure dimension n, ω_X is concentrated in degree $-n$ and $H^{-n}(\omega_X)$ is a locally constant sheaf with fiber $\mathbb{Z}$.*

We denote by or_X the sheaf $H^{-n}(\omega_X)$. Recall that X is orientable if and only if or_X is constant. In that case, an orientation on X is a chosen isomorphism $\mathbb{Z}_X \xrightarrow{\sim} \mathrm{or}_X$.

Definition 1.3.3. Let X be a topological locally compact space. *The Borel-Moore homology* (resp. *Borel-Moore homology with compact support*) of degree k is defined by

$${}^{\mathrm{BM}}H_k(X) = H^{-k}(X, \omega_X) \quad \left(\text{resp. } {}^{\mathrm{BM}}H^c_k(X) = H^{-k}_c(X, \omega_X)\right).$$

If X is homologically locally connected (which is for example the case if X is a topological manifold), then $\mathrm{R}\Gamma_c(X, \omega_X)$ is canonically isomorphic to the complex of singular chains on X. Hence, ${}^{\mathrm{BM}}H^c_k(X)$ is isomorphic to the usual singular homology group of degree k, $H_k(X)$ (see [16]).

Definition 1.3.4. Let X be an oriented topological manifold of pure dimension n. *The orientation class* of X is the class

$$[X] \in {}^{\mathrm{BM}}H_n(X) \simeq H^{-n}(X, \mathbb{Z}_X[n]) \simeq H^0(X, \mathbb{Z}_X)$$

corresponding to the constant section 1 of $\mathbb{Z}_X$.

Now, let K be a compact subset of X and consider the two canonical excision distinguished triangles

$$\mathrm{R}\Gamma_{X\backslash K}(X, \omega_X) \to \mathrm{R}\Gamma(X, \omega_X) \to \mathrm{R}\Gamma(K, \omega_X) \xrightarrow{+}$$

and

$$\mathrm{R}\Gamma_c(X\backslash K, \omega_X) \to \mathrm{R}\Gamma_c(X, \omega_X) \to \mathrm{R}\Gamma(K, \omega_X) \xrightarrow{+} .$$

The second triangle implies that $H^{-n}(K, \omega_X)$ is canonically isomorphic to the relative singular homology group $H_n(X, X\backslash K)$. Hence, we get a sequence of morphisms

$${}^{\mathrm{BM}}H_n(X) \to H^{-n}(K, \omega_X) \xrightarrow{\sim} H_n(X, X\backslash K)$$

and $[X] \in {}^{\mathrm{BM}}H_n(X)$ induces a relative orientation class $[X]_K \in H_n(X, X\backslash K)$.

1.3.2 Index of a complex 1-cycle

In this section, we take $X = \mathbb{C}$. Let $z \in \mathbb{C}$. We have a relative exact sequence

$$H_2(\mathbb{C}) \to H_2(\mathbb{C}, \mathbb{C}\setminus\{z\}) \to H_1(\mathbb{C}\setminus\{z\}) \to H_1(\mathbb{C}).$$

Since $\mathbb{C}$ is contractible, $H_2(\mathbb{C}) \simeq H_1(\mathbb{C}) \simeq 0$ and one gets a canonical isomorphism

$$H_1(\mathbb{C}\setminus\{z\}) \xrightarrow{\sim} H_2(\mathbb{C}, \mathbb{C}\setminus\{z\}) \xrightarrow{\sim} \mathbb{Z}, \tag{1.1}$$

where the second arrow is given by the orientation of $\mathbb{C}$.

Definition 1.3.5. Let $z \in \mathbb{C}$ and c be a complex 1-cycle which avoids z, i.e. an element of $Z_1(\mathbb{C}\setminus\{z\})$. The *index* of c at z is the integer which is the image of $[c] \in H_1(\mathbb{C}\setminus\{z\})$ through (1.1). It is noted $\mathrm{Ind}(c, z)$.

Remark 1.3.6. There are other classical definitions of $\mathrm{Ind}(c, z)$ (see e.g. [98]). For example, if c is a cycle with C^1-regularity, one has

$$\mathrm{Ind}(c, z) = \frac{1}{2i\pi}\int_c \frac{d\zeta}{\zeta - z}.$$

Informally, one sees that $\mathrm{Ind}(c, z)$ counts the number of times that c travels counter-clockwise around the point z.

Proposition 1.3.7. *Let Ω be a proper open subset of $\mathbb{C}$ and let $F = \mathbb{C} \setminus \Omega$. There is a canonical isomorphism*

$$H_1(\Omega) \xrightarrow{\sim} H_c^0(F, \mathbb{Z}_F)$$

given by

$$[c] \mapsto (z \mapsto \mathrm{Ind}_z(c)) .$$

Proof. Let us consider the excision distinguished triangle

$$\mathrm{R}\Gamma_c(\Omega, \omega_{\mathbb{C}}) \to \mathrm{R}\Gamma_c(\mathbb{C}, \omega_{\mathbb{C}}) \to \mathrm{R}\Gamma_c(F, \omega_{\mathbb{C}}) \xrightarrow{+1} . \tag{1.2}$$

It induces a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H_2(\Omega) \arrow[r] & H_2(\mathbb{C}) \arrow[r] & H^{-2}\mathrm{R}\Gamma_c(F, \omega_{\mathbb{C}}) \arrow[dll, rounded corners] \\
& H_1(\Omega) \arrow[r] & H_1(\mathbb{C}) \arrow[r] & H^{-1}\mathrm{R}\Gamma_c(F, \omega_{\mathbb{C}}) \arrow[r] & \cdots
\end{tikzcd}$$

One has $H_2(\mathbb{C}) \simeq H_1(\mathbb{C}) \simeq \{0\}$. Moreover, if one denotes by $j : F \to \mathbb{C}$ the inclusion map, one has $j^{-1}\omega_{\mathbb{C}} \simeq \mathbb{Z}_F[2]$. Therefore one gets a canonical isomorphism

$$\delta : H_c^0(F, \mathbb{Z}_F) \xrightarrow{\sim} H_1(\Omega).$$

Let $z \in F$. Applying (1.2) with $\mathbb{C} \setminus \{z\}, \mathbb{C}$ and $\{z\}$, one gets an isomorphism

$$\delta_z : \mathbb{Z} \simeq H_c^0(\{z\}, \mathbb{Z}_{\{z\}}) \xrightarrow{\sim} H_1(\mathbb{C} \setminus \{z\}).$$

Clearly, $\delta_z^{-1}([c]) = \mathrm{Ind}_z(c)$. Moreover, by Proposition 1.3.6 in [55], there is a commutative diagram

$$\begin{tikzcd}
H_c^0(F, \mathbb{Z}_F) \arrow[r, "\delta"] \arrow[d, "i_z"'] & H_1(\Omega) \arrow[d, "j_z"] \\
H_c^0(\{z\}, \mathbb{Z}_{\{z\}}) \arrow[r, "\delta_z"'] & H_1(\mathbb{C} \setminus \{z\})
\end{tikzcd}$$

where $i_z(f) = f(z)$ and $j_z([c]) = [c]$. Hence, one sees that $\delta^{-1}([c])(z) = \mathrm{Ind}_z(c)$. Since this argument is valid for all $z \in F$, the conclusion follows. □

1.4 Operations on distributional forms

In this section, we recall some classical constructions on manifolds involving distributional forms (see e.g. [21], [24], [37] and [96]). For the sake of simplicity, we will always assume that the real manifolds are oriented. Hence we do not have to make a distinction between distributional forms and currents. This assumption is not restrictive since we will only work with complex manifolds in the main sections.

1.4.1 Bi-type decomposition

For all $r \in \mathbb{Z}$, we denote by $\mathcal{C}^r_{\infty,M}$ (resp. $\mathcal{D}b^r_M$) the sheaf of infinitely differentiable complex differential r-forms (resp. distributional r-forms) on a real manifold M.

Let X be a complex manifold of complex dimension d_X and $r \in \mathbb{Z}$. Recall that $\mathcal{C}^r_{\infty,X}$ admits a decomposition in bi-types

$$\mathcal{C}^r_{\infty,X} \simeq \bigoplus_{p+q=r} \mathcal{C}^{p,q}_{\infty,X}$$

which induces a decomposition of the exterior derivative d as

$$d = \partial + \overline{\partial},$$

where

$$\partial : \mathcal{C}^{p,q}_{\infty,X} \to \mathcal{C}^{p+1,q}_{\infty,X} \quad \text{and} \quad \overline{\partial} : \mathcal{C}^{p,q}_{\infty,X} \to \mathcal{C}^{p,q+1}_{\infty,X} .$$

Similarly, $\mathcal{D}b^r_X$ admits a decomposition in bi-types

$$\mathcal{D}b^r_X \simeq \bigoplus_{p+q=r} \mathcal{D}b^{p,q}_X$$

and an associated decomposition of the distributional exterior derivative. Moreover, for any open subset U of X, we have a canonical isomorphism

$$\mathcal{D}b^r_X(U) \simeq \Gamma_c(U, \mathcal{C}^{2d_X - r}_{\infty,X})'$$

between the space of complex distributional r-forms and the topological dual of the space of infinitely differentiable complex differential $(2d_X - r)$-forms with compact support, which induces the similar isomorphism

$$\mathcal{D}b^{p,q}_X(U) \simeq \Gamma_c(U, \mathcal{C}^{d_X - p, d_X - q}_{\infty,X})'.$$

In the sequel, we denote by Ω^p_X the sheaf of holomorphic differential p-forms on X. Of course, Ω^p_X is canonically isomorphic to both the kernel of

$$\overline{\partial} : \mathcal{C}^{p,0}_{\infty,X} \to \mathcal{C}^{p,1}_{\infty,X}$$

and the kernel of

$$\overline{\partial} : \mathcal{D}b^{p,0}_X \to \mathcal{D}b^{p,1}_X .$$

We set for short $\mathcal{O}_X = \Omega^0_X$ and $\Omega_X = \Omega^{d_X}_X$.

The double complex $\mathcal{C}^{\bullet,\bullet}_{\infty,X}$ (resp. $\mathcal{D}b^{\bullet,\bullet}_X$) is the infinitely differentiable (resp. distributional) Dolbeault complex of X. By construction, the associated simple complex is the infinitely differentiable (resp. distributional) de Rham complex $\mathcal{C}^\bullet_{\infty,X}$ (resp. $\mathcal{D}b^\bullet_X$) of X. Moreover, we have the following chains of canonical quasi-isomorphisms :

$$\mathbb{C}_X \simeq \mathcal{C}^\bullet_{\infty,X} \simeq \mathcal{D}b^\bullet_X \quad \text{and} \quad \Omega^p_X \simeq \mathcal{C}^{p,\bullet}_{\infty,X} \simeq \mathcal{D}b^{p,\bullet}_X,$$

which are given by the de Rham and Dolbeault lemmas.

1.4.2 Integration

Definition 1.4.1. Let M (resp. N) be a real manifold of real dimension d_M (resp. d_N) and let $f : M \to N$ be a $\mathcal{C}_\infty$-map. Let also V be an open subset of N and $u \in \Gamma(f^{-1}(V), \mathcal{D}b_M^p)$ be a distributional form with f-proper support. *The integral of u along the fibers of f* (or *the pushforward of u by f*), noted $\int_f u$, is an element of $\Gamma(V, \mathcal{D}b_N^{d_N-d_M+p})$ defined by

$$\left\langle \int_f u, \omega \right\rangle = \langle u, f^*\omega\rangle$$

for all $\omega \in \Gamma_c(V, \mathcal{C}_{\infty,N}^{d_M-p})$. Hence, we get a morphism of sheaves

$$\int_f : f_! \, \mathcal{D}b_M^{p+d_M} \to \mathcal{D}b_N^{p+d_N}$$

for each $p \in \mathbb{Z}$.

Now, let $f : X \to Y$ be a holomorphic map between complex manifolds of complex dimension d_X and d_Y. By the same definition, we get integration morphisms

$$\int_f : f_! \, \mathcal{D}b_X^{p+d_X,q+d_X} \to \mathcal{D}b_Y^{p+d_Y,q+d_Y}$$

for all $(p, q) \in \mathbb{Z}^2$. Since the pullback f^* of differentiable forms commutes with ∂ and $\overline{\partial}$, the integration morphisms also commute with ∂ and $\overline{\partial}$ and thus give rise to a morphism of double complex

$$\int_f : f_! \, \mathcal{D}b_X^{\bullet+d_X,\bullet+d_X} \to \mathcal{D}b_Y^{\bullet+d_Y,\bullet+d_Y} .$$

Hence, by the Dolbeault lemma, we get a morphism

$$\int_f : \mathrm{R}f_!\Omega_X^{p+d_X}[d_X] \to \Omega_Y^{p+d_Y}[d_Y]$$

for each $p \in \mathbb{Z}$. These morphisms are called *the holomorphic integration maps along the fibers of f*.

1.4.3 Pullback

As we explained previously, it is natural to define the pushforward of a distributional form by duality, using the pullback of differential forms. Conversely, it is not always possible to define by duality a pullback on distributional forms. It is however a classical result that it is possible if the application f is a submersion (see e.g. Theorem 11 in [96]).

Proposition 1.4.2. *Let M (resp. N) be a real manifold of real dimension d_M (resp. d_N) and let $f : M \to N$ be a $\mathcal{C}_\infty$-submersion. Let V be an open subset of N and let $u \in \Gamma(f^{-1}(V), \mathcal{D}b^p_M)$ be a distributional form with f-proper support associated to a p-form ω. Then $\int_f u$ is associated to a $d_N - d_M + p$ form $\int_f \omega$ which can be computed by integrating ω over the fibers of f.*

Example 1.4.3. Let $p_1 : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^k$ be the first projection and consider a top-form $\omega = \varphi(x,y)dx \wedge dy$ on $\mathbb{R}^k \times \mathbb{R}^l$ with p_1-proper support. Then

$$\int_{p_1} \omega = \left(\int_{\mathbb{R}^l} \varphi(x,y)dy \right) dx.$$

Definition 1.4.4. Let $f : M \to N$ be a $\mathcal{C}_\infty$-submersion between real manifolds and let U be an open subset of M. Let $v \in \Gamma(V, \mathcal{D}b^p_N)$ where V is an open subset of N such that $f(U) \subset V$. *The pullback of v by f* is an element $f^*v \in \Gamma(U, \mathcal{D}b^p_M)$ defined by

$$\langle f^*v, \omega \rangle = \left\langle v, \int_f \omega \right\rangle$$

for all $\omega \in \Gamma_c(U, \mathcal{C}^{d_M - p}_{\infty,M})$. Hence, we get a morphism of sheaves

$$f^* : f^{-1}\mathcal{D}b^p_N \to \mathcal{D}b^p_M$$

for each $p \in \mathbb{Z}$.

Now, let $f : X \to Y$ be a submersive holomorphic map between complex manifolds. By the same definition, we get morphisms

$$f^* : f^{-1}\mathcal{D}b^{p,q}_Y \to \mathcal{D}b^{p,q}_X$$

for all $(p,q) \in \mathbb{Z}^2$. Since they commute with ∂ and $\bar\partial$, they give rise to a morphism of double complex

$$f^* : f^{-1}\mathcal{D}b^{\bullet,\bullet}_Y \to \mathcal{D}b^{\bullet,\bullet}_X .$$

Hence, by the Dolbeault lemma we get a morphism

$$f^* : f^{-1}\Omega^p_Y \to \Omega^p_X \tag{1.3}$$

for each $p \in \mathbb{Z}$.

Remark 1.4.5. Note that the morphism (1.3) still exists when f is not a submersion. Indeed, the pullback of differential forms gives a morphism of double complexes

$$f^* : f^{-1}\mathcal{C}^{\bullet,\bullet}_{\infty,Y} \to \mathcal{C}^{\bullet,\bullet}_{\infty,X}$$

which induces the desired morphism in the derived category.

1.5 Convex geometry

Convex sets will play an essential role in chapters 3 and 5. The following review of basic convex geometry is made from [2] and [97].

1.5.1 Legendre transform and support functions

Let V be a finite-dimensional real vector space and V^* its real dual. Let us note

$$\langle -, - \rangle : V \times V^* \to \mathbb{R}$$

the real duality bracket.

Definition 1.5.1. Let $f : V \to \mathbb{R} \cup \{+\infty\}$ be a function.

(i) One says that f is a *closed proper convex function* on V if its epigraph

$$\{(x, t) \in V \times \mathbb{R} : t \geq f(x)\}$$

is closed, convex and non-empty.

(ii) One denotes by $\mathrm{Conv}(V)$ the set of closed proper convex functions on V.

(iii) For any $f \in \mathrm{Conv}(V)$, one sets $\mathrm{dom}(f) = f^{-1}(\mathbb{R})$ and call it *the domain of* f. This set is convex and non-empty.

(iv) For any $f \in \mathrm{Conv}(V)$, one defines a function $f^* : V^* \to \mathbb{R} \cup \{+\infty\}$ by setting

$$f^*(y) = \sup_{x \in \mathrm{dom}(f)} (\langle x, y \rangle - f(x)).$$

It is called *the Legendre transform of* f. It is an element of $\mathrm{Conv}(V^*)$.

Definition 1.5.2. Let S be a non-empty closed convex subset of V. *The support fonction of* S is the function $h_S : V^* \to \mathbb{R} \cup \{+\infty\}$ defined by

$$h_S(y) = \sup_{x \in S} \langle x, y \rangle.$$

Remark 1.5.3. By definition, h_S is the Legendre transform of the indicator function $f_S : V \to \mathbb{R} \cup \{+\infty\}$ which is equal to 0 on S and to $+\infty$ on $V \backslash S$. Hence h_S is a closed proper convex function. Moreover, one can easily check that h_S is positively homogeneous. That is to say

$$h_S(\lambda y) = \lambda h_S(y) \quad \forall \lambda \geq 0, y \in V^*.$$

If $f \in \mathrm{Conv}(V)$, one can prove that $f^{**} = f$. This allows to obtain the following characterisation of support functions :

Theorem 1.5.4 ([97], Theorem 13.2)**.** *Let $h \in \mathrm{Conv}(V^*)$ be a positively homogeneous function. Then, there is a non-empty closed convex subset S of V such that $h = h_S$. The convex S can be explicitly described by*

$$S = \bigcap_{y \in V^*} \{x \in V : \langle x, y\rangle \leq h(y)\}.$$

Definition 1.5.5. The sets $\{x \in V : \langle x, y\rangle \leq h_S(y)\}$ are called the *supporting half-spaces* of S and the subsets $\{x \in V : \langle x, y\rangle = h_S(y)\}$ are called the *supporting hyperplanes* of S.

Example 1.5.6. Let $||\cdot||$ be a norm on V and $||\cdot||^*$ the dual norm on V^*. Let $B(0,\varepsilon)$ be the open ball of center 0 and radius $\varepsilon > 0$ on V. Then

$$h_{B(0,\varepsilon)}(y) = \varepsilon||y||^*$$

for all $y \in V^*$.

Proposition 1.5.7. *If S_1 and S_2 are two non-empty closed convex subsets of V, then*

$$h_{S_1+S_2} = h_{S_1} + h_{S_2},$$

where

$$S_1 + S_2 = \{x_1 + x_2 : x_1 \in S_1, x_2 \in S_2\}$$

is the Minkowski sum of S_1 and S_2.

1.5.2 Asymptotic cones and duality

Definition 1.5.8. A subset C of V is a *cone* if $\lambda C \subset C$ for all $\lambda > 0$. It is a *convex cone* if $C + C \subset C$. It is a *proper cone* if $\{0\} \neq C \neq V$. It is a *salient cone* if $C \cap -C \subset \{0\}$ and it is a *pointed cone* if $C \cap -C \supset \{0\}$.

The polar cone of a cone $C \subset V$, noted C^*, is defined by

$$C^* = \{y \in V^* : \langle x, y\rangle \leq 0, \forall x \in C\}.$$

It is a cone of V^*.

The set of the asymptotic directions of a subset of V can be described by a cone.

Definition 1.5.9. Let S be a non-empty subset of V. *The asymptotic cone of* S, noted S_∞, is the set of vectors $d \in V$ such that there is a sequence $(t_k)_{k\in\mathbb{N}}$ of strictly positive real numbers and a sequence $(x_k)_{k\in\mathbb{N}}$ of S such that

$$\lim_{k\to+\infty} t_k = +\infty \qquad \text{and} \qquad \lim_{k\to+\infty} \frac{x_k}{t_k} = d.$$

Proposition 1.5.10. *The asymptotic cone verifies the following properties :*

1. *For all non-empty subset S of V, S_∞ is a closed pointed cone. If S is convex, S_∞ is a convex cone.*

2. *If C is a non-empty cone of V, then $C_\infty = \overline{C}$.*

3. *A non-empty subset S of V is bounded if and only if $S_\infty = \{0\}$.*

4. *If $(S_i)_{i\in I}$ is a family of non-empty subsets of V, then*

$$\left(\bigcap_{i\in I} S_i\right)_\infty \subset \bigcap_{i\in I}(S_i)_\infty.$$

If the S_i have a non-empty intersection, the inclusion becomes an equality.

5. *If S_1 and S_2 are two non-empty subsets of V, then*

$$S_1 \subset S_2 \text{ implies } (S_1)_\infty \subset (S_2)_\infty.$$

6. *If S is a non-empty subset of V and $x \in V$, then $(x+S)_\infty = S_\infty$.*

7. *If S is a non-empty closed convex subset of V, then*

$$S_\infty = \{x \in V : x + S \subset S\}.$$

We shall need the important following theorem :

Theorem 1.5.11 ([2], Theorem 2.3.4 and [100], Section 19, Theorem 3.1)**.** *If S_1 and S_2 are two non-empty closed subsets of V such that*

$$(S_1)_\infty \cap -(S_2)_\infty = \{0\},$$

then $S_1 + S_2$ is closed and

$$(S_1 + S_2)_\infty \subset (S_1)_\infty + (S_2)_\infty.$$

Moreover, the inclusion becomes an equality if S_1 and S_2 are convex.

Example 1.5.12. Let S be a non-empty closed subset of V and $\varepsilon > 0$. Consider the thickening $S_\varepsilon = S + \overline{B}(0,\varepsilon)$ of S for a certain norm on V. Since $S \subset S_\varepsilon$, one has $S_\infty \subset (S_\varepsilon)_\infty$. Moreover, by Theorem 1.5.11, one also has

$$(S_\varepsilon)_\infty \subset S_\infty + \overline{B}(0,\varepsilon)_\infty = S_\infty.$$

Hence $S_\infty = (S_\varepsilon)_\infty$.

If C is a closed convex cone of V, then $C^{**} = C$. This allows to get a refinement of Theorem 1.5.4.

Theorem 1.5.13 ([2], Theorem 2.2.1)**.** *Let S be a non-empty closed convex subset of V. Then* $\operatorname{dom}(h_S)$ *is a cone $C \subset V^*$ such that*

$$(S_\infty^*)^\circ \subset C \subset S_\infty^*.$$

Conversely, if $h \in \operatorname{Conv}(V^*)$ is a positively homogeneous function whose domain is the cone $C \subset V^*$, then $h = h_S$ for a non-empty closed convex subset S of V such that $S_\infty = C^* \subset V^{**} \simeq V$.

Definition 1.5.14. In the context of Theorem 1.5.13, we say that (h, C) and S are in *convex duality.*

Remark 1.5.15. If (h, C) and S are in convex duality, h is actually continuous on C°.

Remark 1.5.16. Let $\mathbb{V}$ be a complex vector space and $\mathbb{V}^*$ its complex dual. Denote by $\langle -, - \rangle : \mathbb{V} \times \mathbb{V}^* \to \mathbb{C}$ the complex duality bracket. Then, all the previous considerations can be transposed into this complex case by replacing everywhere V (resp. V^*) by $\mathbb{V}$ (resp. $\mathbb{V}^*$) and $\langle x, y \rangle$ with $x \in V, y \in V^*$ by $\Re\langle z, w \rangle$ with $z \in \mathbb{V}, w \in \mathbb{V}^*$.

Chapter 2

Holomorphic cohomological convolution

2.1 Motivation : The Hadamard product

2.1.1 Classical definition

Definition 2.1.1. Let $A(z) = \sum_{n=0}^{+\infty} a_n z^n$ and $B(z) = \sum_{n=0}^{+\infty} b_n z^n$ be two formal power series with complex coefficients. *The Hadamard product* of A and B is the formal power series $A \star B$ defined by

$$(A \star B)(z) = \sum_{n=0}^{+\infty} a_n b_n z^n.$$

Remark 2.1.2. If r_A (resp. r_B) is the radius of convergence of A (resp. B), it is clear (for instance by using the root test) that the radius of convergence $r_{A\star B}$ of $A\star B$ is greater or equal than $r_A \cdot r_B$.

This classical definition appeared for the first time in [42]. It has then been actively studied in [3], [44], [89] and [105]. These authors notably remarked the following fact. If f (resp. g) is a holomorphic function defined by A (resp. B) on the disk $D(0, r_A)$ (resp. $D(0, r_B)$) and if $r \in (0, r_A)$, then

$$\begin{aligned}
\sum_{n=0}^{+\infty} a_n b_n z^n &= \sum_{n=0}^{+\infty} \left(\frac{1}{2i\pi} \int_{C(0,r)^+} \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta \right) b_n z^n \\
&= \frac{1}{2i\pi} \int_{C(0,r)^+} f(\zeta) \left(\sum_{n=0}^{+\infty} b_n \left(\frac{z}{\zeta}\right)^n \right) \frac{d\zeta}{\zeta} \\
&= \frac{1}{2i\pi} \int_{C(0,r)^+} f(\zeta) g\left(\frac{z}{\zeta}\right) \frac{d\zeta}{\zeta},
\end{aligned}$$

for any $z \in D(0, r \cdot r_B)$.

Using this integral representation, it is easy to define the Hadamard product between holomorphic functions defined on open subsets of $\mathbb{C}$ containing the origin (see e.g. [85] for some applications).

2.1.2 Extension of T. Pohlen

In his thesis [88] (see also [87]), Timo Pohlen introduced the more general notion of Hadamard product for holomorphic functions defined on open subsets of the Riemann sphere $\mathbb{P} = \mathbb{C} \cup \{\infty\}$ which do not necessarily contain the origin. This new definition led to interesting applications, (e.g. [86] and [70]). In this section, we shall recall the construction and the results of T. Pohlen.

Definition 2.1.3. Let $\mathbb{P}$ be the Riemann sphere equipped with its canonical structure of complex manifold. Let Ω be an open subset of $\mathbb{P}$. One sets

$$\mathcal{H}(\Omega) = \{f \in \mathcal{O}(\Omega) : f(\infty) = 0\}$$

if $\infty \in \Omega$ and $\mathcal{H}(\Omega) = \mathcal{O}(\Omega)$ otherwise.

Definition 2.1.4. We set $M = (\mathbb{P} \times \mathbb{P}) \setminus \{(0, \infty), (\infty, 0)\}$ and extend the complex multiplication continuously as a map $\cdot : M \to \mathbb{P}$. We then have

$$\infty \cdot a = a \cdot \infty = \infty$$

if $a \in \mathbb{P}$ is not equal to zero. If A, B are subsets of $\mathbb{P}$ such that $A \times B \subset M$, one sets

$$A \cdot B = \{a \cdot b : a \in A, b \in B\}.$$

One also extends the inversion $z \mapsto z^{-1}$ continuously from $\mathbb{C}^*$ to $\mathbb{P}$ by setting $0^{-1} = \infty$ and $\infty^{-1} = 0$. If $S \subset \mathbb{P}$, one sets

$$S^{-1} = \{z : z^{-1} \in S\}.$$

For the rest of the thesis, we shall often drop the point and write the multiplication as a concatenation.

Definition 2.1.5. Two open subsets $\Omega_1, \Omega_2 \subset \mathbb{P}$ are called *star-eligible* if

1. Ω_1 and Ω_2 are proper subsets of $\mathbb{P}$,
2. $(\mathbb{P} \setminus \Omega_1) \times (\mathbb{P} \setminus \Omega_2) \subset M$,
3. $(\mathbb{P} \setminus \Omega_1)(\mathbb{P} \setminus \Omega_2) \neq \mathbb{P}$.

In this case, the *star product* of Ω_1 and Ω_2, noted $\Omega_1 \star \Omega_2$, is defined by

$$\Omega_1 \star \Omega_2 = \mathbb{P} \setminus ((\mathbb{P} \setminus \Omega_1)(\mathbb{P} \setminus \Omega_2)).$$

Recall Definition 1.3.5. For any cycle c in $\mathbb{C}$, one sets $\mathrm{Ind}(c, \infty) = 0$.

Definition 2.1.6. Let Ω be a non-empty open subset of $\mathbb{P}$, K be a non-empty compact subset of Ω and c be a cycle in $\Omega\backslash(K \cup \{0\} \cup \{\infty\})$. If $\infty \notin K$ and

$$\operatorname{Ind}(c, z) = \begin{cases} 1 & \text{if } \ z \in K \\ 0 & \text{if } \ z \in \mathbb{P}\backslash\Omega \end{cases},$$

then c is called a *Cauchy cycle* for K in Ω. If $\infty \in \Omega$ and

$$\operatorname{Ind}(c, z) = \begin{cases} 0 & \text{if } \ z \in K \\ -1 & \text{if } \ z \in \mathbb{P}\backslash\Omega \end{cases},$$

then c is called a *anti-Cauchy cycle* for K in Ω.

In [88], Lemma 2.3.1, T. Pohlen refers to ad hoc explicit constructions which ensure that Cauchy and anti-Cauchy cycles always exist for any Ω and any K. However, one can notice that Proposition 1.3.7 easily gives this existence.

Let Ω_1 and Ω_2 be two star-eligible open subsets of $\mathbb{P}$. Note that, if $z \in \Omega_1 \star \Omega_2$, then $z(\mathbb{P}\backslash\Omega_2)^{-1}$ is a closed subset of Ω_1.

Definition 2.1.7. Let $z \in (\Omega_1 \star \Omega_2)\backslash\{0, \infty\}$. A *Hadamard cycle* for $z(\mathbb{P}\backslash\Omega_2)^{-1}$ in Ω_1 is a cycle c in $\Omega_1\backslash(z(\mathbb{P}\backslash\Omega_2)^{-1} \cup \{0\} \cup \{\infty\})$ which satisfies the condition given in the table

Ω_2 \ Ω_1	$0, \infty$	∞	0	
$0, \infty$	cc^+ or acc^-	acc^-	cc^+	cc
∞	acc^-	acc^-	/	/
0	cc^+	/	cc^+	/
	acc	/	/	/

This table should be understood in the following way : The elements in the first row and the first column tell which of these elements are in Ω_1 and Ω_2 respectively. The abbreviation cc (resp. acc) means that c is a Cauchy (resp. anti-Cauchy) cycle for $z(\mathbb{P}\backslash\Omega_2)^{-1}$ in Ω_1. The abbreviation cc^+ (resp. acc^-) means that c is a Cauchy (resp. anti-Cauchy) cycle with the extra condition $\operatorname{Ind}(c, 0) = 1$ (resp. $\operatorname{Ind}(c, 0) = -1$). A "/" means that this case cannot occur.

One can now extend the standard Hadamard product.

Definition 2.1.8. Let $f_1 \in \mathcal{H}(\Omega_1)$ and $f_2 \in \mathcal{H}(\Omega_2)$. For each $z \in (\Omega_1 \star \Omega_2) \backslash \{0, \infty\}$ one sets

$$(f_1 \star f_2)(z) = \frac{1}{2i\pi} \int_{c_z} f_1(\zeta) f_2\left(\frac{z}{\zeta}\right) \frac{d\zeta}{\zeta},$$

where c_z is a Hadamard cycle for $z(\mathbb{P} \backslash \Omega_2)^{-1}$ in Ω_1. One can check that this integral does not depend on the chosen Hadamard cycle (see Lemma 3.4.2 in [88]). The function $f_1 \star f_2$ is called the *Hadamard product* of f_1 and f_2.

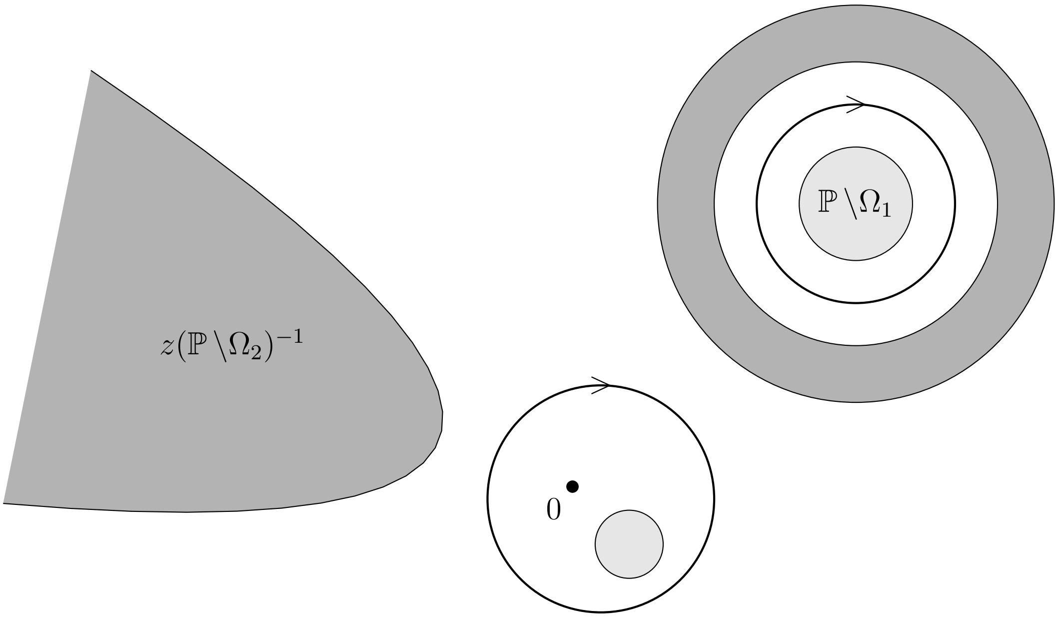

Figure 2.1: A Hadamard cycle for $z(\mathbb{P} \backslash \Omega_2)^{-1}$ in Ω_1, in the case where $0, \infty \in \Omega_1$ and $\infty \in \Omega_2, 0 \notin \Omega_2$.

Proposition 2.1.9 ([88], Lemma 3.4.5 and Proposition 3.6.4)**.** *The Hadamard product $f_1 \star f_2$ can be continuously extended to $\Omega_1 \star \Omega_2$. If $0 \in \Omega_1 \star \Omega_2$ (resp. $\infty \in \Omega_1 \star \Omega_2$), one has $(f_1 \star f_2)(0) = f_1(0) f_2(0)$ (resp. $(f_1 \star f_2)(\infty) = 0$). Moreover, $f_1 \star f_2$ is an element of $\mathcal{H}(\Omega_1 \star \Omega_2)$.*

Proposition 2.1.10 ([88], Proposition 3.6.1)**.** *The Hadamard product is commutative.*

In all this framework, the hypothesis $f(\infty) = 0$, when $\infty \in \Omega$, is highly used. In the next section, we shall provide a more general definition of Hadamard cycles and Hadamard product, based on singular homology theory, which does not require the vanishing condition at infinity.

2.1.3 Generalized Hadamard product

To introduce our definition of generalized Hadamard cycles, we have to be in the same setting as T. Pohlen. However, looking at Definition 2.1.5, we find it more natural to start with closed subsets instead of open ones.

Definition 2.1.11. Two closed subsets S_1 and S_2 of $\mathbb{P}$ are *star-eligible* if S_1, S_2 and S_1S_2 are proper and if $S_1 \times S_2 \subset M$.

For the rest of the section we fix S_1 and S_2, two star-eligible closed subsets of $\mathbb{P}$. If $z \in \mathbb{C}^* \setminus S_1S_2$, S_1 is a compact subset of $\mathbb{P} \setminus zS_2^{-1}$ and, thus, a compact subset of $\mathbb{P} \setminus (zS_2^{-1} \cup (\{0, \infty\} \setminus S_1))$. Moreover, one has

$$(\mathbb{P} \setminus (zS_2^{-1} \cup (\{0, \infty\} \setminus S_1))) \setminus S_1 = \mathbb{P} \setminus (S_1 \cup zS_2^{-1} \cup \{0\} \cup \{\infty\}).$$

Let $z \in \mathbb{C}^* \setminus S_1S_2$.

Definition 2.1.12. A *generalized Hadamard cycle* for S_1 in $\mathbb{P} \setminus (zS_2^{-1} \cup (\{0, \infty\} \setminus S_1))$ is a representative c of the class in $H_1(\mathbb{P} \setminus (S_1 \cup zS_2^{-1} \cup \{0\} \cup \{\infty\}))$ which is the image of

$$-[\mathbb{P} \setminus (zS_2^{-1} \cup (\{0, \infty\} \setminus S_1))]_{S_1} \in H_2(\mathbb{P} \setminus (zS_2^{-1} \cup (\{0, \infty\} \setminus S_1)), \mathbb{P} \setminus (S_1 \cup zS_2^{-1} \cup \{0\} \cup \{\infty\}))$$

by the canonical map

$$\begin{array}{c} H_2(\mathbb{P} \setminus (zS_2^{-1} \cup (\{0, \infty\} \setminus S_1)), \mathbb{P} \setminus (S_1 \cup zS_2^{-1} \cup \{0\} \cup \{\infty\})) \\ \downarrow \\ H_1(\mathbb{P} \setminus (S_1 \cup zS_2^{-1} \cup \{0\} \cup \{\infty\})). \end{array}$$

Our aim is now to define a product

$$\mathcal{O}(\mathbb{P} \setminus S_1) \times \mathcal{O}(\mathbb{P} \setminus S_2) \to \mathcal{O}(\mathbb{C}^* \setminus S_1S_2)$$

which generalizes the extended Hadamard product of T. Pohlen.

Definition 2.1.13. Let $f_1 \in \mathcal{O}(\mathbb{P} \setminus S_1)$ and $f_2 \in \mathcal{O}(\mathbb{P} \setminus S_2)$. For each $z \in \mathbb{C}^* \setminus S_1S_2$ we set

$$(f_1 \star f_2)(z) = \frac{1}{2i\pi} \int_{c_z} f_1(\zeta) f_2\left(\frac{z}{\zeta}\right) \frac{d\zeta}{\zeta},$$

where c_z is a generalized Hadamard cycle for S_1 in $\mathbb{P} \setminus (zS_2^{-1} \cup (\{0, \infty\} \setminus S_1))$. Since two generalized Hadamard cycles are homologous, the definition does not depend on the chosen generalized Hadamard cycle. The function $f_1 \star f_2$ is called the *generalized Hadamard product* of f_1 and f_2.

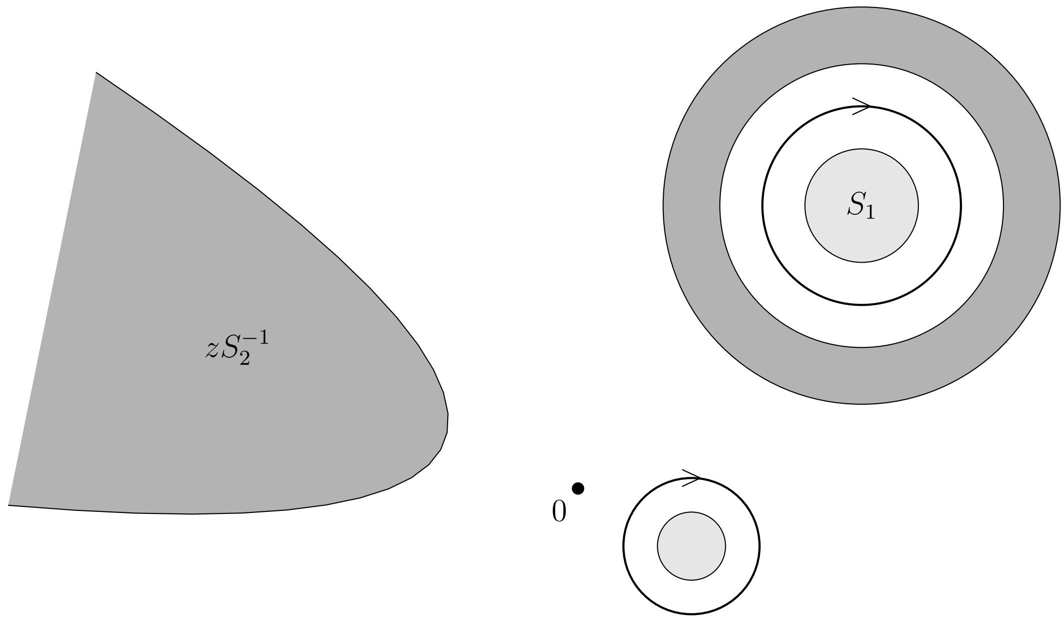

Figure 2.2: A generalized Hadamard cycle for S_1 in $\mathbb{P}\setminus(zS_2^{-1}\cup(\{0,\infty\}\setminus S_1))$, in the case where $0,\infty\notin S_1$ and $0\in S_2,\infty\notin S_2$.

Lemma 2.1.14. *Let $f_1\in\mathcal{O}(\mathbb{P}\setminus S_1)$ and $f_2\in\mathcal{O}(\mathbb{P}\setminus S_2)$. For each compact subset K of $\mathbb{C}^*\setminus S_1S_2$, there is a cycle c_K in $\mathbb{P}\setminus(S_1\cup KS_2^{-1}\cup\{0\}\cup\{\infty\})$ such that*

$$(f_1\star f_2)(z)=\frac{1}{2i\pi}\int_{c_K} f_1(\zeta)f_2\left(\frac{z}{\zeta}\right)\frac{d\zeta}{\zeta},$$

for all $z\in K$.

Proof. There is a relative orientation class $[\mathbb{P}\setminus(KS_2^{-1}\cup(\{0,\infty\}\setminus S_1))]_{S_1}$ in

$$H_2(\mathbb{P}\setminus(KS_2^{-1}\cup(\{0,\infty\}\setminus S_1)),\mathbb{P}\setminus(S_1\cup KS_2^{-1}\cup\{0\}\cup\{\infty\})).$$

We choose c_K to be a representative of the class in $H_1(\mathbb{P}\setminus(S_1\cup KS_2^{-1}\cup\{0\}\cup\{\infty\}))$ which is the image of $-[\mathbb{P}\setminus(KS_2^{-1}\cup(\{0,\infty\}\setminus S_1))]_{S_1}$ by the canonical map

$$\begin{array}{c} H_2(\mathbb{P}\setminus(KS_2^{-1}\cup(\{0,\infty\}\setminus S_1)),\mathbb{P}\setminus(S_1\cup KS_2^{-1}\cup\{0\}\cup\{\infty\})) \\ \big\downarrow \\ H_1(\mathbb{P}\setminus(S_1\cup KS_2^{-1}\cup\{0\}\cup\{\infty\})). \end{array}$$

For each $z\in K$, there is a canonical commutative diagram

$$\begin{tikzcd}
H_2(\mathbb{P}\setminus(KS_2^{-1}\cup(\{0,\infty\}\setminus S_1)),\mathbb{P}\setminus(S_1\cup KS_2^{-1}\cup\{0\}\cup\{\infty\})) \arrow[dd] \arrow[r] & H_1(\mathbb{P}\setminus(S_1\cup KS_2^{-1}\cup\{0\}\cup\{\infty\})) \arrow[d] \\
 & H_1(\mathbb{P}\setminus(S_1\cup zS_2^{-1}\cup\{0\}\cup\{\infty\})) \\
H_2(\mathbb{P}\setminus(zS_2^{-1}\cup(\{0,\infty\}\setminus S_1)),\mathbb{P}\setminus(S_1\cup zS_2^{-1}\cup\{0\}\cup\{\infty\})) \arrow[ur] &
\end{tikzcd} .$$

Obviously, $[\mathbb{P}\setminus(zS_2^{-1}\cup(\{0,\infty\}\setminus S_1))]_{S_1}$ is the image of $[\mathbb{P}\setminus(KS_2^{-1}\cup(\{0,\infty\}\setminus S_1))]_{S_1}$ by the left vertical map. Therefore, by the commutativity of the diagram, one can deduce that c_K is a generalized Hadamard cycle for S_1 in $\mathbb{P}\setminus(zS_2^{-1}\cup(\{0,\infty\}\setminus S_1))$, for all $z\in K$. Hence the conclusion. □

Proposition 2.1.15. *The generalized Hadamard product is a well-defined map*

$$\mathcal{O}(\mathbb{P}\setminus S_1)\times\mathcal{O}(\mathbb{P}\setminus S_2)\to\mathcal{O}(\mathbb{C}^*\setminus S_1S_2).$$

Proof. Let $f_1\in\mathcal{O}(\mathbb{P}\setminus S_1)$ and $f_2\in\mathcal{O}(\mathbb{P}\setminus S_2)$. We have to check that $f_1\star f_2$ is holomorphic on $\mathbb{C}^*\setminus S_1S_2$. Since it is a local property, it is enough to prove that $f_1\star f_2$ is holomorphic on any small open disk $D\subset\mathbb{C}^*\setminus S_1S_2$. Let D be such a disk. By Lemma 2.1.14 there is a cycle $c_{\overline{D}}$ such that

$$(f_1\star f_2)(z)=\frac{1}{2i\pi}\int_{c_{\overline{D}}}f_1(\zeta)f_2\left(\frac{z}{\zeta}\right)\frac{d\zeta}{\zeta}$$

for all $z\in D$. We conclude by derivation under the integral sign. □

We shall now prove that our product is a good generalization of the extended Hadamard product of T. Pohlen. By doing so, the reader shall see why we chose such a sign convention in Definition 2.1.12.

Proposition 2.1.16. *Let $f_1\in\mathcal{H}(\mathbb{P}\setminus S_1)$ and $f_2\in\mathcal{H}(\mathbb{P}\setminus S_2)$. Let $z\in\mathbb{C}^*\setminus S_1S_2$. Let c_z be a generalized Hadamard cycle for S_1 in $\mathbb{P}\setminus(zS_2^{-1}\cup(\{0,\infty\}\setminus S_1))$ and d_z be a Hadamard cycle for zS_2^{-1} in $\mathbb{P}\setminus S_1$. Then,*

$$\frac{1}{2i\pi}\int_{c_z}f_1(\zeta)f_2\left(\frac{z}{\zeta}\right)\frac{d\zeta}{\zeta}=\frac{1}{2i\pi}\int_{d_z}f_1(\zeta)f_2\left(\frac{z}{\zeta}\right)\frac{d\zeta}{\zeta}.$$

Proof. We treat the case where $0,\infty\notin S_1$ and $0\in S_2,\infty\notin S_2$ and leave the other ones to the reader. By construction, it is clear that c_z verifies

$$\mathrm{Ind}(c_z,w)=\begin{cases}0 & \text{if } w\in zS_2^{-1}\cup\{0\}\\ -1 & \text{if } w\in S_1.\end{cases}$$

Let c'_z be a cycle $\mathbb{P}\setminus(S_1 \cup zS_2^{-1} \cup \{0\} \cup \{\infty\})$ such that

$$\mathrm{Ind}(c'_z, w) = \begin{cases} 0 & \text{if } \ w \in zS_2^{-1} \cup S_1 \\ -1 & \text{if } \ w = 0. \end{cases}$$

Since d_z is acc$^-$, it is clear, by Proposition 1.3.7, that d_z is homologous to $c_z + c'_z$ in $\mathbb{P}\setminus(S_1 \cup zS_2^{-1} \cup \{0\} \cup \{\infty\})$. We then have

$$\int_{c_z} f_1(\zeta) f_2\left(\frac{z}{\zeta}\right)\frac{d\zeta}{\zeta} = \int_{d_z} f_1(\zeta) f_2\left(\frac{z}{\zeta}\right)\frac{d\zeta}{\zeta} - \int_{c'_z} f_1(\zeta) f_2\left(\frac{z}{\zeta}\right)\frac{d\zeta}{\zeta}.$$

Moreover, by the residue theorem,

$$\begin{aligned} -\int_{c'_z} f_1(\zeta) f_2\left(\frac{z}{\zeta}\right)\frac{d\zeta}{\zeta} &= 2i\pi \mathrm{Res}_{\zeta=0}\left(\frac{f_1(\zeta)}{\zeta} f_2\left(\frac{z}{\zeta}\right)\right) = 2i\pi \lim_{\zeta\to 0}\left(f_1(\zeta) f_2\left(\frac{z}{\zeta}\right)\right) \\ &= 2i\pi f_1(0) f_2(\infty) = 0. \end{aligned}$$

Hence the conclusion. $\square$

Remark 2.1.17. Of course, the generalized Hadamard product is no longer commutative if the functions do not vanish at infinity. For example, let S_1 and S_2 be as in the proof of the previous proposition. Let $f_1 \in \mathcal{O}(\mathbb{P}\setminus S_1)$ and $f_2 \in \mathcal{O}(\mathbb{P}\setminus S_2)$. By a similar computation, one sees that

$$f_1 \star f_2 - f_2 \star f_1 = f_1(0) f_2(\infty).$$

Despite the lack of commutativity, the generalized Hadamard cycles are more symmetric with respect to 0 and ∞. In the next section, we shall explain how one can define a convolution between 1-forms which have (not necessarily isolated) singularities at 0 and ∞. Generalized Hadamard cycles are key ingredients to compute such a convolution (see also Section 2.2.3). Moreover, the commutativity will eventually be obtained thanks to quotient spaces that naturally occur in this context.

2.2 Holomorphic cohomological convolution

The concept of holomorphic cohomological convolution has originally been introduced in our master thesis [26]. We will first recall its definition and then fully treat the case of $\mathbb{C}^*$ to understand the link with the (generalized) Hadamard product.

2.2.1 General definition

Definition 2.2.1. Let (G, μ) be a locally compact complex Lie group of complex dimension n. Two closed subsets S_1 and S_2 of G are said to be *convolvable* if $S_1 \times S_2$ is μ-proper, i.e. if

$$(S_1 \times S_2) \cap \mu^{-1}(K)$$

is a compact subset of $G \times G$ for any compact subset K of G.

Remark 2.2.2. A proper map on a locally compact topological space is universally closed, in particular closed (see e.g. [12]). Hence, if S_1 and S_2 are convolvable closed subsets of G, then $\mu|_{S_1\times S_2}$ is a proper map and $S_1 + S_2 = \mu|_{S_1\times S_2}(S_1 \times S_2)$ is closed.

Recall Sections 1.4.2 and 1.4.3.

Definition 2.2.3. Two distributional $2n$-forms u_1 and u_2 of G are *convolvable* if the support S_1 of u_1 and the support S_2 of u_2 are convolvable. In that case, the convolution product of u_1 and u_2 is a distributional $2n$-form on G defined by

$$u_1 \star u_2 = \int_\mu (u_1 \boxtimes u_2) := \int_\mu (p_1^* u_1 \wedge p_2^* u_2),$$

where $p_1, p_2 : G \times G \to G$ are the two canonical projections.

Remark 2.2.4. By choosing a Haar form ν on G, one can define the convolution product of two distributions by means of the isomorphism $\mathcal{D}b_G \simeq \mathcal{D}b_G^{2n}$ given by ν (see e.g. [21]).

Remark 2.2.5. If we define

$$\phi : G\times G \to G\times G \quad \text{and} \quad \psi : G\times G \to G\times G$$

by setting $\phi(g_1, g_2) = (g_1, \mu(g_1, g_2))$ and $\psi(g_1, g_2) = (g_1, \mu(g_1^{-1}, g_2))$, we see that ϕ and ψ are reciprocal biholomorphic bijections and that the diagram

$$\begin{tikzcd} G\times G \arrow[rr, "\sim", "\phi"'] \arrow[dr, "\mu"'] & & G\times G \arrow[dl, "p_2"] \\ & G & \end{tikzcd}$$

is commutative. This shows in particular that μ is a surjective submersion and that the preceding procedure allows us also to define the convolution product of $2n$-differential forms.

Let S_1 and S_2 be two convolvable closed subsets of G. By construction, the convolution of distributions on G is the composition of the external product of distributions

$$\Gamma_{S_1}(G, \mathcal{D}b_G^{2n}) \otimes \Gamma_{S_2}(G, \mathcal{D}b_G^{2n}) \to \Gamma_{S_1\times S_2}(G\times G, \mathcal{D}b_{G\times G}^{4n})$$

and the map

$$\int_\mu : \Gamma_{S_1\times S_2}(G\times G, \mathcal{D}b_{G\times G}^{4n}) \to \Gamma_{\mu(S_1\times S_2)}(G, \mathcal{D}b_G^{2n})$$

induced by the holomorphic integration map along the fibers of μ

$$\int_\mu : \Gamma_{\mu-\mathrm{proper}}(G\times G, \mathcal{D}b_{G\times G}^{4n}) \to \Gamma(G, \mathcal{D}b_G^{2n})$$

and the fact that S_1 and S_2 are convolvable if and only if $S_1 \times S_2$ is μ-proper. It is thus natural to define the convolution of cohomology classes of holomorphic forms on G as follows :

Definition 2.2.6. Let S_1 and S_2 be two convolvable closed subsets of G. Consider the external product morphisms

$$\mathrm{R}\Gamma_{S_1}(G, \Omega_G^{p+n})[n] \otimes \mathrm{R}\Gamma_{S_2}(G, \Omega_G^{q+n})[n] \to \mathrm{R}\Gamma_{S_1 \times S_2}(G \times G, \Omega_{G\times G}^{p+q+2n})[2n]$$

and the morphisms

$$\int_\mu : \mathrm{R}\Gamma_{S_1 \times S_2}(G \times G, \Omega_{G\times G}^{p+q+2n})[2n] \to \mathrm{R}\Gamma_{\mu(S_1 \times S_2)}(G, \Omega_G^{p+q+n})[n].$$

induced by the holomorphic integration map and the fact that $S_1 \times S_2$ is μ-proper. By composition, these morphisms give derived category morphisms

$$\star_{(G,\mu)} : \mathrm{R}\Gamma_{S_1}(G, \Omega_G^{p+n})[n] \otimes \mathrm{R}\Gamma_{S_2}(G, \Omega_G^{q+n})[n] \to \mathrm{R}\Gamma_{\mu(S_1 \times S_2)}(G, \Omega_G^{p+q+n})[n],$$

that we call the *holomorphic convolution morphisms of G*. Going to cohomology groups, these morphisms give rise to the morphisms

$$\star_{(G,\mu)} : H_{S_1}^{r+n}(G, \Omega_G^{p+n}) \otimes H_{S_2}^{s+n}(G, \Omega_G^{q+n}) \to H_{\mu(S_1 \times S_2)}^{r+s+n}(G, \Omega_G^{p+q+n}),$$

that we call the *holomorphic cohomological convolution morphisms of G*.

Remark 2.2.7. Consider the diagram

$$\begin{tikzcd}
H^n_{S_1}(G,\Omega_G) \otimes H^n_{S_2}(G,\Omega_G) \arrow[r] & H^n_{\mu(S_1\times S_2)}(G,\Omega_G) \\
\Gamma_{S_1}(G,\mathcal{D}b_G^{2n}) \otimes \Gamma_{S_2}(G,\mathcal{D}b_G^{2n}) \arrow[r] \arrow[u] & \Gamma_{\mu(S_1\times S_2)}(G,\mathcal{D}b_G^{2n}) \arrow[u]
\end{tikzcd}$$

where the vertical arrows are given by the Dolbeault complex of Ω_G and the top (resp. the bottom) horizontal arrow is given by the holomorphic cohomological morphism of G with $p = q = r = s = 0$ (resp. the convolution product of distributions). Obviously, by the definitions, this diagram is commutative. This remark will allow to perform explicit computations in the next section.

2.2.2 Multiplicative convolution on $\mathbb{C}^*$

In this section, we will consider the case where the group G is the group $\mathbb{C}^*$ formed by the set of non-zero complex numbers endowed with the complex multiplication (noted as a concatenation). We will assume that S_1, S_2 are convolvable proper closed subsets of $\mathbb{C}^*$ (remark that this means that $S_1 \cap KS_2^{-1}$ is compact for any compact subset K of $\mathbb{C}^*$) such that S_1S_2 is also a proper subset of $\mathbb{C}^*$ and we will show how to compute the holomorphic cohomological convolution morphism

$$\star : H^1_{S_1}(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \otimes H^1_{S_2}(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \to H^1_{S_1S_2}(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \qquad (2.1)$$

by means of path integral formulas.

In order to lighten the notations, we will write $\Omega(U)$ instead of $\Omega_{\mathbb{C}^*}(U)$ if U is an open subset of $\mathbb{C}^*$.

Proposition 2.2.8. *Let S be a proper closed subset of $\mathbb{C}^*$, then there is a canonical isomorphism*

$$H^r_S(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \simeq \begin{cases} \Omega(\mathbb{C}^* \setminus S)/\Omega(\mathbb{C}^*) & \textit{if } r = 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Consider the following distinguished triangle, obtained by excision :

$$\mathrm{R}\Gamma_S(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \to \mathrm{R}\Gamma(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \to \mathrm{R}\Gamma(\mathbb{C}^* \setminus S, \Omega_{\mathbb{C}^*}) \xrightarrow{+1} .$$

It induces a long exact sequence :

$$\begin{tikzcd} 0 \arrow[r] & H^0_S(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \arrow[r] & H^0(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \arrow[r] & H^0(\mathbb{C}^* \setminus S, \Omega_{\mathbb{C}^*}) \arrow[dll, rounded corners] \\ & H^1_S(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \arrow[r] & H^1(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \arrow[r] & H^1(\mathbb{C}^* \setminus S, \Omega_{\mathbb{C}^*}) \arrow[r] & \cdots \end{tikzcd}$$

Since $\mathcal{C}^{1,\bullet}_{\infty,\mathbb{C}^*}$ is a soft resolution of $\Omega_{\mathbb{C}^*}$, one gets

$$\mathrm{R}\Gamma(U, \Omega_{\mathbb{C}^*}) \simeq \Gamma(U, \mathcal{C}^{1,\bullet}_{\infty,\mathbb{C}^*})$$

for all open subset U of $\mathbb{C}^*$. Therefore, using the fact that $\bar{\partial}$ is globally surjective, one deduces that $\mathrm{R}\Gamma(\mathbb{C}^*, \Omega_{\mathbb{C}^*})$ and $\mathrm{R}\Gamma(\mathbb{C}^* \setminus S, \Omega_{\mathbb{C}^*})$ are concentrated in degree 0. This shows that $H^r_S(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \simeq 0$ for all $r \geq 2$. If $r = 0$, it is clear that $H^0_S(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \simeq 0$. Indeed, a holomorphic function supported by a proper closed subset of $\mathbb{C}^*$ admits an identical zero and is thus equal to 0 by the identity theorem. Hence, the long exact sequence becomes

$$0 \to \Omega(\mathbb{C}^*) \to \Omega(\mathbb{C}^* \setminus S) \to H^1_S(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \to 0$$

and the conclusion follows. □

Thanks to this proposition, one can see that (2.1) can be interpreted as a bilinear map

$$\star : \Omega(\mathbb{C}^* \setminus S_1)/\Omega(\mathbb{C}^*) \times \Omega(\mathbb{C}^* \setminus S_2)/\Omega(\mathbb{C}^*) \to \Omega(\mathbb{C}^* \setminus S_1 S_2)/\Omega(\mathbb{C}^*).$$

Now, let $\omega_1 \in \Omega(\mathbb{C}^* \setminus S_1)$ and $\omega_2 \in \Omega(\mathbb{C}^* \setminus S_2)$ be two given holomorphic forms. Ideally, we would like to obtain a formula of the form

$$[\omega_1] \star [\omega_2] = [\omega]$$

where ω is a holomorphic form on $\mathbb{C}^* \setminus S_1 S_2$ which can be computed from ω_1 and ω_2 by some path integral.

It is in general not possible to find such a nice formula. However, we shall show that for any relatively compact open subset U of $\mathbb{C}^*$ and any open neighbourhood V of $S_1 S_2$ in $\mathbb{C}^*$, there is a holomorphic form ω on $U \setminus \overline{V}$ which can be computed from ω_1 and ω_2 by some path integral and which is such that

$$[\omega] \in \Omega(U \setminus \overline{V})/\Omega(U) \simeq H^1_{\overline{V} \cap U}(U, \Omega_{\mathbb{C}^*})$$

coincides with the image of $[\omega_1] \star [\omega_2]$ by the canonical restriction morphism

$$H^1_{S_1 S_2}(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \to H^1_{\overline{V} \cap U}(U, \Omega_{\mathbb{C}^*}).$$

Thanks to the next lemma, this is in fact sufficient to completely compute $[\omega_1] \star [\omega_2]$.

Lemma 2.2.9. *Let S be a closed subset of $\mathbb{C}^*$. Then*

$$H^1_S(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \simeq \varprojlim_{U \in \mathcal{U}_{rc}, V \in \mathcal{V}_S} H^1_{\overline{V} \cap U}(U, \Omega_{\mathbb{C}^*})$$

where $\mathcal{U}_{rc}$ denotes the set of relatively compact open subsets of $\mathbb{C}^*$ ordered by $\subset$ and $\mathcal{V}_S$ denotes the set of open neighbourhoods of S in $\mathbb{C}^*$ ordered by $\supset$.

Proof. This follows from Corollary 1.2.4. $\square$

To be able to specify the kind of path integral we need, let us first introduce the following definition :

Definition 2.2.10. Let F and G be two closed subsets of $\mathbb{C}^*$ which have a compact intersection and let W be an open neighbourhood of $F \cap G$. A *relative Hadamard cycle for F with respect to G in W* is a relative 1-cycle

$$c \in Z_1(W \setminus F, (W \setminus F) \cap (W \setminus G))$$

such that its class

$$[c] \in H_1(W \setminus F, (W \setminus F) \cap (W \setminus G))$$

is the image of the relative orientation class

$$[W]_{F \cap G} \in H_2(W, W \setminus (F \cap G))$$

by the Mayer-Vietoris morphism

$$H_2(W, W \setminus (F \cap G)) \to H_1(W \setminus F, (W \setminus F) \cap (W \setminus G))$$

associated with the decomposition

$$(W, W \setminus (F \cap G)) = ((W \setminus F) \cup W, (W \setminus F) \cup (W \setminus G)).$$

Remark 2.2.11. Let $c \in Z_1(W \setminus F, (W \setminus F) \cap (W \setminus G))$ such that the associated class $[c] \in H_1(W \setminus F, (W \setminus F) \cap (W \setminus G))$ is the image of $[W]_{F \cap G}$ by the sequence of canonical maps

$$\begin{aligned} H_2(W, W \setminus (F \cap G)) &\to H_1(W \setminus (F \cap G)) \\ &= H_1((W \setminus F) \cup (W \setminus G)) \\ &\to H_1((W \setminus F) \cup (W \setminus G), W \setminus G) \\ &\to H_1(W \setminus F, (W \setminus F) \cap (W \setminus G)). \end{aligned}$$

By construction, c is a relative Hadamard cycle for F with respect to G in W.

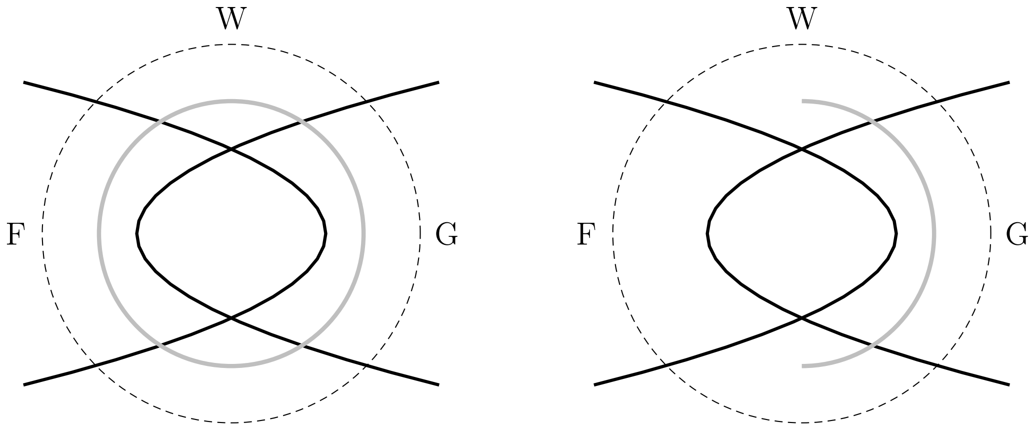

Figure 2.3: On the left, in grey, the boundary of a representative of $[W]_{F\cap G}$. On the right, in grey, a piece of this boundary which is a relative Hadamard cycle for F with respect to G in W.

With this definition at hand, we can now state the main result of this section.

Theorem 2.2.12. *Let S_1 and S_2 be two convolvable proper closed subsets of $\mathbb{C}^*$ such that $S_1S_2 \neq \mathbb{C}^*$ and let us consider $\omega_1 = f_1 dz$ (resp. $\omega_2 = f_2 dz$) with $f_1 \in \mathcal{O}(\mathbb{C}^* \setminus S_1)$ (resp. $f_2 \in \mathcal{O}(\mathbb{C}^* \setminus S_2)$). Fix a relatively compact open subset U of $\mathbb{C}^*$ and an open neighbourhood V of S_1S_2 in $\mathbb{C}^*$. Then, the image of*

$$[\omega_1] \star [\omega_2] \in \Omega(\mathbb{C}^* \setminus S_1S_2)/\Omega(\mathbb{C}^*) \simeq H^1_{S_1S_2}(\mathbb{C}^*, \Omega_{\mathbb{C}^*})$$

in

$$\Omega(U \setminus \overline{V})/\Omega(U) \simeq H^1_{\overline{V}\cap U}(U, \Omega_{\mathbb{C}^*})$$

is the class of the form $\omega = f dz \in \Omega(U \setminus \overline{V})$ where

$$f(z) = \int_c f_1(\zeta) f_2\left(\frac{z}{\zeta}\right) \frac{d\zeta}{\zeta}$$

and c is a relative Hadamard cycle for S_1 with respect to $\overline{U}S_2^{-1}$ in $\mathbb{C}^* \setminus (\overline{U} \setminus V)S_2^{-1}$.

Lemma 2.2.13. *Let S_1 and S_2 be two convolvable closed subsets of $\mathbb{C}^*$ and let $\mathcal{W}$ be a fundamental system of compact neighbourhoods of 1 in $\mathbb{C}^*$. Then*

1. *The set $S_1^W = WS_1$ (resp. $S_2^W = WS_2$, $S_1^W S_2^W = W^2 S_1 S_2$) is a closed neighbourhood of S_1 (resp. S_2, S_1S_2) in $\mathbb{C}^*$ for any $W \in \mathcal{W}$.*
2. *The closed subsets S_1^W et S_2^W are convolvable in $\mathbb{C}^*$ for any $W \in \mathcal{W}$.*
3. *One has $\bigcap_{W\in\mathcal{W}} S_1^W = S_1$, $\bigcap_{W\in\mathcal{W}} S_2^W = S_2$ and $\bigcap_{W\in\mathcal{W}} S_1^W S_2^W = S_1S_2$.*
4. *In particular, if S_1 and S_2 are two convolvable proper closed subsets of $\mathbb{C}^*$ such that $S_1S_2 \neq \mathbb{C}^*$, if U is a relatively compact open subset of $\mathbb{C}^*$ and if V is an open neighbourhood of S_1S_2 in $\mathbb{C}^*$, then there is $W \in \mathcal{W}$ such that S_1^W and S_2^W are convolvable proper closed subsets of $\mathbb{C}^*$ such that $S_1^W S_2^W \neq \mathbb{C}^*$ and $S_1^W S_2^W \cap \overline{U} \subset V$.*

Proof. (1) This follows from the fact that FK is closed in $\mathbb{C}^*$ if F (resp. K) is closed (resp. compact) in $\mathbb{C}^*$ and from the fact that zW is a neighbourhood of z for all $z \in \mathbb{C}$ and all $W \in \mathcal{W}$.
(2) This follows from the inclusion

$$S_1^W \cap K(S_2^W)^{-1} = WS_1 \cap KW^{-1}S_2^{-1} \subset W(S_1 \cap KW^{-2}S_2^{-1})$$

which is satisfied for any compact subset K of $\mathbb{C}^*$.
(3) This is clear since for any closed subset F of $\mathbb{C}^*$ and any $z \notin F$ there is $W \in \mathcal{W}$ such that $zW^{-1} \cap F = \emptyset$.
(4) By contradiction, assume that

$$S_1^W S_2^W \cap \overline{U} \cap (\mathbb{C}^* \setminus V) \neq \emptyset$$

for all $W \in \mathcal{W}$. Then, by compactness,

$$\bigcap_{W \in \mathcal{W}} (S_1^W S_2^W \cap \overline{U} \cap (\mathbb{C}^* \setminus V)) = S_1 S_2 \cap \overline{U} \cap (\mathbb{C}^* \setminus V) \neq \emptyset,$$

but this contradicts the fact that $S_1 S_2 \cap \overline{U} \subset V$. □

Lemma 2.2.14. *Let S be a proper closed subset of $\mathbb{C}^*$ and let $\omega \in \Omega(\mathbb{C}^* \setminus S)$. Assume that ω admits an infinitely differentiable extension to $\mathbb{C}^*$ and denote by $\underline{\omega}$ such an extension. Then $[\omega]$, seen as an element of $H^1_S(\mathbb{C}^*, \Omega_{\mathbb{C}^*})$, is the image of*

$$[\overline{\partial}\underline{\omega}] \in H^1(\Gamma_S(\mathbb{C}^*, \mathcal{C}^{1,\bullet}_{\infty,\mathbb{C}^*}))$$

by the canonical morphism obtained by applying H^1 to the composition in the derived category of the canonical morphism

$$\Gamma_S(\mathbb{C}^*, \mathcal{C}^{1,\bullet}_{\infty,\mathbb{C}^*}) \to \mathrm{R}\Gamma_S(\mathbb{C}^*, \mathcal{C}^{1,\bullet}_{\infty,\mathbb{C}^*})$$

and the inverse of the canonical isomorphism

$$\mathrm{R}\Gamma_S(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \xrightarrow{\sim} \mathrm{R}\Gamma_S(\mathbb{C}^*, \mathcal{C}^{1,\bullet}_{\infty,\mathbb{C}^*}).$$

Proof. It follows from the distinguished triangle

$$\mathrm{R}\Gamma_S(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \to \mathrm{R}\Gamma(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \to \mathrm{R}\Gamma(\mathbb{C}^* \setminus S, \Omega_{\mathbb{C}^*}) \xrightarrow{+1}$$

that $\mathrm{R}\Gamma_S(\mathbb{C}^*, \Omega_{\mathbb{C}^*})$ is canonically isomorphic to the mapping cone $M(\rho_S)$ of the restriction morphism

$$\rho_S : \mathcal{C}^{1,\bullet}_{\infty,\mathbb{C}^*}(\mathbb{C}^*) \to \mathcal{C}^{1,\bullet}_{\infty,\mathbb{C}^*}(\mathbb{C}^* \setminus S)$$

shifted by -1. We know that $M(\rho_S)[-1]$ is a complex concentrated in degrees 0, 1 and 2 of the form

$$\mathcal{C}^{1,0}_{\infty,\mathbb{C}^*}(\mathbb{C}^*) \to \mathcal{C}^{1,1}_{\infty,\mathbb{C}^*}(\mathbb{C}^*) \oplus \mathcal{C}^{1,0}_{\infty,\mathbb{C}^*}(\mathbb{C}^* \setminus S) \to \mathcal{C}^{1,1}_{\infty,\mathbb{C}^*}(\mathbb{C}^* \setminus S)$$

where the differentials in degree 0 and 1 are given by the matrices

$$\begin{pmatrix} \overline{\partial} \\ -\rho_S \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\rho_S & -\overline{\partial} \end{pmatrix}.$$

What we have to show is that

$$\begin{pmatrix} \overline{\partial}\underline{\omega} \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ \omega \end{pmatrix}$$

are two 1-cycles of this complex which are in the same cohomology class. This is clear since

$$\begin{pmatrix} \overline{\partial} \\ -\rho_S \end{pmatrix} \underline{\omega} + \begin{pmatrix} 0 \\ \omega \end{pmatrix} = \begin{pmatrix} \overline{\partial}\underline{\omega} \\ 0 \end{pmatrix}.$$

□

Proof of Theorem 2.2.12. Let U and V be as in the statement of the theorem. Thanks to Lemma 2.2.13, we know that it is possible to find a closed neighbourhood $\underline{S}_1$ of S_1 and a closed neighbourhood $\underline{S}_2$ of S_2 in $\mathbb{C}^*$ such that $\underline{S}_1$ and $\underline{S}_2$ are convolvable and

$$\underline{S}_1\underline{S}_2 \cap \overline{U} \subset V.$$

Let $\underline{f}_1$ (resp. $\underline{f}_2$) be an infinitely differentiable function on $\mathbb{C}^*$ which coincides with f_1 (resp. f_2) on $\mathbb{C}^* \setminus \underline{S}_1$ (resp. $\mathbb{C}^* \setminus \underline{S}_2$) and set

$$\underline{\omega}_1 = \underline{f}_1(z)\, dz \quad \text{and} \quad \underline{\omega}_2 = \underline{f}_2(z)\, dz.$$

It follows from Lemma 2.2.14 that the image of

$$[\omega_1] \in \Omega(\mathbb{C}^* \setminus S_1)/\Omega(\mathbb{C}^*) \simeq H^1_{S_1}(\mathbb{C}^*, \Omega_{\mathbb{C}^*})$$

by the canonical morphism

$$H^1_{S_1}(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \to H^1_{\underline{S}_1}(\mathbb{C}^*, \Omega_{\mathbb{C}^*})$$

is the same as the image of

$$[\overline{\partial}\underline{\omega}_1] \in H^1(\Gamma_{\underline{S}_1}(\mathbb{C}^*, \mathcal{C}^{(1,\bullet)}_{\infty,\mathbb{C}^*}))$$

by the canonical morphism

$$H^1(\Gamma_{\underline{S}_1}(\mathbb{C}^*, \mathcal{C}^{(1,\bullet)}_{\infty,\mathbb{C}^*})) \to H^1_{\underline{S}_1}(\mathbb{C}^*, \Omega_{\mathbb{C}^*})$$

considered in this lemma. A similar conclusion is true for the image of

$$[\omega_2] \in \Omega(\mathbb{C}^* \setminus S_2)/\Omega(\mathbb{C}^*) \simeq H^1_{S_2}(\mathbb{C}^*, \Omega_{\mathbb{C}^*})$$

in $H^1_{\underline{S}_2}(\mathbb{C}^*, \Omega_{\mathbb{C}^*})$. Therefore, the image of

$$[\omega_1] \star [\omega_2] \in \Omega(\mathbb{C}^* \setminus S_1S_2)/\Omega(\mathbb{C}^*) \simeq H^1_{S_1S_2}(\mathbb{C}^*, \Omega_{\mathbb{C}^*})$$

in $H^1_{\underline{S}_1\underline{S}_2}(\mathbb{C}^*, \Omega_{\mathbb{C}^*})$ is the same as the image of $[\overline{\partial}\underline{\omega}_1 \star \overline{\partial}\underline{\omega}_2]$ by the canonical morphism

$$H^1(\Gamma_{\underline{S}_1\underline{S}_2}(\mathbb{C}^*, \mathcal{C}^{(1,\bullet)}_{\infty,\mathbb{C}^*})) \to H^1_{\underline{S}_1\underline{S}_2}(\mathbb{C}^*, \Omega_{\mathbb{C}^*}).$$

Let us note $p_1, p_2 : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^*$ the two canonical projections and μ the complex multiplication. Consider the commutative diagram

$$\begin{tikzcd}
\mathbb{C}^* \times \mathbb{C}^* \arrow[rr, shift left, "\phi"] \arrow[dr, "\mu"'] & & \mathbb{C}^* \times \mathbb{C}^* \arrow[ll, shift left, "\psi"] \arrow[dl, "p_2"] \\
& \mathbb{C}^* &
\end{tikzcd}$$

where $\phi(z_1, z_2) = (z_1, z_1 z_2)$ and $\psi(\zeta, z) = (\zeta, z/\zeta)$. Since $\phi \circ \psi = \mathrm{id} = \psi \circ \phi$, we have

$$\int_\mu = \int_{p_2} \circ \int_\phi = \int_{p_2} \circ\, \psi^*.$$

Therefore,

$$\begin{aligned}
\overline{\partial}\underline{\omega}_1 \star \overline{\partial}\underline{\omega}_2 &= \int_\mu (\overline{\partial}\underline{\omega}_1 \boxtimes \overline{\partial}\underline{\omega}_2) \\
&= \int_{p_2} (\psi^*(p_1^*\overline{\partial}\underline{\omega}_1 \wedge p_2^*\overline{\partial}\underline{\omega}_2)) \\
&= \int_{p_2} (p_1^*\overline{\partial}\underline{\omega}_1 \wedge h^*\overline{\partial}\underline{\omega}_2)),
\end{aligned}$$

where $h(\zeta, z) = z/\zeta$. Since

$$\overline{\partial}\underline{\omega}_1 = \frac{\partial \underline{f}_1}{\partial \overline{z}}(z) d\overline{z} \wedge dz \quad \text{and} \quad \overline{\partial}\underline{\omega}_2 = \frac{\partial \underline{f}_2}{\partial \overline{z}}(z) d\overline{z} \wedge dz,$$

we have

$$\begin{aligned}
h^*\overline{\partial}\underline{\omega}_2 &= \frac{\partial \underline{f}_2}{\partial \overline{z}}\left(\frac{z}{\zeta}\right) d\left(\frac{\overline{z}}{\overline{\zeta}}\right) \wedge d\left(\frac{z}{\zeta}\right) \\
&= \frac{\partial \underline{f}_2}{\partial \overline{z}}\left(\frac{z}{\zeta}\right) \frac{\overline{\zeta} d\overline{z} - \overline{z} d\overline{\zeta}}{\overline{\zeta}^2} \wedge \frac{\zeta dz - z d\zeta}{\zeta^2}
\end{aligned}$$

and

$$p_1^*\overline{\partial}\underline{\omega}_1 \wedge h^*\overline{\partial}\underline{\omega}_2 = \frac{\partial \underline{f}_1}{\partial \overline{z}}(\zeta) \frac{\partial \underline{f}_2}{\partial \overline{z}}\left(\frac{z}{\zeta}\right) \frac{d\overline{\zeta}}{\overline{\zeta}} \wedge \frac{d\zeta}{\zeta} \wedge d\overline{z} \wedge dz.$$

Therefore,

$$\overline{\partial}\underline{\omega}_1 \star \overline{\partial}\underline{\omega}_2 = \left(\int_{\mathbb{C}^*} \frac{\partial \underline{f}_1}{\partial \overline{z}}(\zeta) \frac{\partial \underline{f}_2}{\partial \overline{z}}\left(\frac{z}{\zeta}\right) \frac{d\overline{\zeta}}{\overline{\zeta}} \wedge \frac{d\zeta}{\zeta}\right) d\overline{z} \wedge dz.$$

Since $\underline{f}_1$ coincides with f_1 on $\mathbb{C}^* \setminus \underline{S}_1$, one has

$$\operatorname{supp}\left(\zeta \mapsto \frac{\partial \underline{f}_1}{\partial \bar{z}}(\zeta)\right) \subset \underline{S}_1.$$

Similarly, one has

$$\operatorname{supp}\left(\zeta \mapsto \frac{\partial \underline{f}_2}{\partial \bar{z}}\left(\frac{z}{\zeta}\right)\right) \subset z\underline{S}_2^{-1}.$$

Hence,

$$\zeta \mapsto \frac{\partial \underline{f}_1}{\partial \bar{z}}(\zeta)\frac{\partial \underline{f}_2}{\partial \bar{z}}\left(\frac{z}{\zeta}\right)$$

is an infinitely differentiable function on $\mathbb{C}^*$ supported by $\underline{S}_1 \cap z\underline{S}_2^{-1}$ which is a compact subset of $\mathbb{C}^*$.

Since U is a relatively compact open subset of $\mathbb{C}^*$ and $\underline{S}_1$ and $\underline{S}_2$ are convolvable closed subsets of $\mathbb{C}^*$,

$$K = \underline{S}_1 \cap \overline{U}\underline{S}_2^{-1}$$

is a compact subset of $\mathbb{C}^*$. Let c be a singular infinitely differentiable 2-chain of $\mathbb{C}^*$ such that

$$[c] \in H_2(\mathbb{C}^*, \mathbb{C}^* \setminus K)$$

is the relative orientation class $[\mathbb{C}^*]_K$. Then, on U, one has

$$\bar{\partial}\underline{\omega}_1 \star \bar{\partial}\underline{\omega}_2 = \left(\int_c \frac{\partial \underline{f}_1}{\partial \bar{z}}(\zeta)\frac{\partial \underline{f}_2}{\partial \bar{z}}\left(\frac{z}{\zeta}\right)\frac{d\bar{\zeta}}{\bar{\zeta}} \wedge \frac{d\zeta}{\zeta}\right) d\bar{z} \wedge dz,$$

since the integrated form is supported by $\underline{S}_1 \cap z\underline{S}_2^{-1} \subset K$ for any $z \in U$. Moreover, the function $\underline{f}_2$ is infinitely differentiable on $\mathbb{C}^*$ and the chain c is supported by a compact subset of $\mathbb{C}^*$. Thus, the function

$$f : z \mapsto \int_c \frac{\partial \underline{f}_1}{\partial \bar{z}}(\zeta)\underline{f}_2\left(\frac{z}{\zeta}\right) d\bar{\zeta} \wedge \frac{d\zeta}{\zeta}$$

is infinitely differentiable on $\mathbb{C}^*$ and

$$\frac{\partial f}{\partial \bar{z}}(z) = \int_c \frac{\partial \underline{f}_1}{\partial \bar{z}}(\zeta)\frac{\partial \underline{f}_2}{\partial \bar{z}}\left(\frac{z}{\zeta}\right)\frac{d\bar{\zeta}}{\bar{\zeta}} \wedge \frac{d\zeta}{\zeta}$$

Therefore, on U, one has

$$\bar{\partial}\underline{\omega}_1 \star \bar{\partial}\underline{\omega}_2 = \bar{\partial}\omega$$

where $\omega = f(z)dz$. Since $\operatorname{supp}(\bar{\partial}\underline{\omega}_1 \star \bar{\partial}\underline{\omega}_2) \subset \underline{S}_1\underline{S}_2$, the function f is holomorphic on $U \setminus \underline{S}_1\underline{S}_2$ and it follows from what precedes that

$$([\omega_1] \star [\omega_2])|_U = [\omega|_U]$$

in

$$\Omega(U\backslash \underline{S}_1\underline{S}_2)/\Omega(U) \simeq H^1_{(\underline{S}_1\underline{S}_2)\cap U}(U,\Omega_{\mathbb{C}^*}).$$

Let us now show how to compute $[\omega|_U]$ in $\Omega(U\setminus \overline{V})/\Omega(U)$ by means of f_1 and f_2 alone. Since V is an open neighbourhood of $\underline{S}_1\underline{S}_2$,

$$\underline{S}_1 \cap (\overline{U}\setminus V)\underline{S}_2^{-1} = \emptyset.$$

Therefore,

$$\mathbb{C}^* = (\mathbb{C}^* \setminus \underline{S}_1) \cup \left(\mathbb{C}^* \setminus \left((\overline{U}\setminus V)\underline{S}_2^{-1}\right)\right)$$

and, replacing if necessary c by a barycentric subdivision, we may assume that

$$c = c_1 + c_2$$

where

$$\operatorname{supp} c_1 \subset \mathbb{C}^* \setminus \underline{S}_1 \quad \text{and} \quad \operatorname{supp} c_2 \subset \mathbb{C}^* \setminus \left((\overline{U}\setminus V)\underline{S}_2^{-1}\right).$$

Since $\operatorname{supp} \frac{\partial \underline{f}_1}{\partial \bar z} \subset \underline{S}_1$, it is then clear that

$$f(z) = \int_{c_2} \frac{\partial \underline{f}_1}{\partial \bar z}(\zeta)\underline{f}_2\left(\frac{z}{\zeta}\right) d\bar\zeta \wedge \frac{d\zeta}{\zeta}.$$

Moreover, for any $z \in \overline{U}\setminus V$ one has

$$\mathbb{C}^* \setminus z\underline{S}_2^{-1} \supset \mathbb{C}^* \setminus \left((\overline{U}\setminus V)\underline{S}_2^{-1}\right) \supset \operatorname{supp} c_2$$

and since the function $\zeta \mapsto \underline{f}_2(z/\zeta)$ is holomorphic on $\mathbb{C}^* \setminus z\underline{S}_2^{-1}$, it follows that

$$\begin{aligned} f(z) &= \int_{c_2} \frac{\partial}{\partial \bar\zeta}\left(\underline{f}_1(\zeta)\underline{f}_2\left(\frac{z}{\zeta}\right)\frac{1}{\zeta}\right) d\bar\zeta \wedge d\zeta \\ &= \int_{\partial c_2} \underline{f}_1(\zeta)\underline{f}_2\left(\frac{z}{\zeta}\right)\frac{d\zeta}{\zeta}. \end{aligned}$$

By construction,

$$\operatorname{supp}(\partial c) \subset \mathbb{C}^* \setminus K = (\mathbb{C}^* \setminus \underline{S}_1) \cup (\mathbb{C}^* \setminus \overline{U}\underline{S}_2^{-1}).$$

Replacing, if necessary, c by a one of its barycentric subdivisions, we may thus assume that $\partial c = c'_1 + c'_2$ where $\operatorname{supp} c'_1 \subset \mathbb{C}^* \setminus \underline{S}_1$ and $\operatorname{supp} c'_2 \subset \mathbb{C}^* \setminus \overline{U}\underline{S}_2^{-1}$. Since

$$\partial c_1 + \partial c_2 = \partial c = c'_1 + c'_2,$$

there is a chain c_3 such that

$$\partial c_2 - c'_2 = c_3 = c'_1 - \partial c_1.$$

Since $\operatorname{supp} c_2' \subset \mathbb{C}^* \setminus \overline{U}\underline{S}_2^{-1}$, the function

$$z \mapsto \int_{c_2'} \underline{f}_1(\zeta)\underline{f}_2\left(\frac{z}{\zeta}\right)\frac{d\zeta}{\zeta}$$

is clearly holomorphic on U. Hence, the image of $[\omega|_U]$ in $\Omega(U\setminus\overline{V})/\Omega(U)$ is $[g(z)dz]$ where g is the holomorphic function on $U \setminus \overline{V}$ defined by setting

$$g(z) = \int_{c_3} \underline{f}_1(\zeta)\underline{f}_2\left(\frac{z}{\zeta}\right)\frac{d\zeta}{\zeta}.$$

Since

$$\operatorname{supp}(\partial c_2 - c_2') \subset \left(\mathbb{C}^* \setminus \left((\overline{U} \setminus V)\underline{S}_2^{-1}\right)\right) \cup \left(\mathbb{C}^* \setminus \overline{U}\underline{S}_2^{-1}\right) = \mathbb{C}^* \setminus \left((\overline{U} \setminus V)\underline{S}_2^{-1}\right)$$

and

$$\operatorname{supp}(c_1' - \partial c_1) \subset (\mathbb{C}^* \setminus \underline{S}_1) \cup (\mathbb{C}^* \setminus \underline{S}_1) = \mathbb{C}^* \setminus \underline{S}_1,$$

it is clear that

$$\operatorname{supp} c_3 \subset (\mathbb{C}^* \setminus \underline{S}_1) \cap \left(\mathbb{C}^* \setminus \left((\overline{U} \setminus V)\underline{S}_2^{-1}\right)\right).$$

Therefore, we have in fact

$$g(z) = \int_{c_3} f_1(\zeta)f_2\left(\frac{z}{\zeta}\right)\frac{d\zeta}{\zeta}$$

for any $z \in U \setminus \overline{V}$. Moreover, since $\partial c_3 = \partial c_1' = -\partial c_2'$, it is clear that

$$\operatorname{supp} \partial c_3 \subset (\mathbb{C}^* \setminus \underline{S}_1) \cap (\mathbb{C}^* \setminus \overline{U}\underline{S}_2^{-1}).$$

So,

$$c_3 \in Z_1\left((\mathbb{C}^* \setminus \underline{S}_1) \cap \left(\mathbb{C}^* \setminus \left((\overline{U} \setminus V)\underline{S}_2^{-1}\right)\right), (\mathbb{C}^* \setminus \underline{S}_1) \cap (\mathbb{C}^* \setminus \overline{U}\underline{S}_2^{-1})\right)$$

and it follows by construction that it is a relative Hadamard cycle for $\underline{S}_1$ with respect to $\overline{U}\underline{S}_2^{-1}$ in $\mathbb{C}^* \setminus (\overline{U} \setminus V)\underline{S}_2^{-1}$ (apply Remark 2.2.11 with $F = \underline{S}_1, G = \overline{U}\underline{S}_2^{-1}$ and $W = \mathbb{C}^* \setminus \left((\overline{U} \setminus V)\underline{S}_2^{-1}\right)$). Thus, c_3 is also a relative Hadamard cycle for S_1 with respect to $\overline{U}S_2^{-1}$ in $\mathbb{C}^* \setminus (\overline{U} \setminus V)S_2^{-1}$.

To conclude, it remains to show that if c_3' is another relative Hadamard cycle for S_1 with respect to $\overline{U}S_2^{-1}$ in $\mathbb{C}^* \setminus (\overline{U} \setminus V)S_2^{-1}$ and if

$$\check{g}(z) = \int_{c_3'} f_1(\zeta)f_2\left(\frac{z}{\zeta}\right)\frac{d\zeta}{\zeta}$$

for any $z \in U \setminus \overline{V}$, then $[g(z)dz] = [\check{g}(z)dz]$ in $\Omega(U\setminus\overline{V})/\Omega(U)$. For such a c_3', we have $[c_3] = [c_3']$ in

$$H_1\left((\mathbb{C}^* \setminus S_1) \cap \left(\mathbb{C}^* \setminus \left((\overline{U} \setminus V)S_2^{-1}\right)\right), (\mathbb{C}^* \setminus S_1) \cap (\mathbb{C}^* \setminus \overline{U}S_2^{-1})\right).$$

Therefore, $c'_3 = c_3 + c_4 + \partial c_5$ where c_4 is a 1-chain of $(\mathbb{C}^* \setminus S_1) \cap (\mathbb{C}^* \setminus \overline{U} S_2^{-1})$ and c_5 is a 2-chain of $(\mathbb{C}^* \setminus S_1) \cap (\mathbb{C}^* \setminus (\overline{U} \setminus V) S_2^{-1})$. It follows that the function

$$\check{g} : z \mapsto \int_{c'_3} f_1(\zeta) f_2\left(\frac{z}{\zeta}\right) \frac{d\zeta}{\zeta}$$

is a holomorphic function on $U \setminus \overline{V}$ and that

$$\check{g}(z) = g(z) + \int_{c_4} f_1(\zeta) f_2\left(\frac{z}{\zeta}\right) \frac{d\zeta}{\zeta}$$

on $U \setminus \overline{V}$. Since

$$z \mapsto \int_{c_4} f_1(\zeta) f_2\left(\frac{z}{\zeta}\right) \frac{d\zeta}{\zeta}$$

is clearly holomorphic on U, we have $[g(z)dz] = [\check{g}(z)dz]$ in $\Omega(U \setminus \overline{V})/\Omega(U)$ as expected. $\square$

2.2.3 Strongly convolvable sets

It is natural to ask whether one can compute the holomorphic cohomological multiplicative convolution on $\mathbb{C}^*$ thanks to a global formula, by adding extra-conditions on S_1 and S_2. Recalling Definition 2.1.11, we are led to introduce the following one :

Definition 2.2.15. Let S_1 and S_2 be two convolvable proper closed subsets of $\mathbb{C}^*$ such that $S_1 S_2 \neq \mathbb{C}^*$. These two closed sets are said to be *strongly convolvable* if, furthermore, $\overline{S}_1$ and $\overline{S}_2$ are star-eligible, that is to say, if $\overline{S}_1 \times \overline{S}_2 \subset M$. (Here $\overline{(.)}$ denotes the closure in $\mathbb{P}$.)

Remark 2.2.16. One can find convolvable proper closed subsets of $\mathbb{C}^*$ which are not strongly convolvable. For example, consider [1]

$$S_1 = \left\{ \frac{1}{(2m+1)!} : m \in \mathbb{N} \right\} \qquad \text{and} \qquad S_2 = \{(2n)! : n \in \mathbb{N}\}.$$

It is clear that $0 \in \overline{S}_1$ and $\infty \in \overline{S}_2$. Moreover, these sets are multiplicatively convolvable since

$$\operatorname{card}\left(S_1 \cap \frac{K}{S_2}\right) < \infty$$

for any compact subset K of $\mathbb{C}^*$. Let us prove it for

$$K = A_R = \{z \in \mathbb{C} : R^{-1} \leq |z| \leq R\}$$

with $R > 0$. (It is actually enough since any compact K of $\mathbb{C}^*$ can be included in such an annulus.)

[1] This example has originally been proposed by Daniel Fischer in `https://math.stackexchange.com/questions/1651882/closed-subsets-of-mathbbc-proper-for-multiplication`.

1) If $\frac{1}{R} \leq \frac{(2n)!}{(2m+1)!}$, then $m < n$ or $(2m+1) \leq R$. Indeed, if $R < 2m+1$ and $n \leq m$, then $(2n)! \leq (2m)!$. By multiplying the first and the third inequalities, one gets $(2n)!R < (2m+1)!$ and thus $\frac{1}{R} > \frac{(2n)!}{(2m+1)!}$.

2) If $\frac{(2n)!}{(2m+1)!} \leq R$ then $m \geq n$ or $2n \leq R$. Indeed, if $R < 2n$ and $m < n$ then $m + \frac{1}{2} \leq n - \frac{1}{2}$ and hence $2m+1 \leq 2n-1$. Therefore $(2m+1)! \leq (2n-1)!$ and, multiplying by the first inequality, we get $R(2m+1)! < (2n)!$ or, equivalently, $R < \frac{(2n)!}{(2m+1)!}$.

3) Now, let us assume that $m > \frac{R-1}{2}$. Using 1), we see that $\frac{(2n)!}{(2m+1)!} \in A_R$ only if $n > m$. But in this case, $n > m + \frac{1}{2}$ and thus $2n > 2m+1 > R$. Using 2), this shows that $\frac{(2n)!}{(2m+1)!} \notin A_R$ and leads to a contradiction. Hence, $\frac{(2n)!}{(2m+1)!} \in A_R$ necessarily implies that $m \leq \frac{R-1}{2}$. This proves that

$$\operatorname{card}\left(S_1 \cap \frac{A_R}{S_2}\right) = \operatorname{card}\left\{m \in \mathbb{N} : \exists n \in \mathbb{N}, \frac{1}{R} \leq \frac{(2n)!}{(2m+1)!} \leq R\right\} \leq \frac{R+1}{2}.$$

We shall now highlight the link with the generalized Hadamard product. Recall Definitions 2.1.12 and 2.1.13.

Proposition 2.2.17. *Let S_1 and S_2 be two strongly convolvable proper closed subsets of $\mathbb{C}^*$ and let us consider $\omega_1 = f_1 dz$ (resp. $\omega_2 = f_2 dz$) with $f_1 \in \mathcal{O}(\mathbb{C}^* \setminus S_1)$ (resp. $f_2 \in \mathcal{O}(\mathbb{C}^* \setminus S_2)$). For all $z \in \mathbb{C}^* \setminus S_1 S_2$, let c_z be a generalized Hadamard cycle for $\overline{S}_1$ in $\mathbb{P} \setminus (z\overline{S}_2^{-1} \cup (\{0,\infty\} \setminus \overline{S}_1))$. Then*

$$[\omega_1] \star [\omega_2] = [f dz] \in \Omega(\mathbb{C}^* \setminus S_1 S_2)/\Omega(\mathbb{C}^*),$$

where

$$f(z) = -\int_{c_z} f_1(\zeta) f_2\left(\frac{z}{\zeta}\right) \frac{d\zeta}{\zeta}$$

for all $z \in \mathbb{C}^* \setminus S_1 S_2$.

Proof. Let U be a relatively compact open subset of $\mathbb{C}^*$ and V an open neighbourhood of $S_1 S_2$ in $\mathbb{C}^*$. Let c be a relative Hadamard cycle for S_1 with respect to $\overline{U} S_2^{-1}$ in $\mathbb{C}^* \setminus (\overline{U} \setminus V) S_2^{-1}$. Then, by a similar argument as in the proof of Lemma 2.1.14, it is clear that the image of $[c_z]$ by the sequence of canonical maps

$$\begin{array}{c}
H_1(\mathbb{P} \setminus (\overline{S}_1 \cup z\overline{S}_2^{-1} \cup \{0\} \cup \{\infty\})) = H_1(\mathbb{C}^* \setminus (S_1 \cup z S_2^{-1})) \\
\downarrow \\
{}^{\mathrm{BM}}H_1(\mathbb{C}^* \setminus (S_1 \cup z S_2^{-1})) \\
\downarrow \\
{}^{\mathrm{BM}}H_1((\mathbb{C}^* \setminus S_1) \cap (\mathbb{C}^* \setminus (\overline{U} \setminus V) S_2^{-1})) \\
\downarrow \\
H_1((\mathbb{C}^* \setminus S_1) \cap (\mathbb{C}^* \setminus (\overline{U} \setminus V) S_2^{-1}), (\mathbb{C}^* \setminus S_1) \cap (\mathbb{C}^* \setminus \overline{U} S_2^{-1}))
\end{array}$$

is $[-c]$ for all $z \in \overline{U} \backslash V$. Hence

$$\int_c f_1(\zeta) f_2\left(\frac{z}{\zeta}\right) \frac{d\zeta}{\zeta} = -\int_{c_z} f_1(\zeta) f_2\left(\frac{z}{\zeta}\right) \frac{d\zeta}{\zeta}, \quad \forall z \in U \backslash \overline{V}.$$

Since this argument is valid for all U and all V, the conclusion follows from Theorem 2.2.12. □

In this context, let us set $(f_1 \star f_2)(z) = \frac{1}{2i\pi} \int_{c_z} f_1(\zeta) f_2\left(\frac{z}{\zeta}\right) \frac{d\zeta}{\zeta}$. If $f_1 \in \mathcal{O}(\mathbb{P} \backslash \overline{S}_1)$ and $f_2 \in \mathcal{O}(\mathbb{P} \backslash \overline{S}_2)$, this really coincides with the generalized Hadamard product.

Remark 2.2.18. Let S_1 and S_2 be two strongly convolvable proper closed subsets of $\mathbb{C}^*$. Let us make an identification $f dz \leftrightarrow -2i\pi f$ between holomorphic 1-forms and holomorphic functions. Then, by the previous proposition, the holomorphic cohomological convolution morphism

$$H^1_{S_1}(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \otimes H^1_{S_2}(\mathbb{C}^*, \Omega_{\mathbb{C}^*}) \to H^1_{S_1 S_2}(\mathbb{C}^*, \Omega_{\mathbb{C}^*})$$

can be seen as a bilinear map

$$\mathcal{O}(\mathbb{C}^* \backslash S_1)/\,\mathcal{O}(\mathbb{C}^*) \times \mathcal{O}(\mathbb{C}^* \backslash S_2)/\,\mathcal{O}(\mathbb{C}^*) \to \mathcal{O}(\mathbb{C}^* \backslash S_1 S_2)/\,\mathcal{O}(\mathbb{C}^*),$$

which can be computed by

$$[f_1] \star [f_2] = [f_1 \star f_2].$$

Example 2.2.19. Let $S = \mathbb{C}^* \backslash D(0, s)$ and $T = \mathbb{C}^* \backslash D(0, t)$ with $s > 0$, $t > 0$ and let

$$f \in \mathcal{O}(\mathbb{C}^* \backslash S) = \mathcal{O}(D(0, s) \backslash \{0\}) \quad \text{and} \quad g \in \mathcal{O}(\mathbb{C}^* \backslash T) = \mathcal{O}(D(0, t) \backslash \{0\})$$

be two holomorphic functions. Then, S and T are strongly convolvable proper closed subsets of $\mathbb{C}^*$ and we can write $f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n$, $g(z) = \sum_{n=-\infty}^{+\infty} b_n z^n$. Since the polar part of f (resp. g) is holomorphic on $\mathbb{C}^*$, we have $[f] = \left[\sum_{n=0}^{+\infty} a_n z^n\right]$ in $\mathcal{O}(D(0, s) \backslash \{0\})/\,\mathcal{O}(\mathbb{C}^*)$ and $[g] = \left[\sum_{n=0}^{+\infty} b_n z^n\right]$ in $\mathcal{O}(D(0, t) \backslash \{0\})/\,\mathcal{O}(\mathbb{C}^*)$. Using the preceding remark, we see that the holomorphic cohomological convolution $[f] \star [g]$ is given by

$$[f \star g] = \left[\sum_{n=0}^{+\infty} a_n b_n z^n\right],$$

since the generalized Hadamard product coincides with the usual one in this case.

Let us now state a trivial proposition :

Proposition 2.2.20. *Let S_1 and S_2 be two convolvable closed subsets of $\mathbb{C}^*$ and $S_1' \subset S_1$, $S_2' \subset S_2$ be two closed subsets. Then, S_1' and S_2' are convolvable and the diagram*

$$\begin{tikzcd}
H^1_{S_1}(\mathbb{C}^*,\Omega_{\mathbb{C}^*}) \otimes H^1_{S_2}(\mathbb{C}^*,\Omega_{\mathbb{C}^*}) \arrow[r] & H^1_{S_1S_2}(\mathbb{C}^*,\Omega_{\mathbb{C}^*}) \\
H^1_{S_1'}(\mathbb{C}^*,\Omega_{\mathbb{C}^*}) \otimes H^1_{S_2'}(\mathbb{C}^*,\Omega_{\mathbb{C}^*}) \arrow[r] \arrow[u] & H^1_{S_1'S_2'}(\mathbb{C}^*,\Omega_{\mathbb{C}^*}), \arrow[u]
\end{tikzcd}$$

where the horizontal arrows are given by the holomorphic cohomological convolution morphisms, is commutative.

Example 2.2.19 combined with Proposition 2.2.20 allows to compute several other examples.

Example 2.2.21. Let $S_1 = S_2 = (-\infty, -1]$. The principal determination of the function $z \mapsto \ln(1+z)$ is holomorphic on $\mathbb{C}^* \setminus S_1$. Moreover, S_1 and S_2 are strongly convolvable and thus, there is $g \in \mathcal{O}(\mathbb{C}^* \setminus [1, +\infty))$ such that

$$[\ln(1+z)] \star [\ln(1+z)] = [g].$$

Using the previous results, one has

$$\begin{aligned}
([\ln(1+z)] \star [\ln(1+z)])|_{D(0,1)} &= [\ln(1+z)|_{D(0,1)}] \star [\ln(1+z)|_{D(0,1)}] \\
&= \left[\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} z^n\right] \star \left[\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} z^n\right] \\
&= \left[\sum_{n=1}^{\infty} \frac{z^n}{n^2}\right] \\
&= [\mathrm{Li}_2(z)]|_{D(0,1)},
\end{aligned}$$

where Li_2 is the principal dilogarithm function, holomorphic on $\mathcal{O}(\mathbb{C}^* \setminus [1, +\infty))$. Hence, there is $h \in \mathcal{O}(\mathbb{C}^*)$ such that

$$g|_{D(0,1)} - \mathrm{Li}_2|_{D(0,1)} = h.$$

By the uniqueness of the analytic continuation, one deduces that $g - \mathrm{Li}_2 = h$ on $\mathbb{C}^* \setminus [1, +\infty)$ and, thus, that

$$[\ln(1+z)] \star [\ln(1+z)] = [\mathrm{Li}_2(z)]$$

in $\mathcal{O}(\mathbb{C}^* \setminus S_1S_2)/\mathcal{O}(\mathbb{C}^*)$.

2.2.4 Additive convolution on $\mathbb{C}$

Of course, the content of Section 2.2.2 can be adapted to study the additive holomorphic convolution on $(\mathbb{C}, +)$, i.e. the map

$$H^1_{S_1}(\mathbb{C}, \Omega_{\mathbb{C}}) \otimes H^1_{S_2}(\mathbb{C}, \Omega_{\mathbb{C}}) \to H^1_{S_1+S_2}(\mathbb{C}, \Omega_{\mathbb{C}}),$$

where S_1 and S_2 are additively convolvable proper closed subsets of $\mathbb{C}$ such that $S_1 + S_2 \neq \mathbb{C}$. This map can be interpreted as a bilinear map

$$\Omega(\mathbb{C} \setminus S_1)/\Omega(\mathbb{C}) \times \Omega(\mathbb{C} \setminus S_1)/\Omega(\mathbb{C}) \to \Omega(\mathbb{C} \setminus (S_1 + S_2))/\Omega(\mathbb{C}).$$

Theorem 2.2.12 becomes

Theorem 2.2.22. *Let S_1 and S_2 be two additively convolvable proper closed subsets of $\mathbb{C}$ such that $S_1 + S_2 \neq \mathbb{C}$ and let us consider $\omega_1 = f_1 dz$ (resp. $\omega_2 = f_2 dz$) with $f_1 \in \mathcal{O}(\mathbb{C} \setminus S_1)$ (resp. $f_2 \in \mathcal{O}(\mathbb{C} \setminus S_2)$). Fix a relatively compact open subset U of $\mathbb{C}$ and an open neighbourhood V of $S_1 + S_2$ in $\mathbb{C}$. Then, the image of*

$$[\omega_1] \star [\omega_2] \in \Omega(\mathbb{C} \setminus (S_1 + S_2))/\Omega(\mathbb{C}) \simeq H^1_{S_1+S_2}(\mathbb{C}, \Omega_{\mathbb{C}})$$

in

$$H^1_{\overline{V} \cap U}(U, \Omega_{\mathbb{C}})$$

is the class of the form $\omega = f dz \in \Omega(U \setminus \overline{V})$ where

$$f(z) = \int_c f_1(\zeta) f_2(z - \zeta) d\zeta$$

and c is a relative Hadamard cycle for S_1 with respect to $\overline{U} - S_2$ in $\mathbb{C} \setminus ((\overline{U} \setminus V) - S_2)$.

Section 2.2.3 can also be adapted in a trivial way. Indeed, it would be natural to say that two proper closed subsets S_1 and S_2 of $\mathbb{C}$ are strongly additively convolvable if $(\infty, \infty) \notin \overline{S}_1 \times \overline{S}_2$. However, this would simply imply that one of the two closed sets is a compact subset of $\mathbb{C}$. In that case, Proposition 2.2.17 becomes

Proposition 2.2.23. *Let S_1 be a non-empty compact subset of $\mathbb{C}$ and S_2 be a proper closed subset of $\mathbb{C}$. Consider $\omega_1 = f_1 dz$ (resp. $\omega_2 = f_2 dz$) with $f_1 \in \mathcal{O}(\mathbb{C} \setminus S_1)$ (resp. $f_2 \in \mathcal{O}(\mathbb{C} \setminus S_2)$). For all $z \in \mathbb{C} \setminus (S_1 + S_2)$, let c_z be a cycle in $\mathbb{C} \setminus (S_1 \cup (z - S_2))$ such that*

$$\mathrm{Ind}(c_z, \zeta) = \begin{cases} 1 & \text{if } \zeta \in S_1 \\ 0 & \text{if } \zeta \in z - S_2. \end{cases}$$

Then

$$[\omega_1] \star [\omega_2] = [f dz] \in \Omega(\mathbb{C} \setminus (S_1 + S_2))/\Omega(\mathbb{C}),$$

where

$$f(z) = \int_{c_z} f_1(\zeta) f_2(z - \zeta) d\zeta$$

for all $z \in \mathbb{C} \setminus (S_1 + S_2)$.

Additive convolution (e.g. of distributions) is in general interesting because of its compatibility with the Laplace transform. It seems thus reasonable to ask whether there is a kind of "contour-integration-type" Laplace transform, which would be compatible with the additive holomorphic cohomological convolution. We introduce such a transform in the next chapter.

Chapter 3

Analytic functionals with convex carrier

This chapter invokes some classical concepts and results of functional analysis. Concerning this matter, we refer to standard books, e.g. [39], [48], [50] and [101].

3.1 The compact case

3.1.1 Polya's theorem

The Polya-Ehrenpreis-Martineau theorem, or simply Polya's theorem, states that three particular functional spaces, built from a convex compact subset K of $\mathbb{C}$, are topologically isomorphic through some integral transforms, including a "contour-integration-type" Laplace transform. The original pieces of this construction can be found in [75] and [89] (see also [3], [11] and [17]). A complete and detailed proof of the theorem can be found in the first chapter of [6]. In this section, we shall mainly rely on this last reference.

Definition 3.1.1. An *analytic functional* T on an open subset U of $\mathbb{C}$ is an element of $\mathcal{O}'(U)$, i.e. a continuous linear map

$$T : \mathcal{O}(U) \to \mathbb{C},$$

where $\mathcal{O}(U)$ has its usual Fréchet topology.

Let K be a compact subset of $\mathbb{C}$. The space of *analytic functionals carried by* K is defined by

$$\mathcal{O}'(K) = \left(\varinjlim_{U \supset K} \mathcal{O}(U) \right)'.$$

Definition 3.1.2. For any compact subset K of $\mathbb{C}$, we set

$$\mathcal{O}^0(\mathbb{C} \setminus K) = \{ f \in \mathcal{O}(\mathbb{C} \setminus K) : \lim_{z \to \infty} f(z) = 0 \}.$$

Let us identify $\mathbb{C}$ and its dual in such a way that $\langle z, w\rangle = zw$.

Definition 3.1.3. For any non-empty convex compact subset K of $\mathbb{C}$, we set

$$\operatorname{Exp}(K) = \{g \in \mathcal{O}(\mathbb{C}) : \forall \varepsilon > 0, \sup_{w\in\mathbb{C}} |g(w)|e^{-h_K(w)-\varepsilon|w|} < \infty\}.$$

These three spaces have a canonical structure of Fréchet space (see section 1.4 of [6]). We can now define some transforms between them. We fix once for all a non-empty convex compact subset K of $\mathbb{C}$.

Definition 3.1.4. Let $T \in \mathcal{O}'(K)$. The *Fourier-Borel transform* of T, noted $\mathcal{F}(T)$, is defined by

$$\mathcal{F}(T) : w \in \mathbb{C} \mapsto \langle T_\zeta, e^{w\zeta}\rangle.$$

Proposition 3.1.5. *For any $T \in \mathcal{O}'(K)$, the Fourier-Borel transform $\mathcal{F}(T)$ is an element of* $\operatorname{Exp}(K)$*. Moreover, the map*

$$\mathcal{F} : \mathcal{O}'(K) \to \operatorname{Exp}(K)$$

is linear and continuous.

Definition 3.1.6. Let $T \in \mathcal{O}'(K)$. The *Cauchy transform* of T, noted $\mathcal{C}(T)$, is defined by

$$\mathcal{C}(T) : z \in \mathbb{C}\setminus K \mapsto \left\langle T_\zeta, \frac{1}{z-\zeta}\right\rangle.$$

Proposition 3.1.7. *For any $T \in \mathcal{O}'(K)$, the Cauchy transform $\mathcal{C}(T)$ is an element of $\mathcal{O}^0(\mathbb{C}\setminus K)$. Moreover, the map*

$$\mathcal{C} : \mathcal{O}'(K) \to \mathcal{O}^0(\mathbb{C}\setminus K)$$

is linear and continuous.

Definition 3.1.8. Let $r > 0$ such that $K \subset D(0, r)$. If $f \in \mathcal{O}^0(\mathbb{C}\setminus K)$, *the Polya transform* of f, noted $\mathcal{P}(f)$, is defined by

$$\mathcal{P}(f) : w \mapsto \frac{1}{2i\pi}\int_{C(0,r)^+} e^{zw} f(z)dz.$$

Obviously, this integral does not depend on the chosen r.

Proposition 3.1.9. *For any $f \in \mathcal{O}^0(\mathbb{C}\setminus K)$, the Polya transform $\mathcal{P}(f)$ is an element of* $\operatorname{Exp}(K)$*. Moreover, the map*

$$\mathcal{P} : \mathcal{O}^0(\mathbb{C}\setminus K) \to \operatorname{Exp}(K)$$

is linear and continuous.

Theorem 3.1.10 (Polya-Ehrenpreis-Martineau)**.**

$$\begin{tikzcd} \mathcal{O}^0(\mathbb{C}\setminus K) \arrow[rr, "\mathcal{P}"] & & \mathrm{Exp}(K) \\ & \mathcal{O}'(K) \arrow[ul, "\mathcal{C}"] \arrow[ur, "\mathcal{F}"'] & \end{tikzcd}$$

is a commutative diagram of topological isomorphisms.

Without recalling the proof in details, let us nonetheless explain how to build the inverse of $\mathcal{P}$, which is often called the *Borel transform*.

Recall Theorem 1.5.4 and Remark 1.5.16. Let $g \in \mathrm{Exp}(K)$. For any $\xi \in \mathbb{C}$ such that $|\xi| = 1$, we set

$$\mathcal{B}^{\xi}(g) : z \mapsto \int_{\xi[0,+\infty)} e^{-zw} g(w)dw.$$

This function is well-defined and holomorphic on $U_\xi = \{z \in \mathbb{C} : \Re(z\xi) > h_K(\xi)\}$. If $\xi \neq \xi'$, it is not difficult to see that $\mathcal{B}^{\xi}(g)(z) = \mathcal{B}^{\xi'}(g)(z)$ for all $z \in U_\xi \cap U_{\xi'}$. Hence, by gluing the $\mathcal{B}^{\xi}(g)$, we obtain a function $\mathcal{B}(g)$ which is holomorphic on

$$\bigcup_{\{\xi\in\mathbb{C}:|\xi|=1\}} \{z \in \mathbb{C} : \Re(z\xi) > h_K(\xi)\} = \mathbb{C}\setminus K.$$

Proposition 3.1.11. *For any $g \in \mathrm{Exp}(K)$, the Borel transform $\mathcal{B}(g)$ is an element of $\mathcal{O}^0(\mathbb{C}\setminus K)$. Moreover, the map*

$$\mathcal{B} : \mathrm{Exp}(K) \to \mathcal{O}^0(\mathbb{C}\setminus K)$$

is linear, continuous and the inverse of $\mathcal{P}$.

In chapter 5, we shall see how the bijectivity of $\mathcal{P}$, which is actually the only crucial point of Polya's theorem, can be obtained thanks to cohomological arguments.

3.1.2 Associated convolution

Convolution of analytic functionals with compact carrier can be defined by mimicking the definition of convolution of Schwartz distributions. This notably leads to the theory of convolution equations (see e.g. [8], [6] and [67]). In this section, we simply recall the definition of this convolution product as well as its compatibility with the Fourier-Borel transform (see [6], section 1.5 for more details).

Let us fix two non-empty compact subsets K_1 and K_2 of $\mathbb{C}$ as well as two compactly-carried analytic functionals $T_1 \in \mathcal{O}'(K_1)$ and $T_2 \in \mathcal{O}'(K_2)$.

Proposition 3.1.12. *Let $\varphi \in \mathcal{O}(U)$ where U is an open neighbourhood of $K_1 + K_2$. Then, the application*

$$\zeta_1 \mapsto \langle (T_2)_{\zeta_2}, \varphi(\zeta_1 + \zeta_2)\rangle$$

is well-defined and holomorphic on an open neighbourhood of K_1.

Definition 3.1.13. *The convolution product* of T_1 and T_2 is defined by

$$\langle T_1 \star T_2, \varphi \rangle = \langle (T_1)_{\zeta_1}, \langle (T_2)_{\zeta_2}, \varphi(\zeta_1 + \zeta_2)\rangle\rangle$$

for all $\varphi \in \mathcal{O}(K_1 + K_2)$.

Proposition 3.1.14. *For any $T_1 \in \mathcal{O}'(K_1)$ and $T_2 \in \mathcal{O}'(K_2)$, the convolution product $T_1 \star T_2$ is an element of $\mathcal{O}'(K_1 + K_2)$. Moreover, the convolution map*

$$\star : \mathcal{O}'(K_1) \times \mathcal{O}'(K_2) \to \mathcal{O}'(K_1 + K_2)$$

is bilinear and continuous on each factor.

The link with the Fourier-Borel transform is of course immediate.

Proposition 3.1.15. *For any $T_1 \in \mathcal{O}'(K_1)$ and $T_2 \in \mathcal{O}'(K_2)$, one has*

$$\mathcal{F}(T_1 \star T_2) = \mathcal{F}(T_1)\mathcal{F}(T_2).$$

Proof. Let $w \in \mathbb{C}$. One has

$$\begin{aligned}
\mathcal{F}(T_1 \star T_2)(w) &= \langle (T_1 \star T_2)_\zeta, e^{w\zeta}\rangle \\
&= \langle (T_1)_{\zeta_1}, \langle (T_2)_{\zeta_2}, e^{w(\zeta_1+\zeta_2)}\rangle\rangle \\
&= \langle (T_1)_{\zeta_1}, e^{w\zeta_1}\rangle \langle (T_2)_{\zeta_2}, e^{w\zeta_2}\rangle \\
&= \mathcal{F}(T_1)(w)\mathcal{F}(T_2)(w),
\end{aligned}$$

hence the conclusion. □

Let us now assume that K_1 and K_2 are convex. In this case, Theorem 3.1.10 is applicable and one sees that the Fourier-Borel transform is an isomorphism which interchanges the convolution product of analytic functionals and the standard product of functions. Similarly, thanks to the Cauchy transform $\mathcal{C}$, the convolution can be carried to a bilinear map

$$\star : \mathcal{O}^0(\mathbb{C} \setminus K_1) \times \mathcal{O}^0(\mathbb{C} \setminus K_2) \to \mathcal{O}^0(\mathbb{C} \setminus (K_1 + K_2)). \tag{3.1}$$

We shall study (3.1) in the next section to prove that the convolution of analytic functionals is another avatar of the holomorphic cohomological convolution.

3.1.3 Link with the holomorphic cohomological convolution

Proposition 3.1.16. *Let $f_1 \in \mathcal{O}^0(\mathbb{C} \setminus K_1)$ and $f_2 \in \mathcal{O}^0(\mathbb{C} \setminus K_2)$. Let us choose a cycle c_z in $\mathbb{C} \setminus (K_1 \cup (z - K_2))$, for all $z \in \mathbb{C} \setminus (K_1 + K_2)$, which verifies*

$$\operatorname{Ind}(c_z, \zeta) = \begin{cases} 1 & \text{if } \zeta \in K_1 \\ 0 & \text{if } \zeta \in z - K_2. \end{cases}$$

Then

$$(f_1 \star f_2)(z) = \frac{1}{2i\pi} \int_{c_z} f_1(\zeta) f_2(z - \zeta) d\zeta$$

for all $z \in \mathbb{C} \setminus (K_1 + K_2)$.

Proof. Let $f_1 \in \mathcal{O}^0(\mathbb{C}\setminus K_1)$ and $f_2 \in \mathcal{O}^0(\mathbb{C}\setminus K_2)$.

1) It is clear that such cycles c_z exist for all $z \in \mathbb{C}\setminus(K_1+K_2)$ and that

$$z \mapsto \frac{1}{2i\pi}\int_{c_z} f_1(\zeta)f_2(z-\zeta)d\zeta$$

is holomorphic on $\mathbb{C}\setminus(K_1+K_2)$. This can be proved by adapting Lemma 2.1.14 to the additive case and then by differentiating under the integral sign.

2) One has

$$\begin{aligned} f_1 \star f_2 &= \mathcal{C}(\mathcal{C}^{-1}(f_1)\star\mathcal{C}^{-1}(f_2)) \\ &= \mathcal{B}(\mathcal{F}(\mathcal{C}^{-1}(f_1)\star\mathcal{C}^{-1}(f_2))) \\ &= \mathcal{B}(\mathcal{F}(\mathcal{C}^{-1}(f_1))\mathcal{F}(\mathcal{C}^{-1}(f_1))) \\ &= \mathcal{B}(\mathcal{P}(f_1)\mathcal{P}(f_2)). \end{aligned}$$

3) Let $r_1, r_2 > 0$ be such that $K_1 \subset D(0,r_1), K_2 \subset D(0,r_2)$. Let $z \in \mathbb{C}\setminus(K_1+K_2)$ be such that $\Re(z) > r_1+r_2$. Then by the definitions, one has

$$\begin{aligned} &(f_1\star f_2)(z) = \mathcal{B}(\mathcal{P}(f_1)\mathcal{P}(f_2))(z) \\ &= \left(\frac{1}{2i\pi}\right)^2 \int_0^{+\infty} e^{-zw}\left(\int_{C(0,r_1)^+} e^{z_1 w} f_1(z_1)dz_1\right)\left(\int_{C(0,r_2)^+} e^{z_2 w} f_2(z_2)dz_2\right)dw \\ &= \left(\frac{1}{2i\pi}\right)^2 \int_{C(0,r_1)^+}\int_{C(0,r_2)^+}\left(\int_0^{+\infty} e^{(z_1+z_2-z)w}dw\right) f_1(z_1)f_2(z_2)dz_2dz_1 \\ &= \left(\frac{1}{2i\pi}\right)^2 \int_{C(0,r_1)^+} f_1(z_1)\left(\int_{C(0,r_2)^+}\frac{f_2(z_2)}{z-(z_1+z_2)}dz_2\right)dz_1 \\ &= \frac{1}{2i\pi}\int_{C(0,r_1)^+} f_1(z_1)f_2(z-z_1)dz_1. \end{aligned}$$

Here, the last equality follows from the residue theorem. Indeed, it is clear that $z - z_1 \in \mathbb{C}\setminus\overline{D}(0,r_2)$ for all $z_1 \in C(0,r_1)$ and thus

$$\begin{aligned} &\int_{C(0,r_2)^+}\frac{f_2(z_2)}{z-(z_1+z_2)}dz_2 \\ &= -2i\pi\left(\mathrm{Res}_{z_2=z-z_1}\left(\frac{f_2(z_2)}{z-(z_1+z_2)}\right) - \mathrm{Res}_{z_2=\infty}\left(\frac{f_2(z_2)}{z-(z_1+z_2)}\right)\right) \\ &= -2i\pi\left(\lim_{z_2\to z-z_1}(z_2-(z-z_1))\frac{f_2(z_2)}{z-(z_1+z_2)} + 0\right) \\ &= 2i\pi f_2(z-z_1), \end{aligned}$$

the second equality following from $\lim_{z_2\to\infty} f_2(z_2) = 0$.

4) For all z such that $\Re z > r_1 + r_2$, it is clear that

$$\frac{1}{2i\pi}\int_{c_z} f_1(\zeta)f_2(z-\zeta)d\zeta = \frac{1}{2i\pi}\int_{C(0,r_1)^+} f_1(\zeta)f_2(z-\zeta)d\zeta$$

because c_z and $C(0,r_1)^+$ share the same winding number conditions with respect to K_1 and $z-K_2$ (recall Proposition 1.3.7). Since $f_1 \star f_2$ is holomorphic on $\mathbb{C}\setminus(K_1+K_2)$, the uniqueness of the holomorphic extension allows to conclude. $\square$

Remark 3.1.17. The equality $\mathcal{P}(f_1 \star f_2) = \mathcal{P}(f_1)\mathcal{P}(f_2)$ can be seen as the contour-integration analogue of the usual compatibility theorem between the Laplace transform and the convolution product of functions/distributions.

Let K be a non-empty compact subset of $\mathbb{C}$. Thanks to Liouville's theorem, the map

$$i_K : \mathcal{O}^0(\mathbb{C}\setminus K) \to \Omega(\mathbb{C}\setminus K)/\Omega(\mathbb{C})$$

defined by $i_K(f) = \left[\frac{1}{2i\pi} f dz\right]$ is injective. Moreover, if $f dz \in \Omega(\mathbb{C}\setminus K)$ and if $r > 0$ is such that $K \subset D(0,r)$, then $f(z) = \sum_{m=-\infty}^{+\infty} a_m z^m$ for all $z \in \mathbb{C}\setminus\overline{D}(0,r)$ and $g(z) = \sum_{m=0}^{+\infty} a_m z^m$ is holomorphic on $\mathbb{C}$. Thus

$$[f dz] = [(f-g)dz]$$

in $\Omega(\mathbb{C}\setminus K)/\Omega(\mathbb{C})$ and in addition $\lim_{z\to\infty}(f(z)-g(z)) = 0$. This proves that i_K is also surjective.

Proposition 3.1.18. *Let K_1 and K_2 be two non-empty convex compact subsets of $\mathbb{C}$. Then the following diagram is commutative :*

$$\begin{tikzcd}
H^1_{K_1}(\mathbb{C},\Omega_{\mathbb{C}}) \times H^1_{K_2}(\mathbb{C},\Omega_{\mathbb{C}}) \arrow[r] & H^1_{K_1+K_2}(\mathbb{C},\Omega_{\mathbb{C}}) \\
\Omega(\mathbb{C}\setminus K_1)/\Omega(\mathbb{C}) \times \Omega(\mathbb{C}\setminus K_2)/\Omega(\mathbb{C}) \arrow[r] \arrow[u, "\wr"'] & \Omega(\mathbb{C}\setminus(K_1+K_2))/\Omega(\mathbb{C}) \arrow[u, "\wr"'] \\
\mathcal{O}^0(\mathbb{C}\setminus K_1) \times \mathcal{O}^0(\mathbb{C}\setminus K_2) \arrow[r] \arrow[u, "\wr"', "i_{K_1}\times i_{K_2}"] & \mathcal{O}^0(\mathbb{C}\setminus(K_1+K_2)) \arrow[u, "\wr"', "i_{K_1+K_2}"] \\
\mathcal{O}'(K_1) \times \mathcal{O}'(K_2) \arrow[r] \arrow[u, "\wr"', "\mathcal{C}\times\mathcal{C}"] & \mathcal{O}'(K_1+K_2) \arrow[u, "\wr"', "\mathcal{C}"]
\end{tikzcd}$$

Here, the two top (resp. bottom) horizontal arrows are given by the additive holomorphic cohomological convolution (resp. convolution of analytic functionals).

Proof. Immediate in view of Propositions 2.2.23 and 3.1.16. $\square$

3.2 The non-compact case

The difficulty dramatically increases if one wants to develop a non-compact analogue of this last result. In order to set a complete theory in the unbounded case, we first need to extend the Polya-Ehrenpreis-Martineau theorem for non-compact convex sets. This has already been done by J.W. De Roever in [99] by using L^2-estimations and the fundamental principle of Ehrenpreis-Palamodov. Later, adapting the considerations developed by M. Morimoto in [80], A. Méril exposed a more explicit and less technical proof of the unbounded case in [77]. We shall present this version in the next section.

3.2.1 Méril's theorem

Recall Theorem 1.5.13. Let us fix a proper non-compact closed convex subset S of $\mathbb{C}$ which does not contain any line. Hence S_∞ is a proper closed convex salient cone and $(S_\infty^*)^\circ \neq \emptyset$. The convex S is in duality with (h_S, C), where $C^\circ = (S_\infty^*)^\circ$. We also fix a reference point $\xi_0 \in C^\circ$.

Definition 3.2.1. Let $\varepsilon, \varepsilon' > 0$. We note $Q_{\varepsilon,\varepsilon'}(S)$ the Banach space of holomorphic functions $\varphi \in \mathcal{O}(S_\varepsilon^\circ)$ such that

$$||\varphi||_{\varepsilon,\varepsilon'} = \sup_{\zeta \in S_\varepsilon^\circ} |e^{-\varepsilon'\xi_0\zeta}\varphi(\zeta)| < \infty.$$

Proposition 3.2.2. *Let $\varepsilon > \varepsilon_1 > 0$. The complex derivation operator*

$$\varphi \mapsto \varphi' = \frac{\partial\varphi}{\partial\zeta}$$

is a continuous application from $Q_{\varepsilon,\varepsilon'}(S)$ to $Q_{\varepsilon_1,\varepsilon'}(S)$.

Proposition 3.2.3. *If $\varepsilon > \varepsilon_1 > 0$ and $\varepsilon' > \varepsilon'_1 > 0$, the canonical restriction*

$$r^{\varepsilon,\varepsilon'}_{\varepsilon_1,\varepsilon'_1} : Q_{\varepsilon,\varepsilon'}(S) \to Q_{\varepsilon_1,\varepsilon'_1}(S)$$

is continuous and compact. We note $Q(S)$ the topological inductive limit of this inductive system of Banach spaces. It is a D.F.S. space.

Definition 3.2.4. The strong dual $Q'(S)$ of $Q(S)$ is called the space of *analytic functionals carried by S.*

Remark 3.2.5. In virtue of [66], Theorem 12, $Q'(S)$ is canonically topologically isomorphic to $\varprojlim\limits_{\varepsilon,\varepsilon' \to 0} Q'_{\varepsilon,\varepsilon'}(S)$.

Definition 3.2.6. Let $\varepsilon' > 0$. We note $R(\mathbb{C} \setminus S, \varepsilon')$ the subspace of $\mathcal{O}(\mathbb{C} \setminus S)$ built with functions f such that

$$q^{\varepsilon'}_{\varepsilon,r}(f) = \sup_{z \in S_r \setminus S_\varepsilon^\circ} |e^{\varepsilon'\xi_0 z} f(z)| < \infty$$

for all $r > \varepsilon > 0$. We also note $R(\mathbb{C}, \varepsilon')$ the space of entire functions f such that

$$p_\varepsilon^{\varepsilon'}(f) = \sup_{z\in S_\varepsilon} |e^{\varepsilon'\xi_0 z} f(z)| < \infty$$

for all $\varepsilon > 0$.

Proposition 3.2.7. *For all $\varepsilon' > 0$, the locally convex spaces $(R(\mathbb{C}\setminus S, \varepsilon'), (q_{\varepsilon,r}^{\varepsilon'})_{r>\varepsilon>0})$ and $(R(\mathbb{C}, \varepsilon'), (p_\varepsilon^{\varepsilon'})_{\varepsilon>0})$ are Fréchet spaces. Moreover, $R(\mathbb{C}, \varepsilon')$ is a closed subset of $R(\mathbb{C}\setminus S, \varepsilon')$ and the quotient*

$$R(\mathbb{C}\setminus S, \varepsilon')/R(\mathbb{C}, \varepsilon')$$

is also a Fréchet space. If $\varepsilon' > \varepsilon'_1$, there is a canonical linear and continuous injection

$$i_{\varepsilon',\varepsilon'_1} : R(\mathbb{C}\setminus S, \varepsilon'_1)/R(\mathbb{C}, \varepsilon'_1) \to R(\mathbb{C}\setminus S, \varepsilon')/R(\mathbb{C}, \varepsilon').$$

The system defined by $(i_{\varepsilon',\varepsilon'_1})_{\varepsilon'>\varepsilon'_1>0}$ is projective.

Definition 3.2.8. We set

$$\mathscr{H}_S(\mathbb{C}, \varepsilon') = R(\mathbb{C}\setminus S, \varepsilon')/R(\mathbb{C}, \varepsilon')$$

and

$$\mathscr{H}_S(\mathbb{C}) = \varprojlim_{\varepsilon'\to 0} \mathscr{H}_S(\mathbb{C}, \varepsilon').$$

Remark 3.2.9. An element $F \in \mathscr{H}_S(\mathbb{C})$ can be seen as a family $([f_{\varepsilon'}])_{\varepsilon'>0}$ where $f_{\varepsilon'} \in R(\mathbb{C}\setminus S, \varepsilon')$ for all $\varepsilon' > 0$ and where $f_{\varepsilon'} - f_{\varepsilon'_1} \in R(\mathbb{C}, \varepsilon')$ for all $\varepsilon' > \varepsilon'_1 > 0$.

Definition 3.2.10. We set

$$\operatorname{Exp}(S) = \{g \in \mathcal{O}((S^*_\infty)^\circ) : \forall \varepsilon, \varepsilon' > 0, \sup_{w\in S^\star_\infty + \varepsilon'\xi_0} |g(w)| e^{-h_S(w) - \varepsilon|w|} < \infty\}.$$

Without giving the details, let us just say that the natural topology on $\operatorname{Exp}(S)$ is nuclear and Fréchet.

Remark 3.2.11. The spaces $Q(S)$, $Q'(S)$, $\mathscr{H}_S(\mathbb{C})$ and $\operatorname{Exp}(S)$ do not depend on the reference point $\xi_0 \in C^\circ$ (see [78]).

We can now define, as in the compact case, the transforms $\mathcal{F}, \mathcal{C}, \mathcal{P}$ and $\mathcal{B}$.

Proposition 3.2.12. *Let $z \in \mathbb{C}\setminus S$. For all $\varepsilon' > 0$, the function*

$$\zeta \mapsto \frac{e^{\varepsilon'\xi_0(\zeta - z)}}{z - \zeta}$$

is an element of $Q_{\varepsilon,\varepsilon'}(S)$ for $\varepsilon > 0$ small enough. Let $T \in Q'(S)$. The function

$$\mathcal{C}^{\varepsilon'}(T) : z \in \mathbb{C}\setminus S \mapsto \left\langle T_\zeta, \frac{e^{\varepsilon'\xi_0(\zeta - z)}}{z - \zeta} \right\rangle$$

is an element of $R(\mathbb{C}\setminus S, \varepsilon')$ *for all* $\varepsilon' > 0$. *Moreover* $\mathcal{C}^{\varepsilon'}(T) - \mathcal{C}^{\varepsilon'_1}(T) \in R(\mathbb{C}, \varepsilon')$ *for all* $\varepsilon' > \varepsilon'_1 > 0$. *Therefore, this family determines a unique element* $\mathcal{C}(T)$ *in* $\mathscr{H}_S(\mathbb{C})$. *The map*

$$\mathcal{C} : Q'(S) \to \mathscr{H}_S(\mathbb{C})$$

is linear and continuous.

Proposition 3.2.13. *Let* $w \in (S^*_\infty)^\circ$. *The function* $\zeta \mapsto e^{w\zeta}$ *is an element of* $Q(S)$. *Let* $T \in Q'(S)$. *The function*

$$w \in (S^*_\infty)^\circ \mapsto \langle T_\zeta, e^{w\zeta}\rangle$$

is an element of $\mathrm{Exp}(S)$ *that we shall denote by* $\mathcal{F}(T)$. *The map*

$$\mathcal{F} : Q'(S) \mapsto \mathrm{Exp}(S)$$

is linear and continuous.

Recall that the boundary of a plane convex set (which is not a strip) is always a rectifiable curve.

Proposition 3.2.14. *Let* $F = ([f_{\varepsilon'}])_{\varepsilon'>0} \in \mathscr{H}_S(\mathbb{C})$. *For all* $w \in (S^*_\infty)^\circ$, *the integral*

$$\int_{\partial S^+_\varepsilon} e^{zw} f_{\varepsilon'}(z)dz$$

is well-defined and independent of $\varepsilon, \varepsilon'$. *The function*

$$w \in (S^*_\infty)^\circ \mapsto \frac{1}{2i\pi}\int_{\partial S^+_\varepsilon} e^{zw} f_{\varepsilon'}(z)dz$$

is an element of $\mathrm{Exp}(S)$ *that we shall denote by* $\mathcal{P}(F)$. *The map*

$$\mathcal{P} : \mathscr{H}_S(\mathbb{C}) \to \mathrm{Exp}(S)$$

is linear and continuous.

Theorem 3.2.15 (Méril)**.**

$$\begin{tikzcd}
\mathscr{H}_S(\mathbb{C}) \arrow[rr, "\mathcal{P}"] & & \mathrm{Exp}(S) \\
& Q'(S) \arrow[ul, "\mathcal{C}"] \arrow[ur, "\mathcal{F}"'] &
\end{tikzcd}$$

is a commutative diagram of topological isomorphisms.

Let $g \in \mathrm{Exp}(S)$ and $\varepsilon' > 0$. For any $\xi \in C$ such that $|\xi| = 1$, we set

$$\mathcal{B}^{\varepsilon',\xi}(g) : z \mapsto \int_{\varepsilon'\xi_0+\xi[0,+\infty)} e^{-zw} g(w)dw.$$

This function is well-defined and holomorphic on $U_\xi = \{z \in \mathbb{C} : \Re(z\xi) > h_S(\xi)\}$. If $\xi \neq \xi'$, one can see that $\mathcal{B}^{\varepsilon',\xi}(g)(z) = \mathcal{B}^{\varepsilon',\xi'}(g)(z)$ for all $z \in U_\xi \cap U_{\xi'}$. Hence, by gluing the $\mathcal{B}^{\varepsilon',\xi}(g)$, we obtain a function $\mathcal{B}^{\varepsilon'}(g)$ which is holomorphic on

$$\bigcup_{\{\xi \in C : |\xi|=1\}} \{z \in \mathbb{C} : \Re(z\xi) > h_S(\xi)\} = \mathbb{C} \setminus S.$$

Proposition 3.2.16. *For any $g \in \mathrm{Exp}(S)$ and $\varepsilon' > 0$, the function $\mathcal{B}^{\varepsilon'}(g)$ is an element of $R(\mathbb{C} \setminus S, \varepsilon')$. Moreover $\mathcal{B}^{\varepsilon'}(T) - \mathcal{B}^{\varepsilon'_1}(T) \in R(\mathbb{C}, \varepsilon')$ for all $\varepsilon' > \varepsilon'_1 > 0$. Therefore, this family determines a unique element $\mathcal{B}(g)$ in $\mathscr{H}_S(\mathbb{C})$. The map*

$$\mathcal{B} : \mathrm{Exp}(S) \to \mathscr{H}_S(\mathbb{C})$$

is linear, continuous and the inverse of $\mathcal{P}$.

As for the compact case, we shall see in chapter 5 that the bijectivity of $\mathcal{P}$ can be derived from a cohomological point of view.

3.2.2 Convolution on compatible convex sets

In our knowledge, the general definition of the convolution product between analytic functionals with non-compact convex carrier has never been discussed in the literature. However, a particular case has been treated in [78] and has already led to interesting applications. In order to define a general convolution map

$$\star : Q'(S_1) \times Q'(S_2) \to Q'(S_1 + S_2),$$

it is pretty natural to introduce the following definition :

Definition 3.2.17. Let S_1 (resp. S_2) be a proper non-compact closed convex subset of $\mathbb{C}$ which contains no lines, in duality with (h_{S_1}, C_1) (resp. (h_{S_2}, C_2)). These two sets are *compatible* if $S_1 + S_2$ is a proper non-compact closed convex subset of $\mathbb{C}$ which contains no lines, in duality with $(h_{S_1} + h_{S_2}, C_1 \cap C_2)$.

Remark 3.2.18. Note that the only thing to check for the compatibility of S_1 and S_2 is that $S_1 + S_2$ is still a closed subset of $\mathbb{C}$ which contains no lines. Indeed, the sum of two non-compact convex subsets of $\mathbb{C}$ is obviously still a non-compact convex subset of $\mathbb{C}$. Moreover, by Proposition 1.5.7, $h_{S_1+S_2} = h_{S_1} + h_{S_2}$. Hence, $S_1 + S_2$ must be in duality with $(h_{S_1} + h_{S_2}, C_1 \cap C_2)$.

From now on, we fix two compatible dualities $S_1 \leftrightarrow (h_{S_1}, C_1)$ and $S_2 \leftrightarrow (h_{S_2}, C_2)$. In order to perform computations, we have to fix reference points $\xi_1 \in C_1^\circ, \xi_2 \in C_2^\circ$ and $\xi_{1,2} \in C_1^\circ \cap C_2^\circ$. Thanks to Remark 3.2.11, we shall actually fix $\xi_{1,2} \in C_1^\circ \cap C_2^\circ$ and then choose $\xi_1 = \xi_2 = \xi_{1,2}$.

Proposition 3.2.19. *Let $\varepsilon, \varepsilon' > 0$ and let $\varphi \in Q_{3\varepsilon,\varepsilon'}(S_1 + S_2)$. Then, for any $\zeta_1 \in (S_1)^\circ_\varepsilon$,*

$$\zeta_2 \in (S_2)^\circ_\varepsilon \mapsto \varphi(\zeta_1 + \zeta_2)$$

is an element of $Q_{\varepsilon,\varepsilon'}(S_2)$.

Proof. Since φ is holomorphic on $(S_1 + S_2)^\circ_{3\varepsilon} \supset (S_1)^\circ_\varepsilon + (S_2)^\circ_\varepsilon$, it is clear that the function $\zeta_2 \mapsto \varphi(\zeta_1 + \zeta_2)$ is holomorphic on $(S_2)^\circ_\varepsilon$ for any fixed $\zeta_1 \in (S_1)^\circ_\varepsilon$. Moreover, we know that

$$\sup_{\zeta \in (S_1+S_2)^\circ_{3\varepsilon}} |e^{-\varepsilon' \xi_{1,2} \zeta} \varphi(\zeta)| < \infty.$$

Hence,

$$\begin{aligned}
\sup_{\zeta_2 \in (S_2)^\circ_\varepsilon} |e^{-\varepsilon' \xi_2 \zeta_2} \varphi(\zeta_1 + \zeta_2)| &= \sup_{\zeta_2 \in (S_2)^\circ_\varepsilon} |e^{\varepsilon' \xi_2 \zeta_1} e^{-\varepsilon' \xi_2 (\zeta_1 + \zeta_2)} \varphi(\zeta_1 + \zeta_2)| \\
&= |e^{\varepsilon' \xi_{1,2} \zeta_1}| \sup_{\zeta_2 \in (S_2)^\circ_\varepsilon} |e^{-\varepsilon' \xi_{1,2}(\zeta_1 + \zeta_2)} \varphi(\zeta_1 + \zeta_2)| \\
&\leq |e^{\varepsilon' \xi_{1,2} \zeta_1}| \sup_{\zeta \in (S_1+S_2)^\circ_{3\varepsilon}} |e^{-\varepsilon' \xi_{1,2} \zeta} \varphi(\zeta)|
\end{aligned}$$

is finite for any $\zeta_1 \in (S_1)^\circ_\varepsilon$ and one gets the conclusion. □

Proposition 3.2.20. *Let $\varepsilon, \varepsilon' > 0, T_2 \in Q'_{\varepsilon,\varepsilon'}(S_2)$ and let $\varphi \in Q_{3\varepsilon,\varepsilon'}(S_1 + S_2)$. The map*

$$\zeta_1 \in (S_1)^\circ_\varepsilon \mapsto \langle (T_2)_{\zeta_2}, \varphi(\zeta_1 + \zeta_2) \rangle$$

is an element of $Q_{\varepsilon,\varepsilon'}(S_1)$.

Proof. Let us first prove that $\zeta_1 \mapsto \langle (T_2)_{\zeta_2}, \varphi(\zeta_1 + \zeta_2) \rangle$ is holomorphic on $(S_1)^\circ_\varepsilon$. We will prove that

$$\lim_{\substack{h \to 0 \\ h \neq 0}} \frac{1}{h} (\langle (T_2)_{\zeta_2}, \varphi(\zeta_1 + h + \zeta_2) \rangle - \langle (T_2)_{\zeta_2}, \varphi(\zeta_1 + \zeta_2) \rangle) = \langle (T_2)_{\zeta_2}, \varphi'(\zeta_1 + \zeta_2) \rangle.$$

Remark that the right hand side exists, thanks to Proposition 3.2.2, since all the derivatives of φ are in $Q_{2\varepsilon,\varepsilon'}(S_1 + S_2)$. We have

$$\frac{\varphi(\zeta_1 + h + \zeta_2) - \varphi(\zeta_1 + \zeta_2))}{h} - \varphi'(\zeta_1 + \zeta_2) = h \int_0^1 (1-t) \varphi''(\zeta_1 + th + \zeta_2) dt$$

for small enough h. Therefore, thanks to the continuity of T_2, there is a positive constant A such that

$$
\begin{aligned}
&\left|\frac{1}{h}(\langle (T_2)_{\zeta_2}, \varphi(\zeta_1+h+\zeta_2)\rangle - \langle (T_2)_{\zeta_2}, \varphi(\zeta_1+\zeta_2)\rangle) - \langle (T_2)_{\zeta_2}, \varphi'(z_1+z_2)\rangle\right| \\
&= |h| \left|\left\langle (T_2)_{\zeta_2}, \int_0^1 (1-t)\varphi''(\zeta_1+th+\zeta_2)dt\right\rangle\right| \\
&\leq A|h| \sup_{\zeta_2\in(S_2)^\circ_\varepsilon} \left|e^{-\varepsilon'\xi_2\zeta_2}\int_0^1 (1-t)\varphi''(\zeta_1+th+\zeta_2)dt\right| \\
&\leq 2A|h| \sup_{\zeta_2\in(S_2)^\circ_\varepsilon} \left|e^{-\varepsilon'\xi_2\zeta_2} \sup_{t\in[0,1]} |\varphi''(\zeta_1+th+\zeta_2)|\right| \\
&\leq 2A|h| \sup_{t\in[0,1]} |e^{\varepsilon'\xi_{1,2}(\zeta_1+th)}| \sup_{\zeta_2\in(S_2)^\circ_\varepsilon} \left|\sup_{t\in[0,1]} |e^{-\varepsilon'\xi_{1,2}(\zeta_1+th+\zeta_2)}\varphi''(\zeta_1+th+\zeta_2)|\right| \\
&\leq 2A|h| \sup_{t\in[0,1]} |e^{\varepsilon'\xi_{1,2}(\zeta_1+th)}| \sup_{\zeta\in(S_1+S_2)^\circ_{2\varepsilon}} |e^{-\varepsilon'\xi_{1,2}\zeta}\varphi''(\zeta)|
\end{aligned}
$$

for small enough h. The conclusion follows from the fact that

$$\lim_{\substack{h\to 0\\ h\neq 0}} |h| \sup_{t\in[0,1]} |e^{\varepsilon'\xi_{1,2}(\zeta_1+th)}| = 0$$

and that $\varphi'' \in Q_{2\varepsilon,\varepsilon'}(S_1+S_2)$.

Secondly, we have to prove that

$$\sup_{\zeta_1\in(S_1)^\circ_\varepsilon} |e^{-\varepsilon'\xi_1\zeta_1}\langle (T_2)_{\zeta_2}, \varphi(\zeta_1+\zeta_2)\rangle| < \infty.$$

As above, there is $A > 0$ such that

$$
\begin{aligned}
\sup_{\zeta_1\in(S_1)^\circ_\varepsilon} |e^{-\varepsilon'\xi_1\zeta_1}\langle (T_2)_{\zeta_2}, \varphi(\zeta_1+\zeta_2)\rangle| &\leq A \sup_{\zeta_1\in(S_1)^\circ_\varepsilon,\zeta_2\in(S_2)^\circ_\varepsilon} |e^{-\varepsilon'\xi_1\zeta_1}e^{-\varepsilon'\xi_2\zeta_2}\varphi(\zeta_1+\zeta_2)| \\
&= A \sup_{\zeta_1\in(S_1)^\circ_\varepsilon,\zeta_2\in(S_2)^\circ_\varepsilon} |e^{-\varepsilon'\xi_{1,2}(\zeta_1+\zeta_2)}\varphi(\zeta_1+\zeta_2)| \\
&\leq A \sup_{\zeta\in(S_1+S_2)^\circ_{3\varepsilon}} |e^{-\varepsilon'\xi_{1,2}\zeta}\varphi(\zeta)|,
\end{aligned}
$$

hence the conclusion. □

Proposition 3.2.21. *Let $\varepsilon, \varepsilon' > 0$ and $T_1 \in Q'_{\varepsilon,\varepsilon'}(S_1), T_2 \in Q'_{\varepsilon,\varepsilon'}(S_2)$. Then, the application*

$$T_1 \star T_2 : \varphi \in Q_{3\varepsilon,\varepsilon'}(S_1+S_2) \mapsto \langle (T_1)_{\zeta_1}, \langle (T_2)_{\zeta_2}, \varphi(\zeta_1+\zeta_2)\rangle\rangle$$

is linear and continuous. It is thus an element of $Q'_{3\varepsilon,\varepsilon'}(S_1+S_2)$.

Proof. Clearly, $T_1 \star T_2$ is linear. By continuity of T_1 and T_2, there are constants $B, A > 0$ such that

$$\begin{aligned}|\langle T_1 \star T_2, \varphi\rangle| &= |\langle (T_1)_{\zeta_1}, \langle (T_2)_{\zeta_2}, \varphi(\zeta_1+\zeta_2)\rangle\rangle| \\ &\leq B \sup_{\zeta_1 \in (S_1)^\circ_\varepsilon} |e^{-\varepsilon'\xi_1\zeta_1}\langle (T_2)_{\zeta_2}, \varphi(\zeta_1+\zeta_2)\rangle| \\ &\leq BA \sup_{\zeta_1 \in (S_1)^\circ_\varepsilon, \zeta_2 \in (S_2)^\circ_\varepsilon} |e^{-\varepsilon'\xi_{1,2}(\zeta_1+\zeta_2)}\varphi(\zeta_1+\zeta_2)| \\ &\leq BA \sup_{\zeta \in (S_1+S_2)^\circ_{3\varepsilon}} |e^{-\varepsilon'\xi_{1,2}\zeta}\varphi(\zeta)|\end{aligned}$$

for any $\varphi \in Q_{3\varepsilon,\varepsilon'}(S_1+S_2)$. This proves that $T_1 \star T_2$ is continuous. □

Definition 3.2.22. The convolution product map

$$\star : Q'(S_1) \times Q'(S_2) \to Q'(S_1+S_2)$$

is defined as the projective limit on $\varepsilon, \varepsilon' \to 0$ of the map

$$\star : Q'_{\varepsilon,\varepsilon'}(S_1) \times Q'_{\varepsilon,\varepsilon'}(S_1) \to Q'_{3\varepsilon,\varepsilon'}(S_1+S_2),$$

introduced in the previous proposition.

Proposition 3.2.23. *The multiplication of functions induces a map*

$$\mathrm{Exp}(S_1) \times \mathrm{Exp}(S_2) \to \mathrm{Exp}(S_1+S_2)$$

which is continuous on each factor.

Proof. Let $g_1 \in \mathrm{Exp}(S_1)$ and $g_2 \in \mathrm{Exp}(S_2)$. Since $g_1 \in \mathcal{O}(C_1^\circ)$ and $g_2 \in \mathcal{O}(C_2^\circ)$, it is clear that $g_1g_2 \in \mathcal{O}((C_1 \cap C_2)^\circ)$. Moreover, for all $\varepsilon, \varepsilon' > 0$ we have

$$\sup_{w \in \varepsilon'\xi_1 + C_1^\circ} |g_1(w)|e^{-h_{S_1}(w)-\varepsilon|w|} < \infty$$

and

$$\sup_{w \in \varepsilon'\xi_2 + C_2^\circ} |g_2(w)|e^{-h_{S_2}(w)-\varepsilon|w|} < \infty.$$

Now let $\varepsilon, \varepsilon' > 0$. Since

$$\varepsilon'\xi_{1,2} + (C_1 \cap C_2)^\circ = (\varepsilon'\xi_1 + C_1^\circ) \cap (\varepsilon'\xi_2 + C_2^\circ),$$

we have

$$\begin{aligned}&\sup_{w \in \varepsilon'\xi_{1,2} + (C_1\cap C_2)^\circ} |(g_1g_2)(w)|e^{-(h_{S_1}+h_{S_2})(w)-\varepsilon|w|} \\ &\leq \left(\sup_{w \in \varepsilon'\xi_1 + C_1^\circ} |g_1(w)|e^{-h_{S_1}(w)-\frac{\varepsilon}{2}|w|}\right)\left(\sup_{w \in \varepsilon'\xi_2 + C_2^\circ} |g_2(w)|e^{-h_{S_2}(w)-\frac{\varepsilon}{2}|w|}\right) \\ &< \infty,\end{aligned}$$

which proves the first part of the assertion. This inequality also shows the continuity on each factor. □

Proposition 3.2.24. *For any $T_1 \in Q'(S_1)$ and $T_2 \in Q'(S_2)$, one has*

$$\mathcal{F}(T_1 \star T_2) = \mathcal{F}(T_1)\mathcal{F}(T_2).$$

Proof. Similar to the proof of Proposition 3.1.15. □

Corollary 3.2.25. *The convolution product map of analytic functionals with non-compact convex carrier is commutative and continuous on each factor.*

Proof. It is enough to combine the two previous propositions with the fact that the Fourier-Borel transform is a topological isomorphism. □

Remark 3.2.26. The reader, who might be interested, could also define in a similar manner a "mixed" convolution between a compactly carried analytic functional and a non-compactly carried one.

3.2.3 Compatibility and convolvability

If one wants to make a link between the holomorphic cohomological convolution and the convolution of non-compactly carried analytic functionals, it is pretty natural to investigate the links between the convolvability condition (see Definition 2.2.1 in the case of $(\mathbb{C}, +)$) and the compatibility condition (see Definition 3.2.17). Actually, we shall see that these conditions are equivalent when considering proper non-compact closed convex subsets of $\mathbb{C}$ which contain no lines.

Proposition 3.2.27. *Two non-empty closed subsets S_1 and S_2 of $\mathbb{C}$ are additively convolvable if and only if*

$$(S_1)_\infty \cap -(S_2)_\infty = \{0\}.$$

Proof. The sets S_1 and S_2 are additively convolvable if and only if

$$S_1 \cap (\overline{D}(0, r) - S_2) = S_1 \cap -(S_2)_r$$

is compact for large enough $r > 0$. (We can thus assume that the intersection is not empty.) Hence, using Proposition 1.5.10 and Example 1.5.12, $S_1 \cap -(S_2)_r$ is compact if and only if

$$\{0\} = (S_1 \cap -(S_2)_r)_\infty = (S_1)_\infty \cap -((S_2)_r)_\infty = (S_1)_\infty \cap -(S_2)_\infty.$$

□

Lemma 3.2.28. *If S_1 and S_2 are two additively convolvable proper closed convex subsets which contain no lines, then $S_1 + S_2$ is proper, closed and does not contain any line.*

Proof. Since S_1 and S_2 are convolvable, $S_1 + S_2$ is closed and one can use Proposition 3.2.27 and Theorem 1.5.11 to get the equality

$$(S_1 + S_2)_\infty = (S_1)_\infty + (S_2)_\infty.$$

First, remark that $(S_1)_\infty$ and $(S_2)_\infty$ are salient cones since they are convex cones which do not contain any line. Now, one can proceed by contradiction. If S_1+S_2 were containing a line, then so do $(S_1+S_2)_\infty = (S_1)_\infty + (S_2)_\infty$. This means that one can find $0 \neq w \in \mathbb{C}$ such that w and $-w$ are elements of $(S_1)_\infty + (S_2)_\infty$. Hence, $w = w_1 + w_2$ and $-w = w_1' + w_2'$ with $w_1, w_1' \in (S_1)_\infty$ and $w_2, w_2' \in (S_2)_\infty$. This implies that $w_1 + w_1' = -(w_2 + w_2')$. This complex number is non-zero, otherwise we would have

$$w_1 = -w_1' \in (S_1)_\infty \cap -(S_1)_\infty \qquad \text{and} \qquad w_2 = -w_2' \in (S_2)_\infty \cap -(S_2)_\infty$$

and thus $w_1 = w_1' = w_2 = w_2' = w = 0$ since $(S_1)_\infty$ and $(S_2)_\infty$ are salient. So we have a non-zero complex number, namely $w_1 + w_1' = -(w_2 + w_2')$, which is in $(S_1)_\infty \cap -(S_2)_\infty$. This violates the convolvability condition and implies that S_1+S_2 cannot contain any line and is, of course, proper. $\square$

Proposition 3.2.29. *Let S_1 and S_2 be two proper non-compact closed convex subsets of $\mathbb{C}$ which contain no lines. Then S_1 and S_2 are additively convolvable if and only if they are compatible.*

Proof. The condition is necessary. It immediately follows from Lemma 3.2.28.

The condition is sufficient. Since S_1 and S_2 are non-empty, closed and convex, remark that the inclusion

$$(S_1)_\infty + (S_2)_\infty \subset (S_1 + S_2)_\infty$$

is true. (For example, use Proposition 1.5.10, item 7.) Now assume that S_1 and S_2 are not convolvable. By Proposition 3.2.27, one can find $0 \neq w \in \mathbb{C}$ such that $w \in (S_1)_\infty$ and $-w \in (S_2)_\infty$. In particular, since these cones are pointed, w and $-w$ are elements of $(S_1)_\infty + (S_2)_\infty$, thus of $(S_1+S_2)_\infty$. This implies that $(S_1+S_2)_\infty$ contains a full line, since it is a pointed cone, and thus that S_1+S_2 also contains a line. Hence S_1 and S_2 cannot be compatible. $\square$

3.2.4 Main conjecture

Let us consider two compatible (or equivalently convolvable) proper non-compact closed convex subsets S_1 and S_2 of $\mathbb{C}$ which contain no lines. On the one hand, the additive holomorphic cohomological convolution provides a bilinear map

$$H^1_{S_1}(\mathbb{C}, \Omega_\mathbb{C}) \times H^1_{S_2}(\mathbb{C}, \Omega_\mathbb{C}) \to H^1_{S_1+S_2}(\mathbb{C}, \Omega_\mathbb{C})$$

which can be seen as a map

$$\Omega(\mathbb{C} \setminus S_1)/\Omega(\mathbb{C}) \times \Omega(\mathbb{C} \setminus S_2)/\Omega(\mathbb{C}) \to \Omega(\mathbb{C} \setminus (S_1+S_2))/\Omega(\mathbb{C}).$$

On the other hand, the convolution of analytic functionals with non-compact convex carrier

$$Q'(S_1) \times Q'(S_2) \to Q'(S_1 + S_2)$$

can be seen, through the Cauchy transform, as a bilinear map

$$\mathscr{H}_{S_1}(\mathbb{C}) \times \mathscr{H}_{S_2}(\mathbb{C}) \to \mathscr{H}_{S_1+S_2}(\mathbb{C}).$$

For all $\varepsilon' > 0$ and all proper non-compact closed convex subset S of $\mathbb{C}$ which contains no lines, we consider the map

$$i_{S,\varepsilon'} : \mathscr{H}_S(\mathbb{C}, \varepsilon') \to \Omega(\mathbb{C} \setminus S)/\Omega(\mathbb{C})$$

defined by $i_{S,\varepsilon'}([f]) = \left[\frac{1}{2i\pi} f dz\right]$. We set

$$i_S = \varprojlim_{\varepsilon' \to 0} i_{S,\varepsilon'} : \mathscr{H}_S(\mathbb{C}) \to \Omega(\mathbb{C} \setminus S)/\Omega(\mathbb{C}).$$

Conjecture 3.2.30. *The following diagram is commutative :*

$$\begin{tikzcd}
H^1_{S_1}(\mathbb{C}, \Omega_{\mathbb{C}}) \times H^1_{S_2}(\mathbb{C}, \Omega_{\mathbb{C}}) \arrow[r] & H^1_{S_1+S_2}(\mathbb{C}, \Omega_{\mathbb{C}}) \\
\Omega(\mathbb{C} \setminus S_1)/\Omega(\mathbb{C}) \times \Omega(\mathbb{C} \setminus S_2)/\Omega(\mathbb{C}) \arrow[r] \arrow[u, "\wr"] & \Omega(\mathbb{C} \setminus (S_1 + S_2))/\Omega(\mathbb{C}) \arrow[u, "\wr"] \\
\mathscr{H}_{S_1}(\mathbb{C}) \times \mathscr{H}_{S_2}(\mathbb{C}) \arrow[r] \arrow[u, "i_{S_1} \times i_{S_2}"'] & \mathscr{H}_{S_1+S_2}(\mathbb{C}) \arrow[u, "i_{S_1+S_2}"'] \\
Q'(S_1) \times Q'(S_2) \arrow[r] \arrow[u, "\wr", "\mathcal{C} \times \mathcal{C}"'] & Q'(S_1 + S_2) \arrow[u, "\wr", "\mathcal{C}"']
\end{tikzcd}$$

Here the two top (resp. bottom) horizontal arrows are given by the additive holomorphic cohomological convolution (resp. convolution of analytic functionals).

It is not surprising that this conjecture seems pretty hard to prove, since the holomorphic cohomological convolution cannot, in general, be made explicit by a nice global formula in the non-compact case. In the final chapter, we shall see that the conjecture is true if one adds some subanalytic conditions on S_1 and S_2.

Chapter 4

Enhanced subanalytic sheaves

In this chapter, we introduce all the tools and some key properties needed to study the enhanced Laplace transform in the final chapter.

4.1 Review on $\mathscr{D}$-modules

For the classical theory of $\mathscr{D}$-modules, we refer to [10], [53] and [103]. In this section we fix some notations that will occur afterwards.

Let X be a complex manifold and $\mathscr{D}_X$ its sheaf (of rings) of linear partial differential operators with holomorphic coefficients. Since these rings are not commutative, we make a distinction between the category of left $\mathscr{D}_X$-modules, noted $\mathrm{Mod}(\mathscr{D}_X)$, and the category of right $\mathscr{D}_X$-modules, noted $\mathrm{Mod}(\mathscr{D}_X^{\mathrm{op}})$. Recall that $\mathcal{O}_X$ (resp. Ω_X) has a canonical structure of left (resp. right) $\mathscr{D}_X$-module. The sheaf Ω_X admits a $\otimes_{\mathcal{O}_X}$-inverse, namely $\Omega_X^{\otimes -1} := \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$. This can be used to prove that

$$\mathrm{Mod}(\mathscr{D}_X) \ni \mathcal{M} \mapsto \Omega_X \otimes_{\mathcal{O}_X} \mathcal{M} \in \mathrm{Mod}(\mathscr{D}_X^{\mathrm{op}})$$

is an equivalence of categories. Hence, it is enough to study $\mathrm{Mod}(\mathscr{D}_X)$.

One denotes by $\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$ (resp. $\mathrm{D}^{\mathrm{b}}_{\mathrm{q\text{-}good}}(\mathscr{D}_X)$) the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)$ whose objects have holonomic (resp. quasi-good) cohomologies.

As usual, there are functors

$$\mathrm{R}\mathcal{H}om_{\mathscr{D}_X}(-,-) : \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)^{\mathrm{op}} \times \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X) \to \mathrm{D}^{+}(\mathbb{C}_X),$$
$$- \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} - : \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X^{\mathrm{op}}) \times \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X) \to \mathrm{D}^{\mathrm{b}}(\mathbb{C}_X),$$

as well as an internal tensor product

$$- \overset{\mathrm{D}}{\otimes} - : \mathrm{D}^{-}(\mathscr{D}_X) \times \mathrm{D}^{-}(\mathscr{D}_X) \to \mathrm{D}^{-}(\mathscr{D}_X)$$

defined by endowing $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ with a structure of left $\mathscr{D}_X$-module, for all left $\mathscr{D}_X$-modules $\mathcal{M}$ and $\mathcal{N}$.

Let $f : X \to Y$ be a morphism of complex manifolds. There is a transfer bimodule $\mathscr{D}_{X\to Y}$ defined by

$$\mathscr{D}_{X\to Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathscr{D}_Y.$$

It has a canonical structure of $(\mathscr{D}_X, f^{-1}\mathscr{D}_Y)$-bimodule and a canonical section $1_{X\to Y}$ defined by $1_X \otimes f^{-1}(1_Y)$. One can also define the reverse transfer bimodule as

$$\mathscr{D}_{Y\leftarrow X} = \Omega_X \otimes_{\mathcal{O}_X} \mathscr{D}_{X\to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^{\otimes -1},$$

which is a $(f^{-1}\mathscr{D}_Y, \mathscr{D}_X)$-bimodule. Thanks to these transfer bimodules, one can define three external operations $\mathrm{D}f_*, \mathrm{D}f^*$ and $\mathrm{D}f_!$ on left $\mathscr{D}_X$-modules by setting

1) $\mathrm{D}f_*\mathcal{M} = \mathrm{R}f_*(\mathscr{D}_{Y\leftarrow X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathcal{M})$ for $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)$,

2) $\mathrm{D}f^*\mathcal{N} = \mathscr{D}_{X\to Y} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}_Y} f^{-1}\mathcal{N}$ for $\mathcal{N} \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_Y)$,

3) $\mathrm{D}f_!\mathcal{M} = \mathrm{R}f_!(\mathscr{D}_{Y\leftarrow X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathcal{M})$ for $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)$.

Let $Y \subset X$ be a complex analytic hypersurface. One denotes by $\mathcal{O}_X(*Y)$ the sheaf of holomorphic functions with poles in Y. For any $\varphi \in \mathcal{O}_X(*Y)$, one sets

$$\mathscr{D}_X e^{\varphi} = \mathscr{D}_X/\{P : Pe^{\varphi} = 0 \text{ on } U\} \qquad \text{and} \qquad \mathscr{E}^{\varphi}_{U|X} = \mathscr{D}_X e^{\varphi} \overset{\mathrm{D}}{\otimes} \mathcal{O}_X(*Y),$$

where $U = X\backslash Y$. Note that $\mathscr{E}^{\varphi}_{U|X}$ has a canonical section given by the equivalence class of the operator $P = 1$. This section is noted e^{φ}.

4.2 Subanalytic sheaves

4.2.1 Subanalytic sets

Subanalytic sets have been introduced by Gabrielov ([33]) and Hironaka ([45]). We refer to [9] for a good exposition. In this thesis, we will only need the properties verified by subanalytic sets, hence we will not recall the complete technical definitions.

Let N be a real analytic manifold. The family of *subanalytic subsets* of N is the smallest family satisfying the following properties :

1. The intersection of two subanalytic subsets is subanalytic.
2. The union of a locally finite family of subanalytic subsets is subanalytic.
3. The complement of a subanalytic subset is subanalytic.
4. For any real analytic manifold M and any proper morphism $f : M \to N$, the image of M is subanalytic.

It can be shown that the interior, the closure and the boundary of a subanalytic subset is subanalytic. If $f : M \to N$ is a morphism of real analytic manifolds, then $f^{-1}(S)$ is a subanalytic subset of M if S is a subanalytic subset of N. If S' is a subanalytic subset of M such that $f|_{\overline{S'}}$ is proper, then $f(S')$ is a subanalytic subset of N.

The subanalytic subsets of $\mathbb{R}^n$ verify a crucial property, called "the Łojasiewicz inequality" (see [68] and [73]).

Theorem 4.2.1. *Let U and V be two subanalytic open subsets of $\mathbb{R}^n$ and K be a compact subset of $\mathbb{R}^n$. Then, there are a positive integer N and a real constant $C > 0$ such that*

$$\mathrm{dist}(x, K\backslash(U \cup V))^N \leq C(\mathrm{dist}(x, K\backslash U) + \mathrm{dist}(x, K\backslash V))$$

for all $x \in K$.

4.2.2 Subanalytic sheaves

Subanalytic sets allow to define a Grothendieck topology on a real analytic manifold and thus give rise to a site and an associated topos (see [1] for the original definition or [59]). In this thesis, we will avoid this general background and simply use the point of view of [54]. More details can be found in [58] and [90].

Let M be a real analytic manifold. We write for short Op_M (resp. $\mathrm{Op}_M^{\mathrm{sub},c}$) the category of open subsets of M (resp. the category of open subanalytic subsets of M which are relatively compact).

Definition 4.2.2. *A subanalytic presheaf* is a contravariant functor

$$F : \mathrm{Op}_M^{\mathrm{sub},c} \to \mathrm{Mod}(\mathbb{C}).$$

A *subanalytic sheaf* is a subanalytic presheaf F which satisfies the following finite gluing condition :

1. $F(\emptyset) = 0$.
2. For any $U, V \in \mathrm{Op}_M^{\mathrm{sub},c}$, the sequence

   $$0 \to F(U \cup V) \to F(U) \oplus F(V) \to F(U \cap V)$$

   is exact. (Here, the first arrow is given by the maps $F(U \cup V) \to F(U)$ and $F(U \cup V) \to F(V)$ and the second arrow by the difference between the maps $F(U) \to F(U \cap V)$ and $F(V) \to F(U \cap V)$.)

The category of subanalytic sheaves on M is noted $\mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}})$.

Proposition 4.2.3. *The category $\mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}})$ is abelian, has enough injectives and admits small inductive limits.*

There are more subanalytic sheaves than usual sheaves given that $\mathrm{Op}_M^{\mathrm{sub},c}$ is a full subcategory of Op_M and that the coverings are only finite. Hence, we get a canonical inclusion

$$\iota_M : \mathrm{Mod}(\mathbb{C}_M) \to \mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}}).$$

It is a fully faithful left exact functor. One has

$$\Gamma(U, F) := F(U) = \mathrm{Hom}_{\mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}})}(\iota_M(\mathbb{C}_U), F)$$

for any $U \in \mathrm{Op}_M^{\mathrm{sub},c}$.

Remark 4.2.4. We simply denote by $\varinjlim$ the inductive limit functor in $\mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}})$ but some authors prefer to use the notation "$\varinjlim$" to highlight the fact that, in general, $\iota_M \circ \varinjlim \neq \varinjlim \circ \iota_M$. (See also [58] where the more general phenomenon of ind-sheaves is explained.)

One can build a left adjoint α_M to ι_M by setting

$$\Gamma(U, \alpha_M(F)) = \varprojlim_{V \in \mathrm{Op}_M^{\mathrm{sub},c}, V \subset\subset U} F(V)$$

for any $U \in \mathrm{Op}_M$ and any $F \in \mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}})$. It is an exact functor.

One can also build a left adjoint β_M to α_M by defining $\beta_M(F)$ as the sheafification of the subanalytic presheaf

$$\mathrm{Op}_M^{\mathrm{sub},c} \ni U \mapsto F(\overline{U})$$

for any $F \in \mathrm{Mod}(\mathbb{C}_M)$. It is an exact functor.

On a complex manifold X, using ring actions (see [54], Section 3.7) one can easily define the category of left (resp right) subanalytic $\mathscr{D}_X$-modules, noted $\mathrm{Mod}(\mathscr{D}_X^{\mathrm{sub}})$ (resp. $\mathrm{Mod}(\mathscr{D}_X^{\mathrm{sub,op}})$). The functor β_X naturally appears in a lot of morphisms involving subanalytic $\mathscr{D}_X$-modules. In order to lighten the notations, we shall adopt the little abuse of notation of [60] and not always write this functor.

4.2.3 Grothendieck operations

The Grothendieck operations can easily be adapted to the context of subanalytic sheaves. Let $f : M \to N$ be a morphism of real analytic manifolds. The four operations

$$\begin{aligned}
\mathscr{I}hom(-,-) &: \mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}})^{\mathrm{op}} \times \mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}}) \to \mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}}), \\
- \otimes - &: \mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}}) \times \mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}}) \to \mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}}), \\
f_* &: \mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}}) \to \mathrm{Mod}(\mathbb{C}_N^{\mathrm{sub}}), \\
f^{-1} &: \mathrm{Mod}(\mathbb{C}_N^{\mathrm{sub}}) \to \mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}})
\end{aligned}$$

are defined as for usual sheaves. The functor f^{-1} is exact. There is also a stacky hom functor defined by $\mathcal{H}om = \alpha_M \circ \mathscr{I}hom$. One has

$$\Gamma(U, \mathcal{H}om(F_1, F_2)) = \mathrm{Hom}_{\mathrm{Mod}(\mathbb{C}_U^{\mathrm{sub}})}(F_1|_U, F_2|_U)$$

for any $U \in \mathrm{Op}_M$ and any $F_1, F_2 \in \mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}})$.

If Z is a subanalytic locally closed subset of M, one sets

$$F_Z = F \otimes \iota_M(\mathbb{C}_Z) \qquad \text{and} \qquad \mathscr{I}\Gamma_Z(F) = \mathscr{I}hom(\iota_M(\mathbb{C}_Z), F)$$

as well as $\Gamma_Z(F) = \mathcal{H}om(\iota_M(\mathbb{C}_Z), F)$ for any $F \in \mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}})$. In this subanalytic setting, one should pay attention to the abbreviation $\Gamma_Z(-, F)$ that will stands for $\Gamma(-, \mathscr{I}\Gamma_Z(F))$ and not for $\Gamma(-, \Gamma_Z(F))$.

These operations verify all the usual properties of Grothendieck operations (e.g. adjunctions).

Definition 4.2.5. Let $f : M \to N$ be a morphism of real analytic manifolds. One defines a functor $f_{!!} : \mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}}) \to \mathrm{Mod}(\mathbb{C}_N^{\mathrm{sub}})$ by setting

$$f_{!!}(F) = \varinjlim_K f_*(\mathscr{I}\Gamma_K(F))$$

for any $F \in \mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}})$, where K ranges through the family of subanalytic compact subsets of M.

Remark 4.2.6. We changed a little bit the usual notation to emphasize the fact that $\iota_N \circ f_! \neq f_{!!} \circ \iota_M$ in general. The category of subanalytic sheaves is actually equivalent to the category of ind-constructible sheaves (see. [58], section 7.1) and the previous morphisms are the restrictions of the more general morphisms defined for ind-sheaves. That's why we chose such notations.

Since $\mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}})$ is an abelian category, one can consider the derived categories $\mathrm{D}(\mathbb{C}_M^{\mathrm{sub}}) := \mathrm{D}(\mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}})), \mathrm{D}^+(\mathbb{C}_M^{\mathrm{sub}}), \mathrm{D}^-(\mathbb{C}_M^{\mathrm{sub}})$ and $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_M^{\mathrm{sub}})$. The Grothendieck operations admit derived functors, namely $-\overset{\mathrm{L}}{\otimes}-, \mathrm{R}\mathscr{I}hom(-,-), \mathrm{R}f_*$ and $\mathrm{R}f_{!!}$. By the Brown representability theorem (see e.g. Corollary 14.3.7 of [59]), the functor $\mathrm{R}f_{!!}$ admits a right adjoint $f^!$.

It is convenient to introduce a family of specific subanalytic sheaves, which play an equivalent role as flabby sheaves in classical sheaf theory.

Definition 4.2.7. Let M be a real analytic manifold. A subanalytic sheaf F on M is *quasi-injective* if the restriction map $\Gamma(V, F) \to \Gamma(U, F)$ is surjective for any $U, V \in \mathrm{Op}_M^{\mathrm{sub},c}$ such that $U \subset V$.

Proposition 4.2.8 ([90], Corollary 1.5.6, Proposition 1.5.10 and Proposition 1.5.11)**.** *The family of quasi-injective subanalytic sheaves is injective for* $f_*, f_{!!}$ *and* $\mathscr{I}\Gamma_Z(-)$.

We let the reader adapt some other classical properties (e.g. Mayer-Vietoris, excision) to the subanalytic case.

4.3 Tempered distributions

4.3.1 Several definitions

Tempered distributions constitute an important example of quasi-injective subanalytic sheaf. Several definitions are possible but we shall mainly use the one introduced by M. Kashiwara in [52] (see also [58] for more details).

Definition 4.3.1. Let M be a real analytic manifold and U an open subset of M. One sets

$$\mathcal{D}b^{\mathrm{t}}_M(U) = \{u \in \mathcal{D}b_M(U) : \exists v \in \mathcal{D}b_M(M), v|_U = u\}$$

and call it the $\mathbb{C}$-vector space of *tempered distributions* on U.

Remark 4.3.2. This definition can be made more explicit. Let U be a relatively compact subset of $\mathbb{R}^n$. Then $u \in \mathcal{D}b^{\mathrm{t}}_M(U)$ if and only if there are positive integers m, N and a constant $C > 0$ such that

$$|\langle u, \varphi\rangle| \leq C \sum_{|\alpha|\leq m} \sup_{x\in U} \big(\mathrm{dist}(x, \partial U)^{-N} |D^\alpha \varphi(x)|\big)$$

for all test function φ on U. (See Lemma 3.3 in [52].)

Remark 4.3.3. Let us denote by S^n the real n-dimensional sphere. In [106, p. 238], it is explained that the classical space $\mathcal{S}'(\mathbb{R}^n)$ can be seen as the space of distributions on $\mathbb{R}^n$ which can be extended to S^n. In other words, one has

$$\mathcal{S}'(\mathbb{R}^n) = \mathcal{D}b^{\mathrm{t}}_{S^n}(\mathbb{R}^n).$$

Hence, we see that Kashiwara's definition is a good extension of the historical definition of L. Schwartz.

Remark 4.3.4. There is a trivial embedding $\mathcal{D}b^{\mathrm{t}}_M \hookrightarrow \iota_M(\mathcal{D}b_M)$ of subanalytic sheaves.

One could also need tempered functions instead of distributions.

Definition 4.3.5. Let M be a real analytic manifold and U an open subset of M. One says that $f \in \mathcal{C}_{\infty,M}(U)$ has *polynomial growth* at $x_0 \in M$ if it satisfies the following condition : for a local coordinate system around x_0, there are a sufficiently small compact neighbourhood K of x_0 and a positive integer N such that

$$\sup_{x\in K\cap U} (\mathrm{dist}(x, K\backslash U))^N |f(x)| < \infty.$$

One says that f is *tempered* if f as well as all its derivatives have polynomial growth at all points of M. One denotes by $\mathcal{C}^{\mathrm{t}}_{\infty,M}(U)$ the $\mathbb{C}$-vector space of tempered functions on U.

One can remark that this condition is non-trivial only on the boundary of U. Thanks to Theorem 4.2.1, we get

Proposition 4.3.6. *The subanalytic presheaves*

$$\mathrm{Op}_M^{\mathrm{sub},c} \ni U \mapsto \mathcal{C}^{\mathrm{t}}_{\infty,M}(U), \mathcal{D}b^{\mathrm{t}}_M(U)$$

are subanalytic sheaves. Moreover, by definition $\mathcal{D}b^{\mathrm{t}}_M$ is quasi-injective.

One obviously has a monomorphism $\mathcal{C}^{\mathrm{t}}_{\infty,M} \to \mathcal{D}b^{\mathrm{t}}_M$ of subanalytic sheaves.

We can easily adapt the previous definitions to consider the subanalytic sheaf of tempered differential r-forms (resp. tempered distributional r-forms), noted $\mathcal{C}^{\mathrm{t},r}_{\infty,M}$ (resp. $\mathcal{D}b^{\mathrm{t},r}_M$) for all $r \in \mathbb{Z}$. On a complex manifold X, these subanalytic sheaves admit a bi-type decomposition

$$\mathcal{C}^{\mathrm{t},r}_{\infty,X} \simeq \bigoplus_{p+q=r} \mathcal{C}^{\mathrm{t},p,q}_{\infty,X} \qquad \text{and} \qquad \mathcal{D}b^{\mathrm{t},r}_X \simeq \bigoplus_{p+q=r} \mathcal{D}b^{\mathrm{t},p,q}_X \,.$$

Definition 4.3.7. Let X be a complex manifold of complex dimension d_X and $p \in \mathbb{Z}$. One defines the complex of *tempered holomorphic p-forms* $\Omega^{\mathrm{t},p}_X \in \mathrm{D^b}(\mathbb{C}^{\mathrm{sub}}_X)$ by the Dolbeault complex

$$0 \to \mathcal{D}b^{\mathrm{t},p,0}_X \xrightarrow{\bar\partial} \mathcal{D}b^{\mathrm{t},p,1}_X \to \cdots \to \mathcal{D}b^{\mathrm{t},p,d_X}_X \to 0$$

or equivalently by the Dolbeault complex

$$0 \to \mathcal{C}^{\mathrm{t},p,0}_{\infty,X} \xrightarrow{\bar\partial} \mathcal{C}^{\mathrm{t},p,1}_{\infty,X} \to \cdots \to \mathcal{C}^{\mathrm{t},p,d_X}_{\infty,X} \to 0.$$

One sets for short $\mathcal{O}^{\mathrm{t}}_X = \Omega^{\mathrm{t},0}_X$ and $\Omega^{\mathrm{t}}_X = \Omega^{\mathrm{t},d_X}_X$.

Proposition 4.3.8. *If $d_X = 1$, then $\mathcal{O}^{\mathrm{t}}_X$ is concentrated in degree* 0.

4.3.2 Integration

Let us now adapt the usual distributional operations of section 1.4 to the tempered case.

Lemma 4.3.9 ([56], Proposition 4.3)**.** *Let M (resp. N) be a real analytic manifold of real dimension d_M (resp. d_N) and let $f : M \to N$ be a real analytic map. Then, the integration of distributions along the fibers of f induces a morphism of complexes*

$$\int_f : f_{!!}\, \mathcal{D}b^{\mathrm{t},\bullet+d_M}_M \to \mathcal{D}b^{\mathrm{t},\bullet+d_N}_N \,. \tag{4.1}$$

Proposition 4.3.10 ([56], Theorem 5.7). *Let $f : X \to Y$ be a holomorphic map between complex manifolds of complex dimension d_X and d_Y. The integration of distributions along the fibers of f induces a morphism of double complexes*

$$\int_f : f_{!!}\, \mathcal{D}b_X^{\mathrm{t},\bullet+d_X,\bullet+d_X} \to \mathcal{D}b_Y^{\mathrm{t},\bullet+d_Y,\bullet+d_Y} \tag{4.2}$$

and thus a morphism

$$\int_f : \mathrm{R}f_{!!}\Omega_X^{\mathrm{t},p+d_X}[d_X] \to \Omega_Y^{\mathrm{t},p+d_Y}[d_Y] \tag{4.3}$$

in $\mathrm{D^b}(\mathbb{C}_Y^{\mathrm{sub}})$, for each $p \in \mathbb{Z}$.

Proposition 4.3.11 ([58], Lemma 7.4.4 and Lemma 7.4.5). *Let $f : X \to Y$ be a holomorphic map between complex manifolds of complex dimension d_X and d_Y. There is a natural isomorphism*

$$\Omega_X^{\mathrm{t}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{D}_{X\to Y}[d_X] \xrightarrow{\sim} f^!\Omega_Y^{\mathrm{t}}[d_Y] \tag{4.4}$$

in $\mathrm{D^b}(\mathbb{C}_X^{\mathrm{sub}})$. Its adjoint morphism

$$\mathrm{R}f_{!!}(\Omega_X^{\mathrm{t}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{D}_{X\to Y})[d_X] \to \Omega_Y^{\mathrm{t}}[d_Y]$$

induces, thanks to the canonical section $1_{X\to Y}$ of $\mathscr{D}_{X\to Y}$, a morphism

$$\mathrm{R}f_{!!}\Omega_X^{\mathrm{t}}[d_X] \to \Omega_Y^{\mathrm{t}}[d_Y],$$

which is equivalent to (4.3) when $p = 0$.

Remark 4.3.12. The morphism (4.2) is of course easily defined when X is compact, which will always be the case in the further applications. Indeed, by definition

$$\begin{aligned}\Gamma(V, f_{!!}\, \mathcal{D}b_X^{\mathrm{t},p+d_X,q+d_X}) &= \varinjlim_K \Gamma(f^{-1}(V), \mathscr{I}\Gamma_K\, \mathcal{D}b_X^{\mathrm{t},p+d_X,q+d_X}) \\ &= \Gamma(f^{-1}(V), \mathcal{D}b_X^{\mathrm{t},p+d_X,q+d_X}),\end{aligned}$$

where $V \in \mathrm{Op}_Y^{\mathrm{sub},c}$ and K ranges through the family of subanalytic compact subsets of X. If $u \in \Gamma(f^{-1}(V), \mathcal{D}b_X^{\mathrm{t},p+d_X,q+d_X})$, then it can be extended to a distributional form $\underline{u}$ on X. By compactness of X, the distributional form $\int_f \underline{u}$ is well-defined on Y and is an extension of $\int_f u$. Hence $\int_f u$ is tempered.

4.3.3 Pullback

Lemma 4.3.13 ([52], Proposition 3.9). *Let M (resp. N) be a real analytic manifold of real dimension d_M (resp. d_N) and let $f : M \to N$ be a submersive real analytic map. Then, the pullback of distributions by f induces a morphism of complexes*

$$f^* : f^{-1}\, \mathcal{D}b_N^{\mathrm{t},\bullet} \to \mathcal{D}b_M^{\mathrm{t},\bullet}. \tag{4.5}$$

Proposition 4.3.14. *Let $f : X \to Y$ be a submersive holomorphic map between complex manifolds. The pullback of distributions by f induces a morphism of double complexes*

$$f^* : f^{-1}\,\mathcal{D}b_Y^{\mathrm{t},\bullet,\bullet} \to \mathcal{D}b_X^{\mathrm{t},\bullet,\bullet} \tag{4.6}$$

and thus a morphism

$$f^* : f^{-1}\Omega_Y^{\mathrm{t},p} \to \Omega_X^{\mathrm{t},p} \tag{4.7}$$

in $\mathrm{D^b}(\mathbb{C}_X^{\mathrm{sub}})$*, for each* $p \in \mathbb{Z}$*.*

Proposition 4.3.15 ([56], Theorem 4.5, Theorem 5.8 and [58], Lemma 7.4.9)**.** *Let $f : X \to Y$ be a holomorphic map between complex manifolds. There is a natural morphism*

$$\mathscr{D}_{X\to Y} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}_Y} f^{-1}\,\mathcal{O}_Y^{\mathrm{t}} \to \mathcal{O}_X^{\mathrm{t}} \tag{4.8}$$

in $\mathrm{D^b}(\mathbb{C}_X^{\mathrm{sub}})$*. The canonical section $1_{X\to Y}$ of $\mathscr{D}_{X\to Y}$ induces a morphism*

$$f^{-1}\,\mathcal{O}_Y^{\mathrm{t}} \to \mathcal{O}_X^{\mathrm{t}},$$

which is equivalent to (4.7) when $p = 0$, if f is submersive.

4.4 Bordered spaces

4.4.1 General definition

Bordered spaces have been introduced in [19]. Let us first recall some general definitions. By *good topological space*, we mean a topological space which is Hausdorff, locally compact, countable at infinity and with finite flabby dimension.

Definition 4.4.1. *A bordered space* is a couple $M_\infty = (M, \widehat{M})$, where $\widehat{M}$ is a good topological space and M an open subset of $\widehat{M}$.

If $M_\infty = (M, \widehat{M})$ and $N_\infty = (N, \widehat{N})$ are two bordered spaces and if $f : M \to N$ is a continuous map, we denote by $\Gamma_f \subset M \times N$ the graph of f and by $\overline{\Gamma}_f$ the closure of Γ_f in $\widehat{M} \times \widehat{N}$.

A morphism of bordered spaces $f : M_\infty \to N_\infty$ is a continuous map $f : M \to N$ such that the canonical projection $\overline{\Gamma}_f \to \widehat{M}$ is proper. The composition of two morphisms is the composition of the underlying continuous maps.

Remark 4.4.2. 1. Any good topological space M can be seen as a particular bordered space by considering the couple (M, M). We shall simply write M instead of (M, M). Hence, there are natural morphisms

$$M \to (M, \widehat{M}) \to \widehat{M}.$$

2. If $\widehat{N}$ is compact, then any continuous map $f : M \to N$ induces a morphism of bordered spaces.

3. If $f : M \to N$ can be extended to a continuous map $\widehat{f} : \widehat{M} \to \widehat{N}$, then it induces a morphism of bordered spaces.

4. In general, if $f : M_\infty \to N_\infty$ is a morphism of bordered spaces, the continuous map $f : M \to N$ does not admit such an extension $\widehat{f}$. However, one can always reduce to that situation by considering
$$(M, \widehat{M}) \xleftarrow{q_1} (\Gamma_f, \overline{\Gamma}_f) \xrightarrow{q_2} (N, \widehat{N}),$$
where the first projection $q_1 : \Gamma_f \to M$ gives rise to an isomorphism of bordered spaces and where the second projection $q_2 : \Gamma_f \to N$ extends to a continuous map $\widehat{q}_2 : \overline{\Gamma}_f \to \widehat{N}$.

5. If $M_\infty = (M, \widehat{M})$ and $N_\infty = (N, \widehat{N})$ are two bordered spaces, the product $M_\infty \times N_\infty = (M \times N, \widehat{M} \times \widehat{N})$ is obviously a bordered space.

Definition 4.4.3. A morphism of bordered spaces $f : (M, \widehat{M}) \to (N, \widehat{N})$ is *semi-proper* if $\widehat{q}_2 : \overline{\Gamma}_f \to \widehat{N}$ is proper.

4.4.2 Subanalytic sheaves on subanalytic bordered spaces

Sections 4.2.2 and 4.2.3 can easily be adapted to the bordered case. However, we shall need to consider spaces like $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ which are not real analytic manifolds. Hence, we have to introduce the more general notion of subanalytic spaces. We refer to [54] for the whole content of this section.

Definition 4.4.4. Let M, N be real analytic manifolds, S a closed subanalytic subset of M and $f : S \to N$ a continuous map. We say that f is *subanalytic* if its graph is a subanalytic subset of $M \times N$. One denotes by $\mathscr{A}_S^{\mathbb{R}}$ the sheaf of $\mathbb{R}$-valued subanalytic continuous map on S.

A subanalytic space $(M, \mathscr{A}_M^{\mathbb{R}})$, or simply M, is a $\mathbb{R}$-ringed space locally isomorphic to $(S, \mathscr{A}_S^{\mathbb{R}})$ for some closed subanalytic subset S of a real analytic manifold.

Definition 4.4.5. *A subanalytic bordered space* is a bordered space $(M, \widehat{M})$ where $\widehat{M}$ is a good subanalytic space and M a subanalytic open subset of $\widehat{M}$.

A morphism $f : (M, \widehat{M}) \to (N, \widehat{N})$ of bordered spaces is a morphism of subanalytic bordered spaces if its graph Γ_f is a subanalytic subset of $\widehat{M} \times \widehat{N}$.

Let $M_\infty = (M, \widehat{M})$ be a subanalytic bordered space. One denotes by $\mathrm{Op}_{M_\infty}^{\mathrm{sub},c}$ the full subcategory of Op_M consisting of open subsets of M which are relatively compact and subanalytic in $\widehat{M}$.

Definition 4.4.6. *A subanalytic presheaf* on M_∞ is a contravariant functor

$$F : \mathrm{Op}^{\mathrm{sub},c}_{M_\infty} \to \mathrm{Mod}(\mathbb{C}).$$

A *subanalytic sheaf* on M_∞ is a subanalytic presheaf F on M_∞ which satisfies the following finite gluing condition :

1. $F(\emptyset) = 0$.
2. For any $U, V \in \mathrm{Op}^{\mathrm{sub},c}_{M_\infty}$, the sequence

   $$0 \to F(U \cup V) \to F(U) \oplus F(V) \to F(U \cap V)$$

   is exact.

The category of subanalytic sheaves on M_∞ is noted $\mathrm{Mod}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$. We write $\mathrm{D}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$, $\mathrm{D}^+(\mathbb{C}^{\mathrm{sub}}_{M_\infty}), \mathrm{D}^-(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$ and $\mathrm{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$ the associated derived categories.

Remark 4.4.7. It can be shown that the canonical map

$$\mathrm{Mod}(\mathbb{C}^{\mathrm{sub}}_{\widehat{M}}) / \mathrm{Mod}(\mathbb{C}^{\mathrm{sub}}_{\widehat{M} \setminus M}) \to \mathrm{Mod}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$$

is an isomorphism. (The same result also holds for the bounded derived category.) Here, the quotient is given by the direct image $i_* : \mathrm{Mod}(\mathbb{C}^{\mathrm{sub}}_{\widehat{M} \setminus M}) \to \mathrm{Mod}(\mathbb{C}^{\mathrm{sub}}_{\widehat{M}})$, where $i : \widehat{M} \setminus M \hookrightarrow \widehat{M}$ is the inclusion. Note also that

$$\mathrm{Mod}(\mathbb{C}_{\widehat{M}}) / \mathrm{Mod}(\mathbb{C}_{\widehat{M} \setminus M}) \xrightarrow{\sim} \mathrm{Mod}(\mathbb{C}_M).$$

Hence, going to the quotient, one gets canonical functors

$$\iota_M : \mathrm{Mod}(\mathbb{C}_M) \to \mathrm{Mod}(\mathbb{C}^{\mathrm{sub}}_{M_\infty}) \qquad \text{and} \qquad \alpha_M : \mathrm{Mod}(\mathbb{C}^{\mathrm{sub}}_{M_\infty}) \to \mathrm{Mod}(\mathbb{C}_M).$$

Of course, the Grothendieck operations $- \otimes -, \mathscr{I}hom, f_*, f^{-1}$ and $f_{!!}$ are well-defined for subanalytic sheaves on subanalytic bordered spaces. For example, if

$$f : M_\infty = (M, \widehat{M}) \to N_\infty = (N, \widehat{N})$$

is a morphism of subanalytic bordered spaces and if $F \in \mathrm{Mod}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$, then

$$(f_*F)(V) = \mathrm{Hom}_{\mathrm{Mod}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})}(\iota_M(\mathbb{C}_{f^{-1}(V)}), F)$$

for all $V \in \mathrm{Op}^{\mathrm{sub},c}_{N_\infty}$. (Note that $f^{-1}(V)$ is not necessarily an element of $\mathrm{Op}^{\mathrm{sub},c}_{M_\infty}$.)

These operations verify all the classical properties of Grothendieck operations and, as usual, $\mathrm{R}f_{!!}$ admits a right adjoint, noted $f^!$.

Remark 4.4.8. If $f : M_\infty \to N_\infty$ is semi-proper, then $\iota_N \circ f_! = f_{!!} \circ \iota_M$.

4.4.3 $\mathscr{D}$-modules on complex bordered spaces

The notion of complex bordered space is defined in section 4.3 of [60].

Definition 4.4.9. A *complex bordered space* $X_\infty = (X, \widehat{X})$ is a bordered space where $\widehat{X}$ is a complex manifold and $\widehat{X} \backslash X$ is a complex analytic subset of $\widehat{X}$.

A morphism $f : (X, \widehat{X}) \to (Y, \widehat{Y})$ of bordered spaces is a morphism of complex bordered spaces if $f : X \to Y$ is holomorphic, if the canonical projection $\overline{\Gamma}_f \to \widehat{X}$ is proper and if Γ_f is a complex analytic subset of $\widehat{X} \times \widehat{Y}$.

If $j : X_\infty \to \widehat{X}$ denotes the canonical inclusion of complex bordered spaces, we set $\mathcal{O}_{X_\infty} = j^{-1}(\mathcal{O}_{\widehat{X}}), \Omega_{X_\infty} = j^{-1}(\Omega_{\widehat{X}})$ and $\Omega_{X_\infty}^{\otimes -1} = \mathcal{H}om_{\mathcal{O}_{X_\infty}}(\Omega_{X_\infty}, \mathcal{O}_{X_\infty})$.

It is easy to adapt the definitions of section 4.1 to complex bordered spaces. First, if $X_\infty = (X, \widehat{X})$ is a complex bordered space we set

1. $\mathrm{D^b}(\mathscr{D}_{X_\infty}) = \mathrm{D^b}(\mathscr{D}_{\widehat{X}})/\{\mathcal{M} : \mathrm{supp}(\mathcal{M}) \subset \widehat{X} \backslash X\} \simeq \mathrm{D^b}(\mathscr{D}_X)$,
2. $\mathrm{D^b_{hol}}(\mathscr{D}_{X_\infty}) = \mathrm{D^b_{hol}}(\mathscr{D}_{\widehat{X}})/\{\mathcal{M} : \mathrm{supp}(\mathcal{M}) \subset \widehat{X} \backslash X\}$,
3. $\mathrm{D^b_{q\text{-}good}}(\mathscr{D}_{X_\infty}) = \mathrm{D^b_{q\text{-}good}}(\mathscr{D}_{\widehat{X}})/\{\mathcal{M} : \mathrm{supp}(\mathcal{M}) \subset \widehat{X} \backslash X\}$

and we denote by $\mathscr{D}_{X_\infty}$ the class of $\mathscr{D}_{\widehat{X}} \in \mathrm{D^b}(\mathscr{D}_{\widehat{X}})$.

Secondly, it is clear that the bifunctors $\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{\widehat{X}}}, \overset{\mathrm{D}}{\otimes}$ and $\mathrm{R}\mathcal{H}om_{\mathscr{D}_{\widehat{X}}}$ factor through the quotients and give well-defined functors $\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X_\infty}}, \overset{\mathrm{D}}{\otimes}$ and $\mathrm{R}\mathcal{H}om_{\mathscr{D}_{X_\infty}}$ in the complex bordered setting.

Thirdly, if $f : X_\infty = (X, \widehat{X}) \to Y_\infty = (Y, \widehat{Y})$ is a morphism of complex bordered spaces, it is possible to define a direct and an inverse image in the bordered sense. However, this presents some complications (see Lemma 4.12 in [60]) and therefore, we shall give the definition only in the case where f extends to a holomorphic map $\hat{f} : \widehat{X} \to \widehat{Y}$. In this case, we denote by $\mathscr{D}_{X_\infty \to Y_\infty}$ (resp. $\mathscr{D}_{Y_\infty \leftarrow X_\infty}$) the $(\mathscr{D}_{X_\infty}, f^{-1}\mathscr{D}_{Y_\infty})$-bimodule (resp. the $(f^{-1}\mathscr{D}_{Y_\infty}, \mathscr{D}_{X_\infty})$-bimodule) represented by $\mathscr{D}_{\widehat{X} \to \widehat{Y}}$ (resp. $\mathscr{D}_{\widehat{Y} \leftarrow \widehat{X}}$).

If $\mathcal{N} \in \mathrm{D^b}(\mathscr{D}_{Y_\infty})$ is represented by $\mathcal{N}' \in \mathrm{D^b}(\mathscr{D}_{\widehat{Y}})$, then one sets

$$\mathrm{D}f^*\mathcal{N} = \left[\mathrm{D}\hat{f}^*\mathcal{N}'\right] = \mathscr{D}_{X_\infty \to Y_\infty} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}_{Y_\infty}} f^{-1}\mathcal{N}.$$

This gives a well defined functor

$$\mathrm{D}f^* : \mathrm{D^b}(\mathscr{D}_{Y_\infty}) \to \mathrm{D^b}(\mathscr{D}_{X_\infty}).$$

Nonetheless, the direct image functor cannot be defined simply by factorisation through the quotient since the support condition is not preserved. One thus has to introduce a slight modification. If $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_{X_\infty})$ is represented by $\mathcal{M}' \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_{\widehat{X}})$, then one sets

$$\mathrm{D}f_*\mathcal{M} = [\mathrm{D}\hat{f}_*(\mathcal{M}' \overset{\mathrm{D}}{\otimes} \Gamma_X\, \mathcal{O}^{\mathrm{t}}_{\widehat{X}})].$$

This gives a well defined functor

$$\mathrm{D}f_* : \mathrm{D}^{\mathrm{b}}(\mathscr{D}_{X_\infty}) \to \mathrm{D}^{\mathrm{b}}(\mathscr{D}_{Y_\infty}).$$

Note that the direct image preserves quasi-goodness if f is semi-proper and that the inverse image preserves quasi-goodness for any f.

4.5 Enhanced subanalytic sheaves

The theory of enhanced ind-sheaves has been extensively developed in [19], inspired by an idea of D. Tamarkin (see [112] and also [40]). In this section, we present the alternative theory of enhanced subanalytic sheaves, developed in [54]. We refer to these articles for a complete exposition.

4.5.1 Main definition

Let us denote by $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ the two-points compactification of $\mathbb{R}$ and by $\mathbb{R}_\infty$ the subanalytic bordered space $(\mathbb{R}, \overline{\mathbb{R}})$.

Definition 4.5.1. Let $M_\infty = (M, \widehat{M})$ be a subanalytic bordered space and let

$$\mu, q_1, q_2 : M_\infty \times \mathbb{R}_\infty \times \mathbb{R}_\infty \to M_\infty \times \mathbb{R}_\infty$$

be the morphisms defined by

$$\mu(x, t_1, t_2) = (x, t_1 + t_2), \qquad q_1(x, t_1, t_2) = (x, t_1), \qquad q_2(x, t_1, t_2) = (x, t_2).$$

We define the two *convolution functors*

$$\begin{aligned} - \overset{+}{\otimes} - &: \mathrm{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty}) \times \mathrm{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty}) \to \mathrm{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty}), \\ \mathscr{I}hom^+(-,-) &: \mathrm{D}^-(\mathbb{C}^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty})^{\mathrm{op}} \times \mathrm{D}^+(\mathbb{C}^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty}) \to \mathrm{D}^+(\mathbb{C}^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty}) \end{aligned}$$

by

$$\begin{aligned} F_1 \overset{+}{\otimes} F_2 &= \mathrm{R}\mu_{!!}(q_1^{-1}F_1 \otimes q_2^{-1}F_2), \\ \mathscr{I}hom^+(F_1, F_2) &= \mathrm{R}q_{1*}\mathrm{R}\mathscr{I}hom(q_2^{-1}F_1, \mu^! F_2). \end{aligned}$$

Proposition 4.5.2. *The following adjunction formulas are true for every* F_1, F_2, $F_3 \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty})$ *:*

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})}(F_1 \overset{+}{\otimes} F_2, F_3) \simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})}(F_1, \mathscr{I}hom^+(F_2, F_3)),$$
$$\mathscr{I}hom^+(F_1 \overset{+}{\otimes} F_2, F_3) \simeq \mathscr{I}hom^+(F_1, \mathscr{I}hom^+(F_2, F_3)),$$
$$\mathrm{R}\pi_{M*}\mathrm{R}\mathscr{I}hom(F_1 \overset{+}{\otimes} F_2, F_3) \simeq \mathrm{R}\pi_{M*}\mathrm{R}\mathscr{I}hom(F_1, \mathscr{I}hom^+(F_2, F_3)),$$

where $\pi_M : M_\infty \times \mathbb{R}_\infty \to M_\infty$ *is the first projection.*

Remark 4.5.3. Let $\varphi : M \to \mathbb{R}$ be a continuous function. One sets for short

$$\mathbb{C}_{\{t\geq\varphi(x)\}} = \iota_M(\mathbb{C}_{\{(x,t)\in M\times\mathbb{R}\,:\,t\geq\varphi(x)\}})$$

and one defines similarly $\mathbb{C}_{\{t\leq\varphi(x)\}}, \mathbb{C}_{\{t>\varphi(x)\}}, \mathbb{C}_{\{t<\varphi(x)\}}$ and $\mathbb{C}_{\{t=\varphi(x)\}}$. Let us also denote by $\mu_\varphi : M_\infty \times \mathbb{R}_\infty \to M_\infty \times \mathbb{R}_\infty$ the map defined by $\mu_\varphi(x,t) = (x, t+\varphi(x))$. Then,

$$\mathbb{C}_{\{t=\varphi(x)\}} \overset{+}{\otimes} F \simeq \mathrm{R}\mu_{\varphi *}F \simeq \mathscr{I}hom^+(\mathbb{C}_{\{t=-\varphi(x)\}}, F)$$

for any $F \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})$.

We can now introduce the category of enhanced subanalytic sheaves.

Definition 4.5.4. On a subanalytic bordered space $M_\infty = (M, \widehat{M})$, one defines the category of (bounded) *enhanced subanalytic sheaves* by setting

$$\mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty}) = \mathrm{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})/\{F : (\mathbb{C}_{\{t\geq 0\}} \oplus \mathbb{C}_{\{t\leq 0\}}) \overset{+}{\otimes} F \simeq 0\}.$$

We denote by

$$Q_M : \mathrm{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty}),$$

or simply Q, the quotient functor. If the context is clear, we shall simply write $\mathbb{C}_S$ instead of $Q(\mathbb{C}_S)$ when S is a subanalytic subset of $M \times \mathbb{R}$, locally closed in $\widehat{M} \times \overline{\mathbb{R}}$.

Proposition 4.5.5. *The quotient functor* Q *admits a left adjoint* L^{E} *and a right adjoint* R^{E} *defined by*

$$\mathrm{L}^{\mathrm{E}}(F) = (\mathbb{C}_{\{t\geq 0\}} \oplus \mathbb{C}_{\{t\leq 0\}}) \overset{+}{\otimes} F,$$
$$\mathrm{R}^{\mathrm{E}}(F) = \mathscr{I}hom^+(\mathbb{C}_{\{t\geq 0\}} \oplus \mathbb{C}_{\{t\leq 0\}}, F)$$

for all $F \in \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$. *Moreover, these functors are fully faithful and hence, through* R^{E}, *one can identify* $\mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$ *to a full subcategory of* $\mathrm{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty\times\mathbb{R}_\infty})$.

4.5.2 Grothendieck operations

In order to define the six Grothendieck operations on $\mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$, it is first necessary to remark that $- \overset{\mathrm{L}}{\otimes} -$ (resp. $\mathrm{R}\mathscr{I}hom(-,-)$) does not factor through the product $\mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty}) \times \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$ (resp. $\mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})^{\mathrm{op}} \times \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$). However,

$$- \overset{+}{\otimes} - : \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty}) \times \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty}),$$
$$\mathscr{I}hom^+(-,-) : \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})^{\mathrm{op}} \times \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$$

are well-defined functors.

Definition 4.5.6. One defines the hom functor

$$\mathrm{R}\mathscr{I}hom^{\mathrm{E}}(-,-) : \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})^{\mathrm{op}} \times \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty}) \to \mathrm{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$$

by

$$\mathrm{R}\mathscr{I}hom^{\mathrm{E}}(F_1, F_2) = \mathrm{R}\pi_{M*}\mathrm{R}\mathscr{I}hom(\mathrm{R}^{\mathrm{E}}F_1, \mathrm{R}^{\mathrm{E}}F_2).$$

One also defines

$$\mathrm{R}\mathcal{H}om^{\mathrm{E}}(-,-) : \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})^{\mathrm{op}} \times \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty}) \to \mathrm{D}^{\mathrm{b}}(\mathbb{C}_M)$$

by $\mathrm{R}\mathcal{H}om^{\mathrm{E}} = \alpha_M \circ \mathrm{R}\mathscr{I}hom^{\mathrm{E}}$. Finally, one sets

$$\mathrm{RHom}^{\mathrm{E}}(-,-) = \mathrm{R}\Gamma(M, \mathrm{R}\mathcal{H}om^{\mathrm{E}}(-,-)).$$

Remark that

$$\mathrm{Hom}_{\mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})}(F_1, F_2) \simeq H^0\mathrm{RHom}^{\mathrm{E}}(F_1, F_2)$$

for all $F_1, F_2 \in \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$.

To define the four other operations, let us consider a morphism $f : M_\infty \to N_\infty$ of subanalytic bordered spaces. We define

$$f_{\mathbb{R}} := f \times \mathrm{id}_{\mathbb{R}} : M_\infty \times \mathbb{R}_\infty \to N_\infty \times \mathbb{R}_\infty$$

by $f_{\mathbb{R}}(x, r) = (f(x), r)$ for all $(x, r) \in M \times \mathbb{R}$. As explained in section 4.4.2, there are four functors

$$\mathrm{R}f_{\mathbb{R}*}, \mathrm{R}f_{\mathbb{R}!!} : \mathrm{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty}) \to \mathrm{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{N_\infty \times \mathbb{R}_\infty}),$$
$$f_{\mathbb{R}}^{-1}, f_{\mathbb{R}}^{!} : \mathrm{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{N_\infty \times \mathbb{R}_\infty}) \to \mathrm{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty}).$$

It can be shown that these functors factor through the quotients. We shall write

$$\mathrm{E}f_*, \mathrm{E}f_{!!} : \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{N_\infty}),$$
$$\mathrm{E}f^{-1}, \mathrm{E}f^{!} : \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{N_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$$

their factorisation. They obviously verify all the classical properties of Grothendieck operations.

Let us finally introduce an important enhanced subanalytic sheaf.

Definition 4.5.7. Let $M_\infty = (M, \widehat{M})$ be a subanalytic bordered space. One sets

$$\mathbb{C}^{\mathrm{E}}_{M_\infty} = Q\left(\varinjlim_{a \to +\infty} \mathbb{C}_{\{t \geq a\}}\right).$$

Proposition 4.5.8 ([19], Corollary 4.7.8)**.** *For any $F_1, F_2 \in \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$, there is an isomorphism*

$$\mathscr{I}hom^{+}(\mathbb{C}^{\mathrm{E}}_{M_\infty} \overset{+}{\otimes} F_1, \mathbb{C}^{\mathrm{E}}_{M_\infty} \overset{+}{\otimes} F_2) \simeq \mathscr{I}hom^{+}(F_1, \mathbb{C}^{\mathrm{E}}_{M_\infty} \overset{+}{\otimes} F_2)$$

in $\mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$.

4.6 Enhanced distributions

Enhanced distributions are the key tools to understand the enhanced Laplace transform. They have been introduced in [19] and also studied in [54] and [60]. From now on, we refer to these three articles. Note that, in this section, we will consider *real analytic bordered spaces* $(M, \widehat{M})$, i.e. bordered spaces where $\widehat{M}$ is a real analytic manifold. The morphisms of real analytic bordered spaces are defined in the obvious manner.

4.6.1 Several definitions

Let $M_\infty = (M, \widehat{M})$ be a real analytic bordered space and let $\mathrm{P} = \mathbb{R} \cup \{\infty\}$ (resp. $\mathbb{P} = \mathbb{C} \cup \{\infty\}$) be the one-point compactification of $\mathbb{R}$ (resp. $\mathbb{C}$). We denote by $t \in \mathbb{R} \subset \mathrm{P}$ and $\tau \in \mathbb{C} \subset \mathbb{P}$ the affine coordinates, with $\tau|_{\mathbb{R}} = t$.

Definition 4.6.1. Let

$$j : M_\infty \times \mathbb{R}_\infty \to \widehat{M} \times \mathrm{P}$$

be the canonical inclusion of bordered spaces. One sets

$$\mathcal{D}b^{\mathrm{T}}_{M_\infty} = j^! \mathrm{R}\mathcal{H}om_{\mathscr{D}_{\mathbb{P}}}(\mathscr{E}^{\tau}_{\mathbb{C}|\mathbb{P}}, \mathcal{D}b^{\mathrm{t}}_{\widehat{M}\times \mathrm{P}})[1] \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty}).$$

Recall that ∂_t extends to a vector field on P. The following proposition is helpful to concretely use $\mathcal{D}b^{\mathrm{T}}_{M_\infty}$.

Proposition 4.6.2. *The complex $\mathcal{D}b^{\mathrm{T}}_{M_\infty}$ is concentrated in degree -1 and*

$$H^{-1}(\mathcal{D}b^{\mathrm{T}}_{M_\infty}) \simeq j^{-1}(\ker(\mathcal{D}b^{\mathrm{t}}_{\widehat{M}\times \mathrm{P}} \xrightarrow{\partial_t - 1} \mathcal{D}b^{\mathrm{t}}_{\widehat{M}\times \mathrm{P}})) \in \mathrm{Mod}(\mathbb{C}^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty}).$$

Definition 4.6.3. The subanalytic sheaf $H^{-1}(\mathcal{D}b^{\mathrm{T}}_{M_\infty})$ is called the sheaf of *enhanced distributions*. If no confusion is possible, we shall simply note it $\mathcal{D}b^{\mathrm{T}}_{M_\infty}$. One similarly introduces the sheaves $\mathcal{D}b^{\mathrm{T},r}_{M_\infty}$ of *enhanced distributional r-forms* for each $r \in \mathbb{Z}$.

Of course, on a complex bordered space $X_\infty = (X, \widehat{X})$, these subanalytic sheaves admit a bi-type decomposition

$$\mathcal{D}b^{\mathrm{T},r}_{X_\infty} \simeq \bigoplus_{p+q=r} \mathcal{D}b^{\mathrm{T},p,q}_{X_\infty}$$

for each $r \in \mathbb{Z}$.

Definition 4.6.4. Let X_∞ be a complex bordered space of complex dimension d_X and let $p \in \mathbb{Z}$. One defines the complex of *enhanced holomorphic p-forms* $\Omega^{\mathrm{E},p}_{X_\infty} \in \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X_\infty})$ by the Dolbeault complex

$$Q(0 \to \mathcal{D}b^{\mathrm{T},p,0}_{X_\infty} \xrightarrow{\bar\partial} \mathcal{D}b^{\mathrm{T},p,1}_{X_\infty} \to \cdots \to \mathcal{D}b^{\mathrm{T},p,d_X}_{X_\infty} \to 0).$$

One sets for short $\mathcal{O}^{\mathrm{E}}_{X_\infty} = \Omega^{\mathrm{E},0}_{X_\infty}$ and $\Omega^{\mathrm{E}}_{X_\infty} = \Omega^{\mathrm{E},d_X}_{X_\infty}$.

Proposition 4.6.5 ([19], Corollary 8.2.3)**.** *There are isomorphisms*

$$\mathbb{C}^{\mathrm{E}}_{X_\infty} \overset{+}{\otimes} \Omega^{\mathrm{E},p}_{X_\infty} \simeq \Omega^{\mathrm{E},p}_{X_\infty} \simeq \mathscr{I}hom^{+}(\mathbb{C}^{\mathrm{E}}_{X_\infty}, \Omega^{\mathrm{E},p}_{X_\infty})$$

in $\mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X_\infty})$*, for all* $p \in \mathbb{Z}$*.*

Since $\mathcal{O}^{\mathrm{E}}_{X_\infty}$ (resp. $\Omega^{\mathrm{E}}_{X_\infty}$) has a canonical structure of left (resp. right) $\mathscr{D}_{X_\infty}$-module, one can also introduce an associated De Rham/solution complex.

Definition 4.6.6. One sets

$$\begin{aligned} \mathcal{DR}^{\mathrm{E}}_{X_\infty} &: \mathrm{D}^{\mathrm{b}}(\mathscr{D}_{X_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X_\infty}), \quad \mathcal{M} \mapsto \Omega^{\mathrm{E}}_{X_\infty} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X_\infty}} \mathcal{M}, \\ \mathcal{S}ol^{\mathrm{E}}_{X_\infty} &: \mathrm{D}^{\mathrm{b}}(\mathscr{D}_{X_\infty})^{\mathrm{op}} \to \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X_\infty}), \quad \mathcal{M} \mapsto \mathrm{R}\mathcal{H}om_{\mathscr{D}_{X_\infty}}(\mathcal{M}, \mathcal{O}^{\mathrm{E}}_{X_\infty}). \end{aligned}$$

The functor $\mathcal{DR}^{\mathrm{E}}_{X_\infty}$ (resp. $\mathcal{S}ol^{\mathrm{E}}_{X_\infty}$) is called the *enhanced De Rham functor* (resp. *enhanced solution functor*) of X_∞.

We can now extend the operations of integration and pullback (recall sections 1.4.2, 1.4.3, 4.3.2 and 4.3.3) to enhanced distributional forms. Using Dolbeault resolutions, we shall also point out that these constructions are encoded in the important results of [60].

4.6.2 Integration

Lemma 4.6.7. *Let* $M_\infty = (M, \widehat{M})$ *(resp.* $N_\infty = (N, \widehat{N})$*) be a real analytic bordered space of real dimension* d_M *(resp.* d_N*) and let* $f : M_\infty \to N_\infty$ *be a morphism of real analytic bordered spaces. Then, the integration of distributions along the fibers of* $f_{\mathbb{R}}$ *induces a morphism of complexes*

$$\int_{f_{\mathbb{R}}} : f_{\mathbb{R}!!}\, \mathcal{D}b^{\mathrm{T},\bullet+d_M}_{M_\infty} \to \mathcal{D}b^{\mathrm{T},\bullet+d_N}_{N_\infty}\,. \tag{4.9}$$

Proof. Using the graph embedding (recall Remark 4.4.2), one can assume that f extends to a map $\hat{f} : \widehat{M} \to \widehat{N}$ and thus $f_{\mathbb{R}}$ extends to $\hat{f}_{\mathrm{P}} = \hat{f} \times \mathrm{id}_{\mathrm{P}}$. Let us write

$$\begin{aligned} \mathcal{K}^r_{\widehat{M}\times\mathrm{P}} &= \ker(\mathcal{D}b^{\mathrm{t},r+d_M}_{\widehat{M}\times\mathrm{P}} \xrightarrow{\partial_t - 1} \mathcal{D}b^{\mathrm{t},r+d_M}_{\widehat{M}\times\mathrm{P}}), \\ \mathcal{K}^r_{\widehat{N}\times\mathrm{P}} &= \ker(\mathcal{D}b^{\mathrm{t},r+d_N}_{\widehat{N}\times\mathrm{P}} \xrightarrow{\partial_t - 1} \mathcal{D}b^{\mathrm{t},r+d_N}_{\widehat{N}\times\mathrm{P}}) \end{aligned}$$

for each $r \in \mathbb{Z}$. Let us also denote by

$$j_{M\times\mathbb{R}} : M_\infty \times \mathbb{R}_\infty \to \widehat{M} \times \mathrm{P}, \quad j_{N\times\mathbb{R}} : N_\infty \times \mathbb{R}_\infty \to \widehat{N} \times \mathrm{P}$$

the canonical inclusions of bordered spaces. One has

$$\mathcal{D}b^{\mathrm{T},r+d_M}_{M_\infty} = j^{-1}_{M\times\mathbb{R}} \mathcal{K}^r_{\widehat{M}\times\mathrm{P}} \qquad \text{and} \qquad \mathcal{D}b^{\mathrm{T},r+d_N}_{N_\infty} = j^{-1}_{N\times\mathbb{R}} \mathcal{K}^r_{\widehat{N}\times\mathrm{P}}.$$

Hence, if we manage to define a morphism

$$\int_{\hat{f}_{\mathrm{P}}} : \hat{f}_{\mathrm{P}!!}\mathcal{K}^r_{\widehat{M}\times\mathrm{P}} \to \mathcal{K}^r_{\widehat{N}\times\mathrm{P}} \tag{4.10}$$

for all $r \in \mathbb{Z}$, we will get the conclusion. Indeed, (4.10) will induce a morphism

$$\int_{\hat{f}_{\mathrm{P}}} : \hat{f}_{\mathrm{P}!!}(\mathbb{C}_{M\times\mathbb{R}} \otimes \mathcal{K}^r_{\widehat{M}\times\mathrm{P}}) \to \mathcal{K}^r_{\widehat{N}\times\mathrm{P}} \tag{4.11}$$

and applying $j^{-1}_{N\times R}$ to (4.11) will be enough since

$$\begin{aligned} j^{-1}_{N\times R}\hat{f}_{\mathrm{P}!!}(\mathbb{C}_{M\times\mathbb{R}} \otimes \mathcal{K}^r_{\widehat{M}\times\mathrm{P}}) &\simeq j^{-1}_{N\times R}\hat{f}_{\mathrm{P}!!}(j_{M\times R!!}j^{-1}_{M\times\mathbb{R}}\mathcal{K}^r_{\widehat{M}\times\mathrm{P}}) \\ &\simeq j^{-1}_{N\times R}\hat{f}_{\mathrm{P}!!}j_{M\times R!!}\,\mathcal{D}b^{\mathrm{T},r+d_M}_{M_\infty} \\ &\simeq f_{\mathbb{R}!!}\,\mathcal{D}b^{\mathrm{T},r+d_M}_{M_\infty}\,. \end{aligned}$$

(See Lemmas 3.3.7 and 3.3.12 in [19].)

To define (4.10), let us take $V \in \mathrm{Op}^{\mathrm{sub},c}_{\widehat{N}\times\mathrm{P}}$ and show that the integration of distributions is a well defined map

$$\varinjlim_K \Gamma(\hat{f}_{\mathrm{P}}^{-1}(V), \mathscr{I}\Gamma_K(\mathcal{K}^r_{\widehat{M}\times\mathrm{P}})) \to \Gamma(V, \mathcal{K}^r_{\widehat{N}\times\mathrm{P}}).$$

Thanks to Lemma 4.3.9, we only have to check that the enhanced condition is preserved. Let us choose $u \in \Gamma(\hat{f}_{\mathrm{P}}^{-1}(V), \mathscr{I}\Gamma_K(\mathcal{K}^r_{\widehat{M}\times\mathrm{P}}))$ for some subanalytic compact subset K of $\widehat{M}\times\mathrm{P}$. Thus, $\partial_t u = u$ and it follows that

$$\begin{aligned} \left\langle \partial_t \int_{\hat{f}_{\mathrm{P}}} u, \omega \right\rangle &= -\left\langle \int_{\hat{f}_{\mathrm{P}}} u, \partial_t\omega \right\rangle = -\langle u, \hat{f}^*_{\mathrm{P}}\partial_t\omega\rangle \\ &\underset{(\star)}{=} -\langle u, \partial_t\hat{f}^*_{\mathrm{P}}\omega\rangle = \langle \partial_t u, \hat{f}^*_{\mathrm{P}}\omega\rangle \\ &= \langle u, \hat{f}^*_{\mathrm{P}}\omega\rangle = \left\langle \int_{\hat{f}_{\mathrm{P}}} u, \omega \right\rangle \end{aligned}$$

for all test-form ω. The equality $(\star)$ is of course obtained thanks to the specific form of $\hat{f}_{\mathrm{P}}$. Hence $\int_{\hat{f}_{\mathrm{P}}} u$ is still an enhanced distributional form and we get the conclusion. □

Let us now prove a second lemma which will play a central role in the further considerations.

Lemma 4.6.8. *Let $M_\infty = (M, \widehat{M})$ be a real analytic bordered space. Then, for each $r \in \mathbb{Z}$, the subanalytic sheaf $\mathcal{D}b^{\mathrm{T},r}_{M_\infty}$ is acyclic for $f_{\mathbb{R}!!}$.*

Proof. We keep the same notations than in previous lemma. Let $r \in \mathbb{Z}$. By Lemma 6.2.4 in [19], we get a short exact sequence

$$0 \longrightarrow \mathcal{K}^r_{\widehat{M}\times\mathrm{P}} \longrightarrow \mathcal{D}b^{\mathrm{t},r}_{\widehat{M}\times\mathrm{P}} \xrightarrow{\partial_t - 1} \mathcal{D}b^{\mathrm{t},r}_{\widehat{M}\times\mathrm{P}} \longrightarrow 0.$$

Applying $j^{-1}_{M\times\mathbb{R}}$, which is exact, to the previous sequence gives a short exact sequence

$$0 \longrightarrow \mathcal{D}b^{\mathrm{T},r}_{M_\infty} \longrightarrow j^{-1}_{M\times\mathbb{R}}\,\mathcal{D}b^{\mathrm{t},r}_{\widehat{M}\times\mathrm{P}} \xrightarrow{\partial_t - 1} j^{-1}_{M\times\mathbb{R}}\,\mathcal{D}b^{\mathrm{t},r}_{\widehat{M}\times\mathrm{P}} \longrightarrow 0$$

and thus, a long exact sequence

$$\begin{aligned} 0 \to f_{\mathbb{R}!!}\,\mathcal{D}b^{\mathrm{T},r}_{M_\infty} &\longrightarrow f_{\mathbb{R}!!}(j^{-1}_{M\times\mathbb{R}}\,\mathcal{D}b^{\mathrm{t},r}_{\widehat{M}\times\mathrm{P}}) \longrightarrow f_{\mathbb{R}!!}(j^{-1}_{M\times\mathbb{R}}\,\mathcal{D}b^{\mathrm{t},r}_{\widehat{M}\times\mathrm{P}}) \\ \to \mathrm{R}^1 f_{\mathbb{R}!!}\,\mathcal{D}b^{\mathrm{T},r}_{M_\infty} &\longrightarrow \mathrm{R}^1 f_{\mathbb{R}!!}(j^{-1}_{M\times\mathbb{R}}\,\mathcal{D}b^{\mathrm{t},r}_{\widehat{M}\times\mathrm{P}}) \longrightarrow \mathrm{R}^1 f_{\mathbb{R}!!}(j^{-1}_{M\times\mathbb{R}}\,\mathcal{D}b^{\mathrm{t},r}_{\widehat{M}\times\mathrm{P}}) \to \dots \end{aligned}$$

Since $\mathcal{D}b^{\mathrm{t},r}_{\widehat{M}\times\mathrm{P}}$ is quasi-injective, we know that $\mathrm{R}^k f_{\mathbb{R}!!}(j^{-1}_{M\times\mathbb{R}}\,\mathcal{D}b^{\mathrm{t},r}_{\widehat{M}\times\mathrm{P}}) \simeq 0$ for all $k \geq 1$. Therefore, for all $k \geq 2$, one has $\mathrm{R}^k f_{\mathbb{R}!!}\,\mathcal{D}b^{\mathrm{T},r}_{M_\infty} \simeq 0$ and it only remains to show that $\mathrm{R}^1 f_{\mathbb{R}!!}\,\mathcal{D}b^{\mathrm{T},r}_{M_\infty} \simeq 0$, that is to say, to show that

$$f_{\mathbb{R}!!}(j^{-1}_{M\times\mathbb{R}}\,\mathcal{D}b^{\mathrm{t},r}_{\widehat{M}\times\mathrm{P}}) \xrightarrow{\partial_t - 1} f_{\mathbb{R}!!}(j^{-1}_{M\times\mathbb{R}}\,\mathcal{D}b^{\mathrm{t},r}_{\widehat{M}\times\mathrm{P}})$$

is an epimorphism. Since $j^{-1}_{N\times\mathbb{R}}$ is exact, it is enough to show that

$$\hat{f}_{\mathrm{P}!!}(\mathbb{C}_{M\times\mathbb{R}} \otimes \mathcal{D}b^{\mathrm{t},r}_{\widehat{M}\times\mathrm{P}}) \xrightarrow{\partial_t - 1} \hat{f}_{\mathrm{P}!!}(\mathbb{C}_{M\times\mathbb{R}} \otimes \mathcal{D}b^{\mathrm{t},r}_{\widehat{M}\times\mathrm{P}}) \tag{4.12}$$

is an epimorphism. Using again Lemma 6.2.4 of [19] and the exactness of $\mathbb{C}_{M\times\mathbb{R}} \otimes -$, it is clear that

$$\hat{f}_{\mathrm{P}*}(\mathbb{C}_{M\times\mathbb{R}} \otimes \mathcal{D}b^{\mathrm{t},r}_{\widehat{M}\times\mathrm{P}}) \xrightarrow{\partial_t - 1} \hat{f}_{\mathrm{P}*}(\mathbb{C}_{M\times\mathbb{R}} \otimes \mathcal{D}b^{\mathrm{t},r}_{\widehat{M}\times\mathrm{P}})$$

is an epimorphism. Hence, we just have to prove that, if u and v are two sections of $\hat{f}_{\mathrm{P}*}(\mathbb{C}_{M\times\mathbb{R}} \otimes \mathcal{D}b^{\mathrm{t},r}_{\widehat{M}\times\mathrm{P}})$ such that $\partial_t v - v = u$ and if u is compactly supported, then v is also compactly supported.

Let us denote by $\pi_{\widehat{M}} : \widehat{M} \times \mathrm{P} \to \widehat{M}$ the first projection. Then, if u and v are two sections of $\hat{f}_{\mathrm{P}*}(\mathbb{C}_{M\times\mathbb{R}} \otimes \mathcal{D}b^{\mathrm{t},r}_{\widehat{M}\times\mathrm{P}})$ such that $\partial_t v - v = u$, we have

$$\mathrm{supp}(v) \subset \pi_{\widehat{M}}(\mathrm{supp}(u)) \times \mathbb{R} \subset \pi_{\widehat{M}}(\mathrm{supp}(u)) \times \mathrm{P}\,. \tag{4.13}$$

Indeed, let ω be a test-form supported by $(\complement\pi_{\widehat{M}}(\mathrm{supp}(u))) \times \mathbb{R}$. Then

$$\begin{aligned} \langle v, \omega\rangle &= \left\langle v, (\partial_t + 1)\left(e^{-t}\int e^t\omega\right)\right\rangle \\ &= \left\langle (-\partial_t + 1)v, e^{-t}\int e^t\omega\right\rangle \\ &= -\left\langle u, e^{-t}\int e^t\omega\right\rangle \\ &= 0. \end{aligned}$$

Hence, (4.13) is true and we get the conclusion. $\square$

Proposition 4.6.9. *Let $f : X_\infty = (X, \widehat{X}) \to Y_\infty = (Y, \widehat{Y})$ be a morphism of complex bordered spaces of complex dimension d_X and d_Y. The integration of distributions along the fibers of $f_\mathbb{R}$ induces a morphism of double complexes*

$$\int_{f_\mathbb{R}} : f_{\mathbb{R}!!}\,\mathcal{D}b^{\mathrm{T},\bullet+d_X,\bullet+d_X}_{X_\infty} \to \mathcal{D}b^{\mathrm{T},\bullet+d_Y,\bullet+d_Y}_{Y_\infty} \tag{4.14}$$

and thus a morphism

$$\int_{f_\mathbb{R}} : \mathrm{E}f_{!!}\Omega^{\mathrm{E},p+d_X}_{X_\infty}[d_X] \to \Omega^{\mathrm{E},p+d_Y}_{Y_\infty}[d_Y] \tag{4.15}$$

in $\mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{Y_\infty})$ for each $p \in \mathbb{Z}$.

Proof. Thanks to Lemma 4.6.7, it is clear that (4.14) is well-defined. Moreover, by the same proof as the one of Lemma 4.6.8, one can show that $\mathcal{D}b^{\mathrm{T},p+d_X,q+d_X}_{X_\infty}$ is $f_{\mathbb{R}!!}$-acyclic for all $(p,q) \in \mathbb{Z}^2$. Hence the conclusion. □

Proposition 4.6.10. *Let $f : X_\infty = (X, \widehat{X}) \to Y_\infty = (Y, \widehat{Y})$ be a morphism of complex bordered spaces of complex dimension d_X and d_Y and let $\mathcal{N} \in \mathrm{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_{Y_\infty})$.*

(i) ([60], Proposition 4.15 (i)) There is a natural isomorphism

$$\mathcal{DR}^{\mathrm{E}}_{X_\infty}(\mathrm{D}f^*\mathcal{N})[d_X] \simeq \mathrm{E}f^!\,\mathcal{DR}^{\mathrm{E}}_{Y_\infty}(\mathcal{N})[d_Y]. \tag{4.16}$$

(ii) If f extends to a holomorphic map $\hat{f} : \widehat{X} \to \widehat{Y}$, then applying (4.16) to $\mathcal{N} = \mathscr{D}_{Y_\infty}$ gives an isomorphism

$$\Omega^{\mathrm{E}}_{X_\infty} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X_\infty}} \mathscr{D}_{X_\infty\to Y_\infty}[d_X] \xrightarrow{\sim} \mathrm{E}f^!\Omega^{\mathrm{E}}_{Y_\infty}[d_Y]. \tag{4.17}$$

(iii) This morphism induces, thanks to $1_{X_\infty\to Y_\infty}$, a morphism

$$\Omega^{\mathrm{E}}_{X_\infty}[d_X] \to \mathrm{E}f^!\Omega^{\mathrm{E}}_{Y_\infty}[d_Y]$$

in $\mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X_\infty})$, which is equivalent to the adjoint of (4.15) when $p = 0$.

Proof. First let us replace $\mathcal{N}$ by $\mathscr{D}_{Y_\infty}$ in (4.16). On the one hand

$$\begin{aligned}\mathcal{DR}^{\mathrm{E}}_{X_\infty}(\mathrm{D}f^*\mathcal{N}) &= \Omega^{\mathrm{E}}_{X_\infty} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X_\infty}} \mathrm{D}f^*\mathscr{D}_{Y_\infty}\\ &\simeq \Omega^{\mathrm{E}}_{X_\infty} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X_\infty}} \mathscr{D}_{X_\infty\to Y_\infty} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}_{Y_\infty}} f^{-1}\mathscr{D}_{Y_\infty}\\ &\simeq \Omega^{\mathrm{E}}_{X_\infty} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X_\infty}} \mathscr{D}_{X_\infty\to Y_\infty}\end{aligned}$$

and on the other hand

$$\mathrm{E}f^!\,\mathcal{DR}^{\mathrm{E}}_{Y_\infty}(\mathcal{N}) = \mathrm{E}f^!(\Omega^{\mathrm{E}}_{Y_\infty} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{Y_\infty}} \mathscr{D}_{Y_\infty}) \simeq \mathrm{E}f^!\Omega^{\mathrm{E}}_{Y_\infty}.$$

By construction of (4.16), the isomorphism (4.17) is built as an enhancement on bordered spaces of (4.4). The conclusion follows from Proposition 4.3.11. □

4.6.3 Pullback

Lemma 4.6.11. *Let $f : M_\infty = (M, \widehat{M}) \to N_\infty = (N, \widehat{N})$ be a morphism of real analytic bordered spaces. Assume that f can be extended to a real analytic submersion $\hat{f} : \widehat{M} \to \widehat{N}$. Then, the pullback of distributions by $\hat{f}_{\mathrm{P}}$ induces a morphism of complexes*

$$f_{\mathbb{R}}^* : f_{\mathbb{R}}^{-1}\, \mathcal{D}b_{N_\infty}^{\mathrm{T},\bullet} \to \mathcal{D}b_{M_\infty}^{\mathrm{T},\bullet}\,. \tag{4.18}$$

Proof. Taking the notations of Lemma 4.6.7, it is enough to prove that the pullback of distributions gives a well-defined map

$$\hat{f}_{\mathrm{P}}^* : \hat{f}_{\mathrm{P}}^{-1}\mathcal{K}^r_{\widehat{N}\times\mathrm{P}} \to \mathcal{K}^r_{\widehat{M}\times\mathrm{P}}$$

for each $r \in \mathbb{Z}$ and then apply $j^{-1}_{M\times\mathbb{R}}$. Thanks to Lemma 4.3.13, we only have to show that the enhanced condition is preserved by pullback. Let $U \in \mathrm{Op}^{\mathrm{sub},c}_{\widehat{M}\times\mathrm{P}}$ and $v \in \Gamma(U, \hat{f}_{\mathrm{P}}^{-1}\mathcal{K}^r_{\widehat{N}\times\mathrm{P}})$. Thus, $\partial_t v = v$ and it follows that

$$\begin{aligned}
\langle \partial_t \hat{f}_{\mathrm{P}}^* v, \omega\rangle &= -\langle \hat{f}_{\mathrm{P}}^* v, \partial_t\omega\rangle = -\left\langle v, \int_{\hat{f}_{\mathrm{P}}} \partial_t\omega \right\rangle \\
&\underset{(\star)}{=} -\left\langle v, \partial_t \int_{\hat{f}_{\mathrm{P}}} \omega \right\rangle = \left\langle \partial_t v, \int_{\hat{f}_{\mathrm{P}}} \omega \right\rangle \\
&= \left\langle v, \int_{\hat{f}_{\mathrm{P}}} \omega \right\rangle = \langle \hat{f}_{\mathrm{P}}^* v, \omega\rangle
\end{aligned}$$

for all test-form ω. Here, $(\star)$ comes from the equality $\hat{f}_{\mathrm{P}}^{-1}(x,t) = \hat{f}^{-1}(x)\times\{t\}$ of fibers. Hence $\hat{f}_{\mathrm{P}}^* v$ is still an enhanced distributional form and we get the conclusion. $\square$

Proposition 4.6.12. *Let $f : X_\infty = (X, \widehat{X}) \to Y_\infty = (Y, \widehat{Y})$ be a morphism of complex bordered spaces and assume that f can be extended to a holomorphic submersion $\hat{f} : \widehat{X} \to \widehat{Y}$. The pullback of distributions by $\hat{f}_{\mathrm{P}}$ induces a morphism of double complexes*

$$f_{\mathbb{R}}^* : f_{\mathbb{R}}^{-1}\, \mathcal{D}b_{Y_\infty}^{\mathrm{T},\bullet,\bullet} \to \mathcal{D}b_{X_\infty}^{\mathrm{T},\bullet,\bullet} \tag{4.19}$$

and thus a morphism

$$f_{\mathbb{R}}^* : \mathrm{E}f^{-1}\Omega^{\mathrm{E},p}_{Y_\infty} \to \Omega^{\mathrm{E},p}_{X_\infty} \tag{4.20}$$

in $\mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X_\infty})$ for each $p \in \mathbb{Z}$.

Proof. The first morphism is well-defined thanks to the previous lemma and the second one is obtained by the exactness of $f_{\mathbb{R}}^{-1}$. $\square$

Proposition 4.6.13. *Let $f : X_\infty = (X, \widehat{X}) \to Y_\infty = (Y, \widehat{Y})$ be a semi-proper morphism of complex bordered spaces and let $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_{X_\infty})$.*

(i) ([60], Proposition 4.15 (ii)) There is a natural isomorphism

$$\mathcal{DR}^{\mathrm{E}}_{Y_\infty}(\mathrm{D}f_*\mathcal{M}) \simeq \mathrm{E}f_*\, \mathcal{DR}^{\mathrm{E}}_{X_\infty}(\mathcal{M}). \tag{4.21}$$

(ii) *([58], Lemma* 7.4.10*) If f extends to a holomorphic map $\hat{f} : \widehat{X} \to \widehat{Y}$, then (4.21) is induced by a morphism*

$$\mathscr{D}_{X_\infty \to Y_\infty} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}_{Y_\infty}} \mathrm{E}f^{-1}\,\mathcal{O}^{\mathrm{E}}_{Y_\infty} \to \mathcal{O}^{\mathrm{E}}_{X_\infty}, \tag{4.22}$$

which is an enhancement of (4.8).

(iii) *The morphism (4.22) induces, thanks to $1_{X_\infty \to Y_\infty}$, a morphism*

$$\mathrm{E}f^{-1}\,\mathcal{O}^{\mathrm{E}}_{Y_\infty} \to \mathcal{O}^{\mathrm{E}}_{X_\infty}$$

in $\mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X_\infty})$, which is equivalent to (4.20) when $p = 0$, if $\hat{f}$ is a holomorphic submersion.

Chapter 5

Enhanced Laplace transform and applications

5.1 The enhanced Laplace transform theorem

In this section, we explain how the usual Laplace transform of tempered distributions can be studied in the enhanced subanalytic sheaves framework. Using the Dolbeault complex of enhanced distributions, we remark that an important isomorphism of [60] can be described explicitly. Throughout this section, we constantly refer to [60].

5.1.1 Multiplication by an exponential kernel

In section 4.6, we introduced two operations on enhanced distributions : integration and pullback. To understand the Laplace transform, one needs a third missing operation, the multiplication by an exponential kernel. Recall Remark 4.5.3.

Proposition 5.1.1. *Let $X_\infty = (X, \widehat{X})$ be a complex bordered space and $\varphi : X \to \mathbb{C}$ a tempered function at infinity, i.e. $\varphi \in \Gamma(X, \mathcal{C}^{\mathrm{t}}_{\infty,\widehat{X}})$. Then, there is a morphism*

$$\mu_{-\Re\varphi_*} \mathcal{D}b^{\mathrm{T},p,q}_{X_\infty} \to \mathcal{D}b^{\mathrm{T},p,q}_{X_\infty} \tag{5.1}$$

defined by $u \mapsto e^{\varphi}u$ for any $(p, q) \in \mathbb{Z}$. If moreover φ is holomorphic, this gives rise to a morphism of complexes

$$\mu_{-\Re\varphi_*} \mathcal{D}b^{\mathrm{T},p,\bullet}_{X_\infty} \to \mathcal{D}b^{\mathrm{T},p,\bullet}_{X_\infty} \tag{5.2}$$

for each $p \in \mathbb{Z}$. This morphism induces itself a morphism

$$\mathbb{C}_{\{t=-\Re\varphi(x)\}} \overset{+}{\otimes} \Omega^{\mathrm{E},p}_{X_\infty} \to \Omega^{\mathrm{E},p}_{X_\infty} \tag{5.3}$$

in $\mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{X_\infty})$ for each $p \in \mathbb{Z}$.

Proof. Let us consider $U \in \mathrm{Op}^{\mathrm{sub},c}_{X_\infty \times \mathbb{R}_\infty}$. Then, for any $(p,q) \in \mathbb{Z}^2$, we define the map

$$\Gamma(U, \mu_{-\Re\varphi *} \mathcal{D}b^{\mathrm{T},p,q}_{X_\infty}) = \Gamma(\mu^{-1}_{-\Re\varphi}(U), \mathcal{D}b^{\mathrm{T},p,q}_{X_\infty}) \to \Gamma(U, \mathcal{D}b^{\mathrm{T},p,q}_{X_\infty})$$

by $u(x,t) \mapsto u(x, t+\Re\varphi(x))$. (This little abuse of notation corresponds to the pullback of u by $\mu_{\Re\varphi}$.) Since φ is tempered, this map is well-defined. Moreover, since u is a solution of $\partial_t u = u$, one can write $u(x,t) = e^t \rho(x)$ for a unique distributional form ρ. Hence

$$u(x, t + \Re\varphi(x)) = e^{t+\Re\varphi(x)}\rho(x) = e^{\Re\varphi(x)} u(x,t).$$

To obtain (5.1), it is now enough to compose this map with

$$\Gamma(U, \mathcal{D}b^{\mathrm{T},(p,q)}_{X_\infty}) \ni u \mapsto e^{i\Im\varphi} u \in \Gamma(U, \mathcal{D}b^{\mathrm{T},(p,q)}_{X_\infty}),$$

which is of course well-defined since $|e^{i\Im\varphi}| = 1$. Then, (5.2) follows from the equality $\overline{\partial}(e^\varphi u) = e^\varphi \overline{\partial} u$ if φ is holomorphic and (5.3) from the exactness of $\mu_{-\Re\varphi *}$. $\square$

Particular cases of functions φ tempered at infinity are the functions with a polar singularity at infinity.

Proposition 5.1.2 ([19], Corollary 9.4.12)**.** *Let $X_\infty = (X, \widehat{X})$ be a complex bordered space and $\varphi \in \mathcal{O}_{\widehat{X}}(*\widehat{X}\backslash X)$. Then,*

$$\mathcal{S}ol^{\mathrm{E}}_{X_\infty}(\mathscr{E}^{\varphi}_{X|\widehat{X}}) = \mathbb{C}^{\mathrm{E}}_{X_\infty} \overset{+}{\otimes} \mathbb{C}_{\{t=-\Re\varphi(x)\}}\,.$$

Proposition 5.1.3. *Let $X_\infty = (X, \widehat{X})$ be a complex bordered space, $\mathcal{L} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_{X_\infty})$ and $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_{X_\infty})$.*

(i) ([60], Proposition 4.15 (iii))There is a natural isomorphism

$$\mathcal{DR}^{\mathrm{E}}_{X_\infty}(\mathcal{L} \overset{\mathrm{D}}{\otimes} \mathcal{M}) \simeq \mathrm{R}\mathscr{I}hom^{+}(\mathcal{S}ol^{\mathrm{E}}_{X_\infty}(\mathcal{L})), \mathcal{DR}^{\mathrm{E}}_{X_\infty}(\mathcal{M})). \tag{5.4}$$

(ii) Let $\varphi \in \mathcal{O}_{\widehat{X}}(*\widehat{X}\backslash X)$. Then, (5.4) applied to $\mathcal{M} = \mathscr{D}_{X_\infty} \otimes_{\mathcal{O}_{X_\infty}} \Omega^{\otimes -1}_{X_\infty}$ and $\mathcal{L} = \mathscr{E}^{\varphi}_{X|\widehat{X}}$ gives an adjoint morphism

$$\mathbb{C}_{\{t=-\Re\varphi(x)\}} \overset{+}{\otimes} \left(\mathscr{E}^{\varphi}_{X|\widehat{X}} \overset{\mathrm{D}}{\otimes} \mathcal{O}^{\mathrm{E}}_{X_\infty} \right) \to \mathcal{O}^{\mathrm{E}}_{X_\infty} \tag{5.5}$$

that induces, thanks to the canonical section e^φ of $\mathscr{E}^{\varphi}_{X|\widehat{X}}$, a morphism

$$\mathbb{C}_{\{t=-\Re\varphi(x)\}} \overset{+}{\otimes} \mathcal{O}^{\mathrm{E}}_{X_\infty} \to \mathcal{O}^{\mathrm{E}}_{X_\infty},$$

which is equivalent to (5.3) when $p = 0$.

Proof. Let us replace $\mathcal{M}$ by $\mathscr{D}_{X_\infty} \otimes_{\mathcal{O}_{X_\infty}} \Omega_{X_\infty}^{\otimes -1}$ and $\mathcal{L}$ by $\mathscr{E}^{\varphi}_{X|\widehat{X}}$ in (5.4). On the one hand,

$$\begin{aligned}\mathcal{DR}^{\mathrm{E}}_{X_\infty}(\mathcal{L} \overset{\mathrm{D}}{\otimes} \mathcal{M}) &= \mathcal{DR}^{\mathrm{E}}_{X_\infty}(\mathscr{E}^{\varphi}_{X|\widehat{X}} \overset{\mathrm{D}}{\otimes} (\mathscr{D}_{X_\infty} \otimes_{\mathcal{O}_{X_\infty}} \Omega_{X_\infty}^{\otimes -1})) \\ &= \Omega^{\mathrm{E}}_{X_\infty} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X_\infty}} (\mathscr{E}^{\varphi}_{X|\widehat{X}} \overset{\mathrm{D}}{\otimes} (\mathscr{D}_{X_\infty} \otimes_{\mathcal{O}_{X_\infty}} \Omega_{X_\infty}^{\otimes -1})) \\ &\simeq \mathscr{E}^{\varphi}_{X|\widehat{X}} \overset{\mathrm{D}}{\otimes} (\Omega^{\mathrm{E}}_{X_\infty} \otimes_{\mathcal{O}_{X_\infty}} \Omega_{X_\infty}^{\otimes -1}) \\ &\simeq \mathscr{E}^{\varphi}_{X|\widehat{X}} \overset{\mathrm{D}}{\otimes} ((\Omega_{X_\infty} \otimes_{\mathcal{O}_{X_\infty}} \mathcal{O}^{\mathrm{E}}_{X_\infty}) \otimes_{\mathcal{O}_{X_\infty}} \Omega_{X_\infty}^{\otimes -1}) \\ &\simeq \mathscr{E}^{\varphi}_{X|\widehat{X}} \overset{\mathrm{D}}{\otimes} \mathcal{O}^{\mathrm{E}}_{X_\infty}\end{aligned}$$

and on the other hand,

$$\begin{aligned}\mathrm{R}\mathscr{I}hom^{+}(\mathcal{S}ol^{\mathrm{E}}_{X_\infty}(\mathcal{L})), \mathcal{DR}^{\mathrm{E}}_{X_\infty}(\mathcal{M})) &\underset{(1)}{\simeq} \mathrm{R}\mathscr{I}hom^{+}(\mathbb{C}^{\mathrm{E}}_{X_\infty} \overset{+}{\otimes} \mathbb{C}_{\{t=-\Re\varphi(x)\}}, \mathcal{O}^{\mathrm{E}}_{X_\infty}) \\ &\underset{(2)}{\simeq} \mathrm{R}\mathscr{I}hom^{+}(\mathbb{C}_{\{t=-\Re\varphi(x)\}}, \mathscr{I}hom^{+}(\mathbb{C}^{\mathrm{E}}_{X_\infty}, \mathcal{O}^{\mathrm{E}}_{X_\infty})) \\ &\underset{(3)}{\simeq} \mathrm{R}\mathscr{I}hom^{+}(\mathbb{C}_{\{t=-\Re\varphi(x)\}}, \mathcal{O}^{\mathrm{E}}_{X_\infty}),\end{aligned}$$

where (1) follows from Proposition 5.1.2, (2) from Proposition 4.5.2 and (3) from Proposition 4.6.5. Thus, one obtains a morphism

$$\mathscr{E}^{\varphi}_{X|\widehat{X}} \overset{\mathrm{D}}{\otimes} \mathcal{O}^{\mathrm{E}}_{X_\infty} \to \mathrm{R}\mathscr{I}hom^{+}(\mathbb{C}_{\{t=-\Re\varphi(x)\}}, \mathcal{O}^{\mathrm{E}}_{X_\infty}), \tag{5.6}$$

adjoint to (5.5). Its construction is done in the part (f)-(1) of the proof of Theorem 4.5 in [60]. This morphism is defined in two steps. Let us first consider the canonical projections

$$X_\infty \times \mathbb{R}_\infty \overset{q}{\leftarrow} X_\infty \times \mathbb{P} \times \mathbb{R}_\infty \overset{\pi_{X\times\mathbb{P}}}{\to} X_\infty \times \mathbb{P}, \quad X_\infty \times \mathbb{R}_\infty \overset{\pi_X}{\to} X_\infty.$$

There is an isomorphism

$$\mathscr{E}^{\varphi}_{X|\widehat{X}} \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_{X_\infty}} \mathcal{D}b^{\mathrm{t}}_{X_\infty} \xrightarrow{\sim} \mathrm{R}\pi_{X*}\mathrm{R}\mathscr{I}hom(\mathbb{C}_{\{t<-\Re\varphi(x)\}}[1], \mathcal{D}b^{\mathrm{T}}_{X_\infty}),$$

defined on sections by $e^{\varphi} \otimes u(x) \mapsto e^{t+\varphi(x)}u(x)$ (see Lemma 9.6.3 in [19]). Therefore, we can derive an isomorphism

$$\mathscr{E}^{\varphi}_{X|\widehat{X}} \overset{\mathrm{D}}{\otimes} \mathcal{O}^{\mathrm{t}}_{X_\infty\times\mathbb{P}} \xrightarrow{\sim} \mathrm{R}\pi_{X\times\mathbb{P}*}\mathrm{R}\mathscr{I}hom(q^{-1}\,\mathbb{C}_{\{t=-\Re\varphi(x)\}}, \mathrm{R}^{\mathrm{E}}\,\mathcal{O}^{\mathrm{E}}_{X_\infty\times\mathbb{P}}). \tag{5.7}$$

Now, if we denote by $i : X_\infty \times \mathbb{R}_\infty \to X_\infty \times \mathbb{P}$ the canonical inclusion, it is enough to apply the functor $i^!((\mathscr{E}^{-\tau}_{\mathbb{C}|\mathbb{P}})^r \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{\mathbb{P}}} -)$ to (5.7) to get (5.6). This clearly shows that the morphism (5.6) is derived from a morphism of sheaves

$$\mathscr{E}^{\varphi}_{X|\widehat{X}} \otimes_{\mathcal{O}_{X_\infty}} \mathcal{D}b^{\mathrm{T}}_{X_\infty} \to \mathscr{I}hom^{+}(\mathbb{C}_{\{t=-\Re\varphi(x)\}}, \mathcal{D}b^{\mathrm{T}}_{X_\infty})$$

such that the morphism

$$\mathcal{D}b^{\mathrm{T}}_{X_\infty} \to \mu_{\Re\varphi *}\mathcal{D}b^{\mathrm{T}}_{X_\infty}$$

induced by the canonical section $e^\varphi \in \mathscr{E}^{\varphi}_{X|\widehat{X}}$ is given by $u \mapsto e^\varphi u$. One gets the conclusion by adjunction, noticing that $\mu_{\Re\varphi}^{-1} = \mu_{-\Re\varphi *}$. □

5.1.2 The Fourier-Sato functors

Let us recall some facts about the enhanced Fourier-Sato functors, introduced in [60]. Let us fix $\mathbb{V}$ a n-dimensional complex vector space and $\mathbb{V}^*$ its complex dual. We consider the bordered spaces $\mathbb{V}_\infty = (\mathbb{V}, \overline{\mathbb{V}})$ and $\mathbb{V}^*_\infty = (\mathbb{V}^*, \overline{\mathbb{V}}^*)$ where $\overline{\mathbb{V}}$ (resp. $\overline{\mathbb{V}}^*$) is the projective compactification of $\mathbb{V}$ (resp $\mathbb{V}^*$). Let us also note $\langle -, - \rangle : \mathbb{V} \times \mathbb{V}^* \to \mathbb{C}$ the duality bracket.

Definition 5.1.4. *The Laplace kernels* are defined by

$$L_{\mathbb{V}} = \mathbb{C}_{\{t = \Re\langle z,w\rangle\}} \in \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{\mathbb{V}_\infty \times \mathbb{V}^*_\infty}),$$
$$L^a_{\mathbb{V}} = \mathbb{C}_{\{t = -\Re\langle z,w\rangle\}} \in \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{\mathbb{V}_\infty \times \mathbb{V}^*_\infty}).$$

Let us consider the correspondence

$$\mathbb{V}_\infty \xleftarrow{p} \mathbb{V}_\infty \times \mathbb{V}^*_\infty \xrightarrow{q} \mathbb{V}^*_\infty$$

where p and q are the canonical projections.

Definition 5.1.5. *The enhanced Fourier-Sato functors*

$${}^{\mathrm{E}}\mathcal{F}_{\mathbb{V}}, {}^{\mathrm{E}}\mathcal{F}^a_{\mathbb{V}} : \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{\mathbb{V}_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{\mathbb{V}^*_\infty})$$

are defined by

$${}^{\mathrm{E}}\mathcal{F}_{\mathbb{V}}(F) = \mathrm{E}q_{!!}(L_{\mathbb{V}} \overset{+}{\otimes} \mathrm{E}p^{-1}F),$$
$${}^{\mathrm{E}}\mathcal{F}^a_{\mathbb{V}}(F) = \mathrm{E}q_{!!}(L^a_{\mathbb{V}} \overset{+}{\otimes} \mathrm{E}p^{-1}F).$$

Remark 5.1.6. In [60], the authors mainly work with ${}^{\mathrm{E}}\mathcal{F}_{\mathbb{V}}$. However, it will be more convenient for us to use ${}^{\mathrm{E}}\mathcal{F}^a_{\mathbb{V}}$ instead.

Theorem 5.1.7 ([60], Theorem 5.2)**.** *The enhanced Fourier-Sato functor ${}^{\mathrm{E}}\mathcal{F}^a_{\mathbb{V}}$ is an equivalence of categories whose inverse is given by ${}^{\mathrm{E}}\mathcal{F}_{\mathbb{V}^*}[2n]$. Moreover, there is an isomorphism*

$$\mathrm{RHom}^{\mathrm{E}}(F_1, F_2) \simeq \mathrm{RHom}^{\mathrm{E}}({}^{\mathrm{E}}\mathcal{F}^a_{\mathbb{V}}(F_1), {}^{\mathrm{E}}\mathcal{F}^a_{\mathbb{V}}(F_2)), \tag{5.8}$$

functorial in $F_1, F_2 \in \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{\mathbb{V}_\infty})$, and the functor ${}^{\mathrm{E}}\mathcal{F}_{\mathbb{V}^*}[2n]$ is isomorphic to the functor $\Psi^{\mathrm{E}}_{L^a_{\mathbb{V}}}$, defined by

$$\Psi^{\mathrm{E}}_{L^a_{\mathbb{V}}}(F) = \mathrm{E}p_*\mathrm{R}\mathscr{I}hom^+(L^a_{\mathbb{V}}, \mathrm{E}q^!(F)),$$

which is the right adjoint of ${}^{\mathrm{E}}\mathcal{F}^a_{\mathbb{V}}$.

5.1.3 The enhanced Laplace isomorphism

We keep the notations of the previous section.

Proposition 5.1.8. *There is a morphism of complexes*

$$q_{\mathbb{R}!!}(\mu_{-\langle z,w\rangle}{}_{*}p_{\mathbb{R}}^{-1}\,\mathcal{D}b^{\mathrm{T},n,\bullet+n}_{\mathbb{V}_\infty}) \to \mathcal{D}b^{\mathrm{T},0,\bullet}_{\mathbb{V}^*_\infty} \tag{5.9}$$

encoding the usual positive Laplace transform of distributions with an extra real parameter, i.e. $u \mapsto \int_{q_{\mathbb{R}}} e^{\langle z,w\rangle}p_{\mathbb{R}}^{*}u$.

This morphism induces a morphism

$${}^{\mathrm{E}}\mathcal{F}^{a}_{\mathbb{V}}(\Omega^{\mathrm{E}}_{\mathbb{V}_\infty})[n] \to \mathcal{O}^{\mathrm{E}}_{\mathbb{V}^*_\infty} \tag{5.10}$$

in $\mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{\mathbb{V}^*_\infty})$.

Proof. Using successively the morphisms (4.19), (5.2) and (4.14), we can define a morphism of complexes

$$\begin{aligned} q_{\mathbb{R}!!}(\mu_{-\langle z,w\rangle}{}_{*}p_{\mathbb{R}}^{-1}\,\mathcal{D}b^{\mathrm{T},n,\bullet+n}_{\mathbb{V}_\infty}) &\to q_{\mathbb{R}!!}(\mu_{-\langle z,w\rangle}{}_{*}\,\mathcal{D}b^{\mathrm{T},n,\bullet+n}_{\mathbb{V}_\infty\times\mathbb{V}^*_\infty}) \\ &\to q_{\mathbb{R}!!}(\mathcal{D}b^{\mathrm{T},n,\bullet+n}_{\mathbb{V}_\infty\times\mathbb{V}^*_\infty}) \\ &\to \mathcal{D}b^{\mathrm{T},0,\bullet}_{\mathbb{V}^*_\infty} \end{aligned}$$

which clearly encodes the usual positive Laplace transform of distributions. By the Propositions 4.6.12, 5.1.1 and 4.6.9, this induces a morphism

$$\mathrm{E}q_{!!}(\mathbb{C}_{\{t=-\langle z,w\rangle\}}\overset{+}{\otimes}\mathrm{E}p^{-1}\Omega^{\mathrm{E}}_{\mathbb{V}_\infty})[n] \to \mathcal{O}^{\mathrm{E}}_{\mathbb{V}^*_\infty}$$

in $\mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{\mathbb{V}^*_\infty})$. Hence the conclusion. □

Definition 5.1.9. The morphism (5.10) is called the *enhanced Laplace transform* from $\mathbb{V}$ to $\mathbb{V}^*$.

Theorem 5.1.10. *The enhanced Laplace transform is an isomorphism in* $\mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{\mathbb{V}^*_\infty})$.

Proof. Theorem 6.3 of [60] states that there is a canonical isomorphism

$${}^{\mathrm{E}}\mathcal{F}^{a}_{\mathbb{V}}(\Omega^{\mathrm{E}}_{\mathbb{V}_\infty})[n] \simeq \mathcal{O}^{\mathrm{E}}_{\mathbb{V}^*_\infty} \tag{5.11}$$

in $\mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{\mathbb{V}^*_\infty})$. It is enough to prove that this isomorphism is equivalent to the enhanced Laplace transform. Let us give the sketch of this proof. By Theorem 5.1.7, the isomorphism (5.11) is equivalent to an isomorphism

$$\Omega^{\mathrm{E}}_{\mathbb{V}_\infty} \simeq \Psi^{\mathrm{E}}_{L^a_{\mathbb{V}}}(\mathcal{O}^{\mathrm{E}}_{\mathbb{V}^*_\infty}[-n]). \tag{5.12}$$

We shall recall the construction of (5.12) and prove that it is equivalent to the adjoint of the enhanced Laplace transform. Let us set $\mathbb{H} = \overline{\mathbb{V}}\setminus\mathbb{V}, \mathbb{H}^* = \overline{\mathbb{V}}^*\setminus\mathbb{V}^*$ as well as $\mathcal{L} = \mathscr{E}^{\langle z,w\rangle}_{\mathbb{V}\times\mathbb{V}^*|\overline{\mathbb{V}}\times\overline{\mathbb{V}}^*}$. One has

$$
\begin{aligned}
&\Omega^{\mathrm{E}}_{\mathbb{V}_\infty} \simeq \Omega^{\mathrm{E}}_{\mathbb{V}_\infty} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{\mathbb{V}_\infty}} \mathscr{D}_{\overline{\mathbb{V}}}(*\mathbb{H}) \\
&\underset{(1)}{\simeq} \Omega^{\mathrm{E}}_{\mathbb{V}_\infty} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{\mathbb{V}_\infty}} \mathrm{D}p_*(\mathcal{L} \overset{\mathrm{D}}{\otimes} \mathrm{D}q^*(\mathscr{D}_{\overline{\mathbb{V}}^*}(*\mathbb{H}^*) \otimes_{\mathcal{O}_{\mathbb{V}^*_\infty}} \Omega^{\otimes -1}_{\mathbb{V}^*_\infty})) \\
&\underset{(2)}{\simeq} \mathrm{E}p_*(\Omega^{\mathrm{E}}_{\mathbb{V}_\infty \times \mathbb{V}^*_\infty} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{\mathbb{V}_\infty \times \mathbb{V}^*_\infty}} (\mathcal{L} \overset{\mathrm{D}}{\otimes} \mathrm{D}q^*(\mathscr{D}_{\overline{\mathbb{V}}^*}(*\mathbb{H}^*) \otimes_{\mathcal{O}_{\mathbb{V}^*_\infty}} \Omega^{\otimes -1}_{\mathbb{V}^*_\infty}))) \\
&\underset{(3)}{\simeq} \mathrm{E}p_*\mathrm{R}\mathscr{I}hom^+(\mathcal{S}ol^{\mathrm{E}}_{\mathbb{V}_\infty \times \mathbb{V}^*_\infty}(\mathcal{L}), \Omega^{\mathrm{E}}_{\mathbb{V}_\infty \times \mathbb{V}^*_\infty} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{\mathbb{V}_\infty \times \mathbb{V}^*_\infty}} \mathrm{D}q^*(\mathscr{D}_{\overline{\mathbb{V}}^*}(*\mathbb{H}^*) \otimes_{\mathcal{O}_{\mathbb{V}^*_\infty}} \Omega^{\otimes -1}_{\mathbb{V}^*_\infty})) \\
&\underset{(4)}{\simeq} \mathrm{E}p_*\mathrm{R}\mathscr{I}hom^+(\mathbb{C}^{\mathrm{E}}_{\mathbb{V}_\infty \times \mathbb{V}^*_\infty} \overset{+}{\otimes} L^a_{\mathbb{V}}, \Omega^{\mathrm{E}}_{\mathbb{V}_\infty \times \mathbb{V}^*_\infty} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{\mathbb{V}_\infty \times \mathbb{V}^*_\infty}} \mathrm{D}q^*(\mathscr{D}_{\overline{\mathbb{V}}^*}(*\mathbb{H}^*) \otimes_{\mathcal{O}_{\mathbb{V}^*_\infty}} \Omega^{\otimes -1}_{\mathbb{V}^*_\infty})) \\
&\underset{(5)}{\simeq} \mathrm{E}p_*\mathrm{R}\mathscr{I}hom^+(L^a_{\mathbb{V}}, \Omega^{\mathrm{E}}_{\mathbb{V}_\infty \times \mathbb{V}^*_\infty} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{\mathbb{V}_\infty \times \mathbb{V}^*_\infty}} \mathrm{D}q^*(\mathscr{D}_{\overline{\mathbb{V}}^*}(*\mathbb{H}^*) \otimes_{\mathcal{O}_{\mathbb{V}^*_\infty}} \Omega^{\otimes -1}_{\mathbb{V}^*_\infty})) \\
&\underset{(6)}{\simeq} \mathrm{E}p_*\mathrm{R}\mathscr{I}hom^+(L^a_{\mathbb{V}}, \mathrm{E}q^!(\Omega^{\mathrm{E}}_{\mathbb{V}^*_\infty} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{\mathbb{V}^*_\infty}} \mathscr{D}_{\overline{\mathbb{V}}^*}(*\mathbb{H}^*) \otimes_{\mathcal{O}_{\mathbb{V}^*_\infty}} \Omega^{\otimes -1}_{\mathbb{V}^*_\infty}[-n])) \\
&\simeq \mathrm{E}p_*\mathrm{R}\mathscr{I}hom^+(L^a_{\mathbb{V}}, \mathrm{E}q^!(\mathcal{O}^{\mathrm{E}}_{\mathbb{V}^*_\infty}[-n])) \\
&= \Psi^{\mathrm{E}}_{L^a_{\mathbb{V}}}(\mathcal{O}^{\mathrm{E}}_{\mathbb{V}^*_\infty}[-n]),
\end{aligned}
$$

where

(1) follows from GAGA (see [107]) and a theorem of Katz and Laumon in [62] (see also [57] and [74]),

(2) follows from (4.21),

(3) follows from (5.4),

(4) follows from Proposition 5.1.2,

(5) follows from Propositions 4.5.8 and 4.6.5,

(6) follows from (4.16).

The conclusion then follows from Propositions 4.6.10, 4.6.13 and 5.1.3. □

5.2 Holomorphic Paley-Wiener-type theorems

We shall now explain how the enhanced Laplace transform isomorphism allows to obtain a bunch of Paley-Wiener-type theorems. In particular, we shall show the link with Polya's and Méril's theorems.

5.2.1 Almost $\mathcal{C}_\infty$-subanalytic functions

In this section, we recall some definitions and propositions of the section 6.2 of [60].

Definition 5.2.1. Let M be a real analytic manifold and U a subanalytic open subset of M. A function $f : U \to \mathbb{R}$ is *globally subanalytic* on M if its graph $\Gamma_f \subset U \times \mathbb{R}$ is subanalytic in $M \times \overline{\mathbb{R}}$. A continuous function $f : U \to \mathbb{R}$ is *almost $\mathcal{C}_\infty$-subanalytic* on M if there is a $\mathcal{C}_\infty$-function $g : U \to \mathbb{R}$, globally subanalytic on M, such that

$$\exists\, C > 0, \forall x \in U : |f(x) - g(x)| < C.$$

In this case, we say that g is in the (ASA)-class of f.

Example 5.2.2. If $f : \mathbb{C}^n \to \mathbb{C}$ is a polynomial, then $\Re f$ is globally subanalytic on $\mathbb{CP}^n$ (see e.g. [4]).

Conjecture 5.2.3 ([60], Conjecture 6.11)**.** *Let M be a real analytic manifold and U a subanalytic open subset of M. Then any continuous globally subanalytic function $f : U \to \mathbb{R}$ on M is almost $\mathcal{C}_\infty$-subanalytic on M.*

Definition 5.2.4. Let $M_\infty = (M, \widehat{M})$ be a real analytic bordered space and let $U \in \mathrm{Op}^{\mathrm{sub},c}_{M_\infty}$. Let $f : U \to \mathbb{R}$ be a continuous almost $\mathcal{C}_\infty$-subanalytic function on $\widehat{M}$. For any open set $V \in \mathrm{Op}^{\mathrm{sub},c}_{M_\infty}$ and any $r \in \mathbb{Z}$, we set

$$e^{-f}\,\mathcal{D}b^{\mathrm{t},r}_{M_\infty}(V) = \{u \in \mathcal{D}b^r_M(U \cap V) : e^g u \in \mathcal{D}b^{\mathrm{t},r}_{\widehat{M}}(U \cap V)\},$$

where g is in the (ASA)-class of f. This definition does not depend on g and the correspondence $V \in \mathrm{Op}^{\mathrm{sub},c}_{M_\infty} \mapsto e^{-f}\,\mathcal{D}b^{\mathrm{t},r}_{M_\infty}(V)$ clearly defines a subanalytic sheaf on M_∞.

Proposition 5.2.5. *Let $M_\infty = (\widehat{M}, \widehat{M})$. For each $r \in \mathbb{Z}$, the sheaf $e^{-f}\,\mathcal{D}b^{\mathrm{t},r}_{\widehat{M}}$ is quasi-injective.*

Proof. Let $V \in \mathrm{Op}^{\mathrm{sub},c}_{\widehat{M}}$. We have to show that the restriction map

$$\{u \in \mathcal{D}b^r_{\widehat{M}}(U) : e^g u \in \mathcal{D}b^{\mathrm{t},r}_{\widehat{M}}(U)\} \to \{u \in \mathcal{D}b^r_{\widehat{M}}(U \cap V) : e^g u \in \mathcal{D}b^{\mathrm{t},r}_{\widehat{M}}(U \cap V)\}$$

is surjective. Let $u \in \mathcal{D}b^r_{\widehat{M}}(U \cap V)$ be a distributional form such that $e^g u$ is in $\mathcal{D}b^{\mathrm{t},r}_{\widehat{M}}(U \cap V)$. Hence, there is $v \in \mathcal{D}b^r_{\widehat{M}}(\widehat{M})$ such that $v|_{U\cap V} = e^g u$. The distributional form $e^{-g}v|_U$ verifies $e^g(e^{-g}v|_U) \in \mathcal{D}b^{\mathrm{t},r}_{\widehat{M}}(U)$ and $(e^{-g}v|_U)|_{U\cap V} = u$, which gives the conclusion. □

Let us now state a key proposition.

Proposition 5.2.6 ([18], Proposition 7.3 and [60], Theorem 6.12)**.** *Let $M_\infty = (M, \widehat{M})$ be a real analytic bordered space and $U \in \mathrm{Op}^{\mathrm{sub},c}_{M_\infty}$. Let $f : U \to \mathbb{R}$ be a continuous almost $\mathcal{C}_\infty$-subanalytic function on $\widehat{M}$. There is an isomorphism*

$$e^{-f}\,\mathcal{D}b^{\mathrm{t},r}_{M_\infty} \simeq \mathrm{R}\mathscr{I}hom^{\mathrm{E}}(\mathbb{C}_{\{t \geq f(x), x \in U\}}, Q(\mathcal{D}b^{\mathrm{T},r}_{M_\infty}))$$

for each $r \in \mathbb{Z}$, which is given on sections by $u \mapsto e^t u$. In particular, the right hand side is concentrated in degree 0.

One can notice an immediate corollary :

Corollary 5.2.7. *Let $M_\infty = (M, \widehat{M})$ be a real analytic bordered space and let also $f : M \to \mathbb{R}$ be a continuous almost $\mathcal{C}_\infty$-subanalytic function on $\widehat{M}$. Let S be a subanalytic closed subset of M. There is an isomorphism*

$$\mathscr{I}\Gamma_S(e^{-f}\,\mathcal{D}b^{\mathrm{t},r}_{M_\infty}) \simeq \mathrm{R}\mathscr{I}hom^{\mathrm{E}}(\mathbb{C}_{\{t\geq f(x), x\in S\}}, Q(\mathcal{D}b^{\mathrm{T},r}_{M_\infty}))$$

for each $r \in \mathbb{Z}$, which is given on sections by $u \mapsto e^t u$. In particular, the right hand side is concentrated in degree 0.

Proof. Let us note $\pi_M : M_\infty \times \mathbb{R}_\infty \to M_\infty$ the canonical projection. By Proposition 5.2.6 and the quasi-injectivity of $e^{-f}\,\mathcal{D}b^{\mathrm{t},r}_{\widehat{M}}$, one gets

$$\begin{aligned}
\mathscr{I}\Gamma_S(e^{-f}\,\mathcal{D}b^{\mathrm{t},r}_{M_\infty}) &\simeq \mathrm{R}\mathscr{I}hom(\iota_M(\mathbb{C}_S), e^{-f}\,\mathcal{D}b^{\mathrm{t},r}_{M_\infty})\\
&\simeq \mathrm{R}\mathscr{I}hom(\iota_M(\mathbb{C}_S), \mathrm{R}\mathscr{I}hom^{\mathrm{E}}(\mathbb{C}_{\{t\geq f(x)\}}, Q(\mathcal{D}b^{\mathrm{T},r}_{M_\infty}))\\
&\simeq \mathrm{R}\mathscr{I}hom(\iota_M(\mathbb{C}_S), \mathrm{R}\pi_{M*}\mathrm{R}\mathscr{I}hom(\mathbb{C}_{\{t\geq f(x)\}}, \mathcal{D}b^{\mathrm{T},r}_{M_\infty})\\
&\simeq \mathrm{R}\pi_{M*}\mathrm{R}\mathscr{I}hom(\pi_M^{-1}(\iota_M(\mathbb{C}_S)), \mathrm{R}\mathscr{I}hom(\mathbb{C}_{\{t\geq f(x)\}}, \mathcal{D}b^{\mathrm{T},r}_{M_\infty})\\
&\simeq \mathrm{R}\pi_{M*}\mathrm{R}\mathscr{I}hom(\pi_M^{-1}(\iota_M(\mathbb{C}_S)) \otimes \mathbb{C}_{\{t\geq f(x)\}}, \mathcal{D}b^{\mathrm{T},r}_{M_\infty})\\
&\simeq \mathrm{R}\pi_{M*}\mathrm{R}\mathscr{I}hom(\mathbb{C}_{\{t\geq f(x), x\in S\}}, \mathcal{D}b^{\mathrm{T},r}_{M_\infty})\\
&\simeq \mathrm{R}\mathscr{I}hom^{\mathrm{E}}(\mathbb{C}_{\{t\geq f(x), x\in S\}}, Q(\mathcal{D}b^{\mathrm{T},r}_{M_\infty})).
\end{aligned}$$

□

Thanks to Proposition 5.2.6, one can introduce the following definition :

Definition 5.2.8. Let $X_\infty = (X, \widehat{X})$ be a complex bordered space of complex dimension d_X and let U be an open subset of X. Let $f : U \to \mathbb{R}$ be a continuous almost $\mathcal{C}_\infty$-subanalytic function on $\widehat{X}$. For each $p \in \mathbb{Z}$, one defines the complex of subanalytic sheaves $e^{-f}\Omega^{\mathrm{t},p}_{X_\infty}$ as the Dolbeault complex

$$0 \to e^{-f}\,\mathcal{D}b^{\mathrm{t},p,0}_{X_\infty} \xrightarrow{\bar\partial} e^{-f}\,\mathcal{D}b^{\mathrm{t},p,1}_{X_\infty} \to \cdots \to e^{-f}\,\mathcal{D}b^{\mathrm{t},p,d_X}_{X_\infty} \to 0.$$

As usual, one sets for short $e^{-f}\,\mathcal{O}^{\mathrm{t}}_{X_\infty} = e^{-f}\Omega^{\mathrm{t},0}_{X_\infty}$ and $e^{-f}\Omega^{\mathrm{t}}_{X_\infty} = e^{-f}\Omega^{\mathrm{t},d_X}_{X_\infty}$.

5.2.2 Laplace and Legendre transforms

Let us now focus on an important application. Recall the definitions of section 1.5.

Lemma 5.2.9 ([60], Theorem. 5.9)**.** *Let $f \in \mathrm{Conv}(\mathbb{V})$ and let $d(f)$ be the real dimension of $H(f^*)^\perp$, where $H(f^*)$ is the affine space generated by* $\mathrm{dom}(f^*)$*. One has an isomorphism*

$$^{\mathrm{E}}\mathcal{F}^a_{\mathbb{V}}(\mathbb{C}_{\{t\geq f(z)\}}) \simeq \mathbb{C}_{\{t\geq -f^*(w), w\in \mathrm{dom}^\circ(f^*)\}} \otimes \mathrm{or}_{H(f^*)^\perp}[-d(f)]. \tag{5.13}$$

Using our explicit interpretation of the enhanced Laplace transform, we can refine Theorem 6.14 and Corollary 6.15 of [60].

Theorem 5.2.10. *Let $f : \mathbb{V} \to \mathbb{R}$ be a continuous almost $\mathcal{C}_\infty$-subanalytic function on $\overline{\mathbb{V}}$ and let S be a non-empty subanalytic closed subset of $\mathbb{V}$. Let us denote by f_S the function which is equal to f on S and to $+\infty$ on $\mathbb{V} \setminus S$. Assume that*

(i) $f_S \in \mathrm{Conv}(\mathbb{V})$,

(ii) $H(f_S^*)^\perp = \{0\}$,

(iii) the convex set $\mathrm{dom}^\circ(f_S^*)$ *is subanalytic,*

(iv) the function $f_S^* : \mathrm{dom}^\circ(f_S^*) \to \mathbb{R}$ *is continuous and almost* $\mathcal{C}_\infty$*-subanalytic on* $\overline{\mathbb{V}}^*$.

Then, one gets an isomorphism

$$H^n_S(\mathbb{V}, e^{-f}\Omega^{\mathrm{t}}_{\overline{\mathbb{V}}}) \xrightarrow{\sim} H^0(\mathbb{V}^*, e^{f_S^*}\,\mathcal{O}^{\mathrm{t}}_{\overline{\mathbb{V}}^*}) \simeq e^{f_S^*}\,\mathcal{D}b^{\mathrm{t}}_{\overline{\mathbb{V}}^*}(\mathrm{dom}^\circ(f_S^*)) \cap \mathcal{O}_{\overline{\mathbb{V}}^*}(\mathrm{dom}^\circ(f_S^*)). \tag{5.14}$$

Moreover, there is a commutative diagram

$$\begin{tikzcd} H^n_S(\mathbb{V}, e^{-f}\Omega^{\mathrm{t}}_{\overline{\mathbb{V}}}) \arrow[r, "\sim"] & H^0(\mathbb{V}^*, e^{f_S^*}\,\mathcal{O}^{\mathrm{t}}_{\overline{\mathbb{V}}^*}) \arrow[d] \\ \Gamma_S(\mathbb{V}, e^{-f}\,\mathcal{D}b^{\mathrm{t},n,n}_{\overline{\mathbb{V}}}) \arrow[u] \arrow[r] & \Gamma(\mathbb{V}^*, e^{f_S^*}\,\mathcal{D}b^{\mathrm{t}}_{\overline{\mathbb{V}}^*}) \end{tikzcd}$$

where the left arrow is defined by the Dolbeault resolution of $e^{-f}\Omega^{\mathrm{t}}_{\overline{\mathbb{V}}}$, the right arrow by the inclusion and the bottom arrow by the classical positive Laplace transform $u \mapsto \mathcal{L}^+ u := \int_q e^{\langle z,w\rangle} p^*\omega$. In particular, the isomorphism (5.14) can be explicitly computed by

$$\frac{\Gamma_S(\mathbb{V}, e^{-f}\,\mathcal{D}b^{\mathrm{t},n,n}_{\overline{\mathbb{V}}})}{\bar\partial\Gamma_S(\mathbb{V}, e^{-f}\,\mathcal{D}b^{\mathrm{t},n,n-1}_{\overline{\mathbb{V}}})} \ni [u] \mapsto \mathcal{L}^+ u \in H^0(\mathbb{V}^*, e^{f_S^*}\,\mathcal{O}^{\mathrm{t}}_{\overline{\mathbb{V}}^*}). \tag{5.15}$$

Proof. By using successively Corollary 5.2.7, the isomorphisms (5.8) and (5.10), Lemma 5.2.9 and finally Proposition 5.2.6, one obtains an isomorphism

$$\begin{aligned} H^n_S(\mathbb{V}, e^{-f}\Omega^{\mathrm{t}}_{\overline{\mathbb{V}}}) &\simeq H^n(\mathrm{RHom}^{\mathrm{E}}(\mathbb{C}_{\{t\geq f(z), z\in S\}}, \Omega^{\mathrm{E}}_{\mathbb{V}_\infty}) \\ &\simeq H^n(\mathrm{RHom}^{\mathrm{E}}({}^{\mathrm{E}}\mathcal{F}^a_{\mathbb{V}}(\mathbb{C}_{\{t\geq f(z), z\in S\}}), {}^{\mathrm{E}}\mathcal{F}^a_{\mathbb{V}}(\Omega^{\mathrm{E}}_{\mathbb{V}_\infty})) \\ &\xrightarrow{\sim} H^0(\mathrm{RHom}^{\mathrm{E}}({}^{\mathrm{E}}\mathcal{F}^a_{\mathbb{V}}(\mathbb{C}_{\{t\geq f_S(z)\}}), \mathcal{O}^{\mathrm{E}}_{\mathbb{V}^*_\infty}) \\ &\simeq H^0(\mathrm{RHom}^{\mathrm{E}}(\mathbb{C}_{\{t\geq -f^*(w), w\in\mathrm{dom}^\circ(f_S^*)\}}, \mathcal{O}^{\mathrm{E}}_{\mathbb{V}^*_\infty}) \\ &\simeq H^0(\mathbb{V}^*, e^{f_S^*}\,\mathcal{O}^{\mathrm{t}}_{\overline{\mathbb{V}}^*}). \end{aligned}$$

Following this construction and using (5.9), the map

$$\begin{aligned} \Gamma_S(\mathbb{V}, e^{-f}\,\mathcal{D}b^{\mathrm{t},n,n}_{\overline{\mathbb{V}}}) &\to H^n_S(\mathbb{V}, e^{-f}\Omega^{\mathrm{t}}_{\overline{\mathbb{V}}}) \\ &\xrightarrow{\sim} H^0(\mathbb{V}^*, e^{f_S^*}\,\mathcal{O}^{\mathrm{t}}_{\overline{\mathbb{V}}^*}) \\ &\to \Gamma(\mathbb{V}^*, e^{f_S^*}\,\mathcal{D}b^{\mathrm{t}}_{\overline{\mathbb{V}}^*}) \end{aligned}$$

is given by $u \mapsto e^{-t}\int_{q_{\mathbb{R}}} e^{\langle z,w\rangle} p^*_{\mathbb{R}}(e^t u) = \mathcal{L}^+ u$. Then, the conclusion follows from the quasi-injectivity of $e^{-f}\,\mathcal{D}b^{\mathrm{t},p,q}_{\overline{\mathbb{V}}}$ for all $(p,q) \in \mathbb{Z}^2$. □

This last result can be seen as a purely holomorphic Paley-Wiener-type theorem and can lead to plenty of Laplace isomorphisms, depending on the chosen f. We decide to focus on Polya's and Méril's theorems in order to show how the contour integration can be derived from this algebraic framework.

5.2.3 Link with Polya's theorem

Let $\mathbb{V}$ be a one-dimensional complex vector space. Let us denote by $\mathbb{P}$ (resp. $\mathbb{P}^*$) the projective compactification of $\mathbb{V}$ (resp. $\mathbb{V}^*$) and recall that $\mathcal{O}^{\mathrm{t}}_{\mathbb{P}}$ (resp. $\Omega^{\mathrm{t}}_{\mathbb{P}}$) is concentrated in degree 0 and is a subanalytic subsheaf of $\mathcal{O}_{\mathbb{P}}$ (resp. $\Omega_{\mathbb{P}}$) (see Proposition 4.3.8). If $U \in \mathrm{Op}^{\mathrm{sub},c}_{\mathbb{P}}$, one simply has $\mathcal{O}^{t}_{\mathbb{P}}(U) = \mathcal{O}_{\mathbb{P}}(U) \cap \mathcal{D}b^{\mathrm{t}}_{\mathbb{P}}(U)$ (resp. $\Omega^{\mathrm{t}}_{\mathbb{P}}(U) = \Omega_{\mathbb{P}}(U) \cap \mathcal{D}b^{\mathrm{t},1,0}_{\mathbb{P}}(U)$). We shall also use the sheaf $\Gamma_{\mathbb{V}}\Omega^{\mathrm{t}}_{\mathbb{P}}$ of holomorphic forms which are tempered only at infinity.

Let us choose a Hermitian norm $||\cdot||$ on $\mathbb{V}$ and denote by $||\cdot||^*$ the dual norm on $\mathbb{V}^*$. Let us fix a non-empty convex compact subset K of $\mathbb{V}$ and let us consider the null function $f = 0$ on $\mathbb{V}$. For all $\varepsilon > 0$, we thus get a function f_{K_ε} defined by

$$f_{K_\varepsilon}(z) = \begin{cases} 0 & \text{if } z \in K_\varepsilon, \\ +\infty & \text{else.} \end{cases}$$

Clearly, this function is convex of domain K_ε. Moreover, its Legendre transform is given by

$$f^*_{K_\varepsilon}(w) = \sup_{z \in K_\varepsilon} \Re\langle z, w\rangle = h_{K_\varepsilon}(w) = h_K(w) + h_{\overline{D}(0,\varepsilon)}(w) = h_K(w) + \varepsilon||w||^*$$

for all $w \in \mathbb{V}^*$. In particular $\mathrm{dom}^\circ(f^*_\varepsilon) = \mathbb{V}^*$. Let $\varepsilon > 0$. In order to apply Theorem 5.2.10, let us assume that K_ε is subanalytic (which implies that h_{K_ε} is globally subanalytic on $\mathbb{P}^*$, see [18]) and that h_{K_ε} is almost $\mathcal{C}_\infty$-subanalytic on $\mathbb{P}^*$. Thus, we get an isomorphism

$$\mathcal{L}^+ : H^1_{K_\varepsilon}(\mathbb{V}, \Omega^{\mathrm{t}}_{\mathbb{P}}) \xrightarrow{\sim} e^{h_{K_\varepsilon}} \mathcal{O}^{\mathrm{t}}_{\mathbb{P}^*}(\mathbb{V}^*) \tag{5.16}$$

given by the positive Laplace transform. We shall show that the projective limit on $\varepsilon \to 0$ of this isomorphism is equivalent to the bijectivity of $\mathcal{P}$ in Polya's theorem. Hence, for the rest of the section, it is enough to assume that our subanalytic conditions are fulfilled for small ε.

Proposition 5.2.11. *Let $\varepsilon > 0$. One has a canonical isomorphism*

$$\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V} \setminus K_\varepsilon)/\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}) \xrightarrow{\sim} H^1_{K_\varepsilon}(\mathbb{V}, \Omega^{\mathrm{t}}_{\mathbb{P}}) \tag{5.17}$$

given by

$$\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V} \setminus K_\varepsilon)/\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}) \ni [u] \mapsto [\bar{\partial}\underline{u}] \in \frac{\Gamma_{K_\varepsilon}(\mathbb{V}, \mathcal{D}b^{\mathrm{t},1,1}_{\mathbb{P}})}{\bar{\partial}\Gamma_{K_\varepsilon}(\mathbb{V}, \mathcal{D}b^{\mathrm{t},1,0}_{\mathbb{P}})},$$

where $\underline{u}$ is a distributional extension of u to $\mathbb{V}$.

Proof. 1) Consider the excision distinguished triangle

$$\mathrm{R}\Gamma_{K_\varepsilon}(\mathbb{V},\Omega^{\mathrm{t}}_{\mathbb{P}}) \to \mathrm{R}\Gamma(\mathbb{V},\Omega^{\mathrm{t}}_{\mathbb{P}}) \to \mathrm{R}\Gamma(\mathbb{V}\setminus K_\varepsilon,\Omega^{\mathrm{t}}_{\mathbb{P}}) \xrightarrow{+1} . \tag{5.18}$$

This gives the following exact sequence :

$$\begin{tikzcd} 0 \arrow[r] & H^0_{K_\varepsilon}(\mathbb{V},\Omega^{\mathrm{t}}_{\mathbb{P}}) \arrow[r] & H^0(\mathbb{V},\Omega^{\mathrm{t}}_{\mathbb{P}}) \arrow[r] & H^0(\mathbb{V}\setminus K_\varepsilon,\Omega^{\mathrm{t}}_{\mathbb{P}}) \arrow[dll, rounded corners] \\ & H^1_{K_\varepsilon}(\mathbb{V},\Omega^{\mathrm{t}}_{\mathbb{P}}) \arrow[r] & H^1(\mathbb{V},\Omega^{\mathrm{t}}_{\mathbb{P}}) \arrow[r] & H^1(\mathbb{V}\setminus K_\varepsilon,\Omega^{\mathrm{t}}_{\mathbb{P}}) \arrow[r] & \cdots \end{tikzcd}$$

First, it is clear that $H^0_{K_\varepsilon}(\mathbb{V},\Omega^{\mathrm{t}}_{\mathbb{P}}) \simeq 0$ since a non-trivial holomorphic form cannot be supported by a compact set. Secondly, the surjectivity of

$$\overline{\partial} : \mathcal{D}b^{\mathrm{t},1,0}_{\mathbb{P}}(\mathbb{V}) \to \mathcal{D}b^{\mathrm{t},1,1}_{\mathbb{P}}(\mathbb{V})$$

(see [46] and [68]) implies that $H^1(\mathbb{V},\Omega^{\mathrm{t}}_{\mathbb{P}}) \simeq 0$. Hence, we get the exact sequence

$$0 \to \Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}) \to \Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}\setminus K_\varepsilon) \to H^1_{K_\varepsilon}(\mathbb{V},\Omega^{\mathrm{t}}_{\mathbb{P}}) \to 0,$$

which proves the first statement. The second statement is proven as in Lemma 2.2.14. □

Corollary 5.2.12. *One has a canonical isomorphism*

$$\Gamma_{\mathbb{V}}\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}\setminus K)/\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}) \xrightarrow{\sim} \varprojlim_{\varepsilon\to 0} H^1_{K_\varepsilon}(\mathbb{V},\Omega^{\mathrm{t}}_{\mathbb{P}}). \tag{5.19}$$

Let $\varepsilon > 0$ and let ψ_ε be a $\mathcal{C}_\infty$-cutoff function which is equal to 1 on $\mathbb{V}\setminus K_\varepsilon$ and to 0 on $K_{\varepsilon/2}$. Let $u \in \Gamma_{\mathbb{V}}\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}\setminus K)$. Then the image of $[u]$ through the canonical map

$$\Gamma_{\mathbb{V}}\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}\setminus K)/\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}) \to H^1_{K_\varepsilon}(\mathbb{V},\Omega^{\mathrm{t}}_{\mathbb{P}})$$

is given by $[\bar\partial(\psi_\varepsilon u)]$.

Proof. There are trivial inclusions

$$\Gamma_{\mathbb{V}}\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}\setminus K_\varepsilon) \subset \Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}\setminus K_{2\varepsilon}) \subset \Gamma_{\mathbb{V}}\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}\setminus K_{3\varepsilon})$$

for all $\varepsilon > 0$, which imply that

$$\varprojlim_{\varepsilon\to 0} \Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}\setminus K_\varepsilon) \simeq \varprojlim_{\varepsilon\to 0} \Gamma_{\mathbb{V}}\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}\setminus K_\varepsilon).$$

Moreover, one has

$$\varprojlim_{\varepsilon\to 0} \Gamma_{\mathbb{V}}\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}\setminus K_\varepsilon) \simeq \Gamma_{\mathbb{V}}\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}\setminus K).$$

Indeed,

$$\begin{aligned}\varprojlim_{\varepsilon\to 0}\Gamma_{\mathbb{V}}\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}\setminus K_\varepsilon) &\simeq \varprojlim_{\varepsilon\to 0}\mathrm{Hom}_{\mathbb{C}_{\mathbb{P}}}(\mathbb{C}_{\mathbb{V}\setminus K_\varepsilon},\Gamma_{\mathbb{V}}\Omega^{\mathrm{t}}_{\mathbb{P}})\\ &\simeq \mathrm{Hom}_{\mathbb{C}_{\mathbb{P}}}(\varinjlim_{\varepsilon\to 0}\mathbb{C}_{\mathbb{V}\setminus K_\varepsilon},\Gamma_{\mathbb{V}}\Omega^{\mathrm{t}}_{\mathbb{P}})\\ &\simeq \mathrm{Hom}_{\mathbb{C}_{\mathbb{P}}}(\mathbb{C}_{\mathbb{V}\setminus K},\Gamma_{\mathbb{V}}\Omega^{\mathrm{t}}_{\mathbb{P}})\\ &\simeq \Gamma_{\mathbb{V}}\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}\setminus K).\end{aligned}$$

Hence, by the Mittag-Leffler theorem for projective systems, one obtains

$$\begin{aligned}\varprojlim_{\varepsilon\to 0}H^1_{K_\varepsilon}(\mathbb{V},\Omega^{\mathrm{t}}_{\mathbb{P}}) &\xleftarrow{\sim} \varprojlim_{\varepsilon\to 0}(\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}\setminus K_\varepsilon)/\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}))\\ &\simeq (\varprojlim_{\varepsilon\to 0}\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}\setminus K_\varepsilon))/\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V})\\ &\simeq \Gamma_{\mathbb{V}}\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}\setminus K)/\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}).\end{aligned}$$

The second part of the statement is clear. □

Remark 5.2.13. Note that in

$$\Gamma_{\mathbb{V}}\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}\setminus K)=\{u\in\Omega_{\mathbb{V}}(\mathbb{V}\setminus K) : u \text{ is tempered at } \infty\},$$

one can replace the condition "u is tempered at infinity" by the condition "u has polynomial growth at infinity". Indeed, thanks to Cauchy's inequalities, the polynomial growth of u at infinity implies the polynomial growth of all its derivatives.

Theorem 5.2.14. *There is a canonical isomorphism of $\mathbb{C}$-vector spaces*

$$\Gamma_{\mathbb{V}}\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}\setminus K)/\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}) \xrightarrow{\sim} \varprojlim_{\varepsilon\to 0} e^{h_{K_\varepsilon}}\,\mathcal{O}^{\mathrm{t}}_{\mathbb{P}^*}(\mathbb{V}^*), \tag{5.20}$$

where the spaces are independent of the chosen norm.

Given a global $\mathbb{C}$-linear coordinate z of $\mathbb{V}$ and its dual coordinate w, this isomorphism can be made explicit by $[f(z)dz]\mapsto g$ with

$$g(w)=\int_{C(0,r)^+} e^{zw}f(z)dz,$$

where $C(0,r)^+$ is a positively oriented circle, which encloses K.

Proof. First, since all norms are equivalent, it is clear that

$$\varprojlim_{\varepsilon\to 0} e^{h_{K_\varepsilon}}\,\mathcal{O}^{\mathrm{t}}_{\mathbb{P}^*}(\mathbb{V}^*) \simeq \{v\in\mathcal{O}_{\mathbb{V}^*}(\mathbb{V}^*) : \forall\varepsilon>0, v\in e^{h_{K_\varepsilon}}\,\mathcal{D}b^{\mathrm{t}}_{\mathbb{P}^*}(\mathbb{V}^*)\}$$

is independent of the chosen norm. Secondly, by applying $\varprojlim_{\varepsilon\to 0}$ to (5.16) as well as (5.19), we get the isomorphisms

$$\Gamma_{\mathbb{V}}\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}\setminus K)/\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}) \xrightarrow{\sim} \varprojlim_{\varepsilon\to 0}H^1_{K_\varepsilon}(\mathbb{V},\Omega^{\mathrm{t}}_{\mathbb{P}}) \xrightarrow{\sim} \varprojlim_{\varepsilon\to 0} e^{h_{K_\varepsilon}}\,\mathcal{O}^{\mathrm{t}}_{\mathbb{P}^*}(\mathbb{V}^*).$$

Let us explicit the composition of these two maps within coordinates. Let $f(z)dz$ be in $\Gamma_{\mathbb{V}}\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}\setminus K)$ and let us fix $r>0$ such that $K \subsetneq D(0,r)$. Let us consider $\varepsilon > 0$ small enough such that $K \subsetneq K_\varepsilon \subsetneq D(0,r)$. Let us also choose a cutoff function ψ_ε as in Corollary 5.2.12. Then, applying this corollary, we see that the image of $[f(z)dz]$ in $e^{h_{K_\varepsilon}}\,\mathcal{O}^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}^*)$ is given by g, where

$$\begin{aligned} g(w) = \mathcal{L}^+_w(\bar\partial(\psi_\varepsilon f(z)dz)) &= \int_{\mathbb{V}} e^{zw}\bar\partial(\psi_\varepsilon f(z)dz) = \int_{\mathbb{V}} \bar\partial(e^{zw}\psi_\varepsilon f(z)dz) \\ &\underset{(1)}{=} \int_{\overline{D}(0,r)} \bar\partial(e^{zw}\psi_\varepsilon f(z)dz) \underset{(2)}{=} \int_{C(0,r)^+} e^{zw}\psi_\varepsilon f(z)dz \\ &\underset{(3)}{=} \int_{C(0,r)^+} e^{zw} f(z)dz. \end{aligned}$$

Here, (1) comes from the fact that $e^{zw}\psi_\varepsilon f(z)dz$ is holomorphic on the open set $\mathbb{V}\setminus K_\varepsilon \supset \mathbb{V}\setminus\overline{D}(0,r)$, (2) from Green's theorem and (3) from the fact that $\psi_\varepsilon = 1$ on $C(0,r) \subset \mathbb{V}\setminus K_\varepsilon$.

To conclude, we remark that this formula remains unchanged for smaller $\varepsilon > 0$. Hence, it is the image of $[f(z)dz]$ in $\varprojlim_{\varepsilon\to 0} e^{h_{K_\varepsilon}}\,\mathcal{O}^{\mathrm{t}}_{\mathbb{P}^*}(\mathbb{V}^*)$. □

Remark 5.2.15. Theorem 5.2.14 is actually nothing more but the "algebraic $\mathcal{P}$-part" of Polya's theorem. First, the canonical map

$$\mathcal{O}^0(\mathbb{C}\setminus K) \ni f \mapsto \left[\frac{1}{2i\pi} f dz\right] \in \Gamma_{\mathbb{C}}\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{C}\setminus K)/\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{C})$$

is clearly bijective. (For the same reasons as the map i_K in section 3.1.3.) Secondly, the inclusion

$$\mathrm{Exp}(K) \subset \{g \in \mathcal{O}(\mathbb{C}) : \forall \varepsilon > 0, g \in e^{h_{K_\varepsilon}}\,\mathcal{D}b^{\mathrm{t}}_{\mathbb{P}}(\mathbb{C})\}$$

is an equality. Indeed, if $e^{-h_{K_\varepsilon}}g$ is tempered at infinity, then $e^{-h_{K_{2\varepsilon}}}g$ is bounded.

5.2.4 Link with Méril's theorem

We keep the same conventions that in the previous section. Let us fix S a proper non-compact closed convex subset of $\mathbb{V}$ which contains no lines and ξ_0 a point in $(S^*_\infty)^\circ$. For all $\varepsilon' > 0$, we consider the function $f_{\varepsilon'} : \mathbb{V} \to \mathbb{R}$ defined by $f_{\varepsilon'}(z) = \Re\langle z, \varepsilon'\xi_0\rangle$, which is globally subanalytic on $\mathbb{P}$. For all $\varepsilon, \varepsilon' > 0$, we thus get a function $f_{\varepsilon,\varepsilon'} := (f_{\varepsilon'})_{S_\varepsilon}$ defined by

$$f_{\varepsilon,\varepsilon'}(z) = \begin{cases} \Re\langle z, \varepsilon'\xi_0\rangle & \text{if } z \in S_\varepsilon, \\ +\infty & \text{else.} \end{cases}$$

Clearly, this function is convex of domain S_ε. Moreover, its Legendre transform is given by

$$f^*_{\varepsilon,\varepsilon'}(w) = \sup_{z\in S_\varepsilon} \Re\langle z, (w - \varepsilon'\xi_0)\rangle = h_{S_\varepsilon}(w - \varepsilon'\xi_0)$$

for all $w \in \mathbb{V}^*$. Thanks to Theorem 1.5.13 one immediately gets that

$$\mathrm{dom}^\circ(f^*_{\varepsilon,\varepsilon'}) = (S^\star_\infty)^\circ + \varepsilon'\xi_0$$

and that $f^*_{\varepsilon,\varepsilon'}$ is continuous on this open cone. In particular, since this cone is not empty, its generated affine space is $\mathbb{V}^*$. In order to apply Theorem 5.2.10, we will assume throughout this section that S_ε is subanalytic and that h_{S_ε} is almost $\mathcal{C}_\infty$-subanalytic on $\mathbb{P}^*$ for all $\varepsilon > 0$. Hence, we get an isomorphism

$$\mathcal{L}^+ : H^1_{S_\varepsilon}(\mathbb{V}, e^{-\langle z,\varepsilon'\xi_0\rangle}\Omega^{\mathrm{t}}_{\mathbb{P}}) \xrightarrow{\sim} e^{h_{S_\varepsilon}(w-\varepsilon'\xi_0)}\,\mathcal{O}^{\mathrm{t}}_{\mathbb{P}^*}((S^\star_\infty)^\circ + \varepsilon'\xi_0) \tag{5.21}$$

given by the positive Laplace transform for all $\varepsilon, \varepsilon' > 0$. (Here, $e^{-\langle z,\varepsilon'\xi_0\rangle}\Omega^{\mathrm{t}}_{\mathbb{P}}$ is defined in the obvious way and is of course equal to $e^{-\Re\langle z,\varepsilon'\xi_0\rangle}\Omega^{\mathrm{t}}_{\mathbb{P}}$.) Among other things, we shall show that the projective limit on $\varepsilon, \varepsilon' \to 0$ of this isomorphism is equivalent to the bijectivity of $\mathcal{P}$ in Méril's theorem.

Since $z \mapsto e^{-\langle z,\varepsilon'\xi_0\rangle}$ is holomorphic on $\mathbb{V}$, the surjectivity of

$$\overline{\partial} : \mathcal{D}b^{\mathrm{t},1,0}_{\mathbb{P}}(\mathbb{V}) \to \mathcal{D}b^{\mathrm{t},1,1}_{\mathbb{P}}(\mathbb{V})$$

implies the surjectivity of

$$\overline{\partial} : e^{-\langle z,\varepsilon'\xi_0\rangle}\,\mathcal{D}b^{\mathrm{t},1,0}_{\mathbb{P}}(\mathbb{V}) \to e^{-\langle z,\varepsilon'\xi_0\rangle}\,\mathcal{D}b^{\mathrm{t},1,1}_{\mathbb{P}}(\mathbb{V}).$$

Hence, one can easily adapt Proposition 5.2.11 and Corollary 5.2.12 to obtain

Proposition 5.2.16. *For all $\varepsilon, \varepsilon' > 0$, there is a canonical isomorphism*

$$e^{-\langle z,\varepsilon'\xi_0\rangle}\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}\setminus S_\varepsilon)/e^{-\langle z,\varepsilon'\xi_0\rangle}\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}) \xrightarrow{\sim} H^1_{S_\varepsilon}(\mathbb{V}, e^{-\langle z,\varepsilon'\xi_0\rangle}\Omega^{\mathrm{t}}_{\mathbb{P}}). \tag{5.22}$$

Let $\varepsilon, \varepsilon' > 0$ and let ψ_ε be a $\mathcal{C}_\infty$-cutoff function which is equal to 1 on $\mathbb{V}\setminus S_\varepsilon$ and to 0 on $S_{\varepsilon/2}$. Let $u \in e^{-\langle z,\varepsilon'\xi_0\rangle}\Gamma_{\mathbb{V}}\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}\setminus S)$. Then the image of $[u]$ through the canonical map

$$\begin{aligned} e^{-\langle z,\varepsilon'\xi_0\rangle}\Gamma_{\mathbb{V}}\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}\setminus S)/e^{-\langle z,\varepsilon'\xi_0\rangle}\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}) &\xrightarrow{\sim} \varprojlim_{\varepsilon\to 0} H^1_{S_\varepsilon}(\mathbb{V}, e^{-\langle z,\varepsilon'\xi_0\rangle}\Omega^{\mathrm{t}}_{\mathbb{P}}) \\ &\to H^1_{S_\varepsilon}(\mathbb{V}, e^{-\langle z,\varepsilon'\xi_0\rangle}\Omega^{\mathrm{t}}_{\mathbb{P}}) \end{aligned}$$

is given by $[\bar{\partial}(\psi_\varepsilon u)]$.

By analogy with Méril's spaces, we are led to introduce the following definitions.

Definition 5.2.17. For all $\varepsilon' > 0$ we set

$$\mathscr{H}^{\mathrm{t}}_S(\mathbb{V}, \varepsilon') = \frac{\{u \in \Omega_{\mathbb{V}}(\mathbb{V}\setminus S) : \forall r > \varepsilon > 0, u \in e^{-\langle z,\varepsilon'\xi_0\rangle}\,\mathcal{D}b^{\mathrm{t},1,0}_{\mathbb{P}}(S^\circ_r \setminus S_\varepsilon)\}}{\{u \in \Omega_{\mathbb{V}}(\mathbb{V}) : \forall r > 0, u \in e^{-\langle z,\varepsilon'\xi_0\rangle}\,\mathcal{D}b^{\mathrm{t},1,0}_{\mathbb{P}}(S^\circ_r)\}}.$$

Remark that $\mathscr{H}^{\mathrm{t}}_S(\mathbb{V}, \varepsilon') \simeq e^{-\langle z,\varepsilon'\xi_0\rangle}\Gamma_{\mathbb{V}}\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V}\setminus S)/e^{-\langle z,\varepsilon'\xi_0\rangle}\Omega^{\mathrm{t}}_{\mathbb{P}}(\mathbb{V})$ for all $\varepsilon' > 0$.

Definition 5.2.18. For all $\varepsilon, \varepsilon' > 0$ we set

$$\mathrm{Exp}^{\mathrm{t}}_{\varepsilon,\varepsilon'}(S) = e^{h_{S_\varepsilon}(w-\varepsilon'\xi_0)}\, \mathscr{O}^{\mathrm{t}}_{\mathbb{P}^*}((S^\star_\infty)^\circ + \varepsilon'\xi_0)$$

as well as

$$\mathrm{Exp}^{\mathrm{t}}_{\varepsilon'}(S) = \varprojlim_{\varepsilon \to 0} \mathrm{Exp}^{\mathrm{t}}_{\varepsilon,\varepsilon'}(S).$$

If $z, w \in \mathbb{C}$, we denote by $\theta(z,w)$ the non-oriented angle between z and w. Recall that our inner product on $\mathbb{C}$ is $\Re(zw) = |z||w|\cos(\theta(\bar{z}, w))$.

Lemma 5.2.19. *Let S be a proper non-compact closed convex subset of $\mathbb{C}$ which contains no lines and $w \in (S^*_\infty)^\circ$. Then, there are $R, \delta > 0$ such that*

$$\cos(\theta(\bar{z}, w)) < -\delta, \quad \forall z \in S \backslash D(0,R).$$

Proof. Since $w \in (S^*_\infty)^\circ$, there is a closed cone C such that $w \in C\backslash\{0\} \subsetneq (S^*_\infty)^\circ$. It is then clear that there is $\delta > 0$ such that for all $z \in S_\infty\backslash\{0\} \subsetneq C^*$, one has $\cos(\theta(\bar{z}, w)) < -\delta$. Now, let us proceed by contradiction and assume that for all $n \in \mathbb{N}_0$, there is $z_n \in S\backslash D(0,n)$ such that $\cos(\theta(\bar{z}_n, w)) \geq -\delta$. Since $C(0,1)$ is compact, one can find a subsequence $z_{k(n)}$ such that

$$\frac{z_{k(n)}}{|z_{k(n)}|} \to z.$$

By construction, z is an element of $S_\infty\backslash\{0\}$ and thus $\cos(\theta(\bar{z}, w)) < -\delta$. However, $\cos\left(\theta\left(\frac{\bar{z}_{k(n)}}{|z_{k(n)}|}, w\right)\right) \geq -\delta$ for all $n \in \mathbb{N}_0$, which leads to a contradiction. $\square$

Theorem 5.2.20. *Let $\varepsilon' > 0$. There is a canonical isomorphism of $\mathbb{C}$-vector spaces*

$$\mathscr{H}^{\mathrm{t}}_S(\mathbb{V}, \varepsilon') \xrightarrow{\sim} \mathrm{Exp}^{\mathrm{t}}_{\varepsilon'}(S). \tag{5.23}$$

These spaces do not depend on the chosen norm.

Given a global $\mathbb{C}$-linear coordinate z of $\mathbb{V}$ and its dual coordinate w, this isomorphism can be made explicit by $\mathscr{H}^{\mathrm{t}}_S(\mathbb{V}, \varepsilon') \ni [f(z)dz] \mapsto g \in \mathrm{Exp}^{\mathrm{t}}_{\varepsilon'}(S)$, with

$$g(w) = \int_{\partial S^+_\varepsilon} e^{zw} f(z) dz,$$

where ∂S^+_ε is the positively oriented boundary of any thickening S_ε.

Proof. We apply $\varprojlim\limits_{\varepsilon\to 0}$ to (5.21) as well as (5.22) to get isomorphisms

$$\mathscr{H}^{\mathrm{t}}_S(\mathbb{V}, \varepsilon') \xrightarrow{\sim} \varprojlim_{\varepsilon\to 0} H^1_{S_\varepsilon}(\mathbb{V}, e^{-\langle z, \varepsilon'\xi_0\rangle}\Omega^{\mathrm{t}}_{\mathbb{P}}) \xrightarrow{\sim} \mathrm{Exp}^{\mathrm{t}}_{\varepsilon'}(S).$$

Let us now compute this map within coordinates. Let $[f(z)dz] \in \mathscr{H}^{\mathrm{t}}_S(\mathbb{V}, \varepsilon')$ and fix $\varepsilon > 0$. Let us choose a cutoff function ψ_ε as in Proposition 5.2.16. Then, the image of $[f(z)dz]$ in $\mathrm{Exp}^{\mathrm{t}}_{\varepsilon,\varepsilon'}(S)$ is given by g, where

$$g(w) = \mathcal{L}^+_w(\bar{\partial}(\psi_\varepsilon f(z)dz)) = \int_{\mathbb{V}} e^{zw}\bar{\partial}(\psi_\varepsilon f(z)dz).$$

One has

$$\begin{aligned}\int_{\mathbb{V}} e^{zw}\bar{\partial}(\psi_\varepsilon f(z)dz) &= \int_{\mathbb{V}} \bar{\partial}(e^{zw}\psi_\varepsilon f(z)dz) = \int_{S_\varepsilon \setminus S^\circ_{\varepsilon/2}} \bar{\partial}(e^{zw}\psi_\varepsilon f(z)dz) \\ &= \lim_{R\to+\infty} \int_{(S_\varepsilon \setminus S^\circ_{\varepsilon/2})\cap \overline{D}(0,R)} \bar{\partial}(e^{zw}\psi_\varepsilon f(z)dz) \\ &= \lim_{R\to+\infty} \int_{\partial((S_\varepsilon \setminus S^\circ_{\varepsilon/2})\cap \overline{D}(0,R))^+} e^{zw}\psi_\varepsilon f(z)dz.\end{aligned}$$

For $R > 0$ big enough, it is clear that $\partial((S_\varepsilon \setminus S^\circ_{\varepsilon/2})\cap \overline{D}(0,R))^+$ is a Jordan rectifiable curve which can be decomposed in four oriented rectifiable curves : $(\partial S_\varepsilon \cap \overline{D}(0,R))^+$, $(\partial S_{\varepsilon/2} \cap \overline{D}(0,R))^-$ and two oriented arcs of circle $\mathcal{I}_R$ and $\mathcal{J}_R$ (see Figure 1 below). By construction of ψ_ε, we have

$$\int_{(\partial S_{\varepsilon/2}\cap \overline{D}(0,R))^-} e^{zw}\psi_\varepsilon f(z)dz = 0$$

and

$$\lim_{R\to+\infty} \int_{(\partial S_\varepsilon \cap \overline{D}(0,R))^+} e^{zw}\psi_\varepsilon f(z)dz = \int_{\partial S^+_\varepsilon} e^{zw} f(z)dz.$$

Let us prove that

$$\lim_{R\to+\infty} \int_{\mathcal{I}_R} e^{zw}\psi_\varepsilon f(z)dz = \lim_{R\to+\infty} \int_{\mathcal{J}_R} e^{zw}\psi_\varepsilon f(z)dz = 0.$$

We do it for $\mathcal{I}_R$. We have

$$\begin{aligned}\left|\int_{\mathcal{I}_R} e^{zw}\psi_\varepsilon(z) f(z)dz\right| &< 2\pi R \sup_{z\in\mathcal{I}_R} |e^{zw} f(z)| \\ &\leq 2\pi R \sup_{z\in\mathcal{I}_R} |e^{z(w-\varepsilon'\xi_0)}| \sup_{z\in\mathcal{I}_R} |e^{z(\varepsilon'\xi_0)} f(z)|.\end{aligned}$$

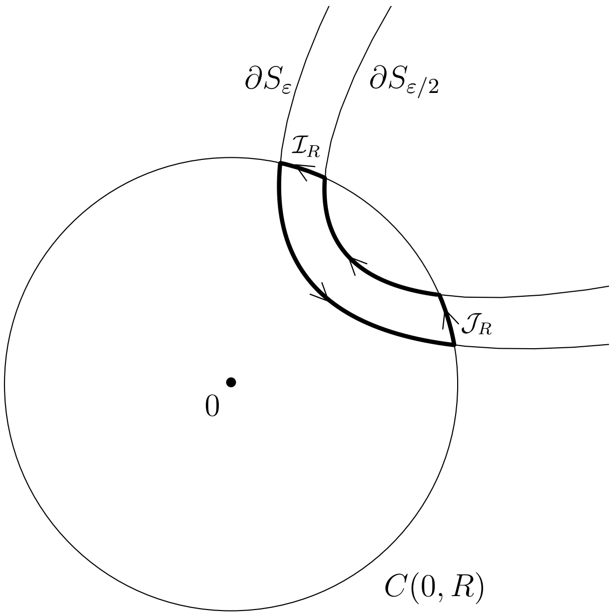

Figure 5.1: The contour $\partial\left(\left(S_\varepsilon\backslash S^\circ_{\varepsilon/2}\right)\cap\overline{D}(0,R)\right)^+$.

On the one hand, thanks to the tempered condition on $e^{z(\varepsilon'\xi_0)}f(z)dz$, one can see that for R big enough, there are $c\in(0,+\infty)$ and $N\in\mathbb{N}$ such that

$$\sup_{z\in\mathcal{I}_R}|e^{z(\varepsilon'\xi_0)}f(z)|\le cR^N.$$

On the other hand, for each $R>0$, there is $z_R\in\mathcal{I}_R$ such that

$$\sup_{z\in\mathcal{I}_R}|e^{z(w-\varepsilon'\xi_0)}|=e^{\Re(z_R(w-\varepsilon'\xi_0))}.$$

Moreover, one can write

$$e^{\Re(z_R(w-\varepsilon'\xi_0))}=e^{|z_R||w-\varepsilon'\xi_0|\cos(\theta(\bar{z}_R,w-\varepsilon'\xi_0))}=e^{R|w-\varepsilon'\xi_0|\cos(\theta(\bar{z}_R,w-\varepsilon'\xi_0))}.$$

By the previous lemma, since we have $w-\varepsilon'\xi_0\in(S^*_\infty)^\circ=(S^*_{\varepsilon,\infty})^\circ$ and $z_R\in S_\varepsilon$, we can find $\delta>0$ such that $\cos(\theta(\bar{z}_R,w-\varepsilon'\xi_0))<-\delta$ for all R big enough. Hence, for R big enough,

$$\left|\int_{\mathcal{I}_R}e^{zw}\psi_\varepsilon(z)f(z)\,dz\right|<2\pi cR^{N+1}e^{-|w-\varepsilon'\xi_0|\delta R}\underset{R\to+\infty}{\longrightarrow}0.$$

We have thus proved that the image of $[f(z)dz]$ in $\mathrm{Exp}^{\mathrm{t}}_{\varepsilon,\varepsilon'}(S)$ is the function g, defined on $(S^*_\infty)^\circ+\varepsilon'\xi_0$ by $g(w)=\int_{\partial S^+_\varepsilon}e^{zw}f(z)dz$. One can check, by a similar proof as above, that this integral remains unchanged with $\varepsilon_1<\varepsilon$. Therefore, it is also the image of $[f(z)dz]$ in $\mathrm{Exp}^{\mathrm{t}}_{\varepsilon'}(S)$ and we get the conclusion. $\square$

Remark 5.2.21. Let $\varepsilon' > \varepsilon'_1 > 0$. Then there is a well defined map

$$\mathscr{H}^{\mathrm{t}}_S(\mathbb{V}, \varepsilon'_1) \to \mathscr{H}^{\mathrm{t}}_S(\mathbb{V}, \varepsilon'),$$

namely $[u] \mapsto [u]$. Indeed, if $e^{\langle z, \varepsilon'_1 \xi_0\rangle} u$ is tempered on $S^\circ_r \backslash S_\varepsilon$ (resp. S°_r), then

$$e^{\langle z, \varepsilon' \xi_0\rangle} u = e^{\langle z, (\varepsilon' - \varepsilon'_1)\xi_0\rangle} e^{\langle z, \varepsilon'_1 \xi_0\rangle} u$$

is also tempered on $S^\circ_r \backslash S_\varepsilon$ (resp. S°_r), since $\Re(\langle z, (\varepsilon' - \varepsilon'_1)\xi_0\rangle) < 0$ for all $z \in S_r$ with big enough module. Hence, this gives rise to a projective system $(\mathscr{H}^{\mathrm{t}}_S(\mathbb{V}, \varepsilon'))_{\varepsilon'>0}$ which is compatible, through the positive Laplace transform, with the projective system $(\mathrm{Exp}^{\mathrm{t}}_{\varepsilon'}(S))_{\varepsilon'>0}$.

Corollary 5.2.22. *There is a canonical isomorphism of $\mathbb{C}$-vector spaces*

$$\varprojlim_{\varepsilon' \to 0} \mathscr{H}^{\mathrm{t}}_S(\mathbb{V}, \varepsilon') \xrightarrow{\sim} \varprojlim_{\varepsilon' \to 0} \mathrm{Exp}^{\mathrm{t}}_{\varepsilon'}(S). \tag{5.24}$$

Given a global $\mathbb{C}$-linear coordinate z of $\mathbb{V}$ and its dual coordinate w, this isomorphism can be made explicit by

$$\varprojlim_{\varepsilon' \to 0} \mathscr{H}^{\mathrm{t}}_S(\mathbb{V}, \varepsilon') \ni ([f_{\varepsilon'}(z)dz])_{\varepsilon'>0} \mapsto g \in \varprojlim_{\varepsilon' \to 0} \mathrm{Exp}^{\mathrm{t}}_{\varepsilon'}(S),$$

with

$$g(w) = \int_{\partial S^+_\varepsilon} e^{zw} f_{\varepsilon'}(z)dz.$$

Proof. Within coordinates, we already know that the image of $([f_{\varepsilon'}(z)dz])_{\varepsilon'>0}$ through (5.24) is given by a family $(g_{\varepsilon'})_{\varepsilon'>0}$, where

$$g_{\varepsilon'}(w) = \int_{\partial S^+_\varepsilon} e^{zw} f_{\varepsilon'}(z)dz$$

on $(S^*_\infty)^\circ + \varepsilon'\xi_0$. To get the conclusion, it is enough to notice that

1. for all $\varepsilon' > 0$, the function $g_{\varepsilon'}$ is well-defined and holomorphic on $(S^*_\infty)^\circ$,
2. for any $\varepsilon' > \varepsilon'_1 > 0$, one has

$$\int_{\partial S^+_\varepsilon} e^{zw} f_{\varepsilon'}(z)dz = \int_{\partial S^+_\varepsilon} e^{zw} f_{\varepsilon'_1}(z)dz.$$

Indeed, since $f_{\varepsilon'} - f_{\varepsilon'_1}$ is entire and verifies a suitable tempered condition, we have

$$\int_{\partial S^+_\varepsilon} e^{zw}(f_{\varepsilon'}(z) - f_{\varepsilon'_1}(z))dz = \lim_{R \to +\infty} \int_{\partial(S_\varepsilon \cap \overline{D}(0,R))^+} e^{zw}(f_{\varepsilon'}(z) - f_{\varepsilon'_1}(z))dz = 0.$$

$\square$

Remark 5.2.23. Corollary 5.2.22 is nothing more but the "algebraic $\mathcal{P}$-part" of Méril's theorem, while Theorem 5.2.20 is a stronger and new result. First, the canonical map

$$\mathscr{H}_S(\mathbb{C},\varepsilon') \to \mathscr{H}_S^{\mathrm{t}}(\mathbb{C},\varepsilon'), \quad [f] \mapsto \left[\frac{1}{2i\pi} f dz\right]$$

is injective for all ε', thanks to the Phragmen-Lindelöf theorem of [44, p. 394]. Hence, it remains injective when applying $\varprojlim_{\varepsilon'\to 0}$. Secondly, the map

$$\mathscr{H}_S(\mathbb{C}) \to \varprojlim_{\varepsilon'\to 0} \mathscr{H}_S^{\mathrm{t}}(\mathbb{C},\varepsilon')$$

is surjective since $e^{\langle z, 2\varepsilon'\xi_0\rangle} f$ is bounded on $S_r \backslash S_\varepsilon^\circ$ if $e^{\langle z,\varepsilon'\xi_0\rangle} f$ is tempered on $S_r\backslash S_\varepsilon^\circ$. Finally, the inclusion

$$\begin{aligned}\operatorname{Exp}(S) &\subset \{g \in \mathcal{O}((S_\infty^\star)^\circ) : \forall \varepsilon, \varepsilon' > 0,\, g \in e^{h_{S_\varepsilon}} \mathcal{D}b_{\mathbb{P}}^{\mathrm{t}}((S_\infty^\star)^\circ + \varepsilon'\xi_0)\} \\ &\simeq \varprojlim_{\varepsilon'\to 0} \operatorname{Exp}^{\mathrm{t}}_{\varepsilon'}(S)\end{aligned}$$

is an equality for the same reasons than in Remark 5.2.15.

5.3 Tempered holomorphic cohomological convolution

Using the link between Méril's spaces and our tempered holomorphic cohomological spaces, we can now prove, in the subanalytic case, the conjecture proposed at the end of chapter 3. For this, we introduce an "intermediary step" between the holomorphic cohomological convolution and the convolution of non-compactly carried analytic functionals.

5.3.1 General definition

Let (G,μ) be a locally compact complex Lie group and recall section 4.2.1.

Proposition 5.3.1. *If S_1 and S_2 are two convolvable subanalytic closed subsets of G, then $\mu(S_1\times S_2)$ is also a subanalytic closed subset of G.*

Proof. We already know that $\mu(S_1 \times S_2)$ is closed. Let us denote by

$$p_1, p_2 : G\times G \to G$$

the two projections. Since they are holomorphic, we know that $p_1^{-1}(S_1) = S_1 \times G$ (resp. $p_2^{-1}(S_2) = G\times S_2$) is a subanalytic subset of $G\times G$. Hence,

$$S_1 \times S_2 = (S_1\times G) \cap (G \times S_2)$$

is a subanalytic subset of $G \times G$. By the convolvability hypothesis, the map

$$\mu|_{S_1\times S_2} : S_1 \times S_2 \to G$$

is proper. Therefore, $\mu(S_1\times S_2)$ is a subanalytic subset of G. □

We can now imitate the definition of section 2.2.1. To simplify things, we will do it for the complex bordered space $\mathbb{V}_\infty = (\mathbb{V}, \overline{\mathbb{V}})$, where $\mathbb{V}$ is a complex vector space of complex dimension n and $\overline{\mathbb{V}}$ the projective compactification of $\mathbb{V}$. Thus, it is clear that the addition on $\mathbb{V}$ induces a semi-proper morphism of bordered spaces

$$+ : \mathbb{V}_\infty \times \mathbb{V}_\infty \to \mathbb{V}_\infty .$$

Let S_1 and S_2 be two convolvable subanalytic closed subsets of $\mathbb{V}$. On the one hand, the external product of tempered distributions is clearly tempered and thus gives a map

$$\Gamma_{S_1}(\mathbb{V}, \mathcal{D}b^{\mathrm{t},2n}_{\overline{\mathbb{V}}}) \otimes \Gamma_{S_2}(\mathbb{V}, \mathcal{D}b^{\mathrm{t},2n}_{\overline{\mathbb{V}}}) \to \Gamma_{S_1\times S_2}(\mathbb{V}\times\mathbb{V}, \mathcal{D}b^{\mathrm{t},4n}_{\overline{\mathbb{V}}\times\overline{\mathbb{V}}}). \tag{5.25}$$

On the other hand, decomposing the addition like a biholomorphism and a projection extendible to $\overline{\mathbb{V}} \times \overline{\mathbb{V}}$, one can use Lemma 4.3.9 to obtain a morphism

$$\int_+ : +_{!!}\, \mathcal{D}b^{\mathrm{t},4n}_{\mathbb{V}_\infty\times\mathbb{V}_\infty} \to \mathcal{D}b^{\mathrm{t},2n}_{\mathbb{V}_\infty},$$

which induces a map

$$\int_+ : \Gamma_{S_1\times S_2}(\mathbb{V}\times\mathbb{V}, \mathcal{D}b^{\mathrm{t},4n}_{\overline{\mathbb{V}}\times\overline{\mathbb{V}}}) \to \Gamma_{S_1+S_2}(\mathbb{V}, \mathcal{D}b^{\mathrm{t},2n}_{\overline{\mathbb{V}}}). \tag{5.26}$$

Obviously, the composition of (5.25) and (5.26) is a linear map

$$\Gamma_{S_1}(\mathbb{V}, \mathcal{D}b^{\mathrm{t},2n}_{\overline{\mathbb{V}}}) \otimes \Gamma_{S_2}(\mathbb{V}, \mathcal{D}b^{\mathrm{t},2n}_{\overline{\mathbb{V}}}) \to \Gamma_{S_1+S_2}(\mathbb{V}, \mathcal{D}b^{\mathrm{t},2n}_{\overline{\mathbb{V}}}),$$

defined by $u_1 \otimes u_2 \mapsto u_1 \star u_2$, where $u_1 \star u_2$ is the usual convolution product of distributions. It is thus natural to define the convolution of cohomology classes of tempered holomorphic forms on $\mathbb{V}_\infty$ as follows :

Definition 5.3.2. Let S_1, S_2 be two convolvable subanalytic closed subsets of $\mathbb{V}$. Consider the external product morphisms

$$\mathrm{R}\Gamma_{S_1}(\mathbb{V}, \Omega^{\mathrm{t},p+n}_{\overline{\mathbb{V}}})[n] \otimes \mathrm{R}\Gamma_{S_2}(\mathbb{V}, \Omega^{\mathrm{t},q+n}_{\overline{\mathbb{V}}})[n] \to \mathrm{R}\Gamma_{S_1\times S_2}(\mathbb{V}\times\mathbb{V}, \Omega^{\mathrm{t},p+q+2n}_{\overline{\mathbb{V}}\times\overline{\mathbb{V}}})[2n]$$

and the morphisms

$$\int_+ : \mathrm{R}\Gamma_{S_1\times S_2}(\mathbb{V}\times\mathbb{V}, \Omega^{\mathrm{t},p+q+2n}_{\overline{\mathbb{V}}\times\overline{\mathbb{V}}})[2n] \to \mathrm{R}\Gamma_{S_1+S_2}(\mathbb{V}, \Omega^{\mathrm{t},p+q+n}_{\overline{\mathbb{V}}})[n]$$

induced by the holomorphic integration map and the fact that $S_1 \times S_2$ is μ-proper. By composition, these morphisms give derived category morphisms

$$\star^{\mathrm{t}}_{\mathbb{V}_\infty} : \mathrm{R}\Gamma_{S_1}(\mathbb{V}, \Omega^{\mathrm{t},p+n}_{\overline{\mathbb{V}}})[n] \otimes \mathrm{R}\Gamma_{S_2}(\mathbb{V}, \Omega^{\mathrm{t},q+n}_{\overline{\mathbb{V}}})[n] \to \mathrm{R}\Gamma_{S_1+S_2}(\mathbb{V}, \Omega^{\mathrm{t},p+q+n}_{\overline{\mathbb{V}}})[n]$$

that we call the *tempered holomorphic convolution morphisms of* $\mathbb{V}_\infty$. Going to cohomology groups, these morphisms give rise to the morphisms

$$\star^{\mathrm{t}}_{\mathbb{V}_\infty} : H^{r+n}_{S_1}(\mathbb{V}, \Omega^{\mathrm{t},p+n}_{\overline{\mathbb{V}}}) \otimes H^{s+n}_{S_2}(\mathbb{V}, \Omega^{\mathrm{t},q+n}_{\overline{\mathbb{V}}}) \to H^{r+s+n}_{S_1+S_2}(\mathbb{V}, \Omega^{\mathrm{t},p+q+n}_{\overline{\mathbb{V}}})$$

that we call the *tempered holomorphic cohomological convolution morphisms of* $\mathbb{V}_\infty$.

Remark 5.3.3. Let us take $p = q = r = s = 0$. Then, by the quasi-injectivity of the tempered distribution sheaves, the tempered holomorphic cohomological convolution morphism

$$\star^{\mathrm{t}}_{\mathbb{V}_\infty} : H^n_{S_1}(\mathbb{V}, \Omega^{\mathrm{t}}_{\overline{\mathbb{V}}}) \otimes H^n_{S_2}(\mathbb{V}, \Omega^{\mathrm{t}}_{\overline{\mathbb{V}}}) \to H^n_{S_1+S_2}(\mathbb{V}, \Omega^{\mathrm{t}}_{\overline{\mathbb{V}}})$$

can be explicitly computed by the linear map

$$\frac{\Gamma_{S_1}(\mathbb{V}, \mathcal{D}b^{\mathrm{t},n,n}_{\overline{\mathbb{V}}})}{\bar\partial\Gamma_{S_1}(\mathbb{V}, \mathcal{D}b^{\mathrm{t},n,n-1}_{\overline{\mathbb{V}}})} \otimes \frac{\Gamma_{S_2}(\mathbb{V}, \mathcal{D}b^{\mathrm{t},n,n}_{\overline{\mathbb{V}}})}{\bar\partial\Gamma_{S_2}(\mathbb{V}, \mathcal{D}b^{\mathrm{t},n,n-1}_{\overline{\mathbb{V}}})} \mapsto \frac{\Gamma_{S_1+S_2}(\mathbb{V}, \mathcal{D}b^{\mathrm{t},n,n}_{\overline{\mathbb{V}}})}{\bar\partial\Gamma_{S_1+S_2}(\mathbb{V}, \mathcal{D}b^{\mathrm{t},n,n-1}_{\overline{\mathbb{V}}})},$$

defined by $[u_1] \otimes [u_2] \mapsto [u_1 \star u_2]$.

Proposition 5.3.4. *There is a canonical commutative diagram*

$$\begin{tikzcd}
H^n_{S_1}(\mathbb{V}, \Omega_{\mathbb{V}}) \otimes H^n_{S_2}(\mathbb{V}, \Omega_{\mathbb{V}}) \arrow[r] & H^n_{S_1+S_2}(\mathbb{V}, \Omega_{\mathbb{V}}) \\
H^n_{S_1}(\mathbb{V}, \Omega^{\mathrm{t}}_{\overline{\mathbb{V}}}) \otimes H^n_{S_2}(\mathbb{V}, \Omega^{\mathrm{t}}_{\overline{\mathbb{V}}}) \arrow[r] \arrow[u] & H^n_{S_1+S_2}(\mathbb{V}, \Omega^{\mathrm{t}}_{\overline{\mathbb{V}}}) \arrow[u]
\end{tikzcd}$$

where the top arrow is given by the holomorphic cohomological convolution $\star_{(\mathbb{V},+)}$ and the bottom arrow by the tempered holomorphic cohomological convolution $\star^{\mathrm{t}}_{\mathbb{V}_\infty}$.

Proof. Consider the trivial embedding $\Omega^{\mathrm{t}}_{\overline{\mathbb{V}}} \hookrightarrow \iota_{\overline{\mathbb{V}}}\Omega_{\overline{\mathbb{V}}}$ of subanalytic sheaves. Then, if S is a closed subanalytic subset of $\mathbb{V}$, one gets a canonical map

$$\begin{aligned}
\mathrm{R}\Gamma_S(\mathbb{V}, \Omega^{\mathrm{t}}_{\overline{\mathbb{V}}}) &\to \mathrm{R}\Gamma_S(\mathbb{V}, \iota_{\overline{\mathbb{V}}}(\Omega_{\overline{\mathbb{V}}})) \\
&\simeq \mathrm{RHom}(\iota_{\overline{\mathbb{V}}}(\mathbb{C}_{\mathbb{V}} \otimes \mathbb{C}_S), \iota_{\overline{\mathbb{V}}}(\Omega_{\overline{\mathbb{V}}})) \\
&\simeq \mathrm{RHom}(\mathbb{C}_{\mathbb{V}} \otimes \mathbb{C}_S, \Omega_{\overline{\mathbb{V}}}) \\
&\simeq \mathrm{R}\Gamma_S(\mathbb{V}, \Omega_{\overline{\mathbb{V}}}) \\
&\simeq \mathrm{R}\Gamma_S(\mathbb{V}, \Omega_{\mathbb{V}})
\end{aligned}$$

thanks to the fully-faithfulness of $\iota_{\overline{\mathbb{V}}}$. This map gives the vertical arrows of the diagram, which is then obviously commutative by construction. (Indeed, by Remark 4.4.8, the semi-properness of $+$ entails that $+_{!!} \circ \iota_{\mathbb{V}\times\mathbb{V}} = \iota_{\mathbb{V}} \circ +_!$.) ☐

5.3.2 Proof of the main conjecture (subanalytic case)

Let us go back to the complex bordered space $(\mathbb{C}, \mathbb{P})$ where $\mathbb{C}$ is equipped with the addition and let us fix two convolvable proper non-compact closed convex subsets of $\mathbb{C}$ which contain no lines. Hence, one gets two compatible dualities $S_1 \leftrightarrow (h_{S_1}, C_1)$ and $S_2 \leftrightarrow (h_{S_2}, C_2)$. Let us also fix a reference point $\xi_{1,2} \in C_1^\circ \cap C_2^\circ$.

Throughout this section, we will assume that $(S_1)_\varepsilon$ (resp. $(S_2)_\varepsilon$) is a subanalytic subset of $\mathbb{C}$ and that $h_{(S_1)_\varepsilon}$ (resp. $h_{(S_2)_\varepsilon}$) is almost $\mathcal{C}_\infty$-subanalytic on $\mathbb{P}$ for all $\varepsilon > 0$. Using an additive version of Lemma 2.2.13, it is clear that $(S_1)_\varepsilon$ and $(S_2)_\varepsilon$ are still convolvable for all $\varepsilon > 0$. Moreover, by Proposition 5.3.1 $(S_1 + S_2)_{2\varepsilon}$ is a subanalytic subset of $\mathbb{C}$ and, obviously, $h_{(S_1+S_2)_{2\varepsilon}} = h_{(S_1)_\varepsilon} + h_{(S_2)_\varepsilon}$ is almost $\mathcal{C}_\infty$-subanalytic on $\mathbb{P}$ for all $\varepsilon > 0$.

Proposition 5.3.5. *Let $\varepsilon, \varepsilon' > 0$. The tempered holomorphic cohomological convolution morphism*

$$\star^{\mathrm{t}}_{(\mathbb{C},\mathbb{P})} : H^1_{(S_1)_\varepsilon}(\mathbb{C}, \Omega^{\mathrm{t}}_{\mathbb{P}}) \otimes H^1_{(S_2)_\varepsilon}(\mathbb{C}, \Omega^{\mathrm{t}}_{\mathbb{P}}) \to H^1_{(S_1+S_2)_{2\varepsilon}}(\mathbb{C}, \Omega^{\mathrm{t}}_{\mathbb{P}})$$

induces a convolution map

$$H^1_{(S_1)_\varepsilon}(\mathbb{C}, e^{-z_1(\varepsilon'\xi_{1,2})}\Omega^{\mathrm{t}}_{\mathbb{P}}) \otimes H^1_{(S_2)_\varepsilon}(\mathbb{C}, e^{-z_2(\varepsilon'\xi_{1,2})}\Omega^{\mathrm{t}}_{\mathbb{P}}) \to H^1_{(S_1+S_2)_{2\varepsilon}}(\mathbb{C}, e^{-z(\varepsilon'\xi_{1,2})}\Omega^{\mathrm{t}}_{\mathbb{P}}).$$

Proof. It is enough to prove that the usual convolution product of distributions is a well defined map from

$$\Gamma_{(S_1)_\varepsilon}(\mathbb{C}, e^{-z_1(\varepsilon'\xi_{1,2})}\,\mathcal{D}b^{\mathrm{t},1,1}_{\mathbb{P}}) \otimes \Gamma_{(S_2)_\varepsilon}(\mathbb{C}, e^{-z_2(\varepsilon'\xi_{1,2})}\,\mathcal{D}b^{\mathrm{t},1,1}_{\mathbb{P}})$$

to

$$\Gamma_{(S_1+S_2)_{2\varepsilon}}(\mathbb{C}, e^{-z(\varepsilon'\xi_{1,2})}\,\mathcal{D}b^{\mathrm{t},1,1}_{\mathbb{P}}).$$

Let u_1 (resp. u_2) be an element of $\mathcal{D}b^{1,1}_{(S_1)_\varepsilon}(\mathbb{C})$ (resp. $\mathcal{D}b^{1,1}_{(S_2)_\varepsilon}(\mathbb{C})$) such that $e^{z_1(\varepsilon'\xi_{1,2})}u_1$ (resp. $e^{z_2(\varepsilon'\xi_{1,2})}u_2$) is tempered at infinity. Then

$$\begin{aligned}\langle u_1 \star u_2, \varphi\rangle &= \langle u_1, \langle u_2, \varphi(z_1+z_2)\rangle\rangle \\ &= \langle e^{z_1(\varepsilon'\xi_{1,2})}u_1, \langle e^{z_2(\varepsilon'\xi_{1,2})}u_2, e^{-(z_1+z_2)(\varepsilon'\xi_{1,2})}\varphi(z_1+z_2)\rangle\rangle \\ &= \langle e^{z_1(\varepsilon'\xi_{1,2})}u_1 \star e^{z_2(\varepsilon'\xi_{1,2})}u_2, e^{-z(\varepsilon'\xi_{1,2})}\varphi\rangle \\ &= \langle e^{-z(\varepsilon'\xi_{1,2})}(e^{z_1(\varepsilon'\xi_{1,2})}u_1 \star e^{z_2(\varepsilon'\xi_{1,2})}u_2), \varphi\rangle\end{aligned}$$

for all test-function φ. Since $e^{z_1(\varepsilon'\xi_{1,2})}u_1 \star e^{z_2(\varepsilon'\xi_{1,2})}u_2$ is tempered at infinity, this implies that $u \star v$ is an element of $\Gamma_{(S_1+S_2)_{2\varepsilon}}(\mathbb{C}, e^{-z(\varepsilon'\xi_{1,2})}\,\mathcal{D}b^{\mathrm{t},1,1}_{\mathbb{P}})$. Hence the conclusion. □

Of course, like in Proposition 5.3.4, this convolution can be embedded in a commutative diagram

$$\begin{tikzcd} H^1_{(S_1)_\varepsilon}(\mathbb{C}, \Omega_{\mathbb{C}}) \otimes H^1_{(S_2)_\varepsilon}(\mathbb{C}, \Omega_{\mathbb{C}}) \arrow[r] & H^1_{(S_1+S_2)_{2\varepsilon}}(\mathbb{C}, \Omega_{\mathbb{C}}) \\ H^1_{(S_1)_\varepsilon}(\mathbb{C}, e^{-z_1(\varepsilon'\xi_{1,2})}\Omega^{\mathrm{t}}_{\mathbb{P}}) \otimes H^1_{(S_2)_\varepsilon}(\mathbb{C}, e^{-z_2(\varepsilon'\xi_{1,2})}\Omega^{\mathrm{t}}_{\mathbb{P}}) \arrow[r] \arrow[u] & H^1_{(S_1+S_2)_{2\varepsilon}}(\mathbb{C}, e^{-z(\varepsilon'\xi_{1,2})}\Omega^{\mathrm{t}}_{\mathbb{P}}) \arrow[u] \end{tikzcd}$$

Taking the projective limit on $\varepsilon \to 0$ and then on $\varepsilon' \to 0$, one finally gets

Proposition 5.3.6. *There is a canonical commutative diagram*

$$\begin{tikzcd} H^1_{S_1}(\mathbb{C}, \Omega_{\mathbb{C}}) \otimes H^1_{S_2}(\mathbb{C}, \Omega_{\mathbb{C}}) \arrow[r] & H^1_{S_1+S_2}(\mathbb{C}, \Omega_{\mathbb{C}}) \\ \mathscr{H}_{S_1}(\mathbb{C}) \otimes \mathscr{H}_{S_2}(\mathbb{C}) \arrow[r] \arrow[u] & \mathscr{H}_{S_1+S_2}(\mathbb{C}) \arrow[u] \end{tikzcd}$$

where the top arrow is given by the holomorphic cohomological convolution and the bottom arrow by the tempered holomorphic cohomological convolution.

Proof. The isomorphism

$$\varprojlim_{\varepsilon \to 0} H^1_{S_\varepsilon}(\mathbb{C}, \Omega_{\mathbb{C}}) \simeq H^1_S(\mathbb{C}, \Omega_{\mathbb{C}})$$

follows from Corollary 1.2.4 and the isomorphisms

$$\mathscr{H}_S(\mathbb{C}) = \varprojlim_{\varepsilon' \to 0} \mathscr{H}_S(\mathbb{C}, \varepsilon') \simeq \varprojlim_{\varepsilon' \to 0} \mathscr{H}^{\mathrm{t}}_S(\mathbb{C}, \varepsilon') \simeq \varprojlim_{\varepsilon, \varepsilon' \to 0} H^1_{S_\varepsilon}(\mathbb{C}, e^{-z(\varepsilon'\xi)}\Omega^{\mathrm{t}}_{\mathbb{P}})$$

from Remark 5.2.23. $\square$

To prove Conjecture 3.2.30 in this subanalytic context, it is thus enough to prove the following theorem :

Theorem 5.3.7. *The following diagram is commutative :*

$$\begin{tikzcd}
\mathscr{H}_{S_1}(\mathbb{C}) \times \mathscr{H}_{S_2}(\mathbb{C}) \arrow[r] & \mathscr{H}_{S_1+S_2}(\mathbb{C}) \\
Q'(S_1) \times Q'(S_2) \arrow[r] \arrow[u, "\wr"] & Q'(S_1+S_2) \arrow[u, "\wr"]
\end{tikzcd}$$

Here, the bottom arrow is given by the convolution of analytic functionals, the top arrow by the tempered holomorphic cohomological convolution and the vertical arrows by the Cauchy transform.

Lemma 5.3.8. *Let*

$$\begin{tikzcd}
A \arrow[r, "f"] & B \\
C \arrow[r, "i"] \arrow[u, "g"] & D \arrow[u, "h"] \\
E \arrow[r, "l"] \arrow[u, "j"] & F \arrow[u, "k"]
\end{tikzcd}$$

be a diagram in a category. Assume that the upper square and the big square are commutative. Then, if h is a monomorphism, the lower square is also commutative.

Proof. One has

$$h \circ i \circ j = f \circ g \circ j = h \circ k \circ l.$$

Since h is a monomorphism, this implies that $i \circ j = k \circ l$. Hence the conclusion. $\square$

Proof of Theorem 5.3.7. Let $\varepsilon, \varepsilon' > 0$. There is a commutative diagram

$$\begin{tikzcd}
\mathrm{Exp}^{\mathrm{t}}_{\varepsilon,\varepsilon'}(S_1) \times \mathrm{Exp}^{\mathrm{t}}_{\varepsilon,\varepsilon'}(S_2) \arrow[r] & \mathrm{Exp}^{\mathrm{t}}_{2\varepsilon,\varepsilon'}(S_1+S_2) \\
H^1_{(S_1)_\varepsilon}(\mathbb{C}, e^{-z_1(\varepsilon'\xi_{1,2})}\Omega^{\mathrm{t}}_{\mathbb{P}}) \times H^1_{(S_2)_\varepsilon}(\mathbb{C}, e^{-z_2(\varepsilon'\xi_{1,2})}\Omega^{\mathrm{t}}_{\mathbb{P}}) \arrow[r] \arrow[u, "\wr"] & H^1_{(S_1+S_2)_{2\varepsilon}}(\mathbb{C}, e^{-z(\varepsilon'\xi_{1,2})}\Omega^{\mathrm{t}}_{\mathbb{P}}) \arrow[u, "\wr"]
\end{tikzcd}$$

where the bottom arrow is given by the tempered holomorphic cohomological convolution, the top arrow by the usual product of functions and the vertical arrows by the isomorphism (5.21). Indeed, one has

$$\mathcal{L}^+([u_1] \star^{\mathrm{t}}_{(\mathbb{C},\mathbb{P})} [u_2]) = \mathcal{L}^+[u_1 \star u_2] = \mathcal{L}^+(u_1 \star u_2) = \mathcal{L}^+(u_1)\mathcal{L}^+(u_2),$$

where the last equality follows from a classical theorem in analysis. Taking the projective limit on $\varepsilon, \varepsilon' \to 0$ and using Corollary 5.2.22 as well as Remark 5.2.23, one gets a commutative diagram

$$\begin{tikzcd}
\mathrm{Exp}(S_1) \times \mathrm{Exp}(S_2) \arrow[r] & \mathrm{Exp}(S_1 + S_2) \\
\mathscr{H}_{S_1}(\mathbb{C}) \times \mathscr{H}_{S_2}(\mathbb{C}) \arrow[r] \arrow[u, "\mathcal{P}\times\mathcal{P}"] & \mathscr{H}_{S_1+S_2}(\mathbb{C}) \arrow[u, "\mathcal{P}"]
\end{tikzcd}$$

It can be embedded in the bigger diagram

$$\begin{tikzcd}
\mathrm{Exp}(S_1) \times \mathrm{Exp}(S_2) \arrow[r] & \mathrm{Exp}(S_1 + S_2) \\
\mathscr{H}_{S_1}(\mathbb{C}) \times \mathscr{H}_{S_2}(\mathbb{C}) \arrow[r] \arrow[u, "\mathcal{P}\times\mathcal{P}"] & \mathscr{H}_{S_1+S_2}(\mathbb{C}) \arrow[u, "\mathcal{P}"] \\
Q'(S_1) \times Q'(S_2) \arrow[r] \arrow[u, "\mathcal{C}\times\mathcal{C}"] & Q'(S_1 + S_2) \arrow[u, "\mathcal{C}"]
\end{tikzcd}$$

where the bottom arrow is given by the convolution of analytic functionals. In the light of the above, the upper square is commutative. Moreover, since $\mathcal{P} \circ \mathcal{C} = \mathcal{F}$, the big square is also commutative by Proposition 3.2.24. The conclusion then follows from the bijectivity of $\mathcal{P}$ and the previous lemma. □

Conclusion

As a conclusion, we would like to point out some possible ways to reinforce the results that were obtained in this thesis and also propose some guidelines for a future research linked to these thematics.

First, one could ask whether it is possible to obtain explicit results as Theorem 2.2.12 for more general complex Lie groups, for example torus. The main difficulty lies in the explicit computation of $H^n_S(G, \Omega_G)$, where S is a proper closed subset of G. One should determine what are the less restrictive conditions to require on G and S in order to treat other interesting cases.

Secondly, one may aim to strengthen as much as possible the statement of Theorem 5.2.10. The first priority would be to prove Conjecture 5.2.3, at least for convex functions, so that one can remove the almost $\mathcal{C}_\infty$-subanalytic condition on f^*_S. Dropping the subanalytic condition on S seems to be more delicate. Of course, we know that Corollary 5.2.22 is true, even without the subanalytic conditions on S, since one can construct an explicit inverse of $\mathcal{P}$, namely $\mathcal{B}$. However, the Borel transform only works as an inverse because one can change the ε'. Hence, it is not an inverse for the stronger map in Theorem 5.2.20. This seems to suggest that our method can provide isomorphisms between bigger spaces than usual, but without any reasonable constructible inverse and this probably is deeply linked to the subanalytic conditions on S. Nonetheless, convex properties are also very strong and it is perhaps possible to preserve our results without any subanalytic assumption. Managing that would in the same time garantee a proof of Conjecture 3.2.30 in the general case. The reader could also want to incorporate a "topological part" so that our isomorphisms of $\mathbb{C}$-vector spaces become isomorphisms of locally convex spaces (nuclear Fréchet in our case). A possible idea is to use subanalytic sheaves valued in the category $\mathcal{I}nd\mathcal{B}an$, studied in [92] and in [104].

Finally, one could replace the tempered conditions by other growth conditions, like Gevrey ones, in order to get a greater diversity of holomorphic Paley-Wiener-type theorems. In [41], the authors study the subanalytic sheaf $\mathcal{O}^{\mathrm{gev}}_X$ of holomorphic functions of Gevrey growth. They also introduce a refinement of the subanalytic site, called the linear subanalytic site, which allows them to deal with the sheaf $\mathcal{O}^{\mathrm{gev}(s)}_X$ of holomorphic functions of Gevrey growth of type $s \in (1, +\infty)$. However, this new site is not well-fit to use Grothendieck operations. For example, it is not possible to define a direct image f_* in general. S. Guillermou and P. Schapira nonetheless explain how it possible to solve the problem if one assumes that f is a submersion. Since the Fourier-Sato functors only involve submersions, it seems that one can adapt them to this new framework in order to study Gevrey growth conditions. The adaptation of Theorem 5.1.10 will of course be the key point and will highly depend on the behaviour of the new sheaves with respect to pullback and integration over the fibers.

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List of symbols

General notations

$\mathbb{N}$	The set of positive integers
$\mathbb{Z}$	The ring of integers
$\mathbb{R}$	The field of real numbers
P	The one-point compactification $\mathbb{R} \cup \{\infty\}$
$\overline{\mathbb{R}}$	The two-points compactification $\mathbb{R} \cup \{-\infty, +\infty\}$
$\mathbb{R}_\infty$	The bordered space $(\mathbb{R}, \overline{\mathbb{R}})$
$\mathbb{C}$	The field of complex numbers
$\mathbb{C}^*$	The multiplicative group of non-zero complex numbers
$\mathbb{P}$	The one-point compactification $\mathbb{C} \cup \{\infty\}$
$\mathrm{card}(A)$	The cardinal of the set A
$A \backslash B$	The set of elements which are in A and not in B
$A \times B$	The cartesian product of A and B
A°	The interior of A
$\overline{A}$	The closure of A
∂A	The boundary of A : $\overline{A} \backslash A^\circ$
$\varprojlim$	The projective limit
$\varinjlim$	The inductive limit
$\{\mathrm{pt}\}$	A set of cardinal 1
a_X	The canonical map $X \to \{\mathrm{pt}\}$

$B(a,r)$	The open ball of center a and radius r in a normed space
$D(a,r)$	Equivalent notation for $B(a,r)$ in $\mathbb{C}$
$C(a,r)$	Circle of center a and radius r in a normed space
$\mathrm{Ind}(c,z)$	Index of a 1-cycle c at z, cf. Definition 1.3.5

Categories

Op_X	The category of open subsets of a topological space X
$\mathrm{Mod}(\mathscr{R})$	The category of sheaves of $\mathscr{R}$-modules
$\mathrm{D}^*(\mathscr{R})$	The derived category of $\mathrm{Mod}(\mathscr{R})$ $(* = \emptyset, +, -, \mathrm{b})$
$\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$	The bounded derived category of left $\mathscr{D}_X$-modules with holonomic cohomologies on a complex manifold X
$\mathrm{D}^{\mathrm{b}}_{\mathrm{q\text{-}good}}(\mathscr{D}_X)$	The bounded derived category of left $\mathscr{D}_X$-modules with quasi-good cohomologies on a complex manifold X
$\mathrm{Op}^{\mathrm{sub},c}_M$	The category of open relatively compact subsets of a subanalytic space M
$\mathrm{Mod}(\mathbb{C}^{\mathrm{sub}}_M)$	The category of subanalytic sheaves on M
$\mathrm{D}^*(\mathbb{C}^{\mathrm{sub}}_M)$	The derived category of $\mathrm{Mod}(\mathbb{C}^{\mathrm{sub}}_M)$ $(* = \emptyset, +, -, \mathrm{b})$
$\mathrm{D}^{\mathrm{b}}(\mathscr{D}_{X_\infty})$	Denotes the category $\mathrm{D}^{\mathrm{b}}(\mathscr{D}_{\widehat{X}})/\{\mathcal{M} : \mathrm{supp}(\mathcal{M}) \subset \widehat{X}\backslash X\}$ on a bordered space $X_\infty = (X, \widehat{X})$
$\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_{X_\infty})$	$\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_{\widehat{X}})/\{\mathcal{M} : \mathrm{supp}(\mathcal{M}) \subset \widehat{X}\backslash X\}$
$\mathrm{D}^{\mathrm{b}}_{\mathrm{q\text{-}good}}(\mathscr{D}_{X_\infty})$	$\mathrm{D}^{\mathrm{b}}_{\mathrm{q\text{-}good}}(\mathscr{D}_{\widehat{X}})/\{\mathcal{M} : \mathrm{supp}(\mathcal{M}) \subset \widehat{X}\backslash X\}$
$\mathrm{Op}^{\mathrm{sub},c}_{M_\infty}$	On a subanalytic bordered space $M_\infty = (M, \widehat{M})$, the category of open subsets of M which are relatively compact in $\widehat{M}$
$\mathrm{Mod}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$	The category of subanalytic sheaves on M_∞
$\mathrm{D}^*(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$	The derived category of $\mathrm{Mod}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$ $(* = \emptyset, +, -, \mathrm{b})$
$\mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$	The category of enhanced subanalytic sheaves on M_∞, cf. Definition 4.5.4

Functors and (co)homologies

$\Gamma(U, -)$	Functor of sections, for sheaves, on an open set U

$\Gamma_Z(-), \Gamma_c(-)$ Functor of sections supported by Z with Z locally closed (resp. sections compactly supported)

$f_*, f^{-1}, f_!, \otimes, \mathcal{H}om$ Grothendieck operations for sheaves

$\mathrm{R}f_*, \mathrm{R}f^{-1}, \mathrm{R}f_!, f^!, \overset{\mathrm{L}}{\otimes}, \mathrm{R}\mathcal{H}om$ Derived Grothendieck operations for sheaves

$\mathrm{D}f_*, \mathrm{D}f^*, \mathrm{D}f_!, \overset{\mathrm{D}}{\otimes}$ Grothendieck operations for $\mathscr{D}$-modules

ι_M The canonical inclusion functor $\mathrm{Mod}(\mathbb{C}_M) \to \mathrm{Mod}(\mathbb{C}_M^{\mathrm{sub}})$ where M is a subanalytic space

α_M, β_M The canonical left adjoint of ι_M (resp. left adjoint of α_M)

$f_*, f^{-1}, f_{!!}, \otimes, \mathscr{I}hom$ Grothendieck operations for subanalytic sheaves

$\mathscr{I}\Gamma_Z(-)$ Internal functor of sections supported by Z for subanalytic sheaves

Q_M Quotient functor $\mathrm{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty \times \mathbb{R}_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$ where M_∞ is a subanalytic bordered space

$\mathrm{L}^{\mathrm{E}}, \mathrm{R}^{\mathrm{E}}$ The canonical left (resp. right) adjoint of Q_M

$\mathrm{E}f_*, \mathrm{E}f^{-1}, \mathrm{E}f_{!!}, \mathrm{E}f^!, \overset{+}{\otimes}, \mathscr{I}hom^+$ Grothendieck operations in $\mathrm{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{M_\infty})$

$\mathrm{R}\mathscr{I}hom^{\mathrm{E}}(-,-)$ Abbreviation for the functor $\mathrm{R}\pi_{M*}\mathrm{R}\mathscr{I}hom(\mathrm{R}^{\mathrm{E}}(-), \mathrm{R}^{\mathrm{E}}(-))$, where $\pi_M : M_\infty \times \mathbb{R}_\infty \to M_\infty$ is the projection

$\mathrm{R}\mathcal{H}om^{\mathrm{E}}(-,-)$ Stands for $\alpha_M \circ \mathrm{R}\mathscr{I}hom^{\mathrm{E}}(-,-)$

$\mathcal{DR}^{\mathrm{E}}_{X_\infty}, \mathcal{S}ol^{\mathrm{E}}_{X_\infty}$ Enhanced de Rham functor (resp. solution functor) on a complex bordered space X_∞, cf. Definition 4.6.6

$[n]$ Shift functor by $n \in \mathbb{Z}$

H^k Cohomological functor of degree k

${}^{\mathrm{BM}}H_k(X)$ Borel-Moore homology of degree k of a topological locally compact space X, cf. Definition 1.3.3

$H_k(X)$ Singular homology of degree k of X

$H_k(X, A)$ Relative singular homology of degree k of X with respect to A

$[X]$ Canonical orientation class of ${}^{\mathrm{BM}}H_n(X)$, if X is an oriented topological manifold of pure dimension n, cf. Definition 1.3.4

$[X]_K$ Canonical relative orientation class of $H_n(X, X\backslash K)$ if K is a compact subset of X

Sheaves

A_X	Constant sheaf of fiber A on a topological space X
ω_X	Orientation complex of X : $a_X^! \mathbb{Z}_{\{\mathrm{pt}\}}$
or_X	Abbreviation of $H^{-n}(\omega_X)$ on an oriented topological manifold of pure dimension n
$\mathcal{C}^r_{\infty,M}, \mathcal{D}b^r_M$	Sheaf of infinitely differentiable complex differential r-forms (resp. distributional r-forms) on a real manifold M
$\mathcal{C}^{p,q}_{\infty,X}, \mathcal{D}b^{p,q}_X$	Sheaf of infinitely differentiable complex differential (p, q)-forms (resp. distributional (p, q)-forms) on a complex manifold X
Ω^p_X	Sheaf of holomorphic p-forms on X ($\mathcal{O}_X = \Omega^0_X$ and $\Omega_X = \Omega^{d_X}_X$)
$\Omega_X^{\otimes -1}$	The inverse sheaf $\mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$ of Ω_X
$\mathscr{D}_X$	Sheaf of linear partial differential operators with holomorphic coefficients on X
$\mathscr{D}_{X\to Y}, \mathscr{D}_{X\leftarrow Y}$	Transfer bimodules associated with a holomorphic map $f : X \to Y$
$\mathcal{C}^{\mathrm{t},r}_{\infty,M}, \mathcal{D}b^{\mathrm{t},r}_M$	Subanalytic sheaf of tempered differential r-forms (resp. tempered distributional r-forms) on a real analytic manifold M, cf. Definitions 4.3.1 and 4.3.5
$e^{-f}\, \mathcal{D}b^{\mathrm{t},r}_{M_\infty}$	Cf. Definition 5.2.4
$\mathcal{C}^{\mathrm{t},p,q}_{\infty,X}, \mathcal{D}b^{\mathrm{t},p,q}_X$	Subanalytic sheaf of tempered differential (p, q)-forms (resp. tempered distributional (p, q)-forms) on a complex manifold X
$\Omega^{\mathrm{t},p}_X$	Complex of tempered holomorphic p-forms on X ($\mathcal{O}^{\mathrm{t}}_X = \Omega^{\mathrm{t},0}_X$ and $\Omega^{\mathrm{t}}_X = \Omega^{\mathrm{t},d_X}_X$)
$\mathcal{O}_X(*Y)$	Sheaf of holomorphic functions with poles in Y, where Y is a complex analytic hypersurface of X
$\mathscr{D}_X e^{\varphi}$	The sheaf $\mathscr{D}_X/\{P : Pe^{\varphi} = 0 \text{ on } X\backslash Y\}$ with $\varphi \in \mathcal{O}_X(*Y)$
$\mathscr{E}^{\varphi}_{U\|X}$	Abbreviation for $\mathscr{D}_X e^{\varphi} \overset{\mathrm{D}}{\otimes} \mathcal{O}_X(*Y)$
$\mathscr{D}_{X_\infty}$	The class of $\mathscr{D}_{\widehat{X}}$ in $\mathrm{D^b}(\mathscr{D}_{\widehat{X}})/\{\mathcal{M} : \mathrm{supp}(\mathcal{M}) \subset \widehat{X}\backslash X\}$ on a complex bordered space $X_\infty = (X, \widehat{X})$
$\mathscr{D}_{X_\infty\to Y_\infty}, \mathscr{D}_{X_\infty\leftarrow Y_\infty}$	Transfer bimodules associated with an extendible morphism of complex bordered spaces $f : X_\infty \to Y_\infty$

$\mathcal{D}b^{\mathrm{T},r}_{M_\infty}$	Subanalytic sheaf of enhanced distributional r-forms on a real analytic bordered space M_∞, cf Definition 4.6.1
$\mathcal{D}b^{\mathrm{T},p,q}_{X_\infty}$	Subanalytic sheaf of enhanced distributional (p,q)-forms on a complex bordered space X_∞
$\Omega^{\mathrm{E},p}_{X_\infty}$	Complex of enhanced holomorphic p-forms on X_∞ ($\mathcal{O}^{\mathrm{E}}_{X_\infty} = \Omega^{\mathrm{E},0}_{X_\infty}$ and $\Omega^{\mathrm{E}}_{X_\infty} = \Omega^{\mathrm{E},d_X}_{X_\infty}$)
$\mathbb{C}_{\{t\mathcal{R}\varphi(x)\}}$	Abbreviation of $\iota_M(\mathbb{C}_{\{(x,t)\in M\times\mathbb{R}\,:\,t\mathcal{R}\varphi(x)\}})$, where $\mathcal{R}$ stands for $=, \leq, \geq, >$ or $<$
$\mathbb{C}^{\mathrm{E}}_{M_\infty}$	The enhanced subanalytic sheaf $Q_M\left(\varinjlim_{a\to+\infty} \mathbb{C}_{\{t\geq a\}}\right)$

Vector spaces

V, V^*	A real vector space V and its real dual
$\mathbb{V}, \mathbb{V}^*$	A complex vector space $\mathbb{V}$ and its complex dual
$\overline{\mathbb{V}}, \overline{\mathbb{V}}^*$	The projective compactification of $\mathbb{V}$ and $\mathbb{V}^*$
$\langle -,-\rangle$	The duality bracket $V\times V^*\to\mathbb{R}$ or $\mathbb{V}\times\mathbb{V}^*\to\mathbb{C}$
$L_{\mathbb{V}}, L^a_{\mathbb{V}}$	The Laplace kernels on $\mathbb{V}$, cf. Definition 5.1.4
${}^{\mathrm{E}}\mathcal{F}_{\mathbb{V}}$, ${}^{\mathrm{E}}\mathcal{F}^a_{\mathbb{V}}$	The enhanced Fourier-Sato functors on $\mathbb{V}_\infty$, cf. Definition 5.1.5
C, C^*	A cone and its polar cone
$\mathrm{Conv}(V)$	The set of closed proper convex functions on V
$\mathrm{dom}(f)$	The domain, $f^{-1}(\mathbb{R})$, of $f\in\mathrm{Conv}(V)$
f^*	The Legendre transform of $f\in\mathrm{Conv}(V)$, cf. Definition 1.5.1
h_S	The support function of a convex set $S\subset V$, cf. Definition 1.5.2
S_ε	The thickening of S by $\varepsilon>0$
S_∞	The asymptotic cone of S, cf. Definition 1.5.9

Functional spaces

$\mathcal{H}(\Omega)$	$\mathcal{O}(\Omega)$ if $\Omega\subset\mathbb{C}$ and $\{f\in\mathcal{O}(\Omega) : f(\infty)=0\}$ if $\infty\in\Omega\subset\mathbb{P}$
$\mathcal{O}'(K)$	The space of analytic functionals carried by the compact subset K of $\mathbb{C}$, i.e. the strong topological dual of $\varinjlim_{U\supset K}\mathcal{O}(U)$

$\mathcal{O}^0(\mathbb{C}\setminus K)$	The space $\{f \in \mathcal{O}(\mathbb{C}\setminus K) : \lim_{z\to\infty} f(z) = 0\}$
$\operatorname{Exp}(K)$	The space $\{g \in \mathcal{O}(\mathbb{C}) : \forall\varepsilon > 0,\ \sup_{w\in\mathbb{C}}\lvert g(w)\rvert e^{-h_K(w)-\varepsilon\lvert w\rvert} < \infty\}$
$Q_{\varepsilon,\varepsilon'}(S)$	The space $\{\varphi \in \mathcal{O}(S^\circ_\varepsilon) : \sup_{\zeta\in S^\circ_\varepsilon}\lvert e^{-\varepsilon'\xi_0\zeta}\varphi(\zeta)\rvert < \infty\}$ where S is a proper non-compact closed convex subset of $\mathbb{C}$ which contains no lines and ξ_0 a point of $(S^*_\infty)^\circ$
$Q(S)$	The space $\varinjlim_{\varepsilon,\varepsilon'\to 0} Q_{\varepsilon,\varepsilon'}(S)$
$Q'(S)$	The space of analytic functionals carried by S, i.e. the strong topological dual of $Q(S)$
$R(\mathbb{C}\setminus S,\varepsilon')$	The space $\{f \in \mathcal{O}(\mathbb{C}\setminus S) : \forall r > \varepsilon > 0,\ \sup_{z\in S_r\setminus S^\circ_\varepsilon}\lvert e^{\varepsilon'\xi_0 z} f(z)\rvert < \infty)\}$
$R(\mathbb{C},\varepsilon')$	The space $\{f \in \mathcal{O}(\mathbb{C}) : \forall r > 0,\ \sup_{z\in S_r}\lvert e^{\varepsilon'\xi_0 z} f(z)\rvert < \infty)\}$
$\mathscr{H}_S(\mathbb{C},\varepsilon')$	The space $R(\mathbb{C}\setminus S,\varepsilon')/R(\mathbb{C},\varepsilon')$
$\mathscr{H}_S(\mathbb{C})$	The space $\varprojlim_{\varepsilon'\to 0}\mathscr{H}_S(\mathbb{C},\varepsilon')$
$\operatorname{Exp}(S)$	The space $$\{g \in \mathcal{O}((S^*_\infty)^\circ) : \forall\varepsilon,\varepsilon' > 0,\ \sup_{w\in S^\star_\infty+\varepsilon'\xi_0}\lvert g(w)\rvert e^{-h_S(w)-\varepsilon\lvert w\rvert} < \infty\}$$
$\mathscr{H}^{\mathrm{t}}_S(\mathbb{V},\varepsilon')$	The space $$\frac{\{u \in \Omega_{\mathbb{V}}(\mathbb{V}\setminus S) : \forall r > \varepsilon > 0, u \in e^{-\langle z,\varepsilon'\xi_0\rangle}\,\mathcal{D}b^{\mathrm{t},1,0}_{\mathbb{P}}(S^\circ_r\setminus S_\varepsilon)\}}{\{u \in \Omega_{\mathbb{V}}(\mathbb{V}) : \forall r > 0, u \in e^{-\langle z,\varepsilon'\xi_0\rangle}\,\mathcal{D}b^{\mathrm{t},1,0}_{\mathbb{P}}(S^\circ_r)\}},$$ where $\mathbb{V}$ is a complex vector space of dimension 1 and $\mathbb{P}$ its one-point compactification
$\operatorname{Exp}^{\mathrm{t}}_{\varepsilon,\varepsilon'}(S)$	The space $e^{h_{S_\varepsilon}(w-\varepsilon'\xi_0)}\,\mathcal{O}^{\mathrm{t}}_{\mathbb{P}^*}((S^\star_\infty)^\circ + \varepsilon'\xi_0)$
$\operatorname{Exp}^{\mathrm{t}}_{\varepsilon'}(S)$	The space $\varprojlim_{\varepsilon\to 0}\operatorname{Exp}^{\mathrm{t}}_{\varepsilon,\varepsilon'}(S)$

Particular transformations

$\int_f \omega, \int_f u$	Integration of an infinitely differentiable form ω (resp. a distributional form u) along the fibers of f, cf. Definition 1.4.1 and Proposition 1.4.2
$f^*\omega, f^*u$	Pullback of an infinitely differentiable form ω (resp. a distributional form u) by f, cf. Definition 1.4.4

$u_1 \boxtimes u_2$	External tensor product of two distributions u and v, cf. Definition 2.2.3
$\star_{(G,\mu)}$	The holomorphic cohomological convolution on a complex Lie group (G,μ), cf. Definition 2.2.6
$\star^{\mathrm{t}}_{\mathbb{V}_\infty}$	The tempered holomorphic cohomological convolution on a bordered complex vector space $\mathbb{V}_\infty = (\mathbb{V}, \overline{\mathbb{V}})$, cf. Definition 5.3.2
$\star$	Indifferently the Hadamard product (cf. Definition 2.1.1), the extended Hadamard product of T. Pohlen (cf. Definition 2.1.8), the generalized Hadamard product (cf. Definition 2.1.13), the additive (resp. multiplicative) holomorphic cohomological convolution on $\mathbb{C}$ (resp. $\mathbb{C}^*$), the convolution of distributions and the convolution of analytic functionals (cf. Definitions 3.1.13 and 3.2.22)
μ_φ	The map $(x,t) \mapsto (x, t+\varphi(x))$
$\mathcal{C}$	The Cauchy transform, cf. Definition 3.1.6 and Proposition 3.2.12
$\mathcal{F}$	The Fourier-Borel transform, cf. Definition 3.1.4 and Proposition 3.2.13
$\mathcal{L}^+$	The positive Laplace transform (of tempered distributions)
$\mathcal{P}$	The Polya transform, cf. Definition 3.1.8 and Proposition 3.2.14
$\mathcal{B}$	The Borel transform $\mathcal{P}^{-1}$, cf. Propositions 3.1.11 and 3.2.16

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