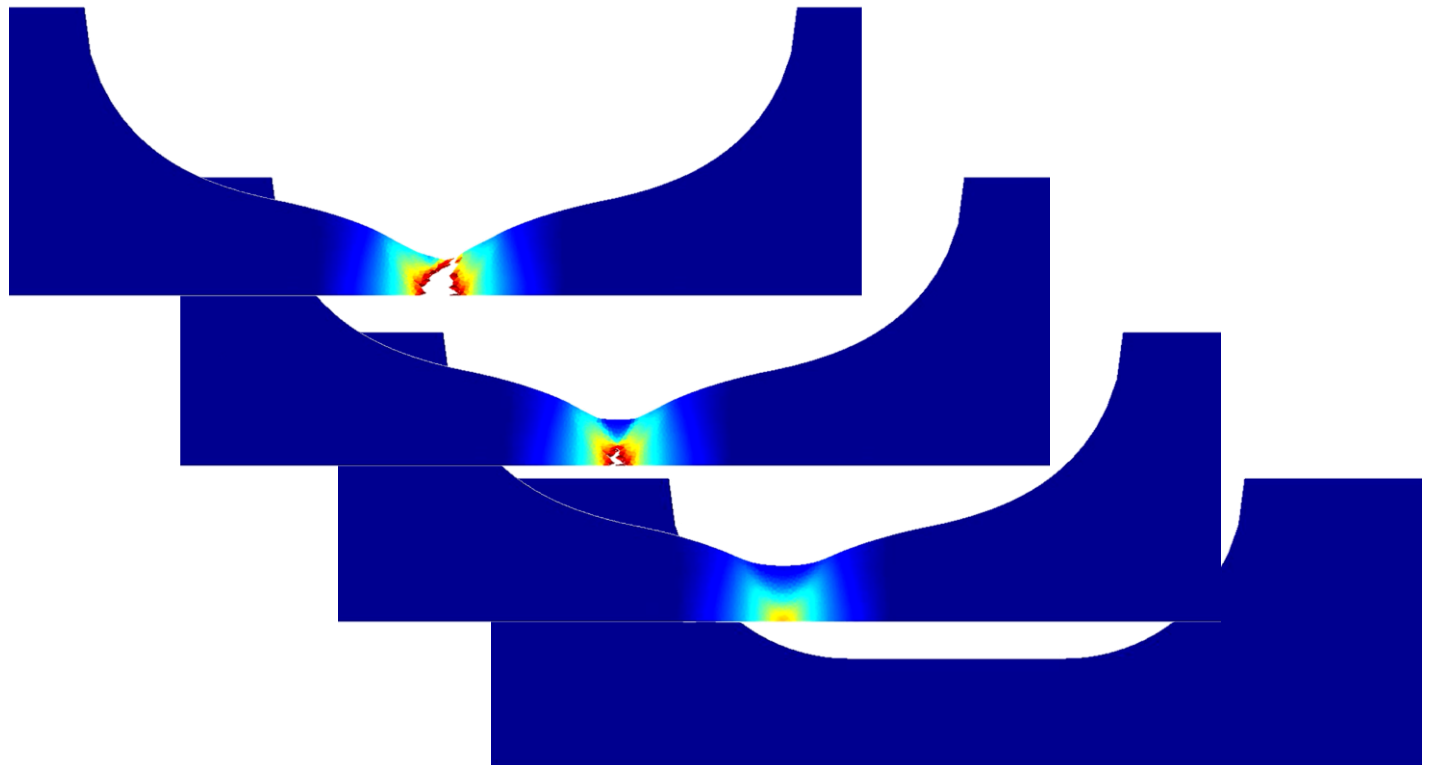


Damage to crack transition for ductile materials using a  
cohesive-band /discontinuous Galerkin framework

*Julien Leclerc, Ling Wu, Van-Dung Nguyen, Ludovic Noels*

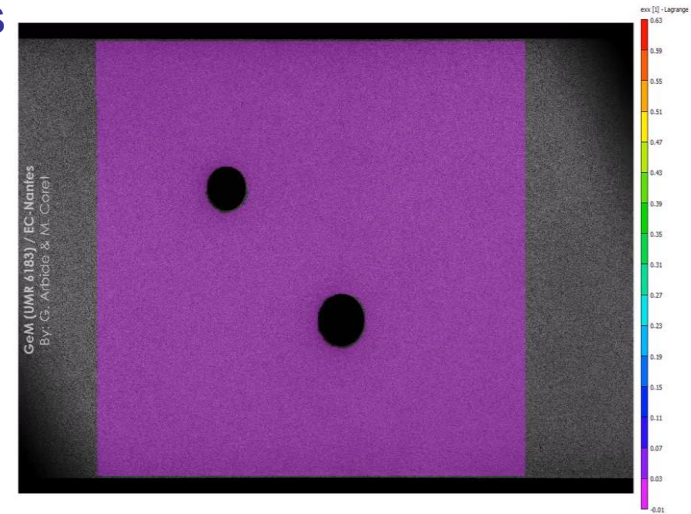


*The research has been funded by the Walloon Region under the agreement  
no.7581-MRIPF in the context of the 16<sup>th</sup> MECATECH call.*

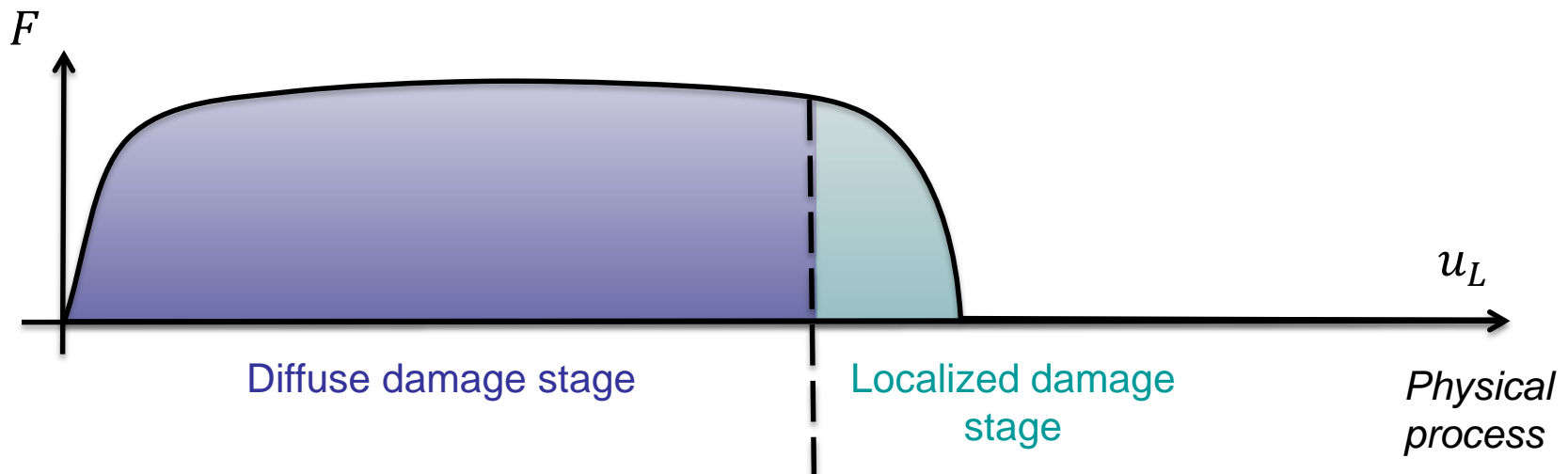


# Introduction

- Goal: To capture the whole ductile failure process
  - A diffuse stage
    - Damage onset / nucleation, growth...
  - Followed by a localized stage
    - Damage coalescence
    - Crack initiation and propagation
    - ...



[<http://radome.ec-nantes.fr/>]



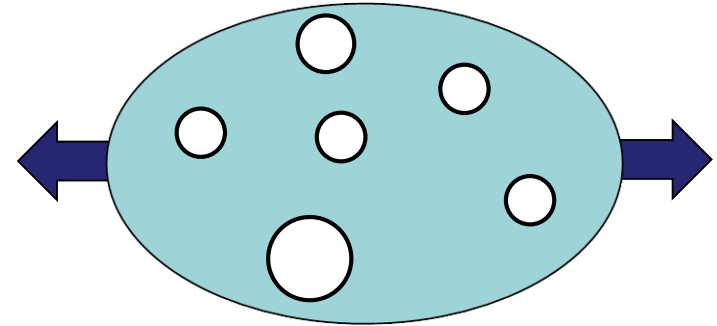
# State of art: Modeling approaches

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- State-of-the-art
  - 2 approaches modeling material failure:

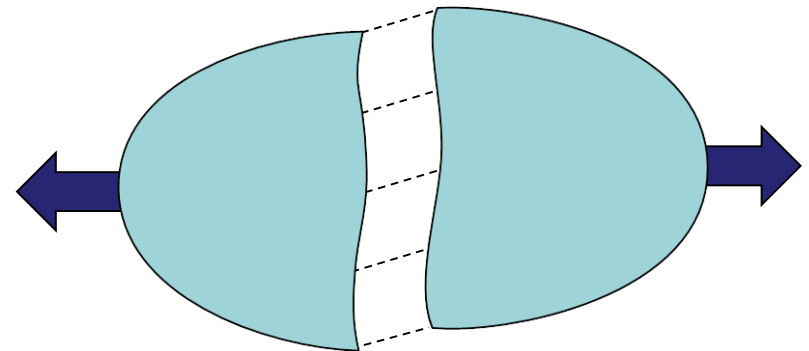
- Continuum Damage Models (CDM)

- Lemaitre-Chaboche,
- Gurson,
- ...



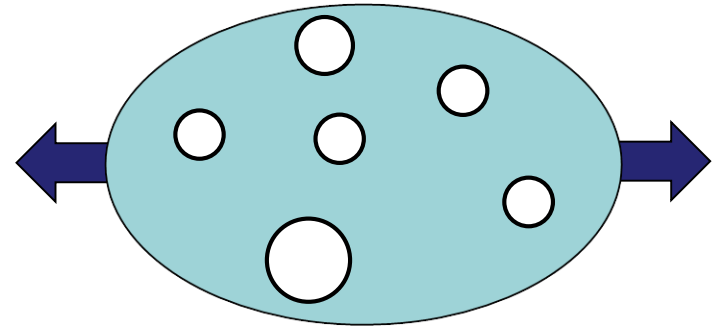
- Discontinuous: Fracture mechanics

- Cohesive zone,
- XFEM
- ...



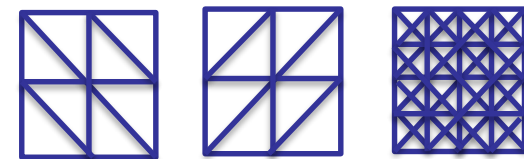
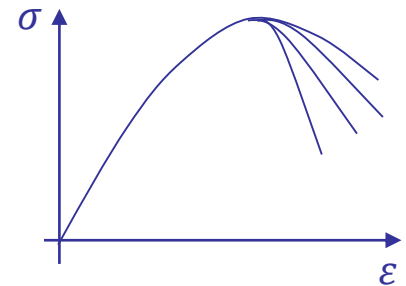
# State of art: Continuous approaches

- Material properties degradation modelled through internal variables evolution
  - Form  $\sigma(\boldsymbol{\varepsilon}; Z(t'))$ 
    - With internal variables history  $Z(t')$
  - Lemaitre-Chaboche model,
  - Gurson model,
    - Porosity evolution  $f_V$



- Continuous Damage Model (CDM) implementation:

- Local form
  - Mesh-dependency
- Non-local form needed [Bažant 1988]
  - Introduction of a characteristic length  $l_c$
  - In terms of Weight functions:  $\tilde{f}_V(\mathbf{x}) = \int_{V_c} W(\mathbf{y}; \mathbf{x}, l_c) f_V(\mathbf{y}) d\mathbf{y}$
- Implicit formulation [Peerlings et al. 1998]
  - New non-local degrees of freedom  $\tilde{f}_V$
  - New Helmholtz type equation to be solved  $\tilde{f}_V - l_c^2 \Delta \tilde{f}_V = f_V$

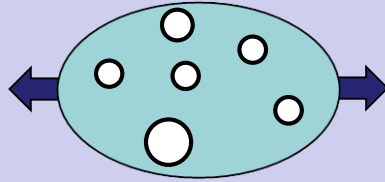


*The numerical results change without convergence*

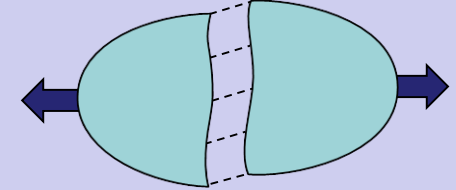
# State of art: Comparison (1)

**Continuous:**

**Continuous Damage Model (CDM)**



**Discontinuous:**

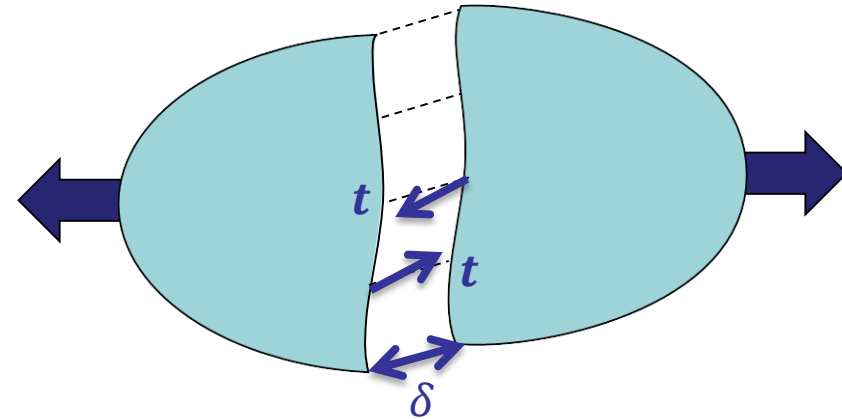


- + Capture the **diffuse damage stage**
- + Capture stress **triaxiality** and **Lode** variable effects
- **Mesh dependency without (implicit) non-local form**
- **Numerical problems** with highly damaged elements
- **Cannot represent cracks** without remeshing / element deletion at  $D \rightarrow 1$  (loss of accuracy, mesh modification ...)
- Crack initiation observed for lower damage values

# State of art: Discontinuous approaches

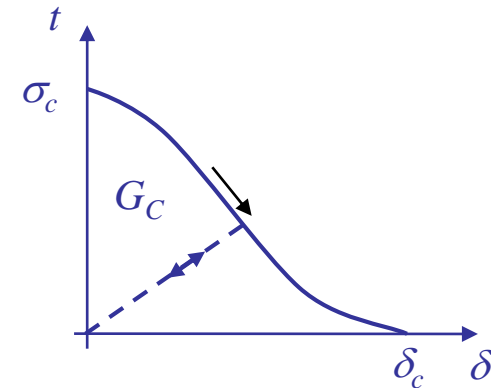
- Based on fracture mechanics concepts

- Characterized by
  - Strength  $\sigma_c$  &
  - Critical energy release rate  $G_C$



- One of the most used methods:

- Cohesive Zone Model (CZM) modelling the crack tip behavior
- Integrate a Traction Separation Law (TSL):
  - At interface elements between two elements
  - Using element enrichment (EFEM) [Armero et al. 2009]
  - Using mesh enrichment (xFEM) [Moes et al. 2002]
  - ...



# State of art: Discontinuous approaches

- Cohesive elements

- Inserted between volume elements

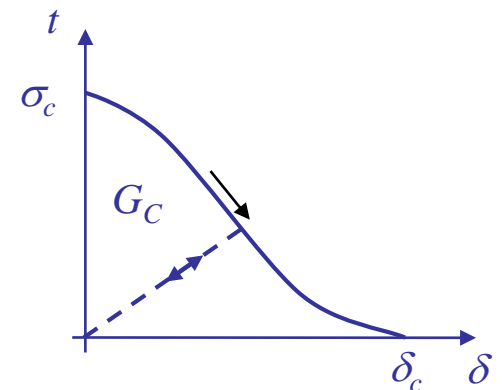
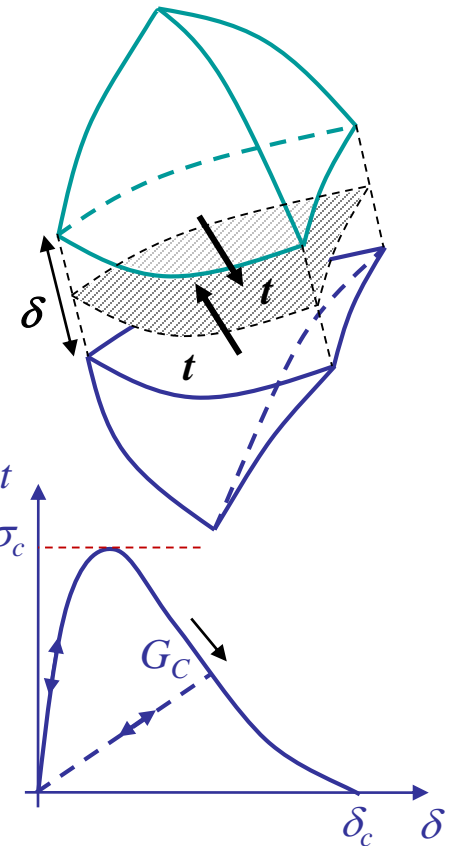
- Zero-thickness  $\Rightarrow$  no triaxiality accounted for

- Intrinsic Cohesive Law (ICL)

- Cohesive elements inserted from the beginning
- Efficient if a priori knowledge of the crack path
- Mesh dependency [Xu & Needleman, 1994]
- Initial slope modifies the effective elastic modulus
- This slope should tend to infinity [Klein et al. 2001]:
  - Alteration of a wave propagation
  - Critical time step is reduced

- Extrinsic Cohesive Law (ECL)

- Cohesive elements inserted on the fly when the failure criterion is verified [Ortiz & Pandolfi 1999]
- Complex implementation in 3D (parallelization)



# State of art: Discontinuous approaches

- Hybrid framework [Radovitzky et al. 2011]
  - Discontinuous Galerkin (DG) framework
    - Test and shape functions discontinuous
    - Consistency, convergence rate, uniqueness** recovered though interface terms

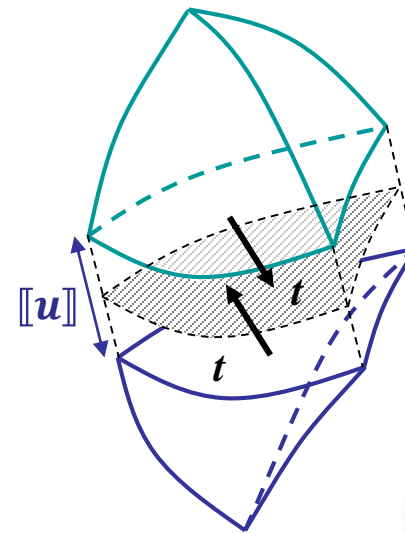
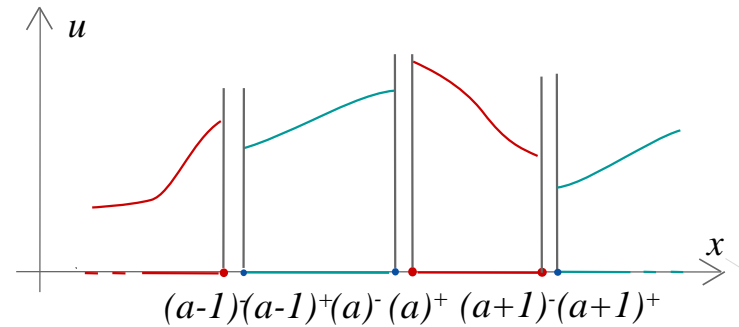
$$\int_{\Omega_0} \mathbf{P} : \nabla_0 \delta \mathbf{u} d\Omega +$$

$$\int_{\partial_I \Omega_0} [[\delta \mathbf{u}]] \cdot \langle \mathbf{P} \rangle \cdot \mathbf{N}^- d\partial\Omega +$$

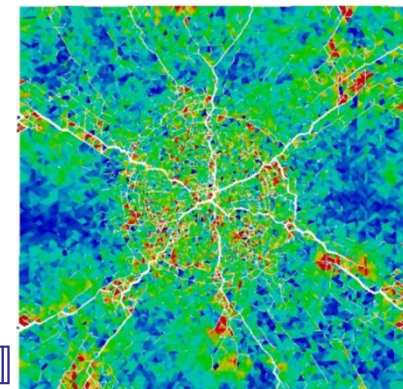
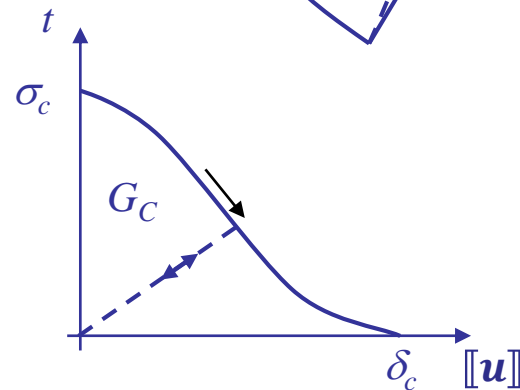
$$\int_{\partial_I \Omega_0} [[\mathbf{u}]] \cdot \langle \mathbf{C}^{el} : \nabla_0 \delta \mathbf{u} \rangle \cdot \mathbf{N}^- d\partial\Omega +$$

$$\int_{\partial_I \Omega_0} [[\mathbf{u}]] \otimes \mathbf{N}^- : \langle \frac{\beta_s \mathbf{C}^{el}}{h^s} \rangle : [[\delta \mathbf{u}]] \otimes \mathbf{N}^- d\partial\Omega = 0$$

- Interface terms integrated on interface elements



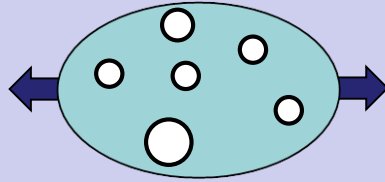
- Combination with extrinsic cohesive laws
  - Interface elements already there
  - Switch to traction separation law
  - Efficient for fragmentation simulations



## State of art: Comparison (2)

### Continuous:

#### Continuous Damage Model (CDM)

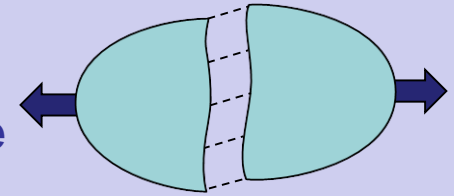


- + Capture the **diffuse damage stage**
- + Capture stress **triaxiality** and **Lode** variable effects

- **Mesh dependency without (implicit) non-local form**
- **Numerical problems** with highly damaged elements
- **Cannot represent cracks** without remeshing / element deletion at  $D \rightarrow 1$  (loss of accuracy, mesh modification ...)
- Crack initiation observed for lower damage values

### Discontinuous:

#### Extrinsic Cohesive Zone Model (CZM)

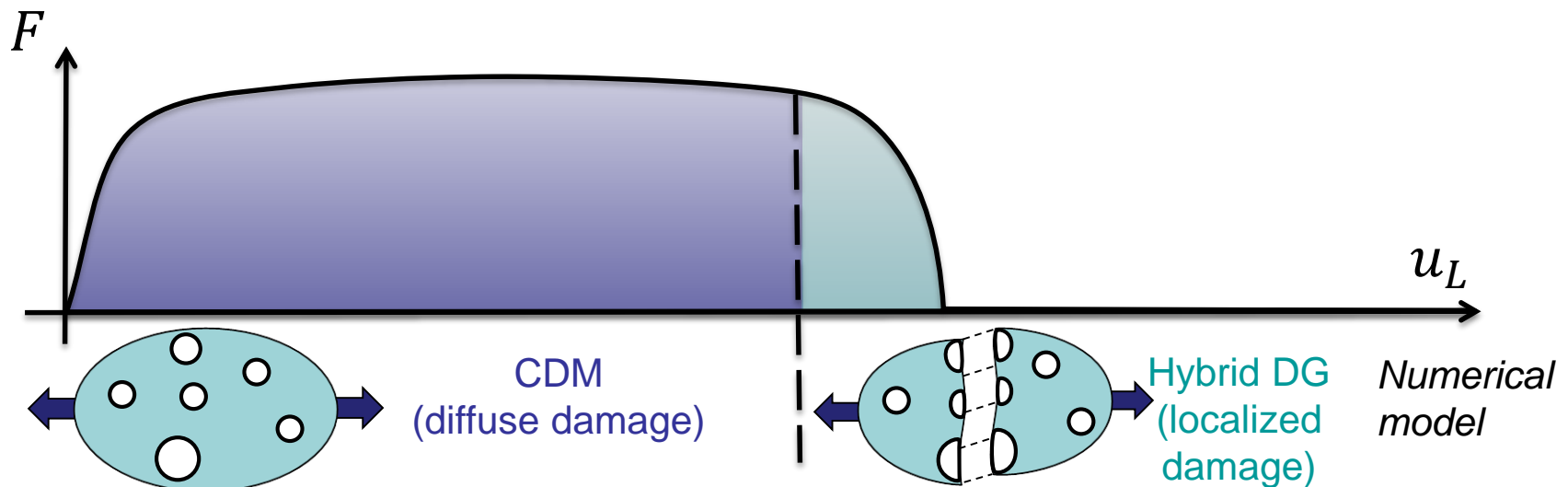


- + **Multiple crack initiation** and propagation naturally managed

- **Cannot capture diffuse damage**
- **No triaxiality effect**
- Currently valid for brittle / small scale yielding elasto-plastic materials

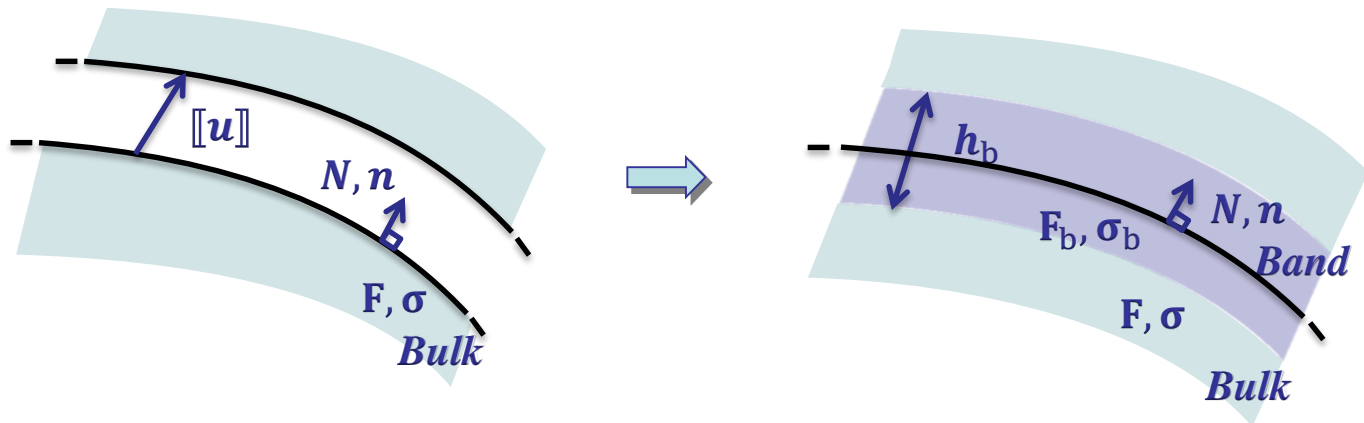
# Goals of research

- Goal:
    - Simulation of the whole ductile failure process with accuracy
  - Main idea:
    - Combination of 2 complementary methods in a single finite element framework:
      - Continuous (non-local damage model)
      - + transition to
      - Discontinuous (Hybrid DG model)
- ⇒ Damage to crack transition
- However triaxiality effects are important ⇒ Cohesive zone model not adequate!!!



# Damage to crack transition – Principles

- Hybrid DG model: use of a Cohesive Band Model (CBM)
  - Hypothesis
    - In the last stage of failure, all damaging process occurs in an uniform thin band
  - Principles
    - Substitute TSL of CZM by the behavior of a uniform band of thickness  $h_b$  [Remmers et al. 2013]



- Methodology [Leclerc et al. 2018]
  1. Compute a band strain tensor  $\mathbf{F}_b = \mathbf{F} + \frac{[[\mathbf{u}]] \otimes \mathbf{N}}{h_b} + \frac{1}{2} \nabla_T [[\mathbf{u}]]$
  2. Compute a band stress tensor  $\boldsymbol{\sigma}_b(\mathbf{F}_b; Z(\tau))$  using the same CDM as bulk elements
  3. Recover a surface traction  $\mathbf{t}([[ \mathbf{u} ]], \mathbf{F}) = \boldsymbol{\sigma}_b \cdot \mathbf{n}$
- What is the effect of  $h_b$  (band thickness)
  - A priori determined with underlying non-local damage model to ensure energy consistency

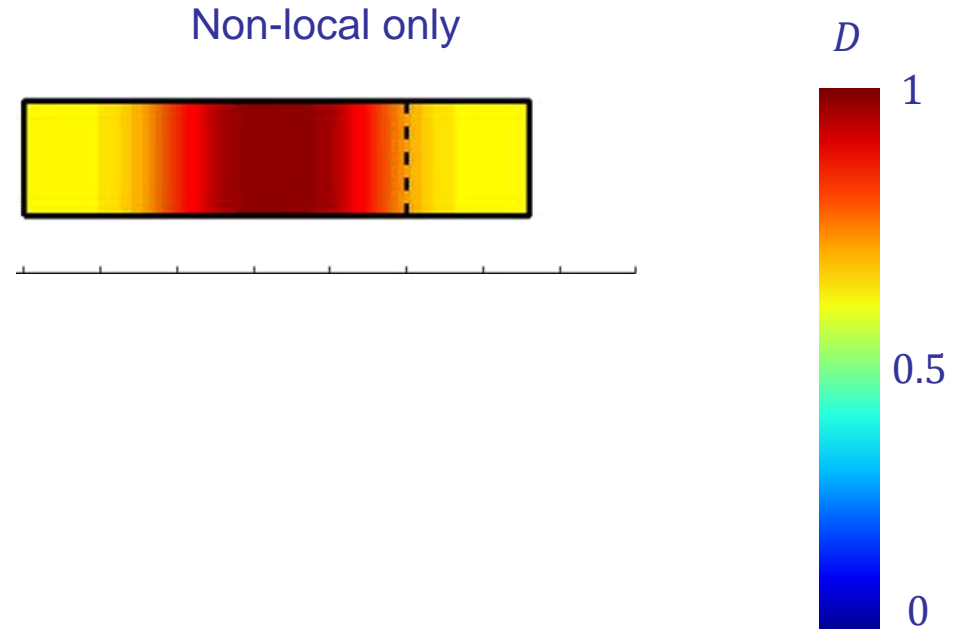
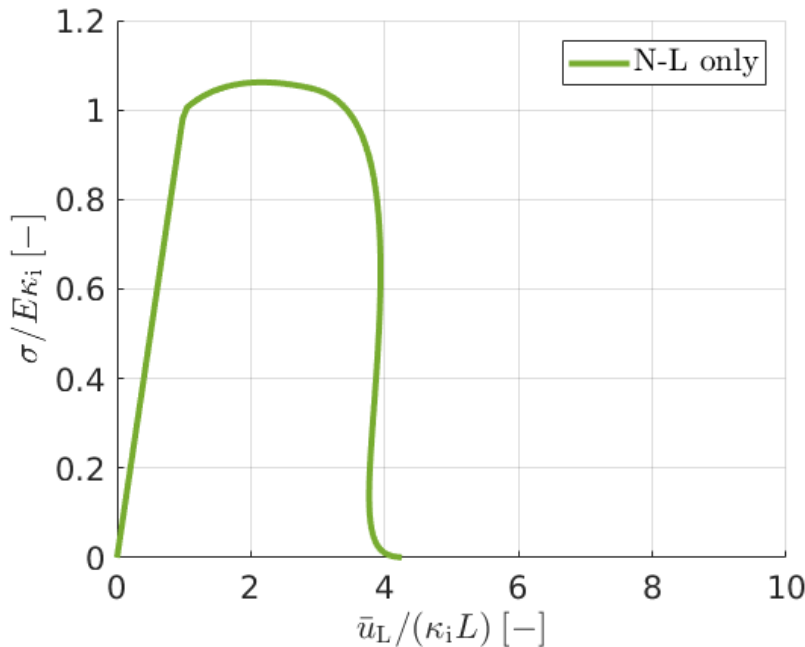
# Damage to crack transition for elastic damage – Proof of concept

- Elastic damage material model

- Constitutive equations

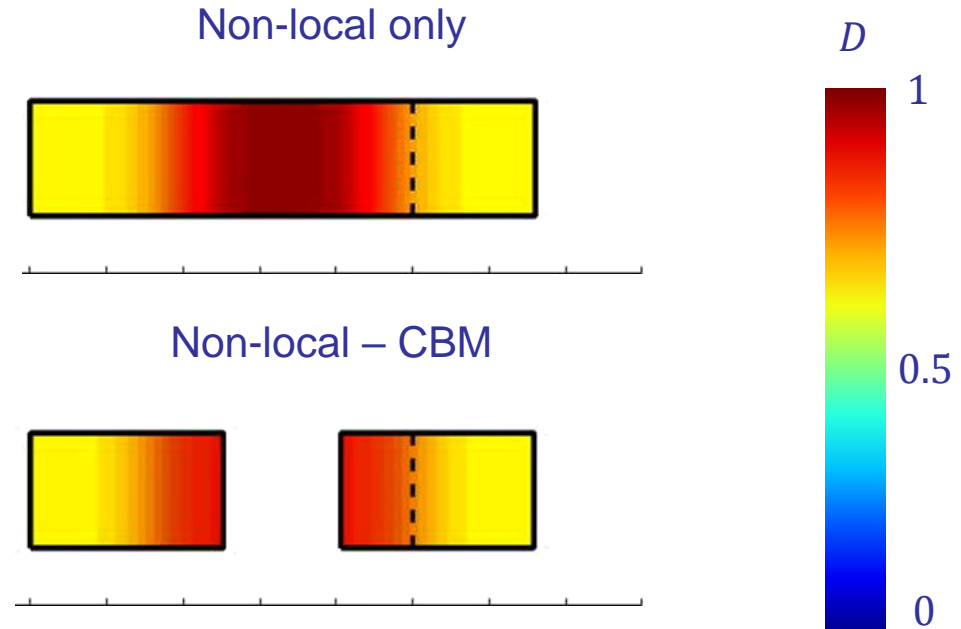
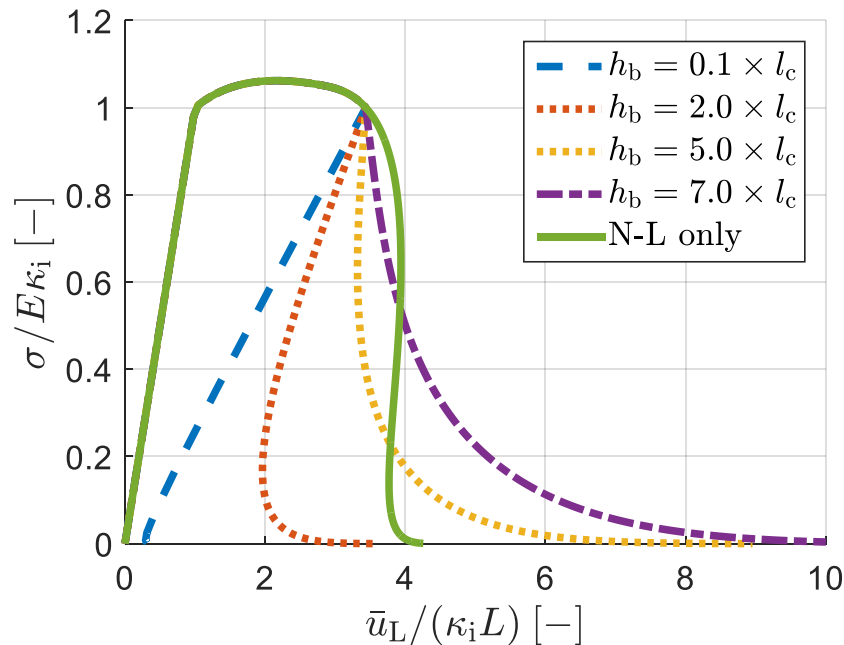
- Helmholtz energy:  $\rho\psi(\boldsymbol{\varepsilon}, D) = \frac{1}{2}(1 - D)\boldsymbol{\varepsilon}:H:\boldsymbol{\varepsilon}$
- Non-local maximum principal strain:  $\tilde{e} - l_c^2\Delta\tilde{e} = e$
- Damage evolution  $\dot{D}(\kappa) = (1 - D)\left(\frac{\beta}{\kappa} + \frac{\alpha}{\kappa_c - \kappa}\right)\dot{\kappa}$  with  $\kappa = \max_{t'} \tilde{e}(t')$

- 1D non-local test



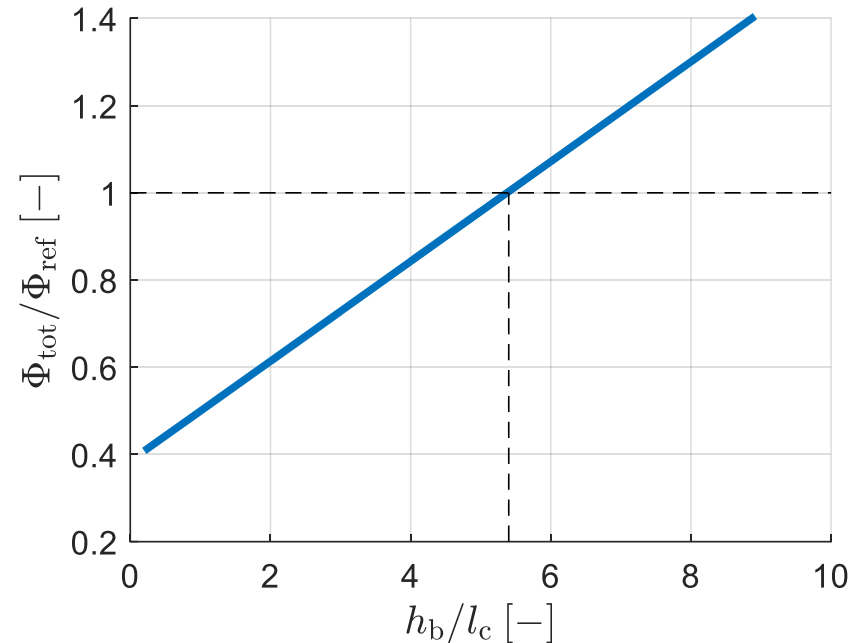
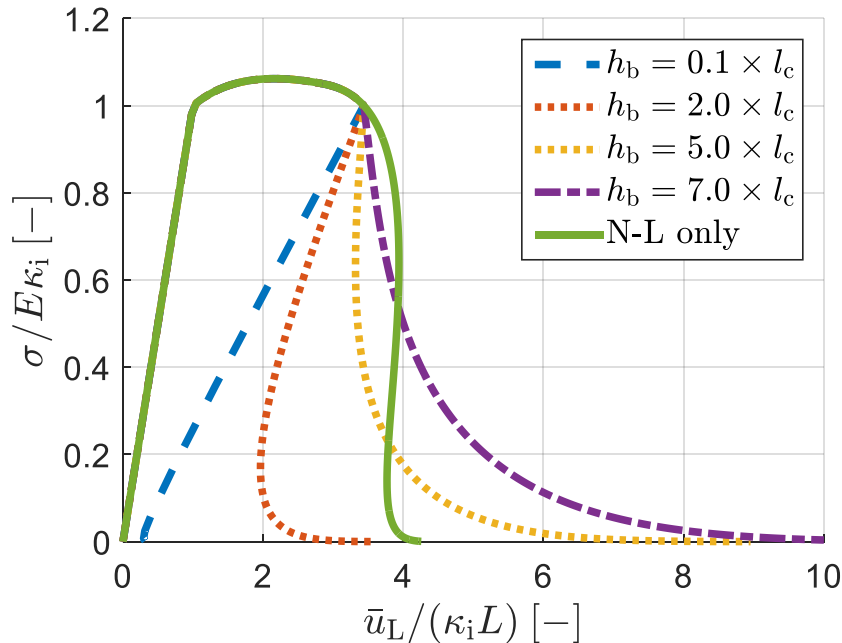
# Damage to crack transition for elastic damage – Proof of concept

- Non-local elastic damage to Cohesive Band Model transition
  - Influence of  $h_b$  (for a given  $l_c$ ) on response in a 1D elastic case [Leclerc et al. 2018]
    - Has effect on the totally dissipated energy  $\Phi$
    - Could be chosen to conserve energy dissipation (physically based)



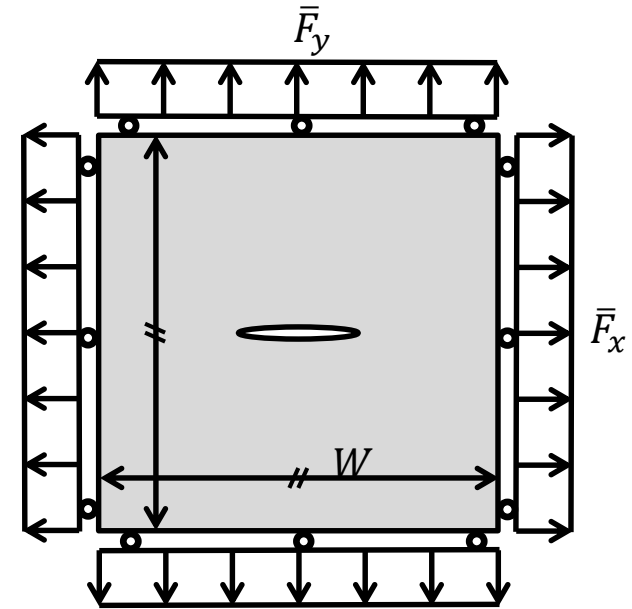
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- Non-local elastic damage to Cohesive Band Model transition
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    - Has effect on the totally dissipated energy  $\Phi$
    - Could be chosen to conserve energy dissipation (physically based)
    - For elastic damage:  $h_b \approx 5.4 l_c$



# Damage to crack transition for elastic damage – Proof of concept

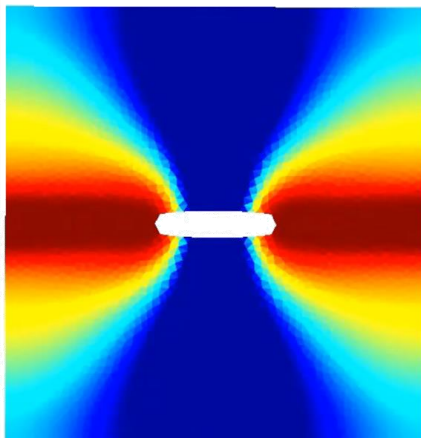
- Study of triaxiality effect on a slit-plate
  - Biaxial loading
    - Loading at constant  $\bar{F}_x/\bar{F}_y$  ratio
    - Plane-strain state
  - Comparison between:
    - Pure non-local
    - Non-local + cohesive zone (CZM)
    - Non-local + cohesive band (CBM)



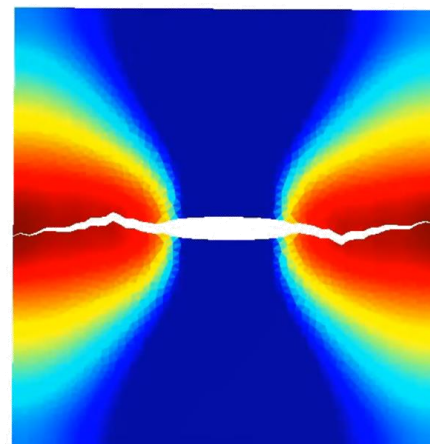
$D$



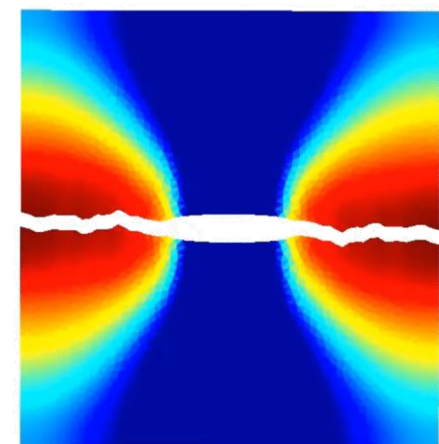
Non-local only



Non-local - CZM



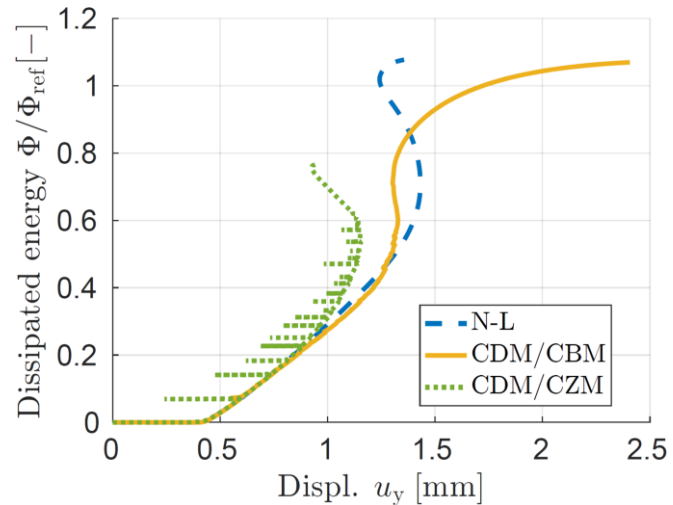
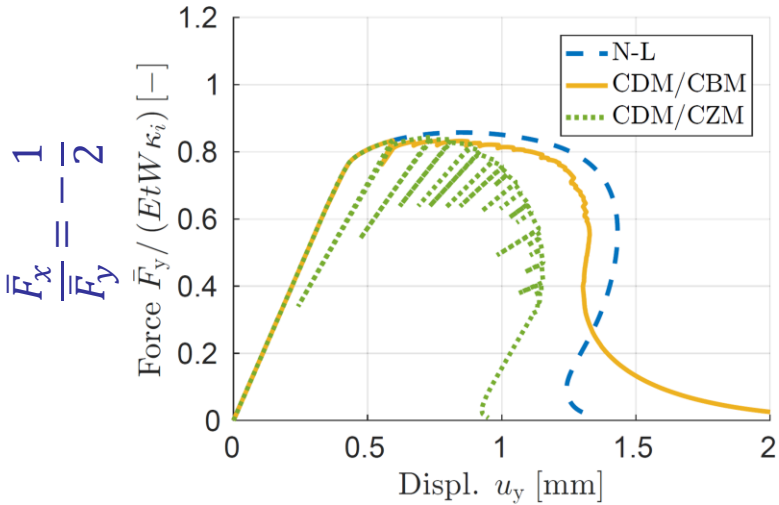
Non-local - CBM



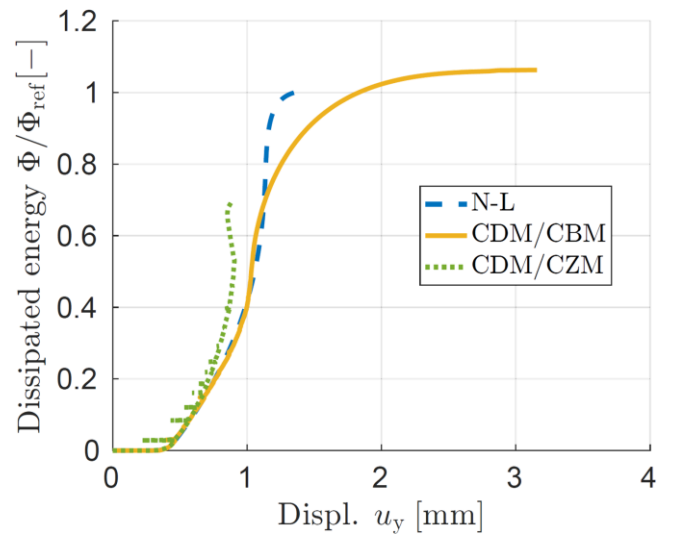
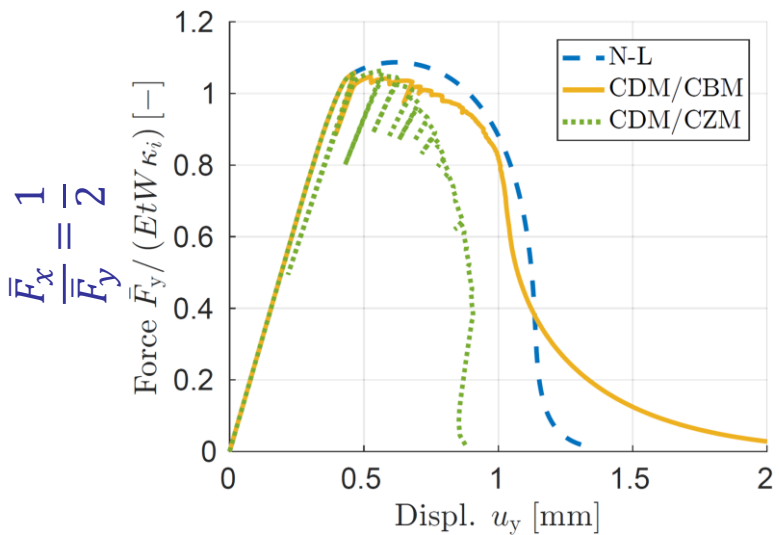
# Damage to crack transition for elastic damage – Proof of concept

- Study of triaxiality effect on a slit-plate

- Reference dissipated energy  $\Phi_{\text{ref}}$  for non-local with  $\bar{F}_x/\bar{F}_y = 0$



Non-Local only —  
 Non-Local - CZM - - -  
 Non-Local - CBM —



*Error on total diss. energy*  
- - - CZM: ~30%  
— CBM: ~3%



# Non-local porous plasticity model

- Hyperelastic-based formulation

- Multiplicative decomposition  
 $\mathbf{F} = \mathbf{F}^e \cdot \mathbf{F}^p$ ,  $\mathbf{C}^e = \mathbf{F}^{eT} \cdot \mathbf{F}^e$ ,  $J^e = \det(\mathbf{F}^e)$

- Stress tensor definition
  - Elastic potential  $\psi(\mathbf{C}^e)$
  - First Piola-Kirchhoff stress tensor

$$\mathbf{P} = 2\mathbf{F}^e \cdot \frac{\partial \psi(\mathbf{C}^e)}{\partial \mathbf{C}^e} \cdot \mathbf{F}^{p-T}$$

- Kirchhoff stress tensors
  - In current configuration  
 $\boldsymbol{\kappa} = \mathbf{P} \cdot \mathbf{F}^T = 2\mathbf{F}^e \cdot \frac{\partial \psi(\mathbf{C}^e)}{\partial \mathbf{C}^e} \cdot \mathbf{F}^{eT}$

- In co-rotational space  
 $\boldsymbol{\tau} = \mathbf{C}^e \cdot \mathbf{F}^{e-1} \cdot \boldsymbol{\kappa} \cdot \mathbf{F}^{e-T} = 2\mathbf{C}^e \cdot \frac{\partial \psi(\mathbf{C}^e)}{\partial \mathbf{C}^e}$

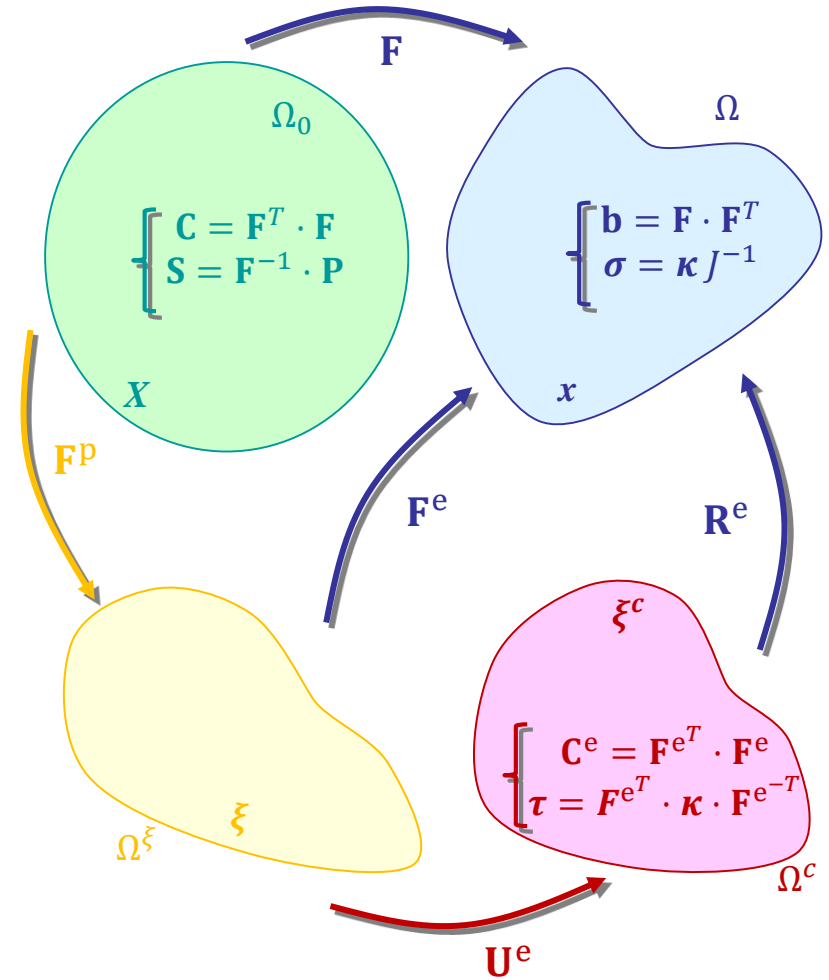
- Logarithmic deformation

- Elastic potential  $\psi$ :  

$$\psi(\mathbf{C}^e) = \frac{K}{2} \ln^2(J^e) + \frac{G}{4} (\ln(\mathbf{C}^e))^{\text{dev}} : (\ln(\mathbf{C}^e))^{\text{dev}}$$

- Stress tensor in co-rotational space

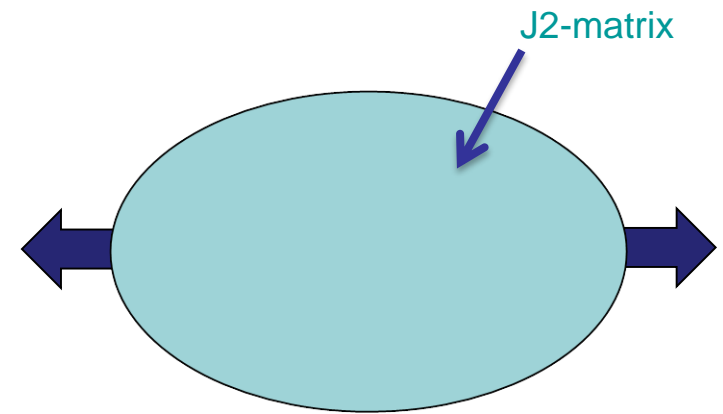
$$\boldsymbol{\tau} = \underbrace{K \ln(J^e)}_p \mathbf{I} + G (\ln(\mathbf{C}^e))^{\text{dev}}$$



# Damage to crack transition in porous elasto-plasticity

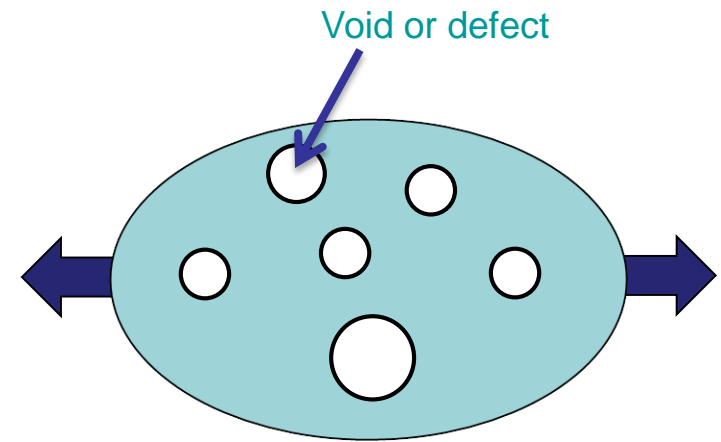
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- Porous plasticity (or Gurson) approach
  - Assuming a  $J_2$ -(visco-)plastic matrix



# Damage to crack transition in porous elasto-plasticity

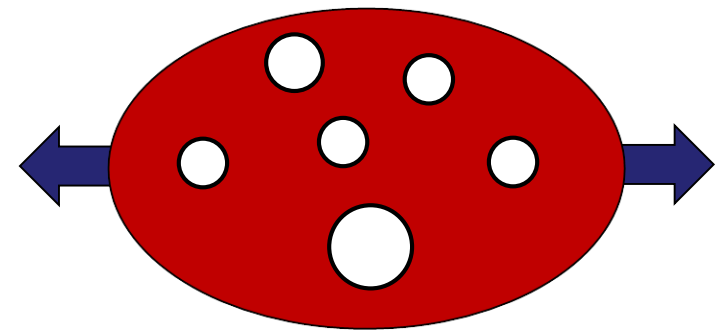
- Porous plasticity (or Gurson) approach
  - Assuming a J2-(visco-)plastic matrix
  - Including effects of void/defect or porosity on plastic behavior
    - Apparent macroscopic yield surface  $f(\tau_{\text{eq}}, p) \leq 0$  due to microstructural state:



# Damage to crack transition in porous elasto-plasticity

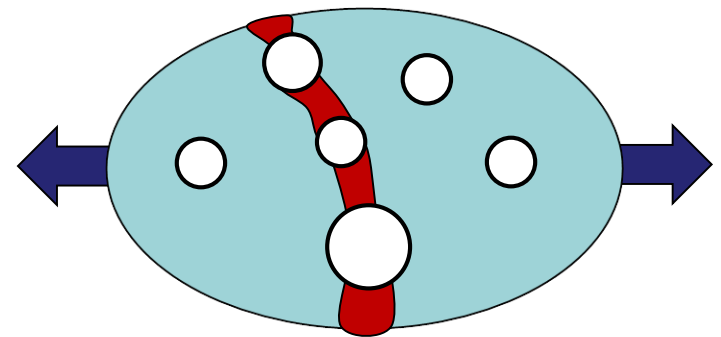
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      - » Diffuse plastic flow spreads in the matrix
      - » Gurson model



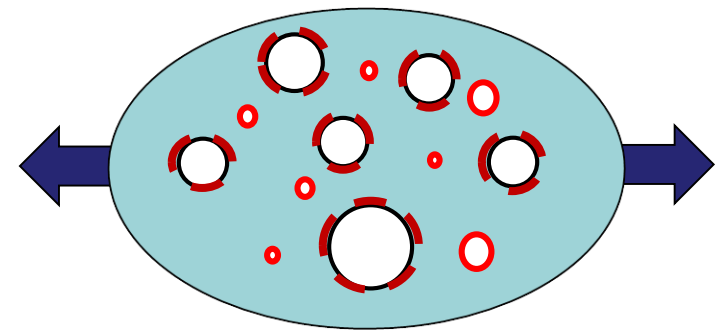
# Damage to crack transition in porous elasto-plasticity

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      - Competition between two deformation modes:
        - » Diffuse plastic flow spreads in the matrix
          - » Gurson model
        - » Before failure: coalescence or localized plastic flow between voids
          - » GTN or Thomason models



# Damage to crack transition in porous elasto-plasticity

- Porous plasticity (or Gurson) approach
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    - Apparent macroscopic yield surface  $f(\tau_{\text{eq}}, p) \leq 0$  due to microstructural state:
      - Competition between two deformation modes:
        - » Diffuse plastic flow spreads in the matrix
          - » Gurson model
        - » Before failure: coalescence or localized plastic flow between voids
          - » GTN or Thomason models
  - Including evolution of microstructure during failure process
    - Void growth by diffuse plastic flow
    - Apparent growth by shearing
    - Nucleation / appearance of new voids
    - Void coalescence until failure



# Non-local porous plasticity model

- Yield surface is considered in the co-rotational space

- Non-local form  $f(\tau_{\text{eq}}, p; \tau_Y, \mathbf{Z}(t'), \tilde{f}_V(t')) \leq 0$

- $\tau_{\text{eq}}$  is the von Mises equivalent Kirchhoff stress and  $p$  the pressure
- $\tau_Y = \tau_Y(\hat{p}, \hat{p})$  is the viscoplastic yield stress in terms of equivalent matrix plastic strain  $\hat{p}$
- $f_V$  is the porosity and  $\tilde{f}_V$ , its non-local counterpart with  $\tilde{f}_V - l_c^2 \Delta \tilde{f}_V = f_V$
- $\mathbf{Z}$  is the vector of internal variables
- $l_c$  is the non-local length

- Normal plastic flow  $\mathbf{D}^p$

$$\mathbf{D}^p = \dot{\mathbf{F}}^p \cdot \mathbf{F}^{p-1} = \dot{\gamma} \frac{\partial f}{\partial \boldsymbol{\tau}} = \hat{d} \frac{\partial \tau_{\text{eq}}}{\partial \boldsymbol{\tau}} + \hat{q} \frac{\partial p}{\partial \boldsymbol{\tau}}$$

- Microstructure evolution (spherical voids):

- Equivalent matrix plastic strain rate

$$\hat{p} = \frac{\boldsymbol{\tau} : \mathbf{D}^p}{(1 - f_{V_0}) \tau_Y}$$

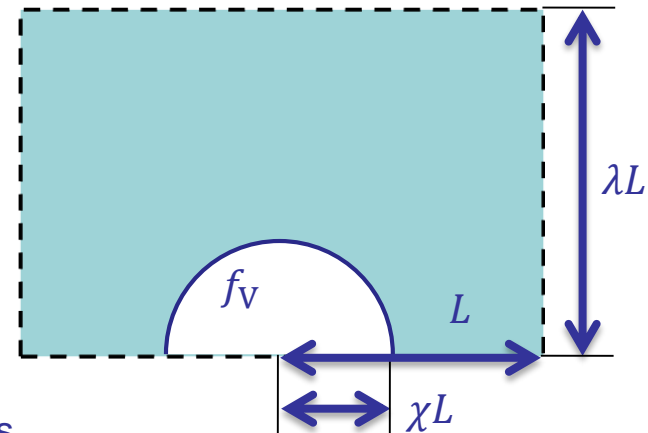
- Porosity:

$$\dot{f}_V = (1 - f_V) \text{tr}(\mathbf{D}^p) + \dot{f}_{\text{nucl}} + \dot{f}_{\text{shear}}$$

- Ligament ratio:

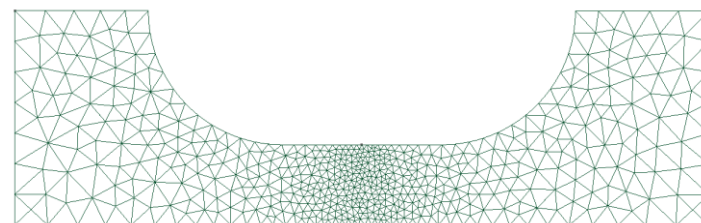
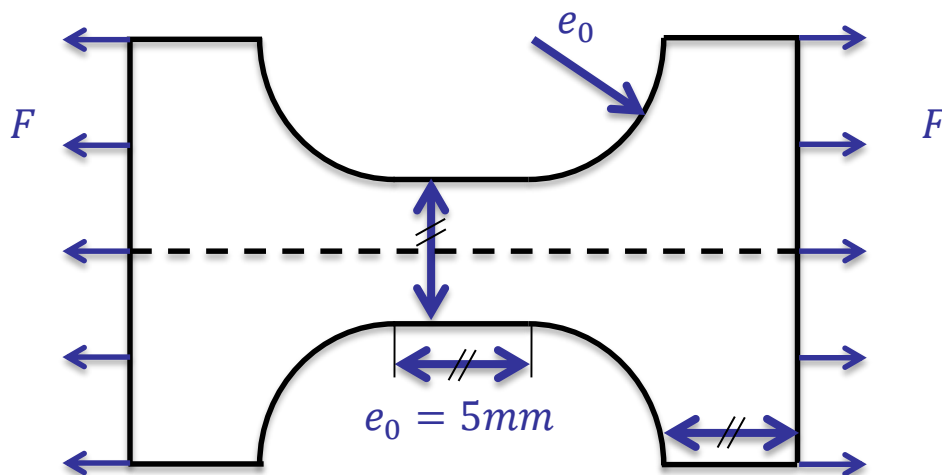
$$\dot{\chi} = \dot{\chi}(\chi, \tilde{f}_V, \kappa, \lambda, \mathbf{Z})$$

Microstructure parameters

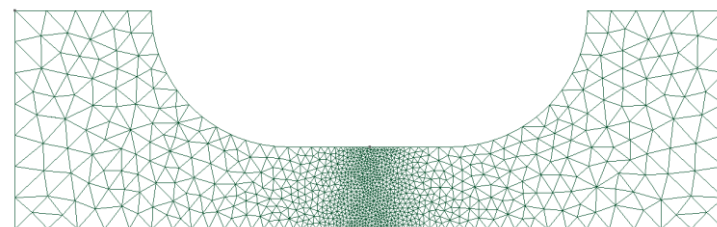


# Non-local porous plasticity – Comparison with literature results

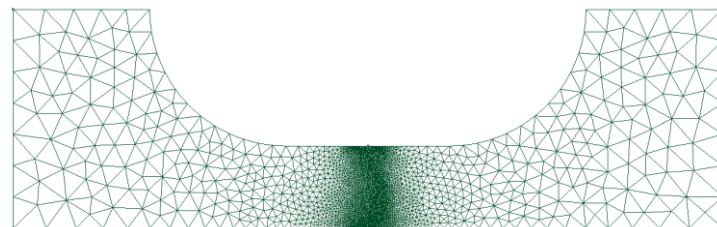
- Plane strain specimen [Besson et al. 2003]
  - Only half specimen is modelled
  - Three  $\neq$  mesh sizes



Coarse mesh  
(~4600 elements,  $l_m \cong 1.12 l_c$ )



Medium mesh  
(~8100 elements,  $l_m \cong 0.75 l_c$ )



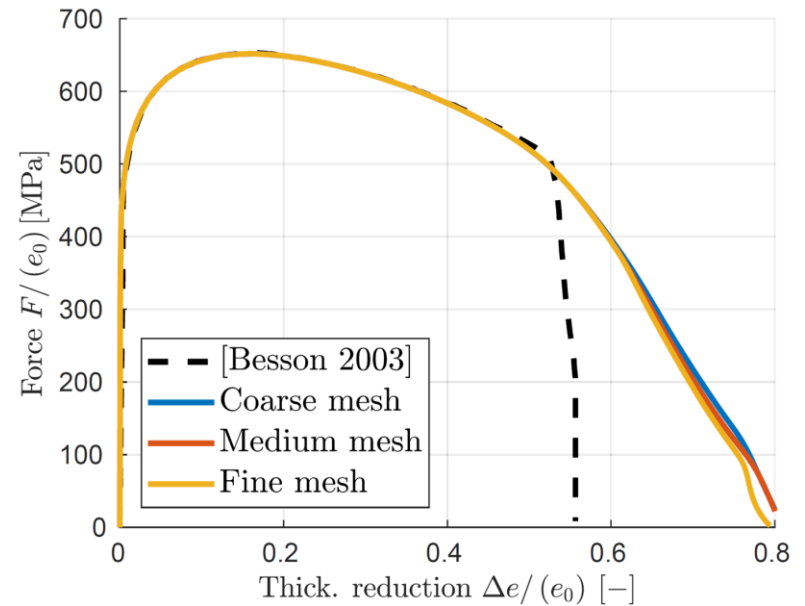
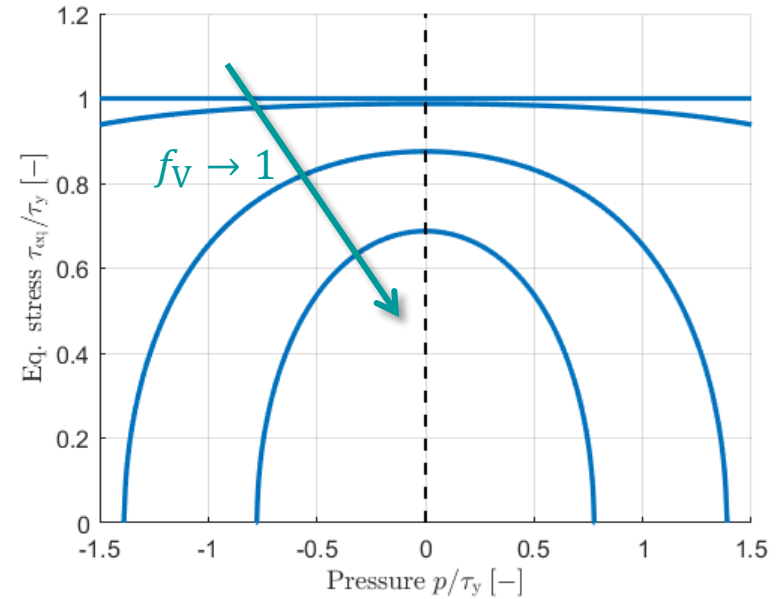
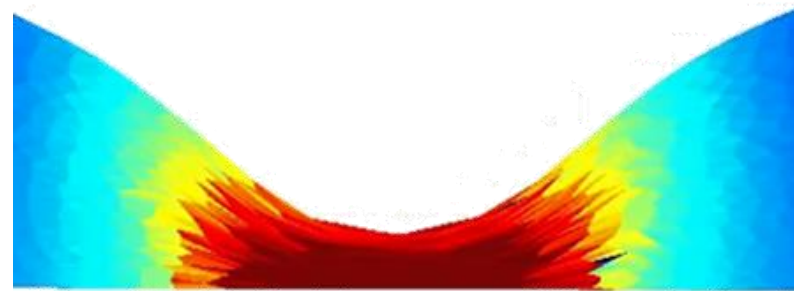
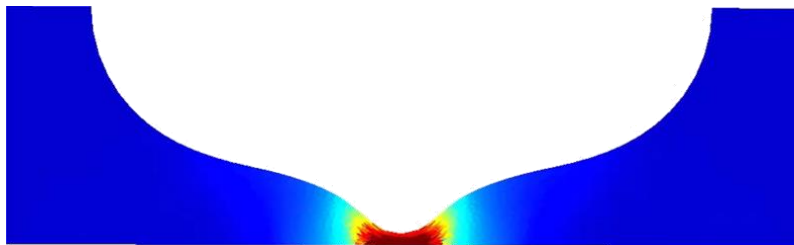
Fine mesh  
(~15500 elements,  $l_m \cong 0.5 l_c$ )

# Non-local porous plasticity – void growth

- Gurson model [Reush et al. 2003]
  - Particularized yield surface

$$f_G = \frac{\tau_{eq}^2}{\tau_Y^2} + 2q_1 \tilde{f}_V \cosh\left(\frac{q_2 p}{2\tau_Y}\right) - 1 - q_3 \tilde{f}_V^2 \leq 0$$

- Verification of non-local model



# Non-local porous plasticity – void coalescence

- Thomason model [Benzerga 2014, Besson 2009]

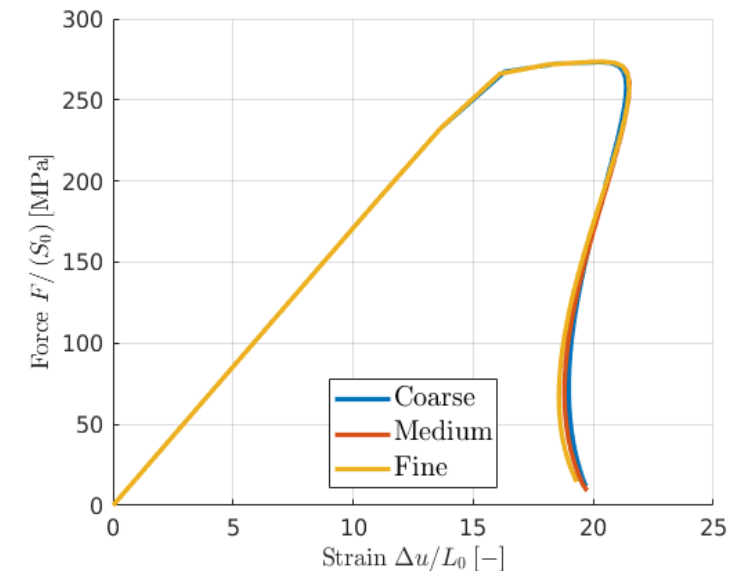
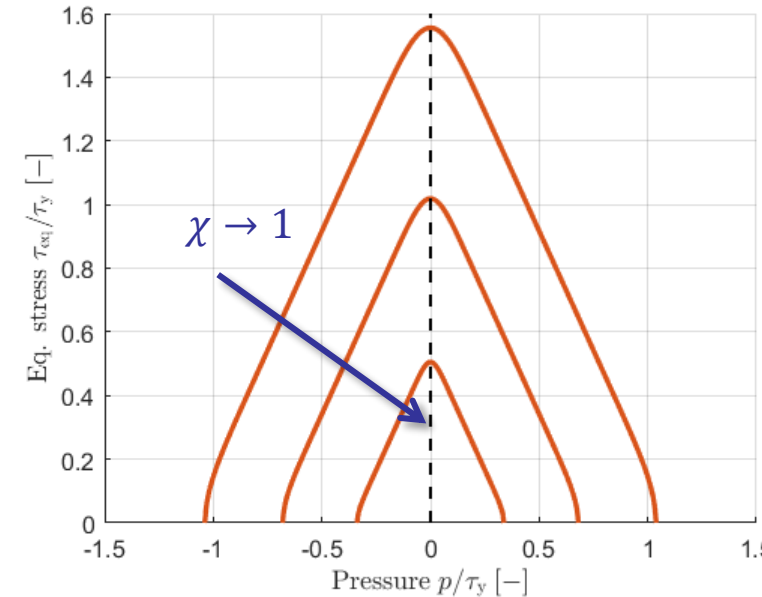
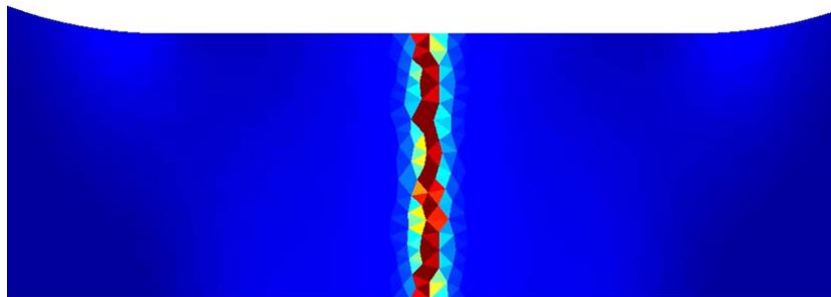
- Particularized yield surface

$$f_T = \frac{2}{3} \tau_{eq} + |p| - C_T^f(\chi) \tau_Y \leq 0$$

- Higher porosity to trigger coalescence
- No lateral contraction due to plasticity

- Verification of non-local model

- For  $\kappa = 0.5$ ;  $\lambda = 0.5$ ;  $l_c = 50 \mu\text{m}$



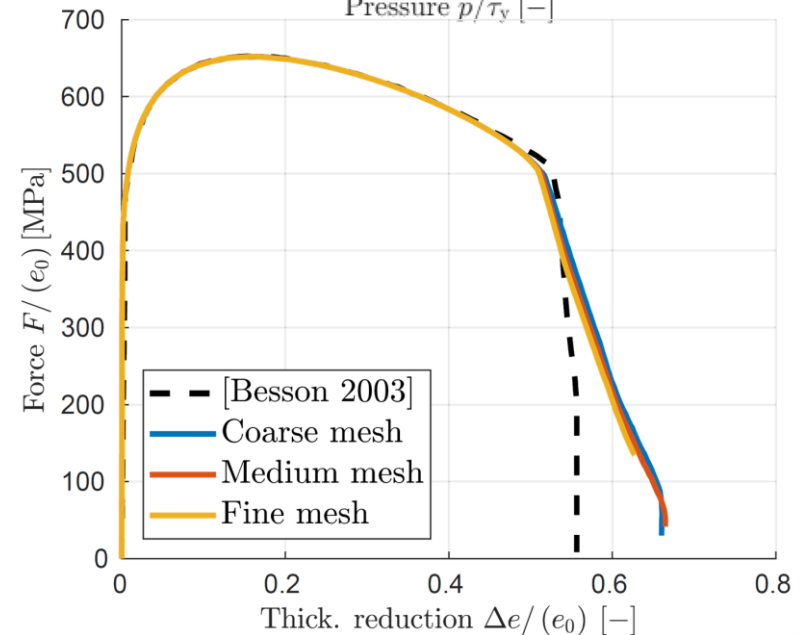
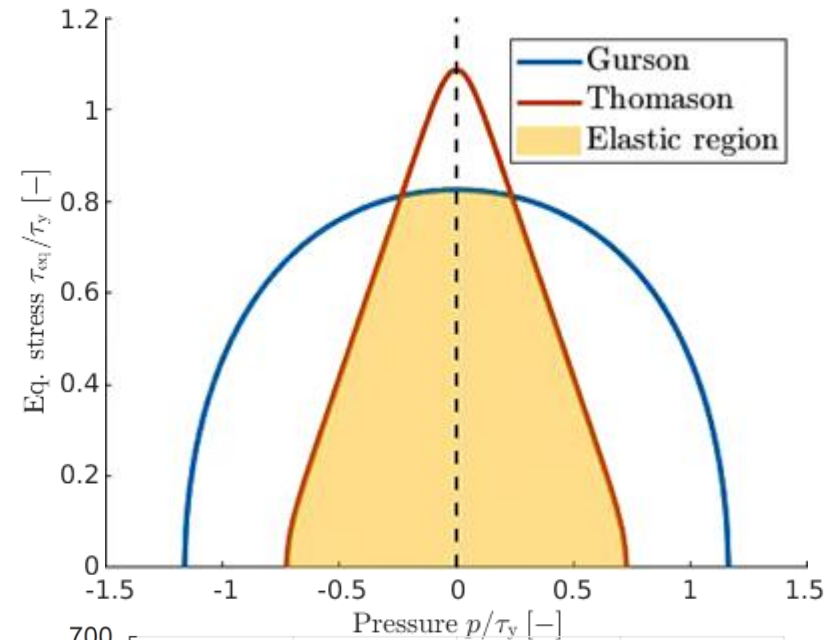
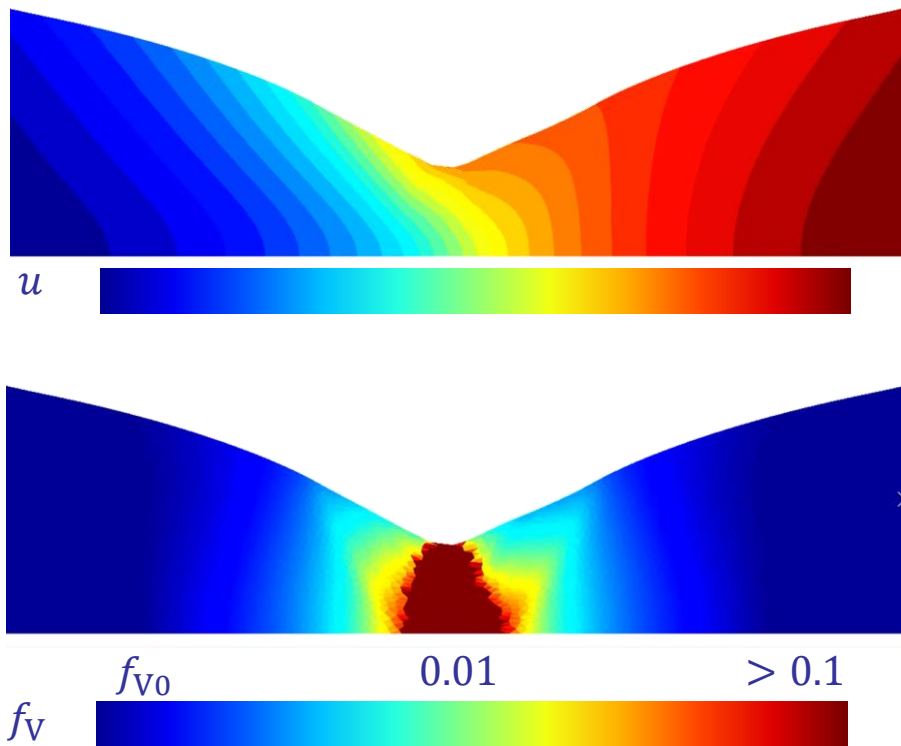
# Non-local porous plasticity – void growth and coalescence

- Coupled non-local Gurson-Thomason

- Competition between  $f_G$  and  $f_T$

$$\begin{cases} f_G = \frac{\tau_{eq}^2}{\tau_Y^2} + 2q_1 \tilde{f}_V \cosh\left(\frac{q_2 p}{2\tau_Y}\right) - 1 - q_3 \tilde{f}_V^2 \leq 0 \\ f_T = \frac{2}{3} \tau_{eq} + |p| - C_T^f(\chi) \tau_Y \leq 0 \end{cases}$$

- For  $\kappa = 0.5; \lambda = 0.5; l_c = 50 \mu\text{m}$



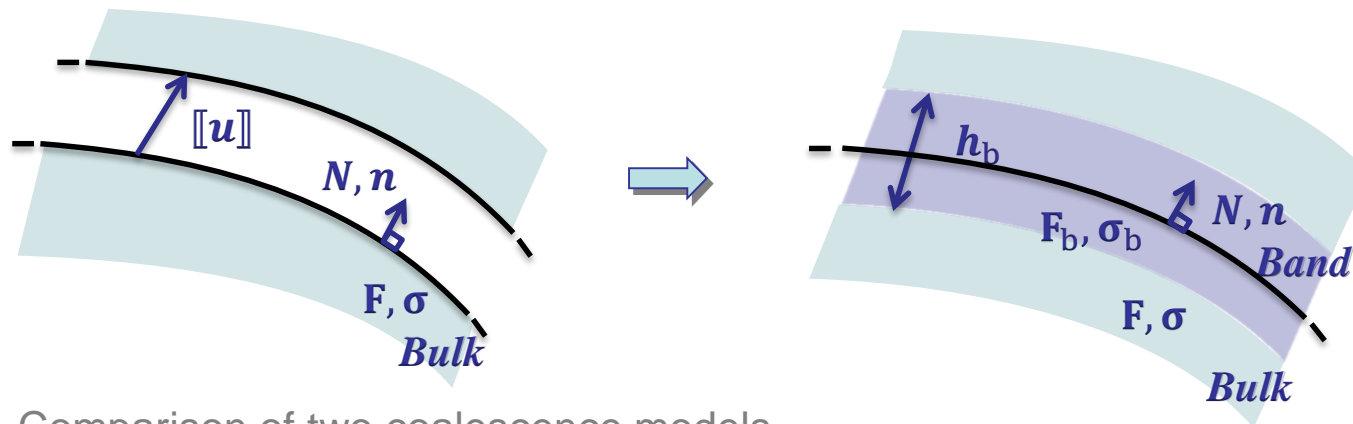
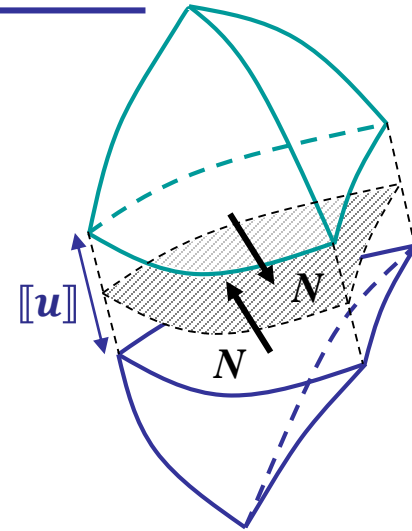
# Damage to crack transition for porous plasticity

- Non-local Gurson model – CBM (arbitrary crack paths)

- Gurson material model  $f_G = \frac{\tau_{eq}^2}{\tau_Y^2} + 2q_1 \tilde{f}_V \cosh\left(\frac{q_2 p}{2\tau_Y}\right) - 1 - q_3 \tilde{f}_V^2 \leq 0$

- Crack insertion at Thomasson criterion  $N \cdot \tau \cdot N - C_T^f(\chi) \tau_Y = 0$

- At crack insertion: Cohesive Band Model



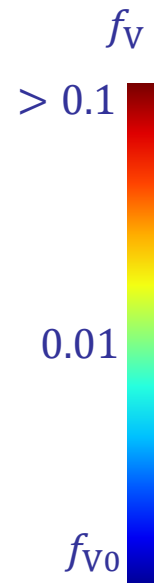
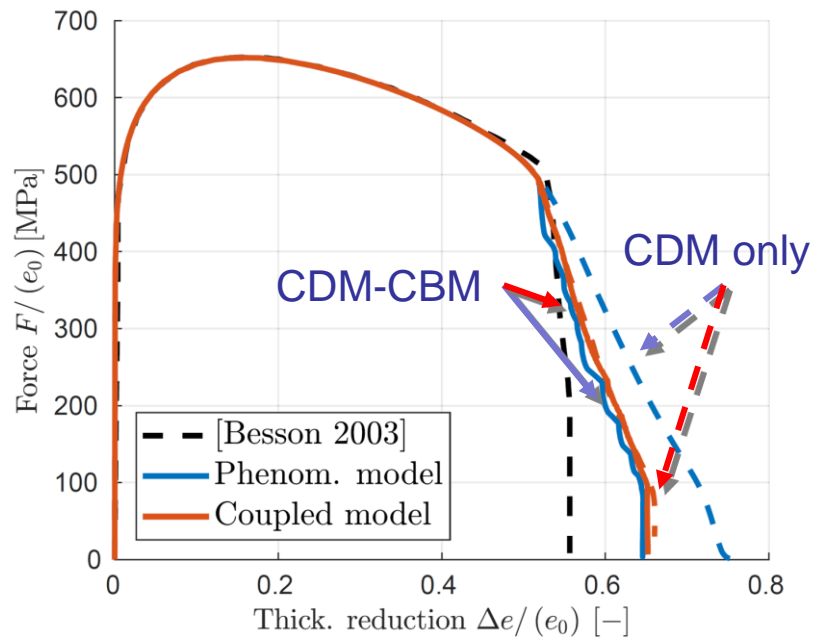
- Comparison of two coalescence models

- Phenomenological approach:  $\tilde{f}_V^* = \begin{cases} \tilde{f}_V & \text{if } \tilde{f}_V \leq f_c \\ f_c + R(\tilde{f}_V - f_c) & \text{if } \tilde{f}_V > f_c \end{cases}$

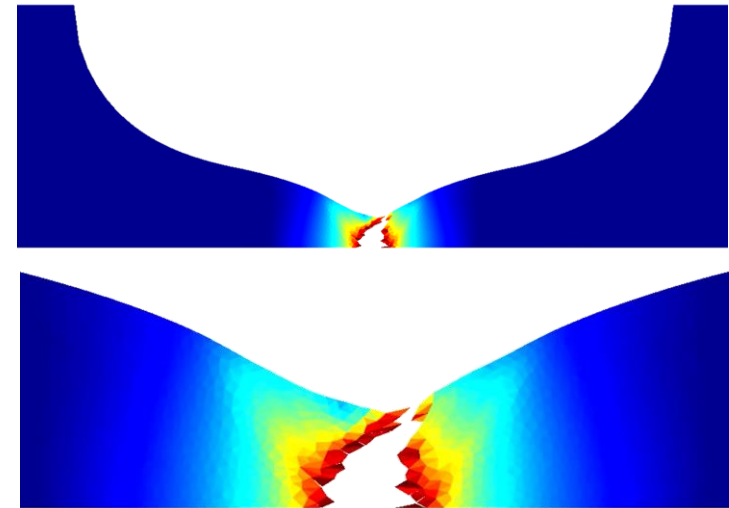
- Thomason model:  $f_T = \frac{2}{3} \tau_{eq} + |p| - C_T^f(\chi) \tau_Y \leq 0$

# Damage to crack transition for porous plasticity

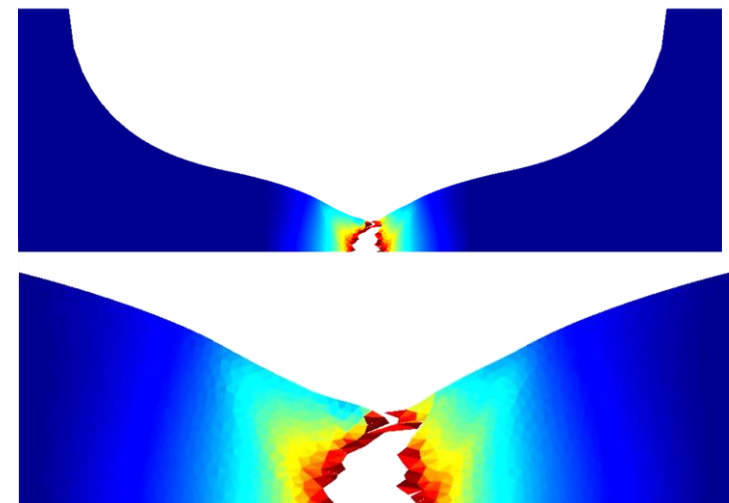
- Non-local Gurson model – CBM
  - CBM insertion at Thomason criterion
  - CBM with coalescence model
    - Comparison of 2 coalescence models
    - For  $\kappa = 0.5; \lambda = 0.5; l_c = 50 \mu\text{m}$



Thomason coalescence



Phenomenological coalescence

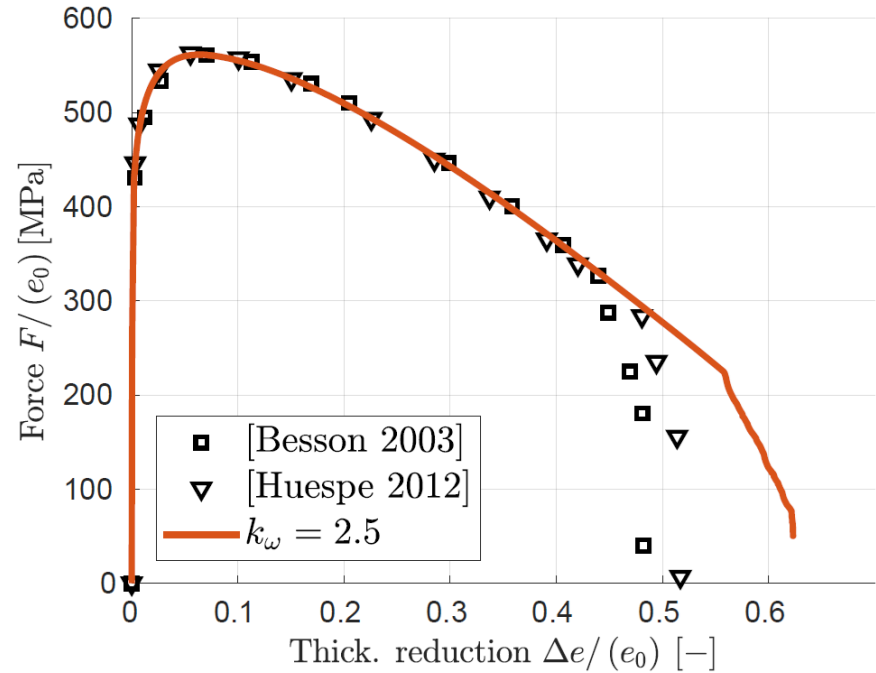


# Damage to crack transition for porous plasticity

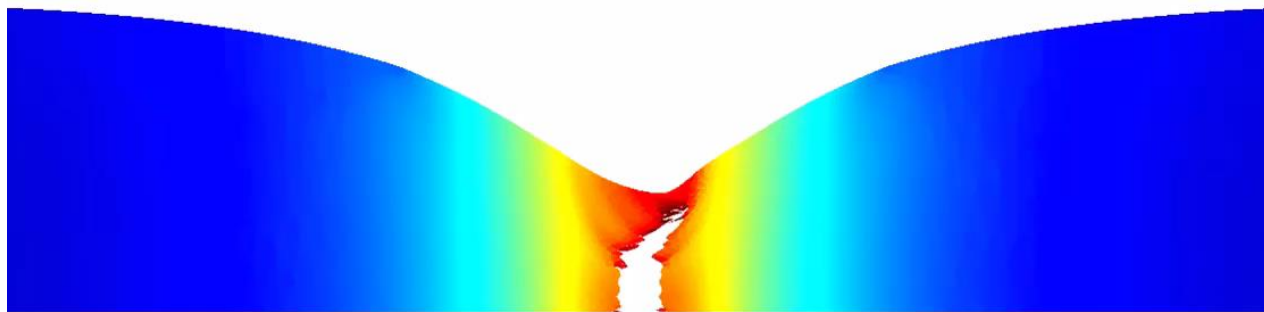
- Round-bar test

- CBM insertion at Thomason criterion
- CBM with Thomason coalescence model
- Nucleation not identified yet
- Crack path in cup-cone shape

[Besson et al. 2001]



## Thomason coalescence



# Conclusions

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- **Objective:**
  - Simulation of material degradation and crack initiation / propagation
- **Methodology**
  - Combination of non-local Continuum Damage Model (CDM)
  - Cohesive Band Model (CBM)
  - Integrated in a Hybrid Discontinuous Galerkin framework
- **Proof of concept**
  - Elastic damage material model
  - Cohesive band thickness controls the failure energy dissipation
- **Ductile materials**
  - Implementation of hyperelastic non-local porous-plastic model
    - Coupled Gurson-Thomason model
  - First results of the CDM-CBM transition
  - Upcoming tasks:
    - Enrichment of nucleation model and coalescence model
    - Calibration of the band thickness
    - Validation/Calibration with literature/experimental tests



Thank you for your attention

Computational & Multiscale Mechanics of Materials – CM3

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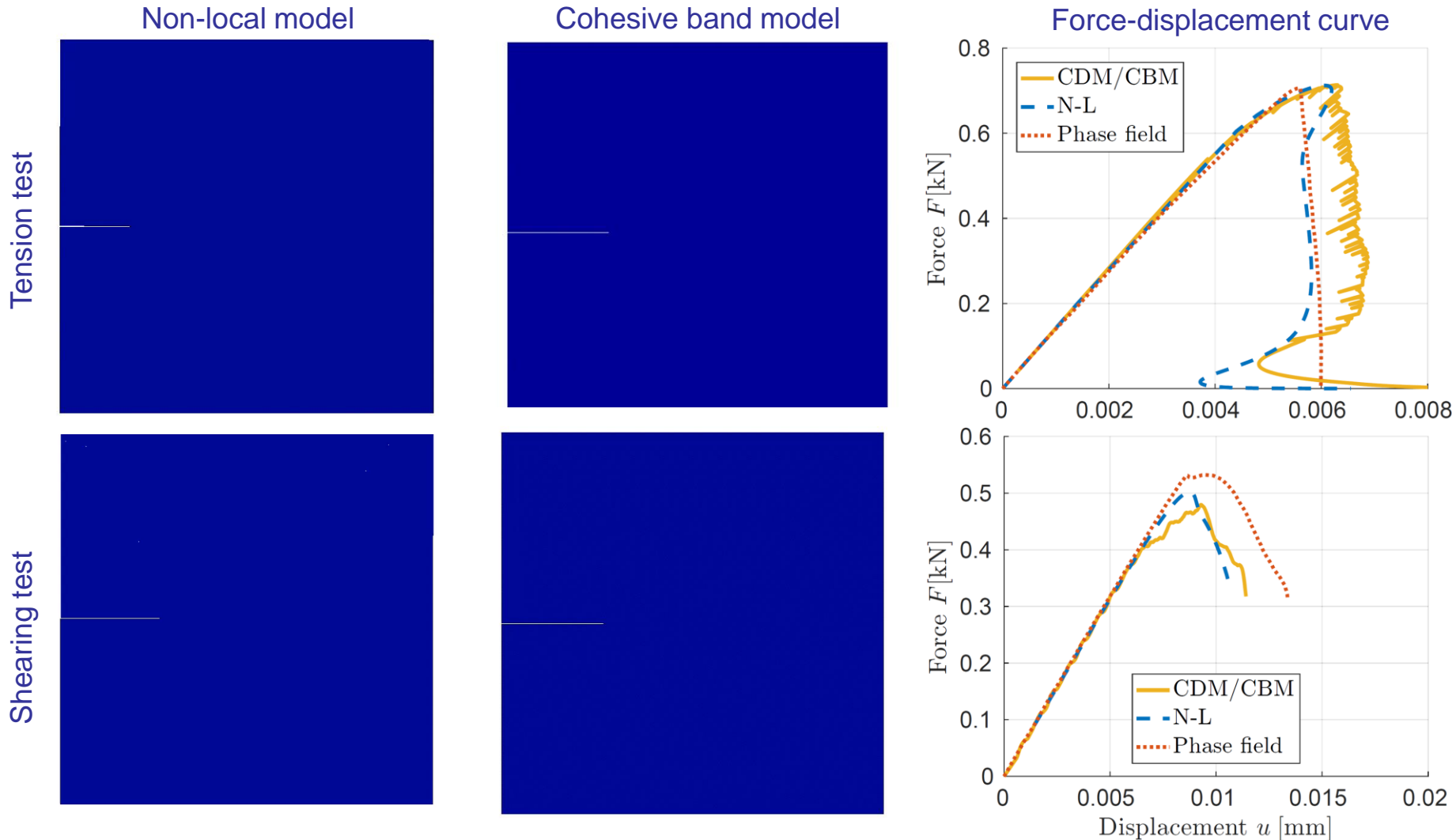
Allée de la découverte 9, B4000 Liège

Julien.Leclerc@ulg.ac.be



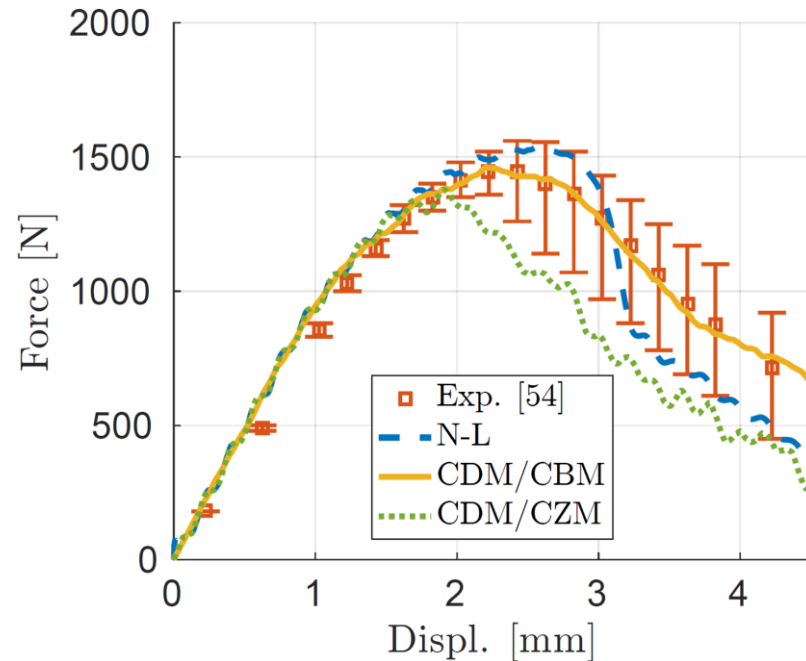
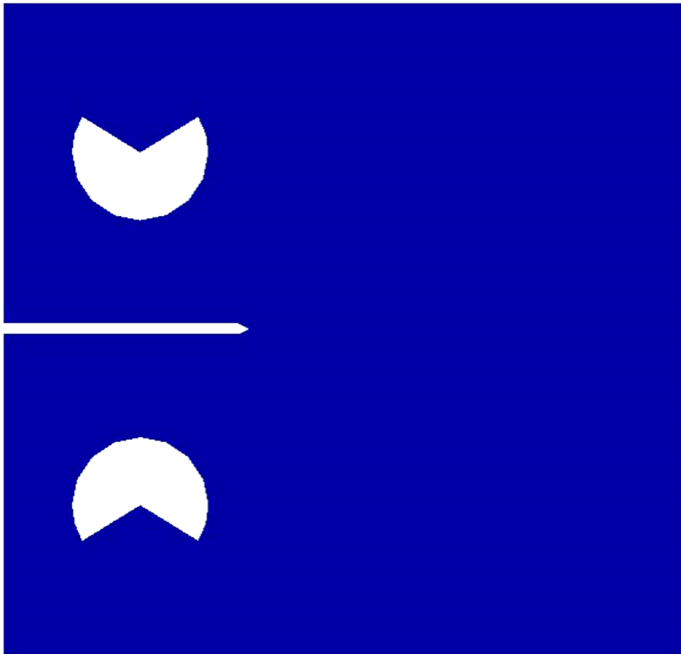
# Damage to crack transition for elastic damage – Proof of concept

- Comparison with phase field
  - Single edge notched specimen [Miehe et al. 2010]
    - Calibration of damage and CBM parameters with 1D case [Leclerc et al. 2018]



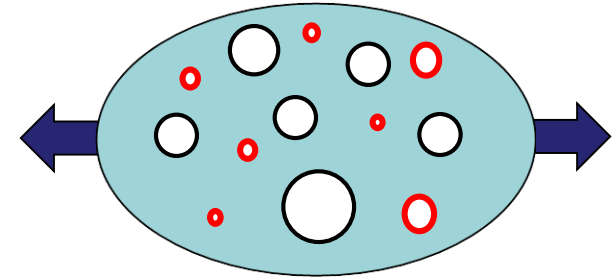
# Damage to crack transition for elastic damage – Proof of concept

- Validation with Compact Tension Specimen [Geers 1997]
  - Better agreement with the cohesive band model than the cohesive zone model or the non-local model alone [Leclerc et al. 2018]



- Evolution of local porosity

$$\dot{f}_V = (1 - f_V)\text{tr}(\mathbf{D}^p) + \dot{f}_{\text{nucl}} + \dot{f}_{\text{shear}}$$



- Voids nucleation  $\dot{f}_{\text{nucl}}$  modifies porosity growth rate

- Linear strain-controlled growth

$$\dot{f}_{\text{nucl}} = A_N \dot{\rho} \quad \text{with} \quad \begin{cases} A_N \neq 0 & \text{if } f_V > f_N \\ A_N = 0 & \text{if } f_V \leq f_N \end{cases}$$

- Gaussian strain-controlled growth

$$\dot{f}_{\text{nucl}} = \frac{f_N}{\sqrt{\{2\pi s_N^2\}}} \exp\left(-\frac{(\dot{\rho} - \epsilon_N)^2}{2s_N^2}\right) \dot{\rho}$$

- where  $A_N$ ,  $f_N$ ,  $\epsilon_N$ ,  $s_N$  are material parameters

# Porous plasticity – Voids nucleation

- Evolution of local porosity

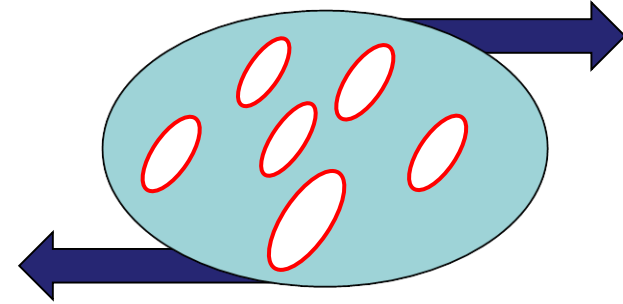
$$\dot{f}_V = (1 - f_V)\text{tr}(\mathbf{D}^p) + \dot{f}_{\text{nucl}} + \dot{f}_{\text{shear}}$$

- Shearing affect voids nucleation:  $\dot{f}_{\text{shear}}$

- Includes Lode variable effect  $\zeta(\boldsymbol{\tau}) = -\frac{27 \det(\boldsymbol{\tau}^{\text{dev}})}{2 \tau_{\text{eq}}^3}$

$$\dot{f}_{\text{shear}} = f_V k_w (1 - \zeta^2(\boldsymbol{\tau})) \frac{\boldsymbol{\tau}^{\text{dev}} : \mathbf{D}^p}{\tau_{\text{eq}}}$$

- where  $k_w$  is a material parameter



# Integration algorithm

- Predictor-corrector procedure

- Elastic predictor

$$\mathbf{F}^{\text{ePr}} = \mathbf{F} \cdot \mathbf{F}_n^{\text{p}^{-1}}$$

- Plastic corrector (radial return-like algorithm)

- 3 equations

- Consistency equation:  $f(\tau_{\text{eq}}, p; \tau_Y, \mathbf{Z}(t'), \tilde{f}_V(t')) = 0$

- Plastic flow rule:  $\mathbf{D}^{\text{p}} = \dot{\mathbf{F}}^{\text{p}} \cdot \mathbf{F}^{\text{p}^{-1}} = \dot{\gamma} \frac{\partial f}{\partial \boldsymbol{\tau}} = \dot{d} \frac{\partial \tau_{\text{eq}}}{\partial \boldsymbol{\tau}} + \dot{q} \frac{\partial p}{\partial \boldsymbol{\tau}}$

- Matrix plastic strain evolution:  $\dot{\hat{p}} = \frac{\boldsymbol{\tau} : \mathbf{D}^{\text{p}}}{(1 - f_{V_0}) \tau_Y}$

- 3 Unknowns  $\Delta \hat{d}, \Delta \hat{q}, \Delta \hat{p}$

- 3 linearized equations

- Consistency equation:  $f(\tau_{\text{eq}}(\Delta \hat{d}), p(\Delta \hat{q}); \tau_Y(\Delta \hat{p}), \mathbf{Z}(\Delta \hat{d}, \Delta \hat{q}, \Delta \hat{p}), \tilde{f}_V) = 0$

- Plastic flow rule:  $\Delta \hat{d} \frac{\partial f}{\partial p} - \Delta \hat{q} \frac{\partial f}{\partial \tau_{\text{eq}}} = 0$

- Matrix plastic strain evolution:  $(1 - f_{V_0}) \tau_Y \Delta \hat{p} = \tau_{\text{eq}} \Delta \hat{d} + p \Delta \hat{q}$