Weighted $U$-frequent hypercyclicity

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Introduction and statement of the problem

Let $T : X \to X$ be a linear operator ($X$ separable Banach space, $\dim X = \infty$).

- $T$ is $U$-frequently hypercyclic (ufhc) if $\exists x \in X$ such that $\forall U \neq \emptyset$ open
  \[\overline{d}(\{n \geq 0 : T^n x \in U\}) > 0.\]
  The set of such points is denoted $UFHC(T)$ (Shkarin 2009).

- $T$ is reiteratively hypercyclic (rhc) if $\exists x \in X$ such that $\forall U \neq \emptyset$ open
  \[\overline{Bd}(\{n \geq 0 : T^n x \in U\}) > 0.\]
  The set of such points is denoted $RHC(T)$ (Bès, Menet, Peris, Puig 2016).
For $E \subseteq \mathbb{N}$, the densities used are

- the upper density $\overline{d}(E) = \limsup_{n \to +\infty} \frac{\#(E \cap [0, n])}{n + 1}$

- the upper Banach density $\overline{Bd}(E) = \lim_{k \to +\infty} \limsup_{n \to +\infty} \frac{\#(E \cap [n, n + k])}{k + 1}$

One has $\overline{d}(E) \leq \overline{Bd}(E)$ hence

$$T \text{ ufhc} \iff T \text{ rhc}$$

but one can prove that $T \text{ rhc} \not\Rightarrow T \text{ ufhc}$ (Bès, Menet, Peris, Puig 2016)

**Question:** Is it possible to find intermediate densities that fill the gap?
Weighted upper densities

Let \( a = (a_n)_{n \in \mathbb{N}} \) be a sequence of positive real numbers such that \( \sum_{n \in \mathbb{N}} a_n = +\infty \). The \( a \)-upper density of a set \( E \subseteq \mathbb{N} \) is defined by

\[
\overline{d}_a(E) = \limsup_{n \to +\infty} \frac{\sum_{k=0}^{n-1} a_k 1_E(k)}{\sum_{k=0}^{n-1} a_k}
\]

An operator \( T \) is \( a \)-upper frequently hypercyclic (ufhc_{\text{a}}) if \( \exists x \in X \) such that \( \forall U \neq \emptyset \) open,

\[
\overline{d}_a(\{n \geq 0 : T^n x \in U\}) > 0.
\]

The set of such points is denoted \( \text{UFHC}_{\text{a}}(T) \) and is a residual set (Bonilla, Grosse-Erdmann 2018)
Remarks

• If $\frac{a_n}{b_n} \searrow 0$, then $\overline{d}_a \leq \overline{d}_b$. Hence ufhc$_a \Rightarrow$ ufhc$_b$  (Ernst, Mouze 2017)

• If $a_n = 1$ for all $n \in \mathbb{N}$, then $\overline{d}_a = \overline{d}$. Hence ufhc$_a \Leftrightarrow$ ufhc

• Let $\nu_n = \frac{a_n}{\sum_{k=0}^{n-1} a_k}$. If there is $C > 0$ such that $\nu_n \geq C$ for all $n \in \mathbb{N}$, then $\overline{d}_a(E) > 0$ as soon as $\#E = +\infty$

Consequences

• ($\mathcal{H}1$) We consider sequences $a = (a_n)_{n \in \mathbb{N}}$ such that $a_n \nearrow +\infty$

• ($\mathcal{H}2$) We impose that $\nu_n \searrow 0$

We denote by $\mathcal{A}$ the set of such sequences $a$. 
Example 1

For any $\alpha > 0$, the sequence $(n^\alpha)_{n \in \mathbb{N}}$ belongs to $\mathcal{A}$:

- $n^\alpha \nearrow +\infty$
- $\nu_n = \frac{n^\alpha}{\sum_{k=1}^{n-1} k^\alpha} \sim \frac{n^\alpha}{n^{\alpha+1}} \searrow 0$

Example 2

For any $\alpha > 1$, the sequence $(\alpha^n)_{n \in \mathbb{N}}$ does not belong to $\mathcal{A}$:

$$\nu_n = \frac{\alpha^n}{\sum_{k=1}^{n-1} \alpha^k} = \frac{\alpha^n}{\alpha - \alpha^{n-1}} = (1 - \alpha) \frac{1}{\alpha^{1-n} - 1} \to \alpha - 1$$
Comparison of the densities

Result

• For every \( a \in A \), one has \( \overline{d} \leq \overline{d}_a \).

• For every \( \alpha > 0 \), the sequence \( a = (n^\alpha)_{n \in \mathbb{N}} \in A \) satisfies

\[
\frac{1}{1 + \alpha} \overline{d}_a \leq \overline{d} \leq \overline{d}_a.
\]

Consequence

For every operator \( T \), there is \( a \in A \) such that \( UFHC(T) = UFHC_a(T) \).

In particular,

\[
UFHC(T) = \bigcap_{a \in A} UFHC_a(T)
\]

and

\[ T \ ufhc <\!\!<\!\! T \ ufhc_a \text{ for every } a \in A \]
Result

- For every $a \in A$, one has $\overline{d_a} \leq B \overline{d}$.
- Let $(E_k)_{k \in \mathbb{N}}$ be a sequence of subsets of $\mathbb{N}$. There exists $a \in A$ such that
  $$B \overline{d}(E_k) \leq e \overline{d_a}(E_k) \quad \forall k \in \mathbb{N}.$$  

Consequence

For every operator $T$,

$$RHC(T) = \bigcup_{a \in A} UFHC_a(T)$$  

and

$$T \text{ rhc } \iff T \text{ ufhc}_a \text{ for some } a \in A$$
If $T$ is rhc, then $RHC(T) = HC(T)$ (Bès, Menet, Peris, Puig 2016)

**Result**

For every $a \in A$, one has $UFHC_a(2B) \neq HC(2B)$.

However, if $T$ is rhc, then

$$HC(T) = \bigcup_{a \in A} UFHC_a(T)$$

**Chaos and ufhc$_a$**

One has (Menet 2017)

- chaos $\implies$ rhc
- chaos $\nRightarrow$ ufhc

**Result**

For every $a \in A$, there is a chaotic operator on $\ell^1$ which is not ufhc$_a$. 
Product of operators

**Question:** Given an operator $T$ which is ufhc / ufhc$_a$ / rhc, what can we say about the operator $T \times \cdots \times T$?

**Result**

If $T$ is ufhc$_a$, then $T \times \cdots \times T$ is ufhc$_a$.

**Keys:** (using ideas from Bayart, Rusza 2015 & Bès, Menet, Peris, Puig 2016)

- ufhc$_a$ $\Rightarrow$ weakly-mixing
- If $\overline{d}_a(E) > 0$, then there is $\delta > 0$ such that the set

$$\{k \geq 0 : \overline{d}_a(E \cap (E - k)) > \delta\}$$

is syndetic

**Consequence**

If $T$ is ufhc (resp. rhc), then $T \times \cdots \times T$ is ufhc (resp. rhc).
Main references

F. Bayart and I. Z. Ruzsa
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