Weighted $\mathcal{U}$-frequent hypercyclicity

Céline ESSER
Celine.Esser@uliege.be

University of Liège – Institute of Mathematics

Lens – October 24, 2018

Joint work with
R. ERNST and Q. MENET

Introduction and statement of the problem

Let $T : X \to X$ be a linear operator (X separable Banach space, $\dim X = \infty$).

- T is $\mathcal{U}$-frequently hypercyclic (ufhc) if $\exists x \in X$ such that $\forall U \neq \emptyset$ open
$$\overline{d}\big(\{n \geq 0 : T^n x \in U\}\big) > 0\,.$$
The set of such points is denoted $UFHC(T)$ (Shkarin 2009).

- T is reiteratively hypercyclic (rhc) if $\exists x \in X$ such that $\forall U \neq \emptyset$ open
$$\overline{Bd}\big(\{n \geq 0 : T^n x \in U\}\big) > 0\,.$$
The set of such points is denoted $RHC(T)$ (Bès, Menet, Peris, Puig 2016).

For $E \subseteq \mathbb{N}$, the densities used are

- the upper density $\overline{d}(E) = \limsup_{n \to +\infty} \dfrac{\#\big(E \cap [0,n]\big)}{n+1}$
- the upper Banach density $\overline{Bd}(E) = \lim_{k \to +\infty} \limsup_{n \to +\infty} \dfrac{\#\big(E \cap [n, n+k]\big)}{k+1}$

One has $\overline{d}(E) \leq \overline{Bd}(E)$ hence

$$\boxed{T \text{ ufhc} \implies T \text{ rhc}}$$

but one can prove that T rhc $\not\Rightarrow T$ ufhc (Bès, Menet, Peris, Puig 2016)

Question: Is it possible to find intermediate densities that fill the gap?

Weighted upper densities

Let $\mathbf{a} = (a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that $\sum_{n\in\mathbb{N}} a_n = +\infty$. The $\mathbf{a}$-upper density of a set $E \subseteq \mathbb{N}$ is defined by

$$\overline{d}_{\mathbf{a}}(E) = \limsup_{n\to+\infty} \frac{\sum_{k=0}^{n} a_k \mathbb{1}_E(k)}{\sum_{k=0}^{n} a_k}$$

An operator T is $\mathbf{a}$-upper frequently hypercyclic (ufhc$_{\mathbf{a}}$) if $\exists x \in X$ such that $\forall U \neq \emptyset$ open,

$$\overline{d}_{\mathbf{a}}\big(\{n \geq 0 : T^n x \in U\}\big) > 0 \,.$$

The set of such points is denoted $UFHC_{\mathbf{a}}(T)$ and is a residual set

(Bonilla, Grosse-Erdmann 2018)

Remarks

- If $\frac{a_n}{b_n} \searrow 0$, then $\overline{d}_{\mathbf{a}} \leq \overline{d}_{\mathbf{b}}$. Hence ufhc$_{\mathbf{a}}$ $\Rightarrow$ ufhc$_{\mathbf{b}}$ (Ernst, Mouze 2017)
- If $a_n = 1$ for all $n \in \mathbb{N}$, then $\overline{d}_{\mathbf{a}} = \overline{d}$. Hence ufhc$_{\mathbf{a}}$ $\Leftrightarrow$ ufhc
- Let $\nu_n = \frac{a_n}{\sum_{k=0}^{n-1} a_k}$. If there is $C > 0$ such that $\nu_n \geq C$ for all $n \in \mathbb{N}$, then $\overline{d}_{\mathbf{a}}(E) > 0$ as soon as $\#E = +\infty$

Consequences

- $(\mathcal{H}1)$ We consider sequences $\mathbf{a} = (a_n)_{n \in \mathbb{N}}$ such that $a_n \nearrow +\infty$
- $(\mathcal{H}2)$ We impose that $\nu_n \searrow 0$

We denote by $\mathcal{A}$ the set of such sequences $\mathbf{a}$.

Example 1

For any $\alpha > 0$, the sequence $(n^\alpha)_{n\in\mathbb{N}}$ belongs to $\mathcal{A}$:

- $n^\alpha \nearrow +\infty$
- $\nu_n = \frac{n^\alpha}{\sum_{k=1}^{n-1} k^\alpha} \sim \frac{n^\alpha}{n^{\alpha+1}} \searrow 0$

Example 2

For any $\alpha > 1$, the sequence $(\alpha^n)_{n\in\mathbb{N}}$ does not belong to $\mathcal{A}$:

$$\nu_n = \frac{\alpha^n}{\sum_{k=1}^{n-1} \alpha^k} = \frac{\alpha^n}{\frac{\alpha-\alpha^n}{1-\alpha}} = (1-\alpha)\frac{1}{\alpha^{1-n}-1} \to \alpha - 1$$

Comparison of the densities

Result

- For every $\mathbf{a} \in \mathcal{A}$, one has $\overline{d} \leq \overline{d}_{\mathbf{a}}$.
- For every $\alpha > 0$, the sequence $\mathbf{a} = (n^\alpha)_{n\in\mathbb{N}} \in \mathcal{A}$ satisfies

$$\frac{1}{1+\alpha}\overline{d}_{\mathbf{a}} \leq \overline{d} \leq \overline{d}_{\mathbf{a}}.$$

Consequence

For every operator T, there is $\mathbf{a} \in \mathcal{A}$ such that $UFHC(T) = UFHC_{\mathbf{a}}(T)$.
In particular,

$$\boxed{UFHC(T) = \bigcap_{a\in\mathcal{A}} UFHC_{\mathbf{a}}(T)}$$

and

$$\boxed{T \text{ ufhc} \Leftrightarrow T \text{ ufhc}_{\mathbf{a}} \text{ for every } \mathbf{a} \in \mathcal{A}}$$

Result

- For every $\mathbf{a} \in \mathcal{A}$, one has $\overline{d}_{\mathbf{a}} \leq \overline{Bd}$.
- Let $(E_k)_{k\in\mathbb{N}}$ be a sequence of subsets of $\mathbb{N}$. There exists $\mathbf{a} \in \mathcal{A}$ such that

$$\overline{Bd}(E_k) \leq e\,\overline{d}_{\mathbf{a}}(E_k) \quad \forall k \in \mathbb{N}\,.$$

Consequence

For every operator T,

$$\boxed{RHC(T) = \bigcup_{a\in\mathcal{A}} UFHC_{\mathbf{a}}(T)}$$

and

$$\boxed{T \text{ rhc} \Leftrightarrow T \text{ ufhc}_{\mathbf{a}} \text{ for some } \mathbf{a} \in \mathcal{A}}$$

If T is rhc, then $RHC(T) = HC(T)$ (Bès, Menet, Peris, Puig 2016)

Result

For every $\mathbf{a} \in \mathcal{A}$, one has $UFHC_{\mathbf{a}}(2B) \neq HC(2B)$.
However, if T is rhc, then

$$HC(T) = \bigcup_{\mathbf{a} \in \mathcal{A}} UFHC_{\mathbf{a}}(T)$$

Chaos and ufhc$_{\mathbf{a}}$

One has (Menet 2017)

- chaos $\Rightarrow$ rhc
- chaos $\not\Rightarrow$ ufhc

Result

For every $\mathbf{a} \in \mathcal{A}$, there is a chaotic operator on ℓ^1 which is not ufhc$_{\mathbf{a}}$.

Product of operators

Question: Given an operator T which is ufhc / ufhc$_{\mathbf{a}}$ / rhc, what can we say about the operator $T \times \cdots \times T$?

Result

If T is ufhc$_{\mathbf{a}}$, then $T \times \cdots \times T$ is ufhc$_{\mathbf{a}}$.

Keys: (using ideas from Bayart, Rusza 2015 & Bès, Menet, Peris, Puig 2016)

- ufhc$_{\mathbf{a}}$ $\Rightarrow$ weakly-mixing
- If $\overline{d}_{\mathbf{a}}(E) > 0$, then there is $\delta > 0$ such that the set

$$\left\{k \geq 0 : \overline{d}_{\mathbf{a}}\big(E \cap (E - k)\big) > \delta\right\}$$

  is syndetic

Consequence

If T is ufhc (resp. rhc), then $T \times \cdots \times T$ is ufhc (resp. rhc).

Main references

F. Bayart and I. Z. Ruzsa
Difference sets and frequently hypercyclic weighted shifts
Ergod. Theory Dyn. Syst., **35**, 691–709, 2015.

J. Bès, Q. Menet, A. Peris and Y. Puig
Recurrence properties of hypercyclic operators
Math. Ann. **366**, 545–572, 2016.

A. Bonilla and K-G. Grosse-Erdmann
Upper frequent hypercyclicity and related notions
Rev. Mat. Complut., **31**, 673–711, 2018.

R. Ernst and A. Mouze
A quantitative interpretation of the frequent hypercyclicity criterion
Ergod. Theory Dyn. Syst., https://doi.org/10.1017/etds.2017.55, 2017.

Q. Menet
Linear chaos and frequent hypercyclicity
Trans. Amer. Math. Soc. **369**, 4977–4994, 2017.

S. Shkarin
On the spectrum of frequently hypercyclic operators
Proc. Amer. Math. Soc. **137**, 123–134, 2009.