The formal inverse of the period-doubling word
Joint work with Narad Rampersad (University of Winnipeg)

Manon Stipulanti (ULiège) FRIA grantee

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Initial problem

Take

- a prime number $p$
- a $p$-automatic sequence $(s_n)_{n \geq 0}$
- its generating function $S(X) = \sum_{n=0}^{+\infty} s_nX^n \in \mathbb{F}_p[[X]]$
- the compositional inverse $T(X) = \sum_{n=0}^{+\infty} t_nX^n \in \mathbb{F}_p[[X]]$ of $S$ (provided it exists), i.e.

$$S(T(X)) = X = T(S(X)).$$

Questions:

1. What can be said about $(t_n)_{n \geq 0}$?
2. What can be said about the sequences

$$\{m \in \mathbb{N} \mid t_m = r\}$$

for $r = 0, 1, \ldots, p - 1$?
History

- Prouhet–Thue–Morse sequence  
  (M. Gawron and M. Ulas, 2016)
- Variations of the Baum–Sweet sequence  
  (Ł. Merta, 2018)
- Generalized Thue–Morse sequences  
  (Ł. Merta, 2018)
- Variations of the Rudin–Shapiro sequence  
  (Ł. Merta, 2018)
- Period-doubling sequence  
  (N. Rampersad and M. S., 2018) ★
Abstract numeration system

An abstract numeration system (ANS) is a triple $S = (L, A, <)$ where $L$ is an infinite regular language over a totally ordered alphabet $(A, <)$.

**$S$-representation:** $\text{rep}_S(n)$ is the $(n + 1)$st word in the genealogically ordered language $L$.

**$S$-numerical value:** inverse map $\text{val}_S : L \to \mathbb{N}$

- base-$k$ numeration system
  $L = \{1, \ldots, k - 1\}\{0, 1, \ldots, k - 1\}^* \cup \{\varepsilon\}$
  $A = \{0, 1, \ldots, k - 1\}, \ 0 < 1 < \cdots < k - 1$

- Zeckendorff numeration system
  based on Fibonacci numbers: $1, 2, 3, 5, 8, 13, 21, 34, \ldots$
  $L_F = 1\{0, 01\}^* \cup \{\varepsilon\}$
  $A_F = \{0, 1\}, \ 0 < 1$
**$S$-automatic sequence**

$S = (L, A, <)$ an ANS

An infinite word $w = w_0w_1w_2 \cdots \in B^\mathbb{N}$ is $S$-automatic if there exists a DFAO $A = (Q, q_0, A, \delta, B, \mu)$ such that

$$w_n = \mu(\delta(q_0, \text{rep}_S(n))) \quad \forall n \geq 0.$$  

The automaton $A$ is called a $S$-DFAO.
Example: Thue–Morse word $t = (t_n)_{n \geq 0}$

$t_n$ counts the number of 1’s (mod 2) in $\text{rep}_2(n)$

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{rep}_2(n)$</td>
<td>$\varepsilon$</td>
<td>1</td>
<td>10</td>
<td>11</td>
<td>100</td>
<td>101</td>
<td>110</td>
<td>111</td>
<td>1000</td>
<td>1001</td>
<td>1010</td>
</tr>
<tr>
<td>$t_n$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
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</tr>
</tbody>
</table>

$t$ is 2-automatic:

$Q = \{q_0, q_1\}$
$A = \{0, 1\}$
$B = \{0, 1\}$
$\mu: q_0 \mapsto 0, q_1 \mapsto 1$

$\mu(\delta(q_0, \text{rep}_2(5))) = \mu(\delta(q_0, 101)) = \mu(q_0) = 0 = t_5$
Period-doubling word $d = (d_n)_{n \geq 0}$

$$d_n = \nu_2(n + 1) \mod 2$$

where $\nu_2$ is the exponent of the highest power of 2 dividing its argument

\[
\begin{array}{c|ccccccccccccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
 n + 1 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
 \nu_2(n + 1) & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 3 & 0 & 1 & 0 & 2 & 0 & 1 & 0 \\
 d_n & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
\end{array}
\]

$d$ is 2-automatic:

\[
\begin{array}{c}
q_0/0 \\
0
\end{array}
\xrightarrow{0}

\begin{array}{c}
q_1/1 \\
1
\end{array}
\xrightarrow{1}

\begin{array}{c}
0, 1
\end{array}

Q = \{q_0, q_1\}
A = \{0, 1\}
B = \{0, 1\}
\mu: q_0 \mapsto 0, q_1 \mapsto 1
Characteristic sequences in $d$

$$d = 0100010100010001000 \cdots$$

Sequence of 1’s: $o = (o_n)_{n \geq 0}$

$$\{o_n \mid n \in N\} = \{m \in N \mid d_m = 1\}$$

$\begin{align*}
o &= 1, 5, 7, 9, 13, 17, 21, 23, 25, 29, 31, 33, 37, 39, 41, 45, 49, 53, 55, 57, \ldots
\end{align*}$

Sequence of 0’s: $z = (z_n)_{n \geq 0}$

$$\{z_n \mid n \in N\} = \{m \in N \mid d_m = 0\}$$

$z = 0, 2, 3, 4, 6, 8, 10, 11, 12, 14, 15, 16, 18, 19, 20, 22, 24, 26, 27, 28, 30, \ldots$
In Sloane’s On-Line Encyclopedia of Integer Sequences (OEIS):

- **A079523**

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
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<tbody>
<tr>
<td>$o_n$</td>
<td>1</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>13</td>
<td>17</td>
<td>21</td>
<td>23</td>
</tr>
<tr>
<td>$\text{rep}_2(o_n)$</td>
<td>1</td>
<td>101</td>
<td>111</td>
<td>1001</td>
<td>1101</td>
<td>10001</td>
<td>10101</td>
<td>10111</td>
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</table>

The binary expansion of $o_n$ ends with an odd number of 1’s.

- **A121539**

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<tr>
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<tbody>
<tr>
<td>$z_n$</td>
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</tr>
<tr>
<td>$\text{rep}_2(z_n)$</td>
<td>ε</td>
<td>10</td>
<td>11</td>
<td>100</td>
<td>110</td>
<td>1000</td>
<td>1010</td>
<td>1011</td>
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</tbody>
</table>

The binary expansion of $z_n$ ends with an even number of 1’s.
How to handle the case of infinite alphabets?

Let \( u = (u_n)_{n \geq 0} \) be an infinite sequence and \( k \geq 2 \) be an integer.
The \( k \)-kernel of \( u \) is the set of subsequences

\[
\mathcal{K}_k(u) = \{(u_{k^i \cdot n + r})_{n \geq 0} \mid i \geq 0 \text{ and } 0 \leq r < k^i\}.
\]

A sequence \( u \) is \( k \)-regular if there exists a finite set \( S \) of sequences such that every sequence in \( \mathcal{K}_k(u) \) is a \( \mathbb{Z} \)-linear combination of sequences of \( S \).
Example: \( S^2 = (S^2_n)_{n \geq 0} \)

\[
S^2_n = \# \left\{ m \in \mathbb{N} \mid \text{rep}_2(m) \text{ is a scattered subword subsequence of } \text{rep}_2(n) \right\}
\]

\( S^2 = 1, 2, 3, 3, 4, 5, 5, 4, 5, 7, 8, 7, 8, 7, 8, 7, 5, 6, 9, 11, 10, 11, 13, \ldots \)

**Theorem [J. Leroy, M. Rigo and M. S. (2017)]**

\( S^2 \) is 2-regular.

**Remark: [J.-P. Allouche and J. Shallit, The Bible (2003)]**

A sequence is \( k \)-regular and takes on only finitely many values \( \iff \) it is \( k \)-automatic.
Proposition

$z$ is not $k$-regular for any $k \in \mathbb{N}_{k \geq 2}$.

Idea of the proof:

- Exchange morphism $E: 0 \mapsto 1, 1 \mapsto 0 \quad \bar{d} = E(d)$
- $\bar{d}$ is the first difference modulo 2 of $t$

\[ \bar{d} = (t_{n+1} - t_n \mod 2)_{n \geq 0} \]

- $z$ describes the positions in $t$ where 0 and 1 alternate

\[ t = 011010011001011010010 \cdots \]
\[ z = 0, 2, 3, 4, 6, 8, 10, 11, 12, 14, 15, 16, 18, 19, \ldots \]

- The first difference of $z$ (= first difference between the positions of 1’s in $\bar{d}$) gives the length of the blocks of consecutive identical letters in $t$ (= sequence of run lengths of $t$).
- $p = $ the sequence of run lengths of $t$
  - $p$ not $k$-regular for any $k \in \mathbb{N}_{k \geq 2}$
- $z$ not $k$-regular for any $k \in \mathbb{N}_{k \geq 2}$
Proposition

$o$ is not $k$-regular for any $k \in \mathbb{N}_{k\geq 2}$.
The formal inverse of the period-doubling word $d$

Generating function of $d$: $D(X) = \sum_{n \geq 0} d_n X^n$

\[
\begin{align*}
  d_0 &= 0 \\
  d_1 &= 1 \text{ invertible in } \mathbb{F}_2
\end{align*}
\Rightarrow D(X) \text{ invertible in } \mathbb{F}_2[[X]],
\]

i.e., there exists a series

\[
U(X) = \sum_{n \geq 0} u_n X^n \in \mathbb{F}_2[[X]]
\]

such that $D(U(X)) = X = U(D(X))$.

**Question:** What does $u = (u_n)_{n \geq 0}$ look like?
Lemma

Over $\mathbb{F}_2[[X]]$, $D(X) = \sum_{n \geq 0} d_n X^n$ satisfies

$$X(1 + X^2)D(X)^2 + (1 + X^2)D(X) + X = 0.$$ 

Proof: $d = h^\omega(0)$ where $h : 0 \mapsto 01, 1 \mapsto 00$

$$\Rightarrow \begin{cases} 
d_{2n} = 0 \ (h : 0 \mapsto 01, 1 \mapsto 00), \\
d_{2n+1} = 1 - d_n \ \forall \ n \geq 0 \ (h : 0 \mapsto 01, 1 \mapsto 00). 
\end{cases}$$

Thus

$$D(X) = \sum_{n \geq 0} d_n X^n = \sum_{n \geq 0} d_{2n} X^{2n} + \sum_{n \geq 0} d_{2n+1} X^{2n+1}$$

$$\quad = X \sum_{n \geq 0} X^{2n} - X \sum_{n \geq 0} d_n X^{2n}.$$ 

We have $1/(1 - X) = \sum_{n \geq 0} X^n$.

Consequently,

$$D(X) = \frac{X}{1 - X^2} - XD(X^2).$$
From
\[ D(X) = \frac{X}{1 - X^2} - XD(X^2), \]
working over \( \mathbb{F}_2[[X]] \) gives
\[ X(1 + X^2)D(X^2) + (1 + X^2)D(X) + X = 0. \]

For any prime \( p \) and for any series \( F(X) \) in \( \mathbb{F}_p[[X]] \), we have \( F(X)^p = F(X^p) \).
Thus
\[ X(1 + X^2)D(X)^2 + (1 + X^2)D(X) + X = 0, \]
as desired.
Proposition

Over $\mathbb{F}_2[[X]]$, $U(X) = \sum_{n \geq 0} u_n X^n$ satisfies

\[
X^2 U(X)^3 + XU(X)^2 + (X^2 + 1)U(X) + X = 0,
\]
\[
X^3 U(X)^4 + X^3 U(X)^2 + U(X) + X = 0.
\]

In particular, $\mathbf{u} = (u_n)_{n \geq 0}$ verifies $u_0 = 0$, $u_1 = 1$, and over $\mathbb{F}_2$,

\[
\begin{cases}
  u_{2n} = 0 & \forall n \geq 0, \\
  u_{4n+1} = u_{2n-1} & \forall n \geq 1, \\
  u_{4n+3} = u_n & \forall n \geq 0.
\end{cases}
\]
First equation: Rewrite the equation

\[ X(1 + X^2)D(X)^2 + (1 + X^2)D(X) + X = 0 \]

in terms of \( X \):

\[ D(X)^2X^3 + D(X)X^2 + (D(X)^2 + 1)X + D(X) = 0. \]

Replace \( X \) by \( U(X) \):

\[ D(U(X))^2U(X)^3 + D(U(X))U(X)^2 + (D(U(X))^2 + 1)U(X) + D(U(X)) = 0. \]

Since \( U(X) \) is the formal inverse of \( D(X) \),

\[ X^2U(X)^3 + Xu(X)^2 + (X^2 + 1)U(X) + X = 0. \]

Second equation: Work a bit.
Recurrence relations for $u$:
Write $U(X) = \sum_{n \geq 0} u_n X^n$ in the second equation

$$X^3 \sum_{n \geq 0} u_n X^{4n} + X^3 \sum_{n \geq 0} u_n X^{2n} + \sum_{n \geq 0} u_n X^n + X = 0$$

$\iff \sum_{n \geq 0} u_n X^{4n+3} + \sum_{n \geq 0} u_n X^{2n+3} + \sum_{n \geq 0} u_n X^n + X = 0.$

Inspection of the coefficients (over $\mathbb{F}_2$):

- $u_0 = 0$ and $u_1 = 1$
- $4n + 3$ and $2n + 3$ odd $\Rightarrow u_{2n} = 0$
- coefficient of $X^{4n+3}$ for $n \geq 0$
  $$u_n + u_{2n} + u_{4n+3} = 0 \Rightarrow u_{4n+3} = u_n$$
- coefficient of $X^{4n+1}$ for $n \geq 1$
  $$u_{2n-1} + u_{4n+1} = 0 \Rightarrow u_{4n+1} = u_{2n-1}$$
The sequence $u = (u_n)_{n \geq 0}$ is referred to as the *inverse period-doubling sequence*, iPd sequence for short.

OEIS tag: A317542

$$u = 01000101000001000001000001000001000001000001000\cdots$$

**Corollary**

$u = (u_n)_{n \geq 0}$ is 2-automatic.

**Proof:** The formal power series $U(X)$ is algebraic over $\mathbb{F}_2(X)$. By Christol’s theorem, $u$ is thus 2-automatic.
This automaton reads its input from least significant digit to most significant digit.
Characteristic sequence in $\mathbf{u}$

Sequence of 1’s: $\mathbf{a} = (a_n)_{n \geq 0}$

$\{a_n \mid n \in \mathbb{N}\} = \{m \in \mathbb{N} \mid u_m = 1\}$

$a = 1, 5, 7, 13, 17, 23, 29, 31, 37, 49, 55, 61, 65, 71, 77, 95, 101, 113, \ldots$

OEIS tag: A317543

Remarks:

- From the previous proposition, $\mathbf{a}$ only contains odd integers.
- In the 2-DFAO generating $\mathbf{u}$, if the states outputting 1 are considered to be final, then

  $$L_a = \{\text{rep}_2(a_n) \mid n \geq 0\} = \{11\}^*1 \cup 1\{1, 00\}^*0\{11\}^*1.$$

Examples: $\text{rep}_2(a_0) = 1, \text{rep}_2(a_1) = 101, \text{rep}_2(a_2) = 111, \text{rep}_2(a_3) = 1101$
\[ L_a = \{1, 101, 111, 1101, 10001, 10111, 11101, 11111, 100101, \ldots \} \]

**Fibonacci numbers:** \((F(n))_{n \geq 0}\)

\[ F(0) = 0, \quad F(1) = 1, \quad F(n) = F(n - 1) + F(n - 2) \quad \forall n \geq 2 \]

**Proposition**

The complexity function \(\rho_{L_a} : \mathbb{N} \to \mathbb{N}\) of the language \(L_a\) satisfies

\[
\begin{align*}
\rho_{L_a}(0) &= 0, \\
\rho_{L_a}(2n) &= F(2n - 1) - 1 \quad \forall n \geq 1, \\
\rho_{L_a}(2n + 1) &= F(2n) + 1 \quad \forall n \geq 0.
\end{align*}
\]

**Idea of the proof:** It follows from the automaton generating \(u\).
Let $n \geq 0$ and let $w_n = \text{rep}_2(a_n)$.

If $w_n \in L_{a,1}$, or if $w_n \in L_{a,2}$ and $|w_n|$ is even, then $a_n \mod 3 \equiv 1$.

If $w_n \in L_{a,2}$ and $|w_n|$ is odd, then $a_n \mod 3 \equiv 2$. 

<table>
<thead>
<tr>
<th>$n$</th>
<th>$a_n$</th>
<th>$\text{rep}_2(a_n)$</th>
<th>$L_a$</th>
<th>$a_n \mod 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$L_{a,1}$</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>101</td>
<td>$L_{a,2}$</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>111</td>
<td>$L_{a,1}$</td>
<td>1</td>
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<tr>
<td>3</td>
<td>13</td>
<td>1101</td>
<td>$L_{a,2}$</td>
<td>1</td>
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<td>4</td>
<td>17</td>
<td>10001</td>
<td>$L_{a,2}$</td>
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<td>5</td>
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<td>10111</td>
<td>$L_{a,2}$</td>
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<td>6</td>
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<td>1</td>
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<tr>
<td>8</td>
<td>37</td>
<td>100101</td>
<td>$L_{a,2}$</td>
<td>1</td>
</tr>
</tbody>
</table>
\[ a = 1, 5, 7, 13, 17, 23, 29, 31, 37, 49, 55, 61, 65, 71, 77, 95, 101, 113, \ldots \]
\[(a_n \mod 3)_{n \geq 0} = 1, 2, 1, 2, 2, 2, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, \ldots \]

**Proposition**

The sequence \((a_n \mod 3)_{n \geq 0}\) is given by the infinite word

\[ 1F(1)2F(2)1F(3)2F(4)1F(5)2F(6)\ldots. \]

In particular, the sequence of run lengths of \((a_n \mod 3)_{n \geq 0}\) is the sequence of Fibonacci numbers \((F(n))_{n \geq 1}\).

**Idea of the proof:** It follows from the complexity result and the previous lemma.
First difference in \((a_n \mod 3)_{n \geq 0}\): \(\delta = (\delta_n)_{n \geq 0}\)

\[
\delta_n = \begin{cases} 
1 & \text{if } (a_{n+1} - a_n) \mod 3 \neq 0 \\
0 & \text{otherwise}
\end{cases}
\]

\((a_n \mod 3)_{n \geq 0} = 1, 2, 1, 1, 2, 2, 2, 1, 1, 1, 2, 2, 2, 2, 2, \ldots\)

\(\delta = 1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, \ldots\)

Characteristic sequence of Fibonacci numbers \((F(n))_{n \geq 2}\): \(x\)

\[
x_n = \begin{cases} 
1 & \text{if } n \text{ is a Fibonacci number} \\
0 & \text{otherwise}
\end{cases}
\]

\(x = 0111010010000100000001\ldots\)

Then \(\delta = (x_n)_{n \geq 2}\).
Particular ANS: $(L_F, \{0, 1\}, <)$ with $0 < 1$ and

$$L_F = \{\varepsilon\} \cup 1\{0, 01\}^*$$ (Fibonacci representations)

A DFA $\mathcal{A}$ accepting the regular language $L_F$:
Lemma

\( x \) is Fibonacci-automatic.

Proof: The following Fibonacci-DFAO \( B \) generates the sequence \( x \) in the Zeckendorff numeration system.

\[
\begin{array}{cccc}
0 & \xrightarrow{1} & 0 & \xrightarrow{1} & 0, 1 \\
0_0 & & 1 & & 0_1 \\
\end{array}
\]

In particular, \( x \) is Fibonacci-automatic.
A morphism \( \sigma : A^* \rightarrow A^* \) is prolongable on a letter \( a \in A \) if

- \( \sigma(a) = au \) with \( u \in A^+ \)
- \( \lim_{n \rightarrow +\infty} |\sigma^n(a)| = +\infty. \)

If \( \sigma \) is prolongable on \( a \), then \( \sigma^n(a) \) is a proper prefix of \( \sigma^{n+1}(a) \)

\[ (\sigma^n(a))_{n \geq 0} \] converges to an infinite word \( w \) (fixed point of \( \sigma \)).

In this case, the word \( w \) is called **pure morphic**.

A **morphic** word is the morphic image of a pure morphic word.

**Examples:**

- Thue–Morse \( t = \tau^\omega(0) \) where \( \tau : 0 \mapsto 01, 1 \mapsto 10 \)
- Period-doubling \( d = h^\omega(0) \) where \( h : 0 \mapsto 01, 1 \mapsto 00 \)
**Theorem** [M. Rigo (2000), M. Rigo and A. Maes (2002)]

An infinite word $w$ is morphic if and only if $w$ is $S$-automatic for some ANS $S$.

**Consequence:** $x$ is morphic

How to build morphisms that generate $x$?

$\Rightarrow$ Constructive proof of the theorem
Lemma

Let $f : \{z, a_0, a_1, \ldots, a_7\}^* \to \{z, a_0, a_1, \ldots, a_7\}^*$ be the morphism defined by $f(z) = za_0$ and

$$
\begin{array}{cccccccc}
 i & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
 f(a_i) & a_1a_2 & a_1a_4 & a_3a_7 & a_3a_6 & a_4a_7 & a_5a_6 & a_5a_7 & a_7a_7 \\
\end{array}
$$

Let $g : \{z, a_0, a_1, \ldots, a_7\}^* \to \{0, 1\}^*$ be the morphism defined by

$$
g(z) = g(a_1) = g(a_4) = g(a_7) = \varepsilon,
g(a_0) = g(a_5) = g(a_6) = 0,
g(a_2) = g(a_3) = 1.
$$

Then $x = g(f^\omega(z))$. 
Proof:
- the DFA $\mathcal{A}$ accepts the language $L_F = \{\varepsilon\} \cup \{0, 01\}^*$
- the Fibonacci-DFAO $\mathcal{B}$ generates $\mathbf{x}$
- product automaton $\mathcal{P} = \mathcal{A} \times \mathcal{B}$:
Set

\[ a_0 = (A, 0_0), a_1 = (E, 0_0), a_2 = (B, 1), a_3 = (C, 1), \]
\[ a_4 = (E, 1), a_5 = (C, 0_1), a_6 = (D, 0_1), a_7 = (E, 0_1). \]

Associated morphism \( \psi_P : \{ z, a_0, a_1, \ldots, a_7 \}^* \rightarrow \{ z, a_0, a_1, \ldots, a_7 \}^* \)
with \( P \) defined by

\[ \psi_P(z) = za_0 \]

and

\[ \psi_P(a_i) = \delta_P(a_i, 0)\delta_P(a_i, 1) \]

<table>
<thead>
<tr>
<th>( i )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \psi_P(a_i) )</td>
<td>( a_1a_2 )</td>
<td>( a_1a_4 )</td>
<td>( a_3a_7 )</td>
<td>( a_3a_6 )</td>
<td>( a_4a_7 )</td>
<td>( a_5a_6 )</td>
<td>( a_5a_7 )</td>
<td>( a_7a_7 )</td>
</tr>
</tbody>
</table>

where \( \delta_P \) is the transition function of \( P \). Then \( \psi_P = f \).

The morphism \( g : \{ z, a_0, a_1, \ldots, a_7 \}^* \rightarrow \{ 0, 1 \}^* \) is defined by

\[ z, a_1, a_4, a_7 \mapsto \varepsilon; a_0, a_5, a_6 \mapsto 0; a_2, a_3 \mapsto 1. \]

Then \( x = g(f^\omega(z)) \) (\( x \) is morphic).
Problem: $g$ is erasing ($\exists a \in \{z, a_0, a_1, \ldots, a_7\}$ s.t. $g(a) = \varepsilon$)

**Lemma** "Getting rid of erasing morphisms" [É Charlier, J. Leroy and M. Rigo (2016)]

Let $w = g(f^\omega(a))$ be a morphic word where $g: B^* \to A^*$ is a (possibly erasing) morphism and $f: B^* \to B^*$ is a non-erasing morphism.

Let $C$ be a subalphabet of $\{b \in B \mid g(b) = \varepsilon\}$ such that $f_C$ is a submorphism of $f$.

Let $\lambda_C: B^* \to B^*$ be the morphism defined by

$$\lambda_C(b) = \begin{cases} 
\varepsilon & \text{if } b \in C \\
 b & \text{otherwise}
\end{cases}$$

The morphisms $f_\varepsilon = (\lambda_C \circ f)\mid_{(B \setminus C)^*}$ and $g_\varepsilon = g\mid_{(B \setminus C)^*}$ are such that $w = g_\varepsilon(f_\varepsilon(a))$. 
Proposition

Let $\phi: \{a, b, c, d, e\}^* \to \{a, b, c, d, e\}^*$ be the morphism defined by

\[
\begin{align*}
a &\mapsto ab, b &\mapsto c, c &\mapsto ce, d &\mapsto de, e &\mapsto d
\end{align*}
\]

and let $\mu: \{a, b, c, d, e\}^* \to \{0, 1\}^*$ be the coding defined by

\[
\begin{align*}
a, d, e &\mapsto 0; b, c &\mapsto 1.
\end{align*}
\]

Then $x = \mu(\phi^\omega(a))$.

Idea of the proof: Making use of the two previous lemmas.
Let $M$ be a matrix with coefficients in $\mathbb{N}$.

$\exists$ permutation matrix $P$ such that

$$P^{-1}MP$$ upper block-triangular matrix

with diagonal square blocks $M_1, \ldots, M_s$ irreducible or zeroes.

The Perron–Frobenius eigenvalue $\lambda_M$ of $M$

$$\lambda_M = \max_{1 \leq i \leq s} \lambda_{M_i}$$

where $\lambda_{M_i}$ is the Perron–Frobenius eigenvalue of the matrix $M_i$.

Let $f : A^* \to A^*$ be a prolongable morphism with fixed point $w$.

Let $\alpha$ be the Perron–Frobenius eigenvalue of $M_f$.

If all letters of $A$ occur in $w$, then $w$ is (pure) $\alpha$-substitutive.

If $g : A^* \to B^*$ is a coding, then $g(w)$ is $\alpha$-substitutive.
Corollary

Let \( \varphi = \frac{1}{2}(\sqrt{5} + 1) \) be the golden ratio.
The word \( \mathbf{x} \) is \( \varphi \)-substitutive.

Proof: Let

\[
M_\varphi = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

be the matrix associated with the morphism \( \varphi \).
The Perron–Frobenius eigenvalue of \( M_\varphi \) is \( \varphi = \frac{1}{2}(\sqrt{5} + 1) \).
All letters \( a, b, c, d, e \) occur in \( \varphi^\omega(a) \)

\[
\varphi^\omega(a) = abcceced \cdots
\]

Thus \( \mathbf{x} \) is \( \varphi \)-substitutive.
Proposition

\( x \) is not \( k \)-automatic for any \( k \in \mathbb{N}_{\geq 2} \).

Proof: Proceed by contradiction and suppose that there exists an integer \( k \geq 2 \) such that \( x \) is \( k \)-automatic.

By Cobham’s theorem, \( x \) is also \( k \)-substitutive.

(Not difficult to see that the Perron–Frobenius eigenvalue of the matrix associated with a \( k \)-uniform morphism is the integer \( k \).)

Clearly, \( k \) and \( \varphi \) are multiplicatively independent.

Thus, by Cobham-Durand’s theorem, \( x \) is ultimately periodic.

This is a contradiction.
Theorem [Cobham (1972)]

An infinite word $w \in B^\mathbb{N}$ is $k$-automatic if and only if there exist a $k$-uniform morphism $f : A^* \to A^*$ prolongable on a letter $a \in A$ and a coding $g : A^* \to B^*$ such that $w = g(f^\omega(a))$.

Two real numbers $\alpha, \beta > 1$ are multiplicatively independent if

$$m, n \in \mathbb{N} \text{ with } \alpha^m = \beta^n \Rightarrow m = n = 0.$$ 

Otherwise, $\alpha$ and $\beta$ are multiplicatively dependent.

Theorem [Durand (2011)]

Let $\alpha, \beta > 1$ be two multiplicatively independent real numbers. Let $u$ be a pure $\alpha$-substitutive word. Let $v$ be a pure $\beta$-substitutive word. Let $g$ and $g'$ be two non-erasing morphisms. If $w = g(u) = g'(v)$, then $w$ is ultimately periodic. In particular, if an infinite word is both $\alpha$-and $\beta$-substitutive, i.e., in the special case where $g$ and $g'$ are codings, then it is ultimately periodic.
Characteristic sequence of 1’s in $u$: $a = (a_n)_{n \geq 0}$

$$\{a_n \mid n \in N\} = \{m \in N \mid u_m = 1\}$$

**Corollary**

$a$ is not $k$-regular for any $k \in \mathbb{N}_{\geq 2}$.

**Proof**: Suppose that $a$ is $k$-regular for some $k \geq 2$. Then the sequence $(a_n \mod 3)_{n \geq 0}$ is $k$-automatic (by stability properties), so is $\delta$ and consequently also $x$. This contradicts the previous proposition.
Characteristic sequence in $u$

**Sequence of 0’s:** $b = (b_n)_{n \geq 0}$

$$\{b_n \mid n \in \mathbb{N}\} = \{m \in \mathbb{N} \mid u_m = 0\}$$

$b = 0, 2, 3, 4, 6, 8, 9, 10, 11, 12, 14, 15, 16, 18, 19, 20, 21, 22, 24, 25, 26, \ldots$

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**Open problem:** Is the sequence $b$ $k$-regular for some $k \geq 2$?


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