The formal inverse of the period-doubling word

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Take

- a prime number p
- a p-automatic sequence $(s_n)_{n\geq 0}$
- its generating function $S(X) = \sum_{n=0}^{+\infty} s_n X^n \in \mathbb{F}_p[[X]]$
- the compositional inverse $T(X) = \sum_{n=0}^{+\infty} t_n X^n \in \mathbb{F}_p[[X]]$ of S (provided it exists), i.e.

$$S(T(X)) = X = T(S(X)).$$

Questions:

1. What can be said about $(t_n)_{n\geq 0}$?
2. What can be said about the sequences

$$\{m \in \mathbb{N} \mid t_m = r\}$$

for $r = 0, 1, \ldots, p-1$?

History

- Prouhet–Thue–Morse sequence
  (M. Gawron and M. Ulas, 2016)
- Variations of the Baum–Sweet sequence
  (Ł. Merta, 2018)
- Generalized Thue–Morse sequences
  (Ł. Merta, 2018)
- Variations of the Rudin–Shapiro sequence
  (Ł. Merta, 2018)
- Period-doubling sequence
  (N. Rampersad and M. S., 2018) ★

Abstract numeration system

An *abstract numeration system* (ANS) is a triple $S = (L, A, <)$ where L is an infinite regular language over a totally ordered alphabet $(A, <)$.

S-representation: $\mathrm{rep}_S(n)$ is the $(n+1)$st word in the genealogically ordered language L.

S-numerical value: inverse map $\mathrm{val}_S\colon L \to \mathbb{N}$

- base-k numeration system
  $L = \{1, \ldots, k-1\}\{0, 1, \ldots, k-1\}^* \cup \{\varepsilon\}$
  $A = \{0, 1, \ldots, k-1\}$, $0 < 1 < \cdots < k-1$
- Zeckendorff numeration system
  based on Fibonacci numbers: $1, 2, 3, 5, 8, 13, 21, 34, \ldots$
  $L_F = 1\{0, 01\}^* \cup \{\varepsilon\}$
  $A_F = \{0, 1\}$, $0 < 1$

S-automatic sequence

$S = (L, A, <)$ an ANS
An infinite word $\boldsymbol{w} = w_0 w_1 w_2 \cdots \in B^{\mathbb{N}}$ is *S-automatic* if there exists a DFAO $\mathcal{A} = (Q, q_0, A, \delta, B, \mu)$ such that

$$w_n = \mu(\delta(q_0, \text{rep}_S(n))) \quad \forall\, n \geq 0.$$

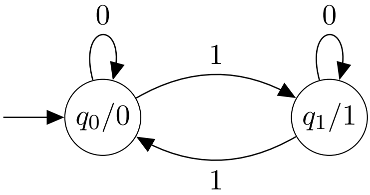
The automaton $\mathcal{A}$ is called a *S-DFAO*.

Example: Thue–Morse word $\boldsymbol{t} = (t_n)_{n\geq 0}$

$$t_n \text{ counts the number of 1's (mod2) in } \mathrm{rep}_2(n)$$

n	0	1	2	3	4	5	6	7	8	9	10
$\mathrm{rep}_2(n)$	ε	1	10	11	100	101	110	111	1000	1001	1010
t_n	0	1	1	0	1	0	0	1	1	0	0

$\boldsymbol{t}$ is 2-automatic:



$$Q = \{q_0, q_1\}$$
$$A = \{0, 1\}$$
$$B = \{0, 1\}$$
$$\mu\colon q_0 \mapsto 0, q_1 \mapsto 1$$

$$\mu(\delta(q_0, \mathrm{rep}_2(5))) = \mu(\delta(q_0, 101)) = \mu(q_0) = 0 = t_5$$

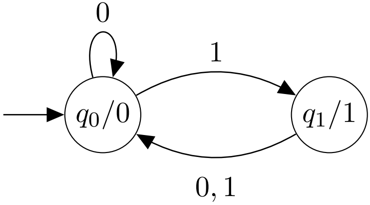
Period-doubling word $\boldsymbol{d} = (d_n)_{n\geq 0}$

$$d_n = \nu_2(n+1) \bmod 2$$

where ν_2 is the exponent of the highest power of 2 dividing its argument

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$n+1$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\nu_2(n+1)$	0	1	0	2	0	1	0	3	0	1	0	2	0	1	0
d_n	0	1	0	0	0	1	0	1	0	1	0	0	0	1	0

$\boldsymbol{d}$ is 2-automatic:



$$Q = \{q_0, q_1\}$$
$$A = \{0, 1\}$$
$$B = \{0, 1\}$$
$$\mu\colon q_0 \mapsto 0, q_1 \mapsto 1$$

Characteristic sequences in $\boldsymbol{d}$

$$\boldsymbol{d} = 010001010100010001000 \cdots$$

Sequence of 1's: $\boldsymbol{o} = (o_n)_{n \geq 0}$

$$\{o_n \mid n \in N\} = \{m \in N \mid d_m = 1\}$$

$\boldsymbol{o} = 1, 5, 7, 9, 13, 17, 21, 23, 25, 29, 31, 33, 37, 39, 41, 45, 49, 53, 55, 57, \ldots$

Sequence of 0's: $\boldsymbol{z} = (z_n)_{n \geq 0}$

$$\{z_n \mid n \in N\} = \{m \in N \mid d_m = 0\}$$

$\boldsymbol{z} = 0, 2, 3, 4, 6, 8, 10, 11, 12, 14, 15, 16, 18, 19, 20, 22, 24, 26, 27, 28, 30, \ldots$

In Sloane's On-Line Encyclopedia of Integer Sequences (OEIS):

- A079523

n	0	1	2	3	4	5	6	7
o_n	1	5	7	9	13	17	21	23
$\mathrm{rep}_2(o_n)$	1	101	111	1001	1101	10001	10101	10111

The binary expansion of o_n ends with an odd number of 1's.

- A121539

n	0	1	2	3	4	5	6	7
z_n	0	2	3	4	6	8	10	11
$\mathrm{rep}_2(z_n)$	ε	10	11	100	110	1000	1010	1011

The binary expansion of z_n ends with an even number of 1's.

Regular sequences

$\rightsquigarrow$ How to handle the case of infinite alphabets?

Let $\boldsymbol{u} = (u_n)_{n\geq 0}$ be an infinite sequence and $k \geq 2$ be an integer.
The *k-kernel* of $\boldsymbol{u}$ is the set of subsequences

$$\mathcal{K}_k(\boldsymbol{u}) = \{(u_{k^i \cdot n + r})_{n\geq 0} \mid i \geq 0 \text{ and } 0 \leq r < k^i\}.$$

k-regular sequence

A sequence $\boldsymbol{u}$ is *k-regular* if there exists a finite set S of sequences such that every sequence in $\mathcal{K}_k(\boldsymbol{u})$ is a $\mathbb{Z}$-linear combination of sequences of S.

Example: $\boldsymbol{S^2} = (S_n^2)_{n\geq 0}$

$$S_n^2 = \#\left\{ m \in \mathbb{N} \mid \text{rep}_2(m) \text{ is a } \begin{cases} \text{scattered subword} \\ \text{subsequence} \end{cases} \text{of } \text{rep}_2(n) \right\}$$

$$\boldsymbol{S^2} = 1, 2, 3, 3, 4, 5, 5, 4, 5, 7, 8, 7, 7, 8, 7, 5, 6, 9, 11, 10, 11, 13, \ldots$$

Theorem [J. Leroy, M. Rigo and M. S. (2017)]

$\boldsymbol{S^2}$ is 2-regular.

Remark: [J.-P. Allouche and J. Shallit, The Bible (2003)]
A sequence is k-regular and takes on only finitely many values $\Leftrightarrow$ it is k-automatic.

Proposition

$\boldsymbol{z}$ is not k-regular for any $k \in \mathbb{N}_{k\geq 2}$.

Idea of the proof:

- Exchange morphism $E\colon 0 \mapsto 1, 1 \mapsto 0$ $\qquad \bar{\boldsymbol{d}} = E(\boldsymbol{d})$
- $\bar{\boldsymbol{d}}$ is the first difference modulo 2 of $\boldsymbol{t}$

$$\bar{\boldsymbol{d}} = (t_{n+1} - t_n \bmod 2)_{n\geq 0}$$

- $\boldsymbol{z}$ describes the positions in $\boldsymbol{t}$ where 0 and 1 alternate

$$\boldsymbol{t} = 01101001100101101001 0\cdots$$

$$\boldsymbol{z} = 0, 2, 3, 4, 6, 8, 10, 11, 12, 14, 15, 16, 18, 19, \ldots$$

- The first difference of $\boldsymbol{z}$ (= first difference between the positions of 1's in $\bar{\boldsymbol{d}}$) gives the length of the blocks of consecutive identical letters in $\boldsymbol{t}$ (= sequence of run lengths of $\boldsymbol{t}$).
- $\boldsymbol{p}$ = the sequence of run lengths of $\boldsymbol{t}$
  $\boldsymbol{p}$ not k-regular for any $k \in \mathbb{N}_{k\geq 2}$
- $\boldsymbol{z}$ not k-regular for any $k \in \mathbb{N}_{k\geq 2}$

Proposition

$\boldsymbol{o}$ is not k-regular for any $k \in \mathbb{N}_{k\geq 2}$.

The formal inverse of the period-doubling word $\boldsymbol{d}$

Generating function of $\boldsymbol{d}$: $D(X) = \sum_{n\geq 0} d_n X^n$

$$\left.\begin{array}{r} d_0 = 0 \\ d_1 = 1 \text{ invertible in } \mathbb{F}_2 \end{array}\right\} \Rightarrow D(X) \text{ invertible in } \mathbb{F}_2[[X]],$$

i.e., there exists a series

$$U(X) = \sum_{n\geq 0} u_n X^n \in \mathbb{F}_2[[X]]$$

such that $D(U(X)) = X = U(D(X))$.

Question: What does $\boldsymbol{u} = (u_n)_{n\geq 0}$ look like?

Lemma

Over $\mathbb{F}_2[[X]]$, $D(X) = \sum_{n\geq 0} d_n X^n$ satisfies

$$X(1+X^2)D(X)^2 + (1+X^2)D(X) + X = 0.$$

Proof: $\boldsymbol{d} = h^\omega(0)$ where $h\colon 0 \mapsto 01, 1 \mapsto 00$

$$\Rightarrow \begin{cases} d_{2n} = 0 \;\; (h\colon 0 \mapsto 01, 1 \mapsto 00), \\ d_{2n+1} = 1 - d_n \;\; \forall\, n \geq 0 \;\; (h\colon 0 \mapsto 01, 1 \mapsto 00). \end{cases}$$

Thus

$$D(X) = \sum_{n\geq 0} d_n X^n = \sum_{n\geq 0} d_{2n} X^{2n} + \sum_{n\geq 0} d_{2n+1} X^{2n+1}$$
$$= X \sum_{n\geq 0} X^{2n} - X \sum_{n\geq 0} d_n X^{2n}.$$

We have $1/(1-X) = \sum_{n\geq 0} X^n$.
Consequently,

$$D(X) = \frac{X}{1-X^2} - XD(X^2).$$

From

$$D(X) = \frac{X}{1 - X^2} - XD(X^2),$$

working over $\mathbb{F}_2[[X]]$ gives

$$X(1 + X^2)D(X^2) + (1 + X^2)D(X) + X = 0.$$

For any prime p and for any series $F(X)$ in $\mathbb{F}_p[[X]]$, we have $F(X)^p = F(X^p)$.
Thus

$$X(1 + X^2)D(X)^2 + (1 + X^2)D(X) + X = 0,$$

as desired.

Proposition

Over $\mathbb{F}_2[[X]]$, $U(X) = \sum_{n\geq 0} u_n X^n$ satisfies

$$X^2U(X)^3 + XU(X)^2 + (X^2+1)U(X) + X = 0,$$
$$X^3U(X)^4 + X^3U(X)^2 + U(X) + X = 0.$$

In particular, $\boldsymbol{u} = (u_n)_{n\geq 0}$ verifies $u_0 = 0$, $u_1 = 1$, and over $\mathbb{F}_2$,

$$\begin{cases} u_{2n} = 0 \quad \forall\, n \geq 0, \\ u_{4n+1} = u_{2n-1} \quad \forall\, n \geq 1, \\ u_{4n+3} = u_n \quad \forall\, n \geq 0. \end{cases}$$

First equation: Rewrite the equation

$$X(1+X^2)D(X)^2+(1+X^2)D(X)+X=0$$

in terms of X:

$$D(X)^2X^3+D(X)X^2+(D(X)^2+1)X+D(X)=0.$$

Replace X by $U(X)$:

$$D(U(X))^2U(X)^3+D(U(X))U(X)^2+(D(U(X))^2+1)U(X) + D(U(X))=0.$$

Since $U(X)$ is the formal inverse of $D(X)$,

$$X^2U(X)^3+XU(X)^2+(X^2+1)U(X)+X=0.$$

Second equation: Work a bit.

Recurrence relations for $\boldsymbol{u}$:

Write $U(X) = \sum_{n\geq 0} u_n X^n$ in the second equation

$$X^3 \sum_{n\geq 0} u_n X^{4n} + X^3 \sum_{n\geq 0} u_n X^{2n} + \sum_{n\geq 0} u_n X^n + X = 0$$

$$\Leftrightarrow \sum_{n\geq 0} u_n X^{4n+3} + \sum_{n\geq 0} u_n X^{2n+3} + \sum_{n\geq 0} u_n X^n + X = 0.$$

Inspection of the coefficients (over $\mathbb{F}_2$):

- $u_0 = 0$ and $u_1 = 1$
- $4n+3$ and $2n+3$ odd $\Rightarrow u_{2n} = 0$
- coefficient of X^{4n+3} for $n \geq 0$

$$u_n + u_{2n} + u_{4n+3} = 0 \Rightarrow u_{4n+3} = u_n$$

- coefficient of X^{4n+1} for $n \geq 1$

$$u_{2n-1} + u_{4n+1} = 0 \Rightarrow u_{4n+1} = u_{2n-1}$$

Automaticity of $\boldsymbol{u}$

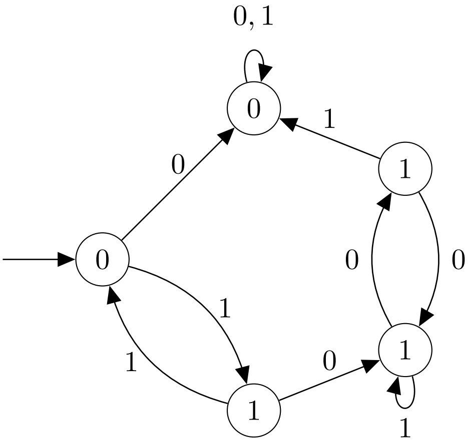
The sequence $\boldsymbol{u} = (u_n)_{n\geq 0}$ is referred to as the *inverse period-doubling sequence*, iPD sequence for short.
OEIS tag: A317542

$$\boldsymbol{u} = 0100010100000100010000001000000101000001000\cdots$$

Corollary

$\boldsymbol{u} = (u_n)_{n\geq 0}$ is 2-automatic.

Proof: The formal power series $U(X)$ is algebraic over $\mathbb{F}_2(X)$.
By Christol's theorem, $\boldsymbol{u}$ is thus 2-automatic.



⚠ This automaton reads its input from least significant digit to most significant digit.

Characteristic sequence in $\boldsymbol{u}$

Sequence of 1's: $\boldsymbol{a} = (a_n)_{n\geq 0}$

$$\{a_n \mid n \in N\} = \{m \in N \mid u_m = 1\}$$

$\boldsymbol{a} = 1, 5, 7, 13, 17, 23, 29, 31, 37, 49, 55, 61, 65, 71, 77, 95, 101, 113, \ldots$

OEIS tag: A317543

Remarks:

- From the previous proposition, $\boldsymbol{a}$ only contains odd integers.
- In the 2-DFAO generating $\boldsymbol{u}$, if the states outputting 1 are considered to be final, then

$$L_a = \{\text{rep}_2(a_n) \mid n \geq 0\} = \{11\}^*1 \cup 1\{1, 00\}^*0\{11\}^*1.$$

Examples: $\text{rep}_2(a_0) = 1$, $\text{rep}_2(a_1) = 101$, $\text{rep}_2(a_2) = 111$, $\text{rep}_2(a_3) = 1101$

$$L_a = \{1, 101, 111, 1101, 10001, 10111, 11101, 11111, 100101, \ldots\}$$

Fibonacci numbers: $(F(n))_{n \geq 0}$
$F(0) = 0,\ F(1) = 1,\ F(n) = F(n-1) + F(n-2)\ \ \forall\, n \geq 2$

Proposition

The complexity function $\rho_{L_a} : \mathbb{N} \to \mathbb{N}$ of the language L_a satisfies

$$\begin{aligned} \rho_{L_a}(0) &= 0, \\ \rho_{L_a}(2n) &= F(2n-1) - 1\ \ \forall\, n \geq 1, \\ \rho_{L_a}(2n+1) &= F(2n) + 1\ \ \forall\, n \geq 0. \end{aligned}$$

Idea of the proof: It follows from the automaton generating $\boldsymbol{u}$.

$$L_a = \{\text{rep}_2(a_n) \mid n \geq 0\} = \underbrace{\{11\}^*1}_{L_{a,1}} \cup \underbrace{1\{1,00\}^*0\{11\}^*1}_{L_{a,2}}$$

n	a_n	$\text{rep}_2(a_n)$	L_a	$a_n \bmod 3$
0	1	1	$L_{a,1}$	1
1	5	101	$L_{a,2}$	2
2	7	111	$L_{a,1}$	1
3	13	1101	$L_{a,2}$	1
4	17	10001	$L_{a,2}$	2
5	23	10111	$L_{a,2}$	2
6	29	11101	$L_{a,2}$	2
7	31	11111	$L_{a,1}$	1
8	37	100101	$L_{a,2}$	1

Lemma

Let $n \geq 0$ and let $w_n = \text{rep}_2(a_n)$.
If $w_n \in L_{a,1}$, or if $w_n \in L_{a,2}$ and $|w_n|$ is even, then $a_n \bmod 3 \equiv 1$.
If $w_n \in L_{a,2}$ and $|w_n|$ is odd, then $a_n \bmod 3 \equiv 2$.

$$\boldsymbol{a} = 1, 5, 7, 13, 17, 23, 29, 31, 37, 49, 55, 61, 65, 71, 77, 95, 101, 113, \ldots$$
$$(a_n \bmod 3)_{n\geq 0} = 1, 2, 1, 1, 2, 2, 2, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, \ldots$$

Proposition

The sequence $(a_n \bmod 3)_{n\geq 0}$ is given by the infinite word

$$1^{F(1)}2^{F(2)}1^{F(3)}2^{F(4)}1^{F(5)}2^{F(6)}\cdots .$$

In particular, the sequence of run lengths of $(a_n \bmod 3)_{n\geq 0}$ is the sequence of Fibonacci numbers $(F(n))_{n\geq 1}$.

Idea of the proof: It follows from the complexity result and the previous lemma.

First difference in $(a_n \bmod 3)_{n\geq 0}$: $\boldsymbol{\delta} = (\delta_n)_{n\geq 0}$

$$\delta_n = \begin{cases} 1 & \text{if } (a_{n+1} - a_n) \bmod 3 \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$(a_n \bmod 3)_{n\geq 0} = 1,2,1,1,2,2,2,1,1,1,1,1,2,2,2,2,2,2,\ldots$$
$$\boldsymbol{\delta} = 1,1,0,1,0,0,1,0,0,0,0,1,0,0,0,0,0,\ldots$$

Characteristic sequence of Fibonacci numbers $(F(n))_{n\geq 2}$: $\boldsymbol{x}$

$$x_n = \begin{cases} 1 & \text{if } n \text{ is a Fibonacci number} \\ 0 & \text{otherwise} \end{cases}$$

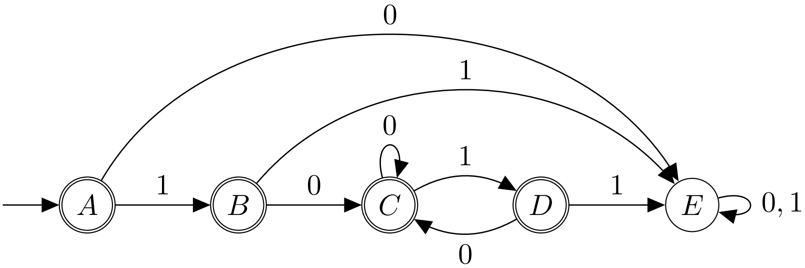
$$\boldsymbol{x} = 0111010010000100000001\cdots$$

Then $\boldsymbol{\delta} = (x_n)_{n\geq 2}$.

Particular ANS: $(L_F, \{0,1\}, <)$ with $0 < 1$ and

$$L_F = \{\varepsilon\} \cup 1\{0,01\}^* \quad \text{(Fibonacci representations)}$$

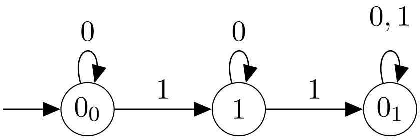
A DFA $\mathcal{A}$ accepting the regular language L_F:



Lemma

$\boldsymbol{x}$ is Fibonacci-automatic.

Proof: The following Fibonacci-DFAO $\mathcal{B}$ generates the sequence $\boldsymbol{x}$ in the Zeckendorff numeration system.



In particular, $\boldsymbol{x}$ is Fibonacci-automatic.

A morphism $\sigma\colon A^* \to A^*$ is *prolongable* on a letter $a \in A$ if

- $\sigma(a) = au$ with $u \in A^+$
- $\lim_{n\to+\infty} |\sigma^n(a)| = +\infty$.

If σ is prolongable on a, then $\sigma^n(a)$ is a proper prefix of $\sigma^{n+1}(a)$ $\Rightarrow$ $(\sigma^n(a))_{n\geq 0}$ converges to an infinite word $\boldsymbol{w}$ (fixed point of σ).

In this case, the word $\boldsymbol{w}$ is called *pure morphic*.
A *morphic* word is the morphic image of a pure morphic word.

Examples:

- Thue–Morse $\boldsymbol{t} = \tau^\omega(0)$ where $\tau\colon 0 \mapsto 01, 1 \mapsto 10$
- Period-doubling $\boldsymbol{d} = h^\omega(0)$ where $h\colon 0 \mapsto 01, 1 \mapsto 00$

Theorem [M. Rigo (2000), M. Rigo and A. Maes (2002)]

An infinite word $\boldsymbol{w}$ is morphic if and only if $\boldsymbol{w}$ is S-automatic for some ANS S.

<u>Consequence</u>: $\boldsymbol{x}$ is morphic

How to build morphisms that generate $\boldsymbol{x}$?
$\rightsquigarrow$ Constructive proof of the theorem

Lemma

Let $f\colon \{z, a_0, a_1, \ldots, a_7\}^* \to \{z, a_0, a_1, \ldots, a_7\}^*$ be the morphism defined by $f(z) = za_0$ and

i	0	1	2	3	4	5	6	7
$f(a_i)$	a_1a_2	a_1a_4	a_3a_7	a_3a_6	a_4a_7	a_5a_6	a_5a_7	a_7a_7

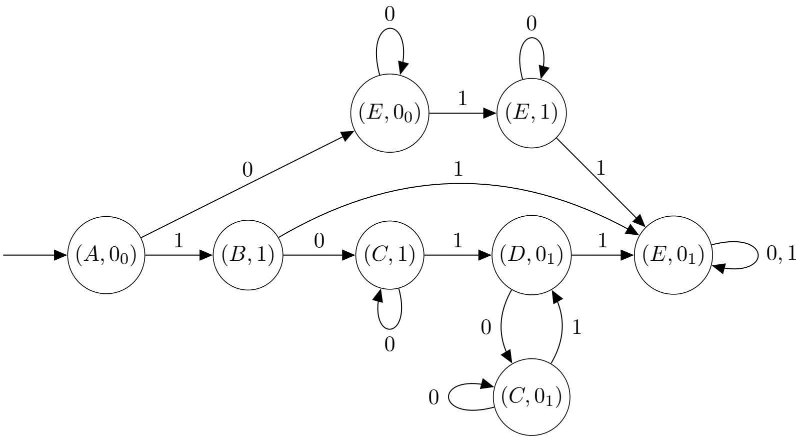
Let $g\colon \{z, a_0, a_1, \ldots, a_7\}^* \to \{0,1\}^*$ be the morphism defined by

$$g(z) = g(a_1) = g(a_4) = g(a_7) = \varepsilon,$$
$$g(a_0) = g(a_5) = g(a_6) = 0,$$
$$g(a_2) = g(a_3) = 1.$$

Then $\boldsymbol{x} = g(f^\omega(z))$.

Proof:

- the DFA $\mathcal{A}$ accepts the language $L_F = \{\varepsilon\} \cup 1\{0, 01\}^*$
- the Fibonacci-DFAO $\mathcal{B}$ generates $\boldsymbol{x}$
- product automaton $\mathcal{P} = \mathcal{A} \times \mathcal{B}$:



Set

$$a_0 = (A, 0_0), a_1 = (E, 0_0), a_2 = (B, 1), a_3 = (C, 1),$$
$$a_4 = (E, 1), a_5 = (C, 0_1), a_6 = (D, 0_1), a_7 = (E, 0_1).$$

Associated morphism $\psi_{\mathcal{P}} \colon \{z, a_0, a_1, \ldots, a_7\}^* \to \{z, a_0, a_1, \ldots, a_7\}^*$ with $\mathcal{P}$ defined by

$$\psi_{\mathcal{P}}(z) = z a_0$$

and

$$\psi_{\mathcal{P}}(a_i) = \delta_{\mathcal{P}}(a_i, 0)\delta_{\mathcal{P}}(a_i, 1)$$

i	0	1	2	3	4	5	6	7
$\psi_{\mathcal{P}}(a_i)$	a_1a_2	a_1a_4	a_3a_7	a_3a_6	a_4a_7	a_5a_6	a_5a_7	a_7a_7

where $\delta_{\mathcal{P}}$ is the transition function of $\mathcal{P}$. Then $\psi_{\mathcal{P}} = f$.
The morphism $g \colon \{z, a_0, a_1, \ldots, a_7\}^* \to \{0, 1\}^*$ is defined by

$$z, a_1, a_4, a_7 \mapsto \varepsilon; a_0, a_5, a_6 \mapsto 0; a_2, a_3 \mapsto 1.$$

Then $\boldsymbol{x} = g(f^\omega(z))$ ($\boldsymbol{x}$ is morphic).

Problem: g is erasing ($\exists\, a \in \{z, a_0, a_1, \ldots, a_7\}$ s.t. $g(a) = \varepsilon$)

Lemma "Getting rid of erasing morphisms" [É Charlier, J. Leroy and M. Rigo (2016)]

Let $\boldsymbol{w} = g(f^\omega(a))$ be a morphic word where $g\colon B^* \to A^*$ is a (possibly erasing) morphism and $f\colon B^* \to B^*$ is a non-erasing morphism.
Let C be a subalphabet of $\{b \in B \mid g(b) = \varepsilon\}$ such that f_C is a submorphism of f.
Let $\lambda_C\colon B^* \to B^*$ be the morphism defined by

$$\lambda_C(b) = \begin{cases} \varepsilon & \text{if } b \in C \\ b & \text{otherwise.} \end{cases}$$

The morphisms $f_\varepsilon = (\lambda_C \circ f)|_{(B\setminus C)^*}$ and $g_\varepsilon = g|_{(B\setminus C)^*}$ are such that $\boldsymbol{w} = g_\varepsilon(f_\varepsilon^\omega(a))$.

Proposition

Let $\phi\colon \{a,b,c,d,e\}^* \to \{a,b,c,d,e\}^*$ be the morphism defined by

$$a \mapsto ab, b \mapsto c, c \mapsto ce, d \mapsto de, e \mapsto d$$

and let $\mu\colon \{a,b,c,d,e\}^* \to \{0,1\}^*$ be the coding defined by

$$a,d,e \mapsto 0; b,c \mapsto 1.$$

Then $\boldsymbol{x} = \mu(\phi^\omega(a))$.

Idea of the proof: Making use of the two previous lemmas.

Let M be a matrix with coefficients in $\mathbb{N}$.
$\exists$ permutation matrix P such that

$$P^{-1}MP \text{ upper block-triangular matrix}$$

with diagonal square blocks $M_1, \ldots, M_s$ irreducible or zeroes.
The *Perron–Frobenius* eigenvalue λ_M of M

$$\lambda_M = \max_{1 \le i \le s} \lambda_{M_i}$$

where λ_{M_i} is the Perron–Frobenius eigenvalue of the matrix M_i.

Let $f \colon A^* \to A^*$ be a prolongable morphism with fixed point $\boldsymbol{w}$.
Let α be the Perron–Frobenius eigenvalue of M_f.
If all letters of A occur in $\boldsymbol{w}$, then $\boldsymbol{w}$ is *(pure) α-substitutive.*
If $g \colon A^* \to B^*$ is a coding, then $g(\boldsymbol{w})$ is *α-substitutive.*

Corollary

Let $\varphi = \frac{1}{2}(\sqrt{5}+1)$ be the golden ratio.
The word $\boldsymbol{x}$ is φ-substitutive.

Proof: Let

$$M_\phi = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

be the matrix associated with the morphism ϕ.
The Perron–Frobenius eigenvalue of M_ϕ is $\varphi = \frac{1}{2}(\sqrt{5}+1)$.
All letters a, b, c, d, e occur in $\phi^\omega(a)$

$$\phi^\omega(a) = abcceced \cdots$$

Thus $\boldsymbol{x}$ is φ-substitutive.

Proposition

$\boldsymbol{x}$ is not k-automatic for any $k \in \mathbb{N}_{\geq 2}$.

Proof: Proceed by contradiction and suppose that there exists an integer $k \geq 2$ such that $\boldsymbol{x}$ is k-automatic.
By Cobham's theorem, $\boldsymbol{x}$ is also k-substitutive.
(Not difficult to see that the Perron–Frobenius eigenvalue of the matrix associated with a k-uniform morphism is the integer k.)
Clearly, k and φ are multiplicatively independent.
Thus, by Cobham-Durand's theorem, $\boldsymbol{x}$ is ultimately periodic. This is a contradiction.

Theorem [Cobham (1972)]

An infinite word $\boldsymbol{w} \in B^{\mathbb{N}}$ is k-automatic if and only if there exist a k-uniform morphism $f\colon A^* \to A^*$ prolongable on a letter $a \in A$ and a coding $g\colon A^* \to B^*$ such that $\boldsymbol{w} = g(f^\omega(a))$.

Two real numbers $\alpha, \beta > 1$ are *multiplicatively independent* if

$$m, n \in \mathbb{N} \text{ with } \alpha^m = \beta^n \Rightarrow m = n = 0.$$

Otherwise, α and β are *multiplicatively dependent.*

Theorem [Durand (2011)]

Let $\alpha, \beta > 1$ be two multiplicatively independent real numbers.
Let $\boldsymbol{u}$ be a pure α-substitutive word
$\boldsymbol{v}$ be a pure β-substitutive word.
Let g and g' be two non-erasing morphisms.
If $\boldsymbol{w} = g(\boldsymbol{u}) = g'(\boldsymbol{v})$, then $\boldsymbol{w}$ is ultimately periodic.
In particular, if an infinite word is both α-and β-substitutive, i.e., in the special case where g and g' are codings, then it is ultimately periodic.

Characteristic sequence of 1's in $\boldsymbol{u}$: $\boldsymbol{a} = (a_n)_{n\geq 0}$

$$\{a_n \mid n \in N\} = \{m \in N \mid u_m = 1\}$$

Corollary

$\boldsymbol{a}$ is not k-regular for any $k \in \mathbb{N}_{\geq 2}$.

Proof: Suppose that $\boldsymbol{a}$ is k-regular for some $k \geq 2$.
Then the sequence $(a_n \bmod 3)_{n\geq 0}$ is k-automatic (by stability properties), so is $\boldsymbol{\delta}$ and consequently also $\boldsymbol{x}$.
This contradicts the previous proposition.

Characteristic sequence in $\boldsymbol{u}$

Sequence of 0's: $\boldsymbol{b} = (b_n)_{n \geq 0}$

$$\{b_n \mid n \in N\} = \{m \in N \mid u_m = 0\}$$

$\boldsymbol{b} = 0, 2, 3, 4, 6, 8, 9, 10, 11, 12, 14, 15, 16, 18, 19, 20, 21, 22, 24, 25, 26, \ldots$

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Open problem: Is the sequence $\boldsymbol{b}$ k-regular for some $k \geq 2$?

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