Extensions of the Pascal Triangle to Words, and Related Counting Problems

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Liège
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### Classical Pascal triangle

**Definition:**

Let \( P : (m, k) \in \mathbb{N} \times \mathbb{N} \mapsto \binom{m}{k} \in \mathbb{N} \)

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**Binomial coefficients of integers:**

\[
\binom{m}{k} = \frac{m!}{(m-k)!k!}
\]
A specific construction

- Grid: first $2^n$ rows and columns of the Pascal triangle

\[
\begin{pmatrix} \left( \binom{m}{k} \mod 2 \right) \end{pmatrix}_{0 \leq m, k < 2^n}
\]
A specific construction

- Grid: first $2^n$ rows and columns of the Pascal triangle
  $$\left( \begin{pmatrix} m \\ k \end{pmatrix} \mod 2 \right)_{0 \leq m, k < 2^n}$$

- Color each square in
  - white if $\left( \begin{pmatrix} m \\ k \end{pmatrix} \right) \equiv 0 \mod 2$
  - black if $\left( \begin{pmatrix} m \\ k \end{pmatrix} \right) \equiv 1 \mod 2$
A specific construction

- Grid: first $2^n$ rows and columns of the Pascal triangle

$$\left( \begin{array}{c} m \\ k \end{array} \right) \mod 2$$

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- Normalize by a homothety of ratio $1/2^n$
  (bring into $[0, 1]^2$)
A specific construction

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- Normalize by a homothety of ratio \( 1/2^n \)
  (bring into \([0, 1]^2\))

\( \rightsquigarrow \) sequence of compact sets belonging to \([0, 1]^2\)
The first six elements of the sequence

\[
\begin{array}{cccc}
0 & 1 & & \\
2 & 2 & & \\
0 & 2 & & \\
2 & 2 & & \\
0 & 2 & & \\
2 & 2 & & \\
\end{array}
\]

Generalized Pascal Triangles, and Related Counting Problems

M. Stipulanti (ULiège)
Folklore fact
The latter sequence of compact sets converges to the Sierpiński gasket (w.r.t. the Hausdorff distance).

Definitions:
• \( \epsilon \)-fattening of a subset \( S \subset \mathbb{R}^2 \)
  \[ [S]_\epsilon = \bigcup_{x \in S} B(x, \epsilon) \]
• \((\mathcal{H}(\mathbb{R}^2), d_h)\) complete space of the non-empty compact subsets of \( \mathbb{R}^2 \) equipped with the Hausdorff distance \( d_h \)
  \[ d_h(S, S') = \inf\{ \epsilon \in \mathbb{R}_{>0} \mid S \subset [S']_\epsilon \quad \text{and} \quad S' \subset [S]_\epsilon \} \]
Theorem (von Haeseler, Peitgen, and Skordev, 1992)

Let $p$ be a prime and $s > 0$.
The sequence of compact sets corresponding to

\[
\left( \left( \binom{m}{k} \mod p^s \right) \right)_{0 \leq m, k < p^n}
\]

converges when $n$ tends to infinity (w.r.t. the Hausdorff distance).

\begin{align*}
p = 2, & \quad s = 1 \\
p = 2, & \quad s = 2 \\
p = 2, & \quad s = 3
\end{align*}
Part I
Let $u, v$ be two finite words over the alphabet $A$. The *binomial coefficient* $\binom{u}{v}$ of $u$ and $v$ is the number of times $v$ occurs as a subsequence of $u$ (meaning as a “scattered” subword).
Binomial coefficient of finite words

Let $u, v$ be two finite words over the alphabet $A$. The binomial coefficient $\binom{u}{v}$ of $u$ and $v$ is the number of times $v$ occurs as a subsequence of $u$ (meaning as a “scattered” subword).

Example: $u = 101001$ \hspace{1cm} $v = 101$

\[
101001, 101001, 101001, 101001, 101001 \Rightarrow \binom{101001}{101} = 6
\]
Binomial coefficient of finite words

Let \( u, v \) be two finite words over the alphabet \( A \).
The \textit{binomial coefficient} \( \binom{u}{v} \) of \( u \) and \( v \) is the number of times \( v \) occurs as a subsequence of \( u \) (meaning as a “scattered” subword).

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Natural generalization:

\[
\binom{a^m}{a^k} = \binom{\underbrace{a \cdots a}_{m \text{ times}}}{\underbrace{a \cdots a}_{k \text{ times}}} = \binom{m}{k} \quad \forall \ m, k \in \mathbb{N}
\]
Let \((A, <)\) be a totally ordered alphabet.
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Let \(L \subseteq A^*\) be an infinite language over \(A\).
The words in \(L\) are genealogically ordered

\[ w_0 <_{\text{gen}} w_1 <_{\text{gen}} w_2 <_{\text{gen}} \cdots. \]
Generalized Pascal triangles

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The \textit{generalized Pascal triangle} \(P_L\) associated with \(L\) is defined by

\[ P_L: (m, k) \in \mathbb{N} \times \mathbb{N} \mapsto \binom{w_m}{w_k} \in \mathbb{N}. \]
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Questions:

- With a similar construction, can we expect the convergence to an analogue of the Sierpiński gasket?
- In particular, where should we cut to normalize a given generalized Pascal triangle?
- Could we describe this limit object?
Definition

A *numeration system* is a sequence $U = (U(n))_{n \geq 0}$ of integers s.t.

- $U$ increasing
- $U(0) = 1$
- $\sup_{n \geq 0} \frac{U(n+1)}{U(n)}$ bounded by a constant $\leadsto$ finite alphabet.
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A numeration system $U$ is \emph{linear} if $\exists k \geq 1, \exists a_0, \ldots, a_{k-1} \in \mathbb{Z}$ s.t.

$$U(n + k) = a_{k-1} U(n + k - 1) + \cdots + a_0 U(n) \quad \forall n \geq 0.$$
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Greedy representation in $(U(n))_{n \geq 0}$:

$$n = \sum_{i=0}^{\ell} c_i U(i) \quad \text{with} \quad \sum_{i=0}^{j-1} c_i U(i) < U(j)$$

$$\text{rep}_U(n) = c_\ell \cdots c_0 \in \mathcal{L}_U = \text{rep}_U(\mathbb{N})$$

numeration language
Parry numbers

\( \beta \in \mathbb{R}_{>1} \quad A_\beta = \{0, 1, \ldots, \lceil \beta \rceil - 1\} \)

\[ x \in [0, 1] \mapsto x = \sum_{j=1}^{+\infty} c_j \beta^{-j}, \quad c_j \in A_\beta \]

Greedy way: \( c_j \beta^{-j} + c_{j+1} \beta^{-j-1} + \cdots < \beta^{-(j-1)} \)

\( \beta \)-expansion of \( x \): \( d_\beta(x) = c_1 c_2 c_3 \cdots \)
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**Definition**

\( \beta \in \mathbb{R}_{>1} \) is a *Parry number* if \( d_\beta(1) \) is ultimately periodic.

**Example:** \( b \in \mathbb{N}_{>1} \): \( d_b(1) = (b - 1)^\omega \)

Golden ratio \( \varphi \): \( d_\varphi(1) = 110^\omega \)
Parry numeration system

Parry number $\beta \in \mathbb{R}_{>1} \rightarrow$ linear numeration system $(U_{\beta}(n))_{n \geq 0}$

- $d_{\beta}(1) = t_1 \cdots t_m 0^\omega$

  \[
  \begin{align*}
  U_{\beta}(0) &= 1 \\
  U_{\beta}(i) &= t_1 U_{\beta}(i-1) + \cdots + t_i U_{\beta}(0) + 1 \quad \forall 1 \leq i \leq m-1 \\
  U_{\beta}(n) &= t_1 U_{\beta}(n-1) + \cdots + t_m U_{\beta}(n-m) \quad \forall n \geq m
  \end{align*}
  \]

- $d_{\beta}(1) = t_1 \cdots t_m (t_{m+1} \cdots t_{m+k})^\omega$

  \[
  \begin{align*}
  U_{\beta}(0) &= 1 \\
  U_{\beta}(i) &= t_1 U_{\beta}(i-1) + \cdots + t_i U_{\beta}(0) + 1 \quad \forall 1 \leq i \leq m+k-1 \\
  U_{\beta}(n) &= t_1 U_{\beta}(n-1) + \cdots + t_{m+k} U_{\beta}(n-m-k) + U_{\beta}(n-k) \\
  &\quad - t_1 U_{\beta}(n-k-1) - \cdots - t_m U_{\beta}(n-m-k)
  \end{align*}
  \]

Examples:

- $b \in \mathbb{N}_{>1} \rightarrow (b^n)_{n \geq 0}$ base $b$
- Golden ratio $\varphi \rightarrow (F(n))_{n \geq 0}$ Fibonacci numeration system
Parry number $\beta \in \mathbb{R}_{>1} \leadsto$ linear numeration system $U_\beta$

$\leadsto P_\beta: (m, k) \in \mathbb{N} \times \mathbb{N} \mapsto \left(\begin{array}{c} \text{rep}_{U_\beta}(m) \\ \text{rep}_{U_\beta}(k) \end{array} \right) \in \mathbb{N}$
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**Examples:**

**Base-2 numeration system**

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$L_2 = 1\{0, 1\}^* \cup \{\varepsilon\}$

**Fibonacci numeration system**

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$L_\varphi = 1\{01, 0\}^* \cup \{\varepsilon\}$
Special case of generalized Pascal triangles

Parry number $\beta \in \mathbb{R}_{>1} \Rightarrow$ linear numeration system $U_\beta$

$\Rightarrow P_\beta : (m, k) \in \mathbb{N} \times \mathbb{N} \mapsto \left(\text{rep}_{U_\beta}(m), \text{rep}_{U_\beta}(k)\right) \in \mathbb{N}$

Examples:

Base-2 numeration system

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</tr>
</tbody>
</table>

$L_\phi = 1\{01, 0\}^* \cup \{\varepsilon\}$

Remark: Copies of the usual Pascal triangle
Sequence of compact sets (first $U_\beta(n)$ rows and columns of $P_\beta$) in $[0,1]^2$:

$$U_\beta^n = \frac{1}{U_\beta(n)} \bigcup_{u,v \in LU_\beta, |u|,|v| \leq n} \text{val}_{U_\beta}(v,u) + [0,1]^2$$

$(\frac{u}{v}) \equiv 1 \mod 2$

Examples: Base-2 numeration system

Generalized Pascal Triangles, and Related Counting Problems

M. Stipulanti (ULiège)
Sequence of compact sets (first $U_\beta(n)$ rows and columns of $P_\beta$) in $[0, 1]^2$:

$$U_\beta^U = \frac{1}{U_\beta(n)} \bigcup_{u,v \in L_{U_\beta}, |u|, |v| \leq n} \text{val}_{U_\beta}(v, u) + [0, 1]^2$$

$(\binom{u}{v}) \equiv 1 \mod 2$

Examples: Fibonacci numeration system

---

**Generalized Pascal Triangles, and Related Counting Problems**

M. Stipulanti (ULiège)
Base-2 numeration system

Lines of slopes: $2^n, n \geq 0$

Fibonacci numeration system

Lines of slopes: $\varphi^n, n \geq 0$

General case: Lines of slopes: $\beta^n, n \geq 0$
\[ p(u, v) \in \mathbb{N} \text{ s.t. } u0^p(u,v)w, v0^p(u,v)w \in L_{U_\beta} \text{ for all } w \in 0^*L_{U_\beta} \]

\[(\star) \]

\((u, v)\) satisfies \((\star)\) iff \(u = v = \varepsilon\) or
\[
\begin{cases}
    u, v \neq \varepsilon \\
    (u0^p(u,v)) \equiv 1 \pmod{2} \\
    (v0^p(u,v)) = 0 \\
    (u0^p(u,v))_a = 0 \quad \forall a \in A_{U_\beta}.
\end{cases}
\]
• The (⋆) condition describes lines of slope 1 in $[0, 1]^2$.

$$(u, v) \in L_{U_\beta} \times L_{U_\beta} \text{ satisfying (⋆) \implies closed segment } S_{u,v}$$

- slope 1
- length $\sqrt{2} \cdot \beta^{-|u|-p(u,v)}$
- origin $A_{u,v} = (0.0|u|-|v|, 0.0)$
The (*) condition describes lines of slope 1 in $[0, 1]^2$.

$(u, v) \in L_{U\beta} \times L_{U\beta}$ satisfying (*)

$\rightsquigarrow$ closed segment $S_{u,v}$

- slope 1
- length $\sqrt{2} \cdot \beta^{-|u|} - p(u,v)$
- origin $A_{u,v} = (0.0|u| - |v|, 0.u)$

New compact set containing those lines:

$A_0^\beta = \bigcup_{(u,v) \text{ satisfying } (\ast)} S_{u,v} \subset [0, 1]^2$
- Two maps $c: (x, y) \mapsto (\frac{x}{\beta}, \frac{y}{\beta})$ and $h: (x, y) \mapsto (x, \beta y)$

Example:
- Two maps \( c : (x, y) \mapsto \left( \frac{x}{\beta}, \frac{y}{\beta} \right) \) and \( h : (x, y) \mapsto (x, \beta y) \)

**Example:**

\[ \begin{array}{c|c|c|c}
\hline
0 & 1 & \beta & 1 \\
\hline
1 & \beta & 1 & \beta \\
\hline
\end{array} \]
Two maps $c: (x, y) \mapsto \left(\frac{x}{\beta}, \frac{y}{\beta}\right)$ and $h: (x, y) \mapsto (x, \beta y)$

Example:
• Two maps $c: (x, y) \mapsto (\frac{x}{\beta}, \frac{y}{\beta})$ and $h: (x, y) \mapsto (x, \beta y)$

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Example:
• Two maps \( c: (x, y) \mapsto \left( \frac{x}{\beta}, \frac{y}{\beta} \right) \) and \( h: (x, y) \mapsto (x, \beta y) \)

Example:

\[
\begin{array}{cccc}
0 & \frac{1}{\beta} & \frac{1}{\beta} & 1 \\
1 & \frac{1}{\beta} & \frac{1}{\beta} & 1 \\
\ldots & \ldots & \ldots & \ldots \\
1 & 1 & 1 & 1
\end{array}
\]

• New compact set containing lines of slopes \( 1, \beta, \ldots, \beta^n \):

\[
\mathcal{A}^\beta_n = \bigcup_{0 \leq i \leq n, 0 \leq j \leq i} h^j(c^i(\mathcal{A}^\beta_0)) \subset [0, 1]^2.
\]
- Two maps \( c: (x, y) \mapsto \left( \frac{x}{\beta}, \frac{y}{\beta} \right) \) and \( h: (x, y) \mapsto (x, \beta y) \)

**Example:**

![Graph showing two maps]  

- New compact set containing lines of slopes \( 1, \beta, \ldots, \beta^n \): 
  
  \[
  \mathcal{A}_n^\beta = \bigcup_{0 \leq i \leq n, 0 \leq j \leq i} h^j(c^i(\mathcal{A}_0^\beta)) \subset [0, 1]^2.
  \]

- The compact sets \((\mathcal{A}_n^\beta)_{n \geq 0}\) are increasingly nested and their union is bounded.

\((\mathcal{A}_n^\beta)_{n \geq 0}\) converges to \(\mathcal{L}_n^\beta = \bigcup_{n \geq 0} \mathcal{A}_n^\beta\) w.r.t. the Hausdorff distance.
(\mathcal{U}_n^\beta)_{n\geq 0} \text{ converges to } \mathcal{L}^\beta \text{ w.r.t. the Hausdorff distance.}

3 \rightsquigarrow \mathcal{L}^3 \quad \varphi \rightsquigarrow \mathcal{L}^\varphi \quad \varphi^2 \rightsquigarrow \mathcal{L}^{\varphi^2}

\beta_1 \approx 2.47098 \rightsquigarrow \mathcal{L}^{\beta_1} \quad \beta_2 \approx 1.38028 \rightsquigarrow \mathcal{L}^{\beta_2} \quad \beta_3 \approx 2.80399 \rightsquigarrow \mathcal{L}^{\beta_3}
Theorem

Let $p$ be a prime and $r \in \{1, \ldots, p - 1\}$. When considering binomial coefficients congruent to $r \pmod{p}$, the sequence $(U_{n,p,r}^\beta)_{n \geq 0}$ converges to a well-defined compact set $L_{p,r}^\beta$ w.r.t. the Hausdorff distance.

Example: $L_{3,1}^2 \cup L_{3,2}^2$
Part II
**Example: \( P_\varphi \) (Fibonacci numeration system)**

<table>
<thead>
<tr>
<th>( \varepsilon )</th>
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<th>10</th>
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<th>1010</th>
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<th>( S_\varphi(n) )</th>
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</tbody>
</table>

**Generalized Pascal Triangles, and Related Counting Problems**

M. Stipulanti (ULiège)
Classical Pascal triangle: $S(n) = n + 1 \quad \forall n \geq 0$
Classical Pascal triangle: \( S(n) = n + 1 \quad \forall n \geq 0 \)

Base-2 numeration system:
\[ S_2 = 1, 2, 3, 3, 4, 5, 5, 4, 5, 7, 8, 7, 7, 8, 7, 5, 6, 9, 11, 10, 11, 13, 12, \ldots \]

OEIS tag: A007306

Fibonacci numeration system:
\[ S_\phi = 1, 2, 3, 4, 4, 5, 6, 6, 6, 8, 9, 8, 8, 7, 10, 12, 12, 12, 10, 12, 12, 8, 12, \ldots \]

OEIS tag: A282717
Example: Sum-of-digits function $\text{Sum}_2$ in base 2

$\text{Sum}_2(n) = \#$ of 1’s in $\text{rep}_2(n)$

<table>
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<tr>
<th>$n$</th>
<th>0</th>
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2-kernel $\mathcal{K}_2(\text{Sum}_2)$:
Example: Sum-of-digits function $\text{Sum}_2$ in base 2

$\text{Sum}_2(n) = \# \text{ of 1's in } \text{rep}_2(n)$

<table>
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2-kernel $\mathcal{K}_2(\text{Sum}_2)$:

<table>
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<tr>
<th>Suffix</th>
<th>$(\text{Sum}<em>2(n))</em>{n \geq 0}$</th>
<th>Suffix 1</th>
<th>$(\text{Sum}<em>2(2n + 1))</em>{n \geq 0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Suffix</td>
<td>$(\text{Sum}<em>2(2n))</em>{n \geq 0}$</td>
<td>Suffix 01</td>
<td>$(\text{Sum}<em>2(4n + 1))</em>{n \geq 0}$</td>
</tr>
<tr>
<td>Suffix 00</td>
<td>$(\text{Sum}<em>2(4n))</em>{n \geq 0}$</td>
<td>Suffix 11</td>
<td>$(\text{Sum}<em>2(4n + 3))</em>{n \geq 0}$</td>
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<tr>
<td>Suffix 10</td>
<td>$(\text{Sum}<em>2(4n + 2))</em>{n \geq 0}$</td>
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</tbody>
</table>

$\text{Sum}_2(2n) = \text{Sum}_2(n) + \text{Sum}_2(n)$

$\text{Sum}_2(2n + 1) = \text{Sum}_2(2n) + 1$

$\text{Sum}_2$ is 2-regular:

Sequences in $\mathcal{K}_2(\text{Sum}_2)$ are $\mathbb{Z}$-linear combinations of $\text{Sum}_2$. For example:

$\text{Sum}_2(4n + 1) = \text{Sum}_2(2(2n) + 1) = \text{Sum}_2(2n) + 1 = \text{Sum}_2(n) + 1$
Regularity

Example: Sum-of-digits function \( \text{Sum}_2 \) in base 2

\( \text{Sum}_2(n) = \# \text{ of 1's in } \text{rep}_2(n) \)

<table>
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<tr>
<th>( n )</th>
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<th>2</th>
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2-kernel \( K_2(\text{Sum}_2) \):

- Suffix \( \varepsilon \) \( (\text{Sum}_2(n))^n \geq 0 \)
- Suffix 0 \( (\text{Sum}_2(2n))^n \geq 0 \)
- Suffix 00 \( (\text{Sum}_2(4n))^n \geq 0 \)
- Suffix 10 \( (\text{Sum}_2(4n + 2))^n \geq 0 \)
- Suffix 1 \( (\text{Sum}_2(2n + 1))^n \geq 0 \)
- Suffix 01 \( (\text{Sum}_2(4n + 1))^n \geq 0 \)
- Suffix 11 \( (\text{Sum}_2(4n + 3))^n \geq 0 \)
Example: Sum-of-digits function $\text{Sum}_2$ in base 2

$\text{Sum}_2(n) = \# \text{ of 1's in } \text{rep}_2(n)$

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2-kernel $\mathcal{K}_2(\text{Sum}_2)$:

- Suffix $\varepsilon \quad (\text{Sum}_2(n))_{n \geq 0}$
- Suffix 0 \quad (\text{Sum}_2(2n))_{n \geq 0}
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- Suffix 11 \quad (\text{Sum}_2(4n + 3))_{n \geq 0}$
Example: Sum-of-digits function \( \text{Sum}_2 \) in base 2

\( \text{Sum}_2(n) = \# \) of 1’s in \( \text{rep}_2(n) \)

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<th>( n )</th>
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2-kernel \( K_2(\text{Sum}_2) \):

- Suffix \( \varepsilon \) \( (\text{Sum}_2(n))_{n \geq 0} \)
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Regularity

Example: Sum-of-digits function $\text{Sum}_2$ in base 2
$\text{Sum}_2(n) = \# \text{ of } 1\text{'s in } \text{rep}_2(n)$

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2-kernel $K_2(\text{Sum}_2)$:

- Suffix $\varepsilon$ \((\text{Sum}_2(n))_{n\geq0}\)
- Suffix 0 \((\text{Sum}_2(2n))_{n\geq0}\)
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- Suffix 11 \((\text{Sum}_2(4n+3))_{n\geq0}\)

$\text{Sum}_2(2n) = \text{Sum}_2(n)$  \hspace{1cm} $\text{Sum}_2(2n+1) = \text{Sum}_2(n)+1$

$\text{Sum}_2$ 2-regular:
sequences in $K_2(\text{Sum}_2)$ are $\mathbb{Z}$-linear combinations of $\text{Sum}_2$, 1

e.g. $\text{Sum}_2(4n+1) = \text{Sum}_2(2(2n)+1) = \text{Sum}_2(2n)+1 = 1 \cdot \text{Sum}_2(n)+1 \cdot 1$
Regularity: general case

Let $U = (U(n))_{n \geq 0}$ be a numeration system.
Let $s = (s(n))_{n \geq 0}$ be a sequence.

- The sequence $(s(i_w(n)))_{n \geq 0}$ is called the subsequence of $s$ with least significant digits equal to $w$ w.r.t. representations in the numeration system $U$.
- The $U$-kernel of $s$ is the set
  \[ K_U(s) = \{(s(i_w(n)))_{n \geq 0} \mid \text{for all suffixes } w\} \].
- A sequence $s$ of integers is $U$-regular if there exists a finite number of sequences $t_1 = (t_1(n))_{n \geq 0}$, ..., $t_\ell = (t_\ell(n))_{n \geq 0}$ s.t. every sequence in the $U$-kernel $K_U(s)$ is a $\mathbb{Z}$-linear combination of the sequences $t_1, \ldots, t_\ell$. 
Method

Define, study, use a new tree structure called *tries of scattered subwords*. 

\[ \Rightarrow \text{easily count/enumerate scattered subwords} \]

**Example:** in base 2

Scattered subwords of 10110:

\[ \varepsilon, 1, 10, 11, 100, 101, 110, 111, \\
   1010, 1011, 1110, 10110 \]

\[ S_2(\text{val}_2(10110)) = S_2(22) = \# \text{ nodes} = 12 \]

Internal structure: subtree of root 11 \( \cong \) subtree of root 101

Consequence: Study trees to deduce properties of \( S_\beta \)
Define, study, use a new tree structure called *tries of scattered subwords*. Easily count/enumerate scattered subwords

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\textbf{Example:} in base 2

\[
\begin{align*}
S_2(\text{val}_2(10110)) &= S_2(22) = \# \text{ nodes} = 12
\end{align*}
\]

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![Tree Diagram](image)

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Results

Integer base case:

- The sequence \((S_b(n))_{n \geq 0}\) satisfies recurrence relations.
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- The sequence \((S_b(n))_{n \geq 0}\) satisfies recurrence relations.
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- The sequence $(S_\varphi(n))_{n \geq 0}$ satisfies recurrence relations.
- The sequence $(S_\varphi(n))_{n \geq 0}$ is Fibonacci-regular.
- We obtain a matrix representation for the sequence $(S_\varphi(n))_{n \geq 0}$. 
• Connection with the Farey tree (every reduced positive rational less than 1 exactly once).

Let \( w \in 1\{0, 1\}^* \) with \( \text{val}_2(w) = 2^k + r \).

\[
\begin{array}{c}
0 & 1 & \cdots & w & S_2(r) \\
1 & & & & S_2(2^k+r)
\end{array}
\]

• The Stern–Brocot sequence \((SB(n))_{n \geq 0}\) contains the numerators and denominators in the Farey tree. Then \( S_2(n) = SB(2n + 1) \) for all \( n \geq 0 \).
Part III
Behavior of summatory functions

Example: Sum-of-digits function $\text{Sum}_2$ in base 2

$\text{Sum}_2(n) = \# \text{ of } 1\text{'s in } \text{rep}_2(n)$

$\text{Sum}_2$ is 2-regular

Summatory function $A$ of $\text{Sum}_2$

$$A(n) = \sum_{j=0}^{n-1} \text{Sum}_2(j) \quad \forall n \geq 0$$

Theorem (Delange, 1975)

There exists a continuous nowhere differentiable periodic function $\mathcal{G}$ of period 1 s.t.

$$\frac{A(n)}{n} = \frac{1}{2} \log_2 n + \mathcal{G}(\log_2 n).$$
Summatory functions of $b$-regular sequences
⇝ algebraic or analytic methods

New method: to tackle the behavior of the summatory function

Let $s = (s(n))_{n \geq 0}$ be a regular sequence

- Find $r = (r(n))_{n \geq 0}$ and $t = (t(n))_{n \geq 0}$ each satisfying a linear recurrence relation,
- verifying $A_s \circ r = t$.

Relevant representations of $A_s$ in some exotic numeration system associated with $t$.

"Exotic" ⇝ possibly unbounded coefficients

The behavior of $A_s$ depends on

- the dominant root of the characteristic polynomial of the linear recurrence relation defining $t$,
- a periodic fluctuation.

Definition: $A_\beta = (A_\beta(n))_{n \geq 0}$ is the summatory function of $S_\beta$.
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New method: to tackle the behavior of the summatory function

$$A_s = (A_s(n))_{n \geq 0}$$

of a regular sequence $s = (s(n))_{n \geq 0}$
Summatory functions of $b$-regular sequences
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New method: to tackle the behavior of the summatory function
\[ A_s = (A_s(n))_{n \geq 0} \] of a regular sequence \[ s = (s(n))_{n \geq 0} \]

- Find \[ r = (r(n))_{n \geq 0} \] and \[ t = (t(n))_{n \geq 0} \]
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\[
\leadsto \text{algebraic or analytic methods}
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\[
A_s = (A_s(n))_{n \geq 0}
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- Find $r = (r(n))_{n \geq 0}$ and $t = (t(n))_{n \geq 0}$
  - each satisfying a linear recurrence relation,
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- Recurrence relation for $s$
\[
\leadsto \text{recurrence relation for } A_s \text{ in which } t \text{ is involved.}
\]
Summatory functions of $b$-regular sequences
⇝ algebraic or analytic methods

**New method:** to tackle the behavior of the summatory function

\[ A_s = (A_s(n))_{n \geq 0} \text{ of a regular sequence } s = (s(n))_{n \geq 0} \]

- Find \( r = (r(n))_{n \geq 0} \) and \( t = (t(n))_{n \geq 0} \)
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  ⇝ recurrence relation for \( A_s \) in which \( t \) is involved.

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  “Exotic” ⇝ possibly unbounded coefficients
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New method: to tackle the behavior of the summatory function
\[ A_s = (A_s(n))_{n \geq 0} \]
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- The behavior of \( A_s \) depends on
  - the dominant root of the characteristic polynomial of the linear
    recurrence relation defining \( t \),
  - a periodic fluctuation.
Summatory functions of $b$-regular sequences
⇝ algebraic or analytic methods

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• The sequence \((A_b(n))_{n \geq 0}\) is \(b\)-regular.
Results in the integer base case

- The sequence \((A_b(n))_{n \geq 0}\) is \(b\)-regular.
- Matrix representation for the sequence \((A_b(n))_{n \geq 0}\).

\(\text{(2b-1)-decomposition of } A_b: \text{mixing the base } b \text{ and the base } 2b-1\)
Results in the integer base case

- The sequence \((A_b(n))_{n \geq 0}\) is \(b\)-regular.
- Matrix representation for the sequence \((A_b(n))_{n \geq 0}\).
- For all \(n \geq 0\), \(A_b\left(\begin{array}{c} b^n \\ r(n) \\ t(n) \end{array}\right) = (2b - 1)^n\).
Results in the integer base case

- The sequence \((A_b(n))_{n \geq 0}\) is \(b\)-regular.
- Matrix representation for the sequence \((A_b(n))_{n \geq 0}\).
- For all \(n \geq 0\), \(A_b(b^n) = (2b - 1)^n\).
- Recurrence relations for \(A_b\) involving \(2b - 1\).

For all \(a, a' \in \{1, \ldots, b - 1\}\) with \(a \neq a'\), all \(\ell \geq 1\) and all \(r \in \{0, \ldots, b^{\ell-1}\}\)

\[
A_b(ab^\ell + r) = (2b - 2) \cdot (2a - 1) \cdot (2b - 1)^{\ell-1} + A_b(ab^{\ell-1} + r) + A_b(r),
\]

\[
A_b(ab^\ell + ab^{\ell-1} + r) = (4ab - 2a - 2b + 2) \cdot (2b - 1)^{\ell-1} + 2A_b(ab^{\ell-1} + r) - A_b(r),
\]

\[
A_b(ab^\ell + a'b^{\ell-1} + r) = \begin{cases} 
(4ab - 4a - 2b + 3) \cdot (2b - 1)^{\ell-1} + A_b(ab^{\ell-1} + r) \\
+2A_b(a'b^{\ell-1} + r) - 2A_b(r), & \text{if } a' < a; \\
(4ab - 4a - 2b + 2) \cdot (2b - 1)^{\ell-1} + A_b(ab^{\ell-1} + r) \\
+2A_b(a'b^{\ell-1} + r) - 2A_b(r), & \text{if } a' > a.
\end{cases}
\]
Results in the integer base case

- The sequence \( (A_b(n))_{n \geq 0} \) is \( b \)-regular.
- Matrix representation for the sequence \( (A_b(n))_{n \geq 0} \).
- For all \( n \geq 0 \), \( A_b(\begin{array}{c} b^n \\ r(n) \\ t(n) \end{array}) = (2b - 1)^n \).

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+ 2A_b(a'b^{\ell-1} + r) - 2A_b(r), & \text{if } a' > a.
\end{cases}
\]

- \( (2b-1) \)-decomposition of \( A_b \): mixing the base \( b \) and the base \( 2b-1 \) numeration systems
Illustration in base 2

\[
\begin{array}{cccc}
2^{n+1} & 2^{n+2} \\
\hline
\alpha & \\
\end{array}
\]

\[\alpha \in [0, 1) \text{ with } d_2(\alpha) = d_1 d_2 d_3 \cdots \]
\[\leadsto e_n(\alpha) = \text{val}_2(1d_1 \cdots d_n1)\]

Example: \(\alpha = \pi - 3\)

<table>
<thead>
<tr>
<th>(n)</th>
<th>(e_n(\pi - 3))</th>
<th>(3\text{dec}(A_2(e_n(\pi - 3))))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>2 7</td>
</tr>
<tr>
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<td>9</td>
<td>2 2 8</td>
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<tr>
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<td>19</td>
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</tr>
<tr>
<td>10</td>
<td>2337</td>
<td>2 6 -6 2 24 -24 6 30 30 30 146</td>
</tr>
</tbody>
</table>

\[3\text{dec}(A_2(e_n(\alpha))) \xrightarrow{n \to +\infty} a_0(\alpha)a_1(\alpha)a_2(\alpha) \cdots\]
Step functions: \( \alpha \in [0, 1) \mapsto \phi_n(\alpha) = A_2(e_n(\alpha))/3^{\log_2(e_n(\alpha))} \)

\[
\phi_n(\alpha) = \begin{cases} 
\frac{1}{3^{1+\{\log_2(e_n(\alpha))\}}} \sum_{i=0}^{n} \frac{a_i(e_n(\alpha))}{3^i} & \text{if } 0 \leq \alpha < \frac{1}{2} \\
\frac{1}{3^{\{\log_2(e_n(\alpha))\}}} \sum_{i=0}^{n+1} \frac{a_i(e_n(\alpha))}{3^i} & \text{if } \frac{1}{2} \leq \alpha < 1 
\end{cases}
\]

\((\phi_n)_{n \geq 1}\) uniformly converges to the function

\[
\Phi_2(\alpha) = \begin{cases} 
\frac{1}{3^{1+\log_2(\alpha+1)}} \sum_{i=0}^{+\infty} \frac{a_i(\alpha)}{3^i} & \text{if } 0 \leq \alpha < \frac{1}{2} \\
\frac{1}{3^{\log_2(\alpha+1)}} \sum_{i=0}^{+\infty} \frac{a_i(\alpha)}{3^i} & \text{if } \frac{1}{2} \leq \alpha < 1 
\end{cases}
\]
• $\Phi_2$ is continuous over $[0, 1)$ s.t. $\Phi_2(0) = 1$ and $\lim_{\alpha \to 1^-} \Phi_2(\alpha) = 1$.

• Define $\mathcal{H}_2$ with the help of $\Phi_2$.

Then $\mathcal{H}_2$ is continuous and 1-periodic s.t. for all large enough $n$

$$A_2(n) = 3^{\log_2 n} \mathcal{H}_2(\log_2 n).$$
Theorem

There exists a continuous and 1-periodic function $\mathcal{H}_b$ s.t. for all large enough $n$

$$A_b(n) = \sum_{j=0}^{n-1} S_b(j) = (2b - 1)^{\log_b n} \mathcal{H}_b(\log_b n).$$
Results in the Fibonacci case

Let \((B(n))_{n\geq 0}\) be defined by \(B(0) = 1\), \(B(1) = 3\), \(B(2) = 6\), and 
\(B(n + 3) = 2B(n + 2) + B(n + 1) - B(n)\) for all \(n \geq 0\).

- For all \(n \geq 0\), \(A_\phi(F(n) - 1) = B(n)\).

- Recurrence relations for \(A_\phi\) involving \(B\).

If \(0 \leq r < F(\ell - 2)\), then

\[ A_\phi(F(\ell) + r) = B(\ell) - B(\ell - 1) + A_\phi(F(\ell - 1) + r) + A_\phi(r). \]

If \(F(\ell - 2) \leq r < F(\ell - 1)\), then

\[ A_\phi(F(\ell) + r) = 2B(\ell) - B(\ell - 1) - B(\ell - 2) + 2A_\phi(r). \]

- \(B\)-decomposition of \(A_\phi\): mixing the Fibonacci numeration system and the numeration system based on \(B\).
Theorem

Let $\lambda$ be the dominant root of the characteristic polynomial $P_B(X) = X^3 - 2X^2 - X + 1$ of $B$.
Let $c$ be a constant s.t. $\lim_{n \to +\infty} B(n)/\lambda^n = c$.

There exists a continuous and 1-periodic function $G$ s.t., for all large enough $n$,

$$A_\varphi(n) = \sum_{j=0}^{n} S_\varphi(j) = c \lambda^{\log_F n} G(\log_F n) + o(\lambda^{\lfloor \log_F n \rfloor}).$$

Remark: There is an error term, but the method allows us to deal with generalized regular sequences.
Summary

Part I: Pascal triangle
- Generalization to Parry numeration systems
- Description of the limit set $\mathcal{L}^\beta$ (segments, maps $c$ and $h$)
- Works for $r \mod p$ ($p$ prime)

Part II: Sequences counting scattered subwords
- $b$-regularity of $S_b$, Fibonacci-regularity of $S_\varphi$
- Method using tries of scattered subwords
- Matrix representations
- In base 2: link with the Farey tree

Part III: Summatory functions
- Asymptotics for $A_b$ and $A_\varphi$
- Method mixing numeration systems (exotic decompositions)
Part I: Generalized Pascal triangles

- Extension to other numeration systems/languages (conditions)
- Other colorings (generalizations of Lucas’ theorem)
- Properties of $L^\beta$: Hausdorff dimension, Minkowski dimension, Hölder exponent, Lebesgue measure, etc.
- Iterated functions systems (IFS)
- Comparison/Classification of limit sets

Part II: Regularity of sequences counting scattered subwords

- Extension to other numeration systems/languages (conditions)
- $b$-regularity of $(S_\varphi(n))_{n \geq 0}$ (Cobham-like theorem)
- Extension of the relation between $(S_2(n))_{n \geq 0}$ and the Farey tree

Part III: Asymptotics of summatory functions

- Extension to other numeration systems/languages (conditions)
- Regularity of $(A_\beta(n))_{n \geq 0}$
- Differentiability of periodic fluctuations
- Comparison/Classification of periodic fluctuations


