Extensions of the Pascal Triangle to Words, and Related Counting Problems

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Classical Pascal triangle

$\mathrm{P}\colon (m,k) \in \mathbb{N}\times\mathbb{N} \mapsto \binom{m}{k} \in \mathbb{N}$

$\binom{m}{k}$		k							
		0	1	2	3	4	5	6	7	$\cdots$
	0	1	0	0	0	0	0	0	0	
	1	1	1	0	0	0	0	0	0	
	2	1	2	1	0	0	0	0	0	
m	3	1	3	3	1	0	0	0	0	
	4	1	4	6	4	1	0	0	0	
	5	1	5	10	10	5	1	0	0	
	6	1	6	15	20	15	6	1	0	
	7	1	7	21	35	35	21	7	1	
	$\vdots$									$\ddots$

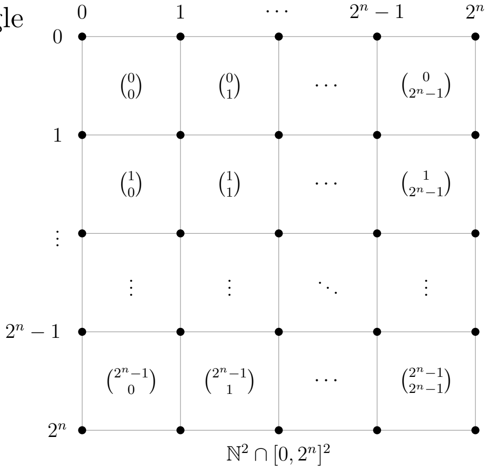
Binomial coefficients of integers:

$$\binom{m}{k} = \frac{m!}{(m-k)!\,k!}$$

A specific construction

- Grid: first 2^n rows and columns of the Pascal triangle

$$\left(\binom{m}{k} \bmod 2\right)_{0\leq m,k<2^n}$$



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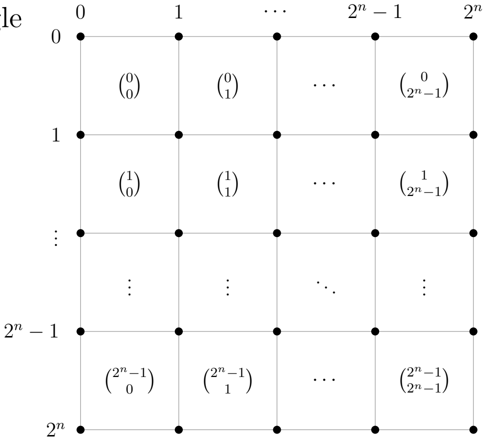

$\mathbb{N}^2 \cap [0, 2^n]^2$

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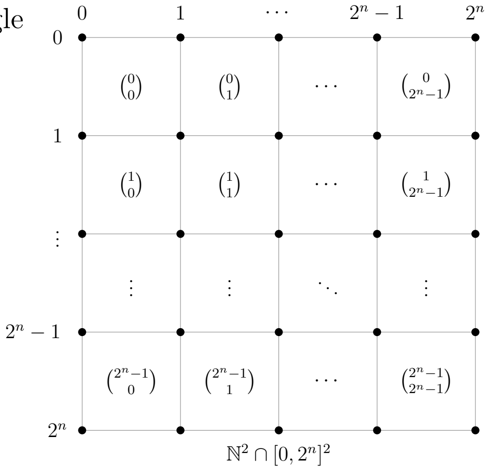

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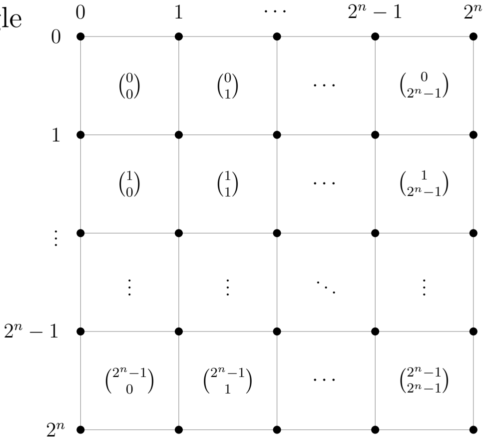
$\rightsquigarrow$ sequence of compact sets belonging to $[0,1]^2$



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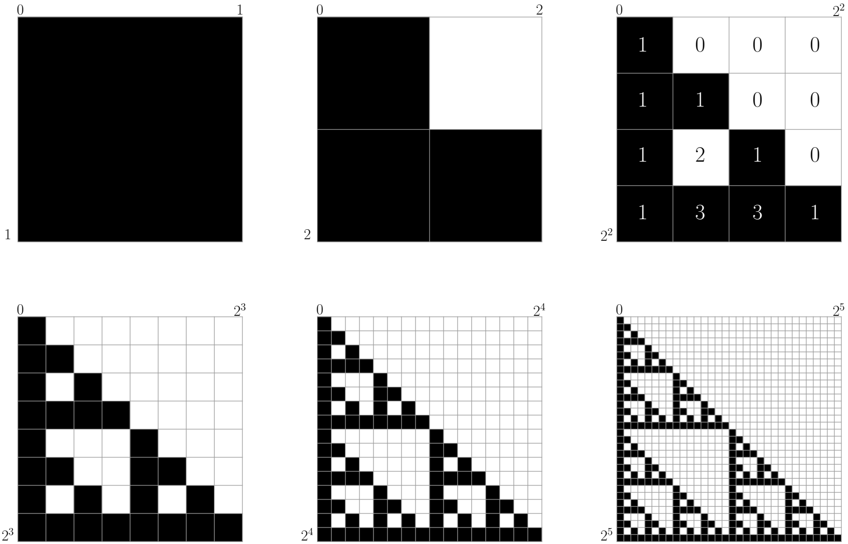
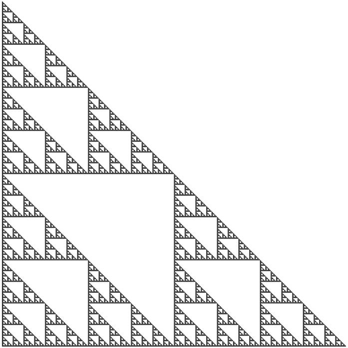

0
1
1
0
2
2
0
2^2
1 0 0 0
1 1 0 0
1 2 1 0
1 3 3 1
2^2
0
2^3
2^3
0
2^4
2^4
0
2^5
2^5



Folklore fact

The latter sequence of compact sets converges to the Sierpiński gasket (w.r.t. the Hausdorff distance).

Definitions:

- *ϵ-fattening* of a subset $S \subset \mathbb{R}^2$

$$[S]_\epsilon = \bigcup_{x \in S} B(x, \epsilon)$$

- $(\mathcal{H}(\mathbb{R}^2), d_h)$ complete space of the non-empty compact subsets of $\mathbb{R}^2$ equipped with the *Hausdorff distance* d_h

$$d_h(S, S') = \inf\{\epsilon \in \mathbb{R}_{>0} \mid S \subset [S']_\epsilon \quad \text{and} \quad S' \subset [S]_\epsilon\}$$

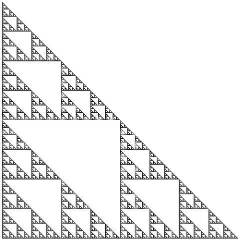
Theorem (von Haeseler, Peitgen, and Skordev, 1992)

Let p be a prime and $s > 0$.
The sequence of compact sets corresponding to

$$\left(\binom{m}{k} \bmod p^s\right)_{0 \le m,k < p^n}$$

converges when n tends to infinity (w.r.t. the Hausdorff distance).

$p = 2,\ s = 1$



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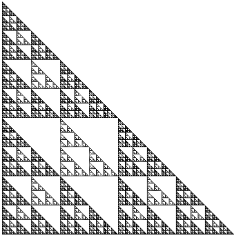
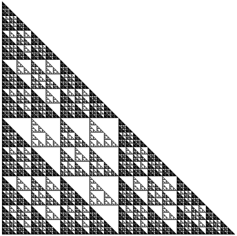


$p = 2,\ s = 3$



Part I

Binomial coefficient of finite words

Binomial coefficient of words

Let u, v be two finite words over the alphabet A.
The *binomial coefficient* $\binom{u}{v}$ of u and v is the number of times v occurs as a subsequence of u (meaning as a "scattered" subword).

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Example: $u = 101001$ $\qquad v = 101$

$$\left.\begin{array}{l} \mathbf{101}001, \mathbf{10}100\mathbf{1}, \mathbf{1}01\mathbf{0}0\mathbf{1}, \\ \mathbf{1}010\mathbf{01}, 10\mathbf{1}\mathbf{0}0\mathbf{1}, 10\mathbf{1}0\mathbf{01} \end{array}\right\} \Rightarrow \binom{101001}{101} = 6$$

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Natural generalization:

$$\binom{a^m}{a^k} = \binom{\overbrace{a \cdots a}^{m \text{ times}}}{\underbrace{a \cdots a}_{k \text{ times}}} = \binom{m}{k} \quad \forall\, m, k \in \mathbb{N}$$

Generalized Pascal triangles

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$$w_0 <_{\text{gen}} w_1 <_{\text{gen}} w_2 <_{\text{gen}} \cdots .$$

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The *generalized Pascal triangle* P_L associated with L is defined by

$$\mathrm{P}_L \colon (m, k) \in \mathbb{N} \times \mathbb{N} \mapsto \binom{w_m}{w_k} \in \mathbb{N} .$$

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Questions:

- With a similar construction, can we expect the convergence to an analogue of the Sierpiński gasket?
- In particular, where should we cut to normalize a given generalized Pascal triangle?
- Could we describe this limit object?

Linear numeration system

Definition

A *numeration system* is a sequence $U = (U(n))_{n \geq 0}$ of integers s.t.

- U increasing
- $U(0) = 1$
- $\sup_{n \geq 0} \frac{U(n+1)}{U(n)}$ bounded by a constant $\rightsquigarrow$ finite alphabet.

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A numeration system U is *linear* if $\exists k \geq 1, \exists a_0, \ldots, a_{k-1} \in \mathbb{Z}$ s.t.

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Greedy representation in $(U(n))_{n\geq 0}$:

$$n = \sum_{i=0}^{\ell} c_i U(i) \qquad \text{with} \qquad \sum_{i=0}^{j-1} c_i U(i) < U(j)$$

$$\mathrm{rep}_U(n) = c_\ell \cdots c_0 \in \underbrace{L_U = \mathrm{rep}_U(\mathbb{N})}_{\text{numeration language}}$$

Parry numbers

$\beta \in \mathbb{R}_{>1} \qquad A_\beta = \{0, 1, \ldots, \lceil \beta \rceil - 1\}$

$$x \in [0,1] \rightsquigarrow x = \sum_{j=1}^{+\infty} c_j \beta^{-j}, \quad c_j \in A_\beta$$

Greedy way: $c_j\beta^{-j} + c_{j+1}\beta^{-j-1} + \cdots < \beta^{-(j-1)}$
β-expansion of x: $d_\beta(x) = c_1c_2c_3\cdots$

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Example: $b \in \mathbb{N}_{>1}$: $d_b(1) = (b-1)^\omega$
Golden ratio φ: $d_\varphi(1) = 110^\omega$

Parry numeration system

Parry number $\beta \in \mathbb{R}_{>1} \rightsquigarrow$ linear numeration system $(U_\beta(n))_{n\geq 0}$

- $d_\beta(1) = t_1 \cdots t_m 0^\omega$

$$
\begin{array}{rcll}
U_\beta(0) & = & 1 & \\
U_\beta(i) & = & t_1 U_\beta(i-1) + \cdots + t_i U_\beta(0) + 1 & \forall\, 1 \leq i \leq m-1 \\
U_\beta(n) & = & t_1 U_\beta(n-1) + \cdots + t_m U_\beta(n-m) & \forall\, n \geq m
\end{array}
$$

- $d_\beta(1) = t_1 \cdots t_m (t_{m+1} \cdots t_{m+k})^\omega$

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U_\beta(n) & = & t_1 U_\beta(n-1) + \cdots + t_{m+k} U_\beta(n-m-k) & \forall\, n \geq m+k \\
 & & +U_\beta(n-k) & \\
 & & -t_1 U_\beta(n-k-1) - \cdots - t_m U_\beta(n-m-k) &
\end{array}
$$

Examples:
$b \in \mathbb{N}_{>1} \rightsquigarrow (b^n)_{n\geq 0}$ base b
Golden ratio $\varphi \rightsquigarrow (F(n))_{n\geq 0}$ Fibonacci numeration system

Special case of generalized Pascal triangles

Parry number $\beta \in \mathbb{R}_{>1} \rightsquigarrow$ linear numeration system U_β

$$\rightsquigarrow \mathrm{P}_\beta \colon (m,k) \in \mathbb{N} \times \mathbb{N} \mapsto \binom{\mathrm{rep}_{U_\beta}(m)}{\mathrm{rep}_{U_\beta}(k)} \in \mathbb{N}$$

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Examples:

Base-2 numeration system

	ε	1	10	11	100	101	110	111
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1	1	1	0	0	0	0	0	0
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100	1	1	2	0	1	0	0	0
101	1	2	1	1	0	1	0	0
110	1	2	2	1	0	0	1	0
111	1	3	0	3	0	0	0	1

$$L_2 = 1\{0,1\}^* \cup \{\varepsilon\}$$

Fibonacci numeration system

	ε	1	10	100	101	1000	1001	1010
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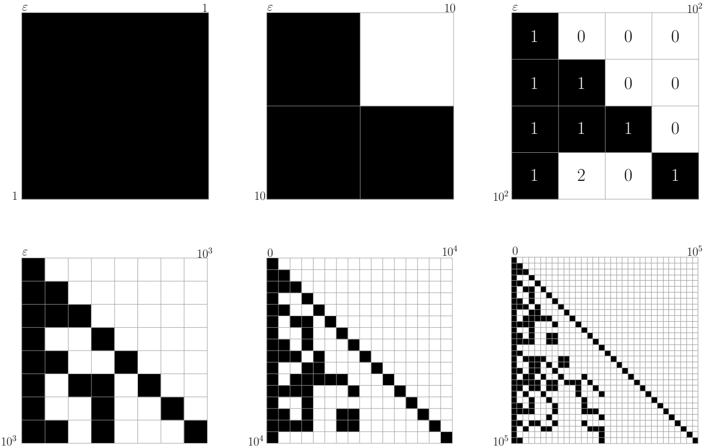
Remark: Copies of the usual Pascal triangle

Sequence of compact sets (first $U_\beta(n)$ rows and columns of P_β) in $[0,1]^2$:

$$\mathcal{U}_n^\beta = \frac{1}{U_\beta(n)} \bigcup_{\substack{u,v \in L_{U_\beta}, |u|,|v| \leq n \\ \binom{u}{v} \equiv 1 \bmod 2}} \mathrm{val}_{U_\beta}(v,u) + [0,1]^2$$

Examples: Base-2 numeration system



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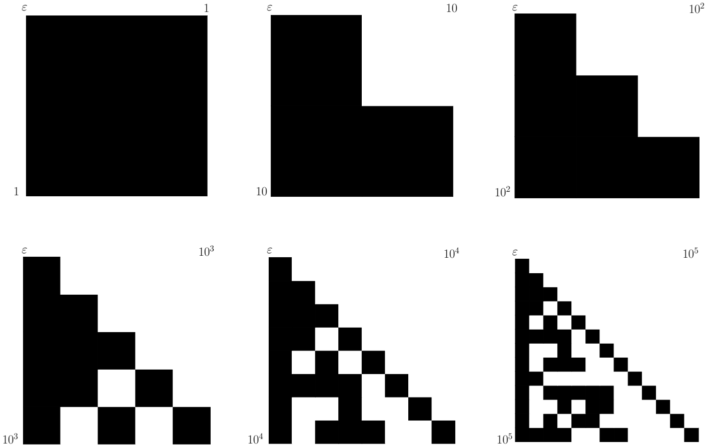
Examples: Fibonacci numeration system



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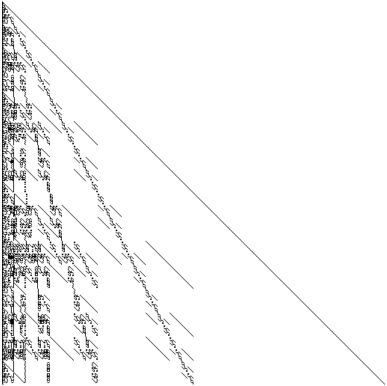
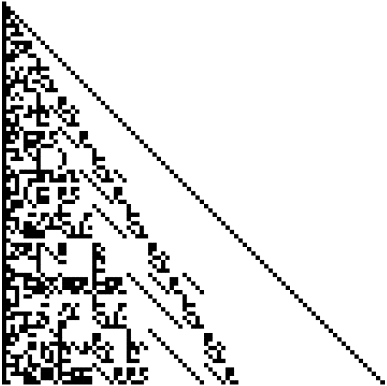


Lines of slopes: 2^n, $n \geq 0$

Fibonacci numeration system



Lines of slopes: φ^n, $n \geq 0$

General case: Lines of slopes: β^n, $n \geq 0$

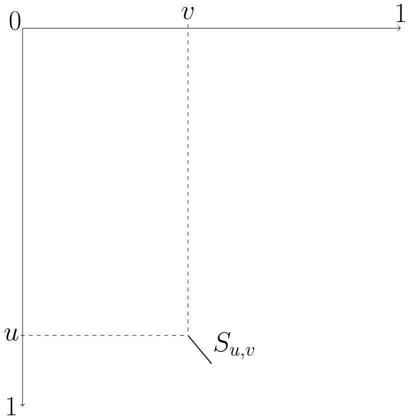
$(\star)$ condition

$p(u,v) \in \mathbb{N}$ s.t. $u0^{p(u,v)}w, v0^{p(u,v)}w \in L_{U_\beta}$ for all $w \in 0^* L_{U_\beta}$

$(\star)$

(u,v) satisfies $(\star)$ iff $u = v = \varepsilon$ or $\begin{cases} u, v \neq \varepsilon \\ \binom{u0^{p(u,v)}}{v0^{p(u,v)}} \equiv 1 \pmod 2 \\ \binom{u0^{p(u,v)}}{v0^{p(u,v)}a} = 0 \quad \forall\, a \in A_{U_\beta}. \end{cases}$

- The $(\star)$ condition describes lines of slope 1 in $[0,1]^2$.



$(u,v) \in L_{U_\beta} \times L_{U_\beta}$ satisfying $(\star)$
$\rightsquigarrow$ closed segment $S_{u,v}$

- slope 1
- length $\sqrt{2} \cdot \beta^{-|u|-p(u,v)}$
- origin $A_{u,v} = (0.0^{|u|-|v|}v, 0.u)$

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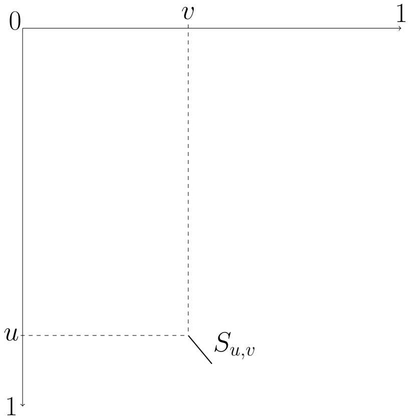

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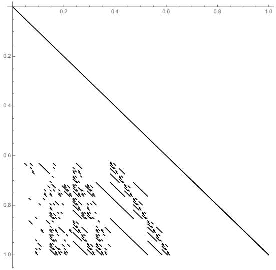
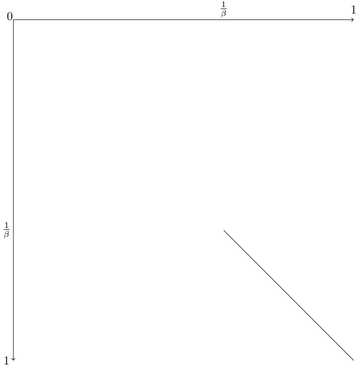
- New compact set containing those lines:

$$\mathcal{A}_0^\beta = \overline{\bigcup_{\substack{(u,v) \\ \text{satisfying}(\star)}} S_{u,v}} \subset [0,1]^2$$



- Two maps $c\colon (x,y) \mapsto (\frac{x}{\beta}, \frac{y}{\beta})$ and $h\colon (x,y) \mapsto (x, \beta y)$
  Example:



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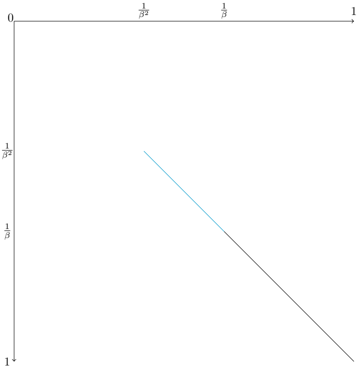

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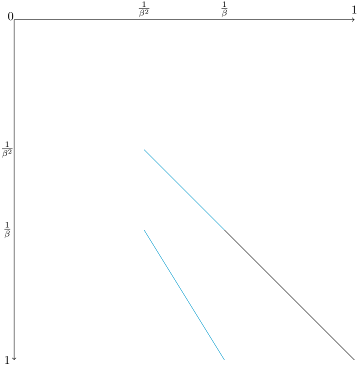

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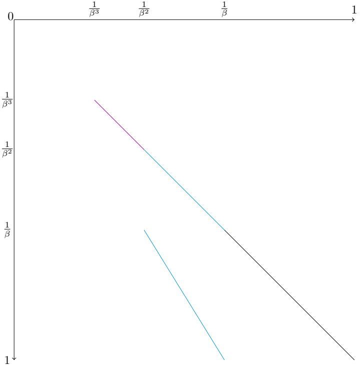

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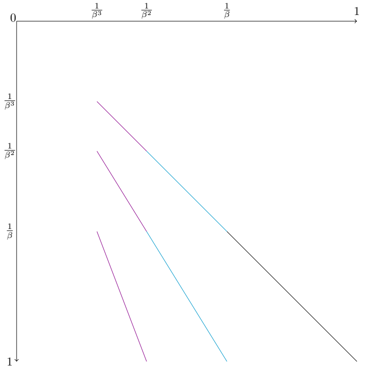

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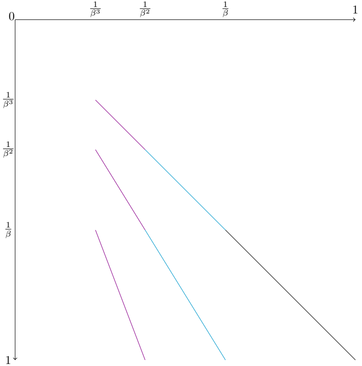

- New compact set containing lines of slopes $1, \beta, \ldots, \beta^n$:

$$\mathcal{A}_n^\beta = \bigcup_{0 \le i \le n,\, 0 \le j \le i} h^j(c^i(\mathcal{A}_0^\beta)) \subset [0,1]^2.$$

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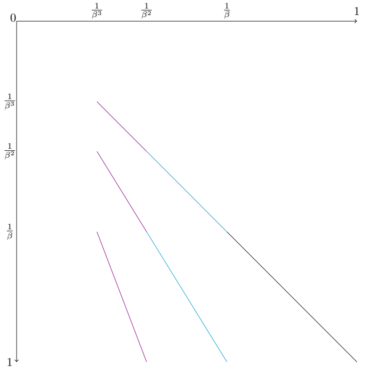

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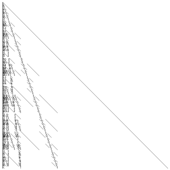
- The compact sets $(\mathcal{A}_n^\beta)_{n \ge 0}$ are increasingly nested and their union is bounded.
  $(\mathcal{A}_n^\beta)_{n \ge 0}$ converges to $\mathcal{L}^\beta = \overline{\bigcup_{n \ge 0} \mathcal{A}_n^\beta}$ w.r.t. the Hausdorff distance.

Results

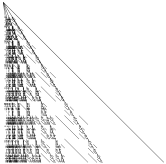
Theorem

$(\mathcal{U}_n^\beta)_{n\geq 0}$ converges to $\mathcal{L}^\beta$ w.r.t. the Hausdorff distance.

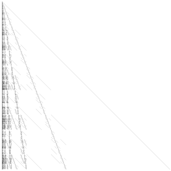
$3 \rightsquigarrow \mathcal{L}^3$



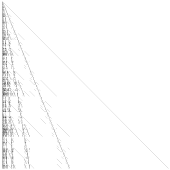
$\varphi \rightsquigarrow \mathcal{L}^\varphi$



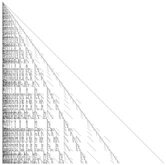
$\varphi^2 \rightsquigarrow \mathcal{L}^{\varphi^2}$



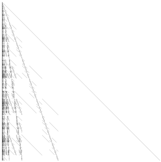
$\beta_1 \approx 2.47098 \rightsquigarrow \mathcal{L}^{\beta_1}$



$\beta_2 \approx 1.38028 \rightsquigarrow \mathcal{L}^{\beta_2}$



$\beta_3 \approx 2.80399 \rightsquigarrow \mathcal{L}^{\beta_3}$

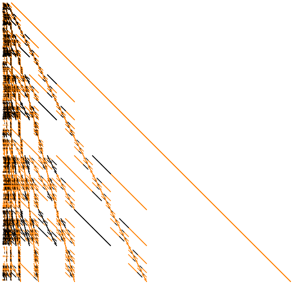


Previous result: even and odd binomial coefficients

Theorem

Let p be a prime and $r \in \{1, \ldots, p-1\}$. When considering binomial coefficients congruent to $r \pmod p$, the sequence $(\mathcal{U}^{\beta}_{n,p,r})_{n\geq 0}$ converges to a well-defined compact set $\mathcal{L}^{\beta}_{p,r}$ w.r.t. the Hausdorff distance.

Example: $\mathcal{L}^2_{3,1} \cup \mathcal{L}^2_{3,2}$

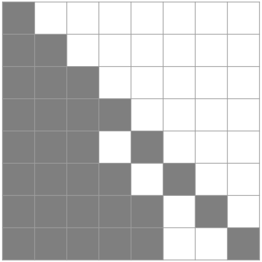


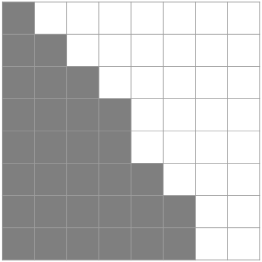
Part II

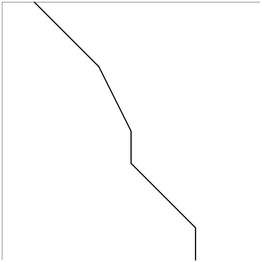
Counting scattered subwords

Example: P_φ (Fibonacci numeration system)

	ε	1	10	100	101	1000	1001	1010	n	$S_\varphi(n)$
ε	1	0	0	0	0	0	0	0	0	**1**
1	1	1	0	0	0	0	0	0	1	**2**
10	1	1	1	0	0	0	0	0	2	**3**
100	1	1	2	1	0	0	0	0	3	**4**
101	1	2	1	0	1	0	0	0	4	**4**
1000	1	1	3	3	0	1	0	0	5	**5**
1001	1	2	2	1	2	0	1	0	6	**6**
1010	1	2	3	1	1	0	0	1	7	**6**







Classical Pascal triangle: $S(n) = n + 1 \quad \forall\, n \geq 0$

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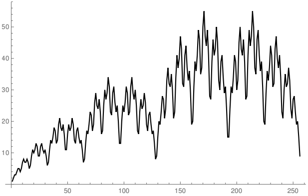
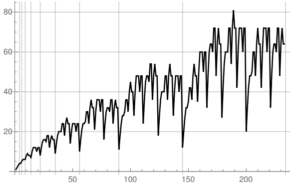
Base-2 numeration system:
$S_2 = 1, 2, 3, 3, 4, 5, 5, 4, 5, 7, 8, 7, 7, 8, 7, 5, 6, 9, 11, 10, 11, 13, 12, \ldots$



OEIS tag: `A007306`

Fibonacci numeration system:
$S_\varphi = 1, 2, 3, 4, 4, 5, 6, 6, 6, 8, 9, 8, 8, 7, 10, 12, 12, 12, 10, 12, 12, 8, 12, \ldots$



OEIS tag: `A282717`

Regularity

Example: Sum-of-digits function Sum_2 in base 2
$\mathsf{Sum}_2(n) = \#$ of 1's in $\mathrm{rep}_2(n)$

n	0	1	2	3	4	5	6	7	8	9	10
$\mathrm{rep}_2(n)$	ε	1	10	11	100	101	110	111	1000	1001	1010
$\mathsf{Sum}_2(n)$	0	1	1	2	1	2	2	3	1	2	2

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$$\mathsf{Sum}_2(2n) = \mathsf{Sum}_2(n) \quad \mathsf{Sum}_2(2n+1) = \mathsf{Sum}_2(n) + 1$$

Sum_2 2-regular:

sequences in $\mathcal{K}_2(\mathsf{Sum}_2)$ are $\mathbb{Z}$-linear combinations of Sum_2, 1

e.g. $\mathsf{Sum}_2(4n+1) = \mathsf{Sum}_2(2(2n)+1) = \mathsf{Sum}_2(2n)+1 = 1\cdot\mathsf{Sum}_2(n)+1\cdot 1$

Regularity: general case

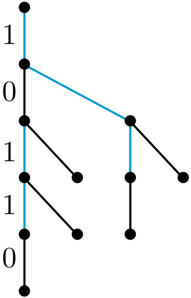
Let $U = (U(n))_{n\geq 0}$ be a numeration system.
Let $s = (s(n))_{n\geq 0}$ be a sequence.

- The sequence $(s(i_w(n)))_{n\geq 0}$ is called the *subsequence of s with least significant digits equal to w* w.r.t. representations in the numeration system U.
- The *U-kernel* of s is the set
$$\mathcal{K}_U(s) = \{(s(i_w(n)))_{n\geq 0} \mid \text{for all suffixes } w\}.$$
- A sequence s of integers is *U-regular* if there exists a finite number of sequences $t_1 = (t_1(n))_{n\geq 0}, \ldots, t_\ell = (t_\ell(n))_{n\geq 0}$ s.t. every sequence in the U-kernel $\mathcal{K}_U(s)$ is a $\mathbb{Z}$-linear combination of the sequences $t_1, \ldots, t_\ell$.

Method

Define, study, use a new tree structure called *tries of scattered subwords*.
$\rightsquigarrow$ easily count/enumerate scattered subwords

Example: in base 2



Scattered subwords of 10110:

$$\varepsilon, 1, 10, 11, 100, 101, 110, 111,$$
$$1010, 1011, 1110, 10110$$

$$S_2(\mathrm{val}_2(10110)) = S_2(22) = \#\ \text{nodes} = 12$$

Internal structure: subtree of root 11 $\approxeq$ subtree of root 101

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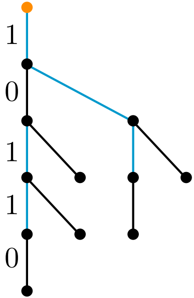

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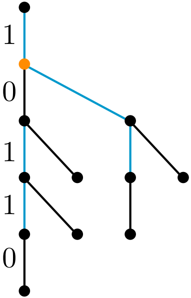

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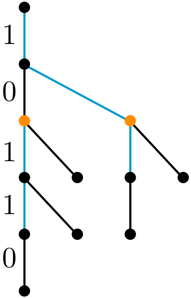

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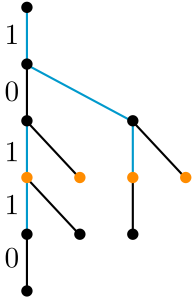

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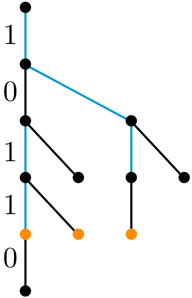

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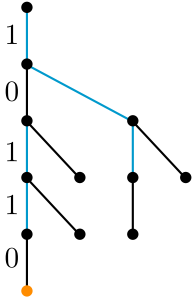

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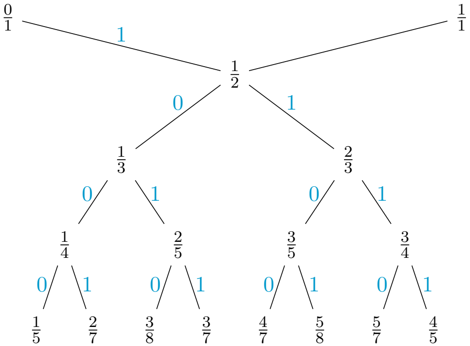
More results in the base-2 case

- Connection with the Farey tree (every reduced positive rational less than 1 exactly once).



Let $w \in 1\{0,1\}^*$ with $\mathrm{val}_2(w) = 2^k + r$.

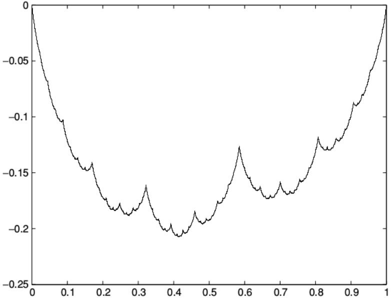
$$\frac{0}{1} \overset{w}{\rightsquigarrow} \frac{S_2(r)}{S_2(2^k+r)}$$

- The Stern–Brocot sequence $(SB(n))_{n\geq 0}$ contains the numerators and denominators in the Farey tree.
  Then $S_2(n) = SB(2n+1)$ for all $n \geq 0$.

Part III

Behavior of summatory functions

Example: Sum-of-digits function Sum_2 in base 2
$\mathsf{Sum}_2(n) = \#$ of 1's in $\mathrm{rep}_2(n)$
Sum_2 is 2-regular



Summatory function A of Sum_2

$$A(n) = \sum_{j=0}^{n-1} \mathsf{Sum}_2(j) \quad \forall n \geq 0$$

Theorem (Delange, 1975)

There exists a continuous nowhere differentiable periodic function $\mathcal{G}$ of period 1 s.t.

$$\frac{A(n)}{n} = \frac{1}{2}\log_2 n + \mathcal{G}(\log_2 n).$$

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$\rightsquigarrow$ algebraic or analytic methods

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Definition: $A_\beta = (A_\beta(n))_{n\geq 0}$ is the summatory function of S_β.

Results in the integer base case

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For all $a, a' \in \{1, \ldots, b-1\}$ with $a \neq a'$, all $\ell \geq 1$ and all $r \in \{0, \ldots, b^{\ell-1}\}$

$$A_b(ab^\ell + r) = (2b-2)\cdot(2a-1)\cdot(2b-1)^{\ell-1} + A_b(ab^{\ell-1}+r) + A_b(r),$$

$$A_b(ab^\ell + ab^{\ell-1} + r) = (4ab-2a-2b+2)\cdot(2b-1)^{\ell-1} + 2A_b(ab^{\ell-1}+r) - A_b(r),$$

$$A_b(ab^\ell + a'b^{\ell-1} + r) = \begin{cases} (4ab-4a-2b+3)\cdot(2b-1)^{\ell-1} + A_b(ab^{\ell-1}+r) & \\ +2A_b(a'b^{\ell-1}+r) - 2A_b(r), & \text{if } a' < a; \\ (4ab-4a-2b+2)\cdot(2b-1)^{\ell-1} + A_b(ab^{\ell-1}+r) & \\ +2A_b(a'b^{\ell-1}+r) - 2A_b(r), & \text{if } a' > a. \end{cases}$$

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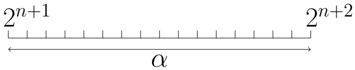
- $(2b-1)$-decomposition of A_b: mixing the base b and the base $2b-1$ numeration systems

Illustration in base 2



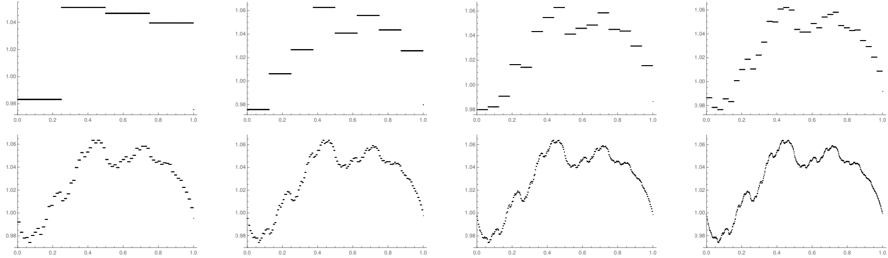
$\alpha \in [0,1)$ with $d_2(\alpha) = d_1 d_2 d_3 \cdots$
$\rightsquigarrow e_n(\alpha) = \mathrm{val}_2(1 d_1 \cdots d_n 1)$

Example: $\alpha = \pi - 3$

n	$e_n(\pi-3)$	$\mathsf{3dec}(A_2(e_n(\pi-3)))$										
1	5	2	7									
2	9	2	2	8								
3	19	2	6	-2	5							
4	37	2	6	-6	6	15						
5	73	2	6	-6	2	8	31					
6	147	2	6	-6	2	24	-8	14				
7	293	2	6	-6	2	24	-24	22	53			
8	585	2	6	-6	2	24	-24	6	30	116		
9	1169	2	6	-6	2	24	-24	6	30	30	131	
10	2337	2	6	-6	2	24	-24	6	30	30	30	146

$$\mathsf{3dec}(A_2(e_n(\alpha))) \xrightarrow{n \to +\infty} a_0(\alpha) a_1(\alpha) a_2(\alpha) \cdots$$

Step functions: $\alpha \in [0,1) \mapsto \phi_n(\alpha) = A_2(e_n(\alpha))/3^{\log_2(e_n(\alpha))}$

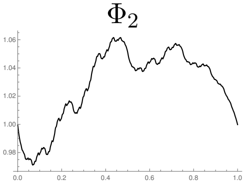


$$\phi_n(\alpha) = \begin{cases} \dfrac{1}{3^{1+\{\log_2(e_n(\alpha))\}}} \displaystyle\sum_{i=0}^{n} \dfrac{a_i(e_n(\alpha))}{3^i} & \text{if } 0 \le \alpha < \frac{1}{2} \\ \dfrac{1}{3^{\{\log_2(e_n(\alpha))\}}} \displaystyle\sum_{i=0}^{n+1} \dfrac{a_i(e_n(\alpha))}{3^i} & \text{if } \frac{1}{2} \le \alpha < 1 \end{cases}$$

$(\phi_n)_{n\geq 1}$ uniformly converges to the function

$$\Phi_2(\alpha) = \begin{cases} \dfrac{1}{3^{1+\log_2(\alpha+1)}} \displaystyle\sum_{i=0}^{+\infty} \dfrac{a_i(\alpha)}{3^i} & \text{if } 0 \le \alpha < \frac{1}{2} \\ \dfrac{1}{3^{\log_2(\alpha+1)}} \displaystyle\sum_{i=0}^{+\infty} \dfrac{a_i(\alpha)}{3^i} & \text{if } \frac{1}{2} \le \alpha < 1 \end{cases}$$

- Φ_2 is continuous over $[0,1)$ s.t $\Phi_2(0)=1$ and $\lim_{\alpha\to 1^-} \Phi_2(\alpha)=1$.



- Define $\mathcal{H}_2$ with the help of Φ_2.
  Then $\mathcal{H}_2$ is continuous and 1-periodic s.t. for all large enough n

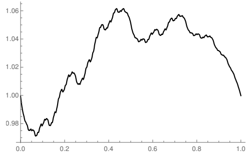
$$A_2(n) = 3^{\log_2 n}\ \mathcal{H}_2(\log_2 n).$$

Theorem

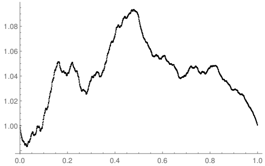
There exists a continuous and 1-periodic function $\mathcal{H}_b$ s.t. for all large enough n

$$A_b(n) = \sum_{j=0}^{n-1} S_b(j) = (2b-1)^{\log_b n}\, \mathcal{H}_b(\log_b n).$$

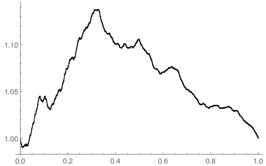
$\mathcal{H}_2$



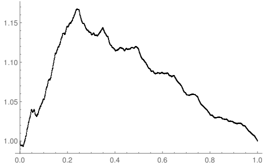
$\mathcal{H}_3$



$\mathcal{H}_4$



$\mathcal{H}_5$



Results in the Fibonacci case

Let $(B(n))_{n\geq 0}$ be defined by $B(0) = 1$, $B(1) = 3$, $B(2) = 6$, and $B(n+3) = 2B(n+2) + B(n+1) - B(n)$ for all $n \geq 0$.

- For all $n \geq 0$, $A_\varphi(\underbrace{F(n)-1}_{r(n)}) = \underbrace{B(n)}_{t(n)}$.
- Recurrence relations for A_φ involving B.

  If $0 \leq r < F(\ell-2)$, then

  $$A_\varphi(F(\ell)+r) = B(\ell) - B(\ell-1) + A_\varphi(F(\ell-1)+r) + A_\varphi(r).$$

  If $F(\ell-2) \leq r < F(\ell-1)$, then

  $$A_\varphi(F(\ell)+r) = 2B(\ell) - B(\ell-1) - B(\ell-2) + 2A_\varphi(r).$$

- B-decomposition of A_φ: mixing the Fibonacci numeration system and the numeration system based on B

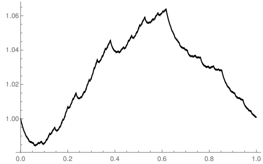
Results in the Fibonacci case (continued)

Theorem

Let λ be the dominant root of the characteristic polynomial $P_B(X) = X^3 - 2X^2 - X + 1$ of B.
Let c be a constant s.t. $\lim_{n \to +\infty} B(n)/\lambda^n = c$.

There exists a continuous and 1-periodic function $\mathcal{G}$ s.t., for all large enough n,

$$A_\varphi(n) = \sum_{j=0}^{n} S_\varphi(j) = c\,\lambda^{\log_F n} \mathcal{G}(\log_F n) + o(\lambda^{\lfloor \log_F n \rfloor}).$$



Remark: There is an error term, **but** the method allows us to deal with generalized regular sequences.

Summary

Part I: Pascal triangle

- Generalization to Parry numeration systems
- Description of the limit set $\mathcal{L}^{\beta}$ (segments, maps c and h)
- Works for r mod p (p prime)

Part II: Sequences counting scattered subwords

- b-regularity of S_b, Fibonacci-regularity of S_{φ}
- Method using tries of scattered subwords
- Matrix representations
- In base 2: link with the Farey tree

Part III: Summatory functions

- Asymptotics for A_b and A_{φ}
- Method mixing numeration systems (exotic decompositions)

Perspectives: natural open problems/questions

Part I: Generalized Pascal triangles

- Extension to other numeration systems/languages (conditions)
- Other colorings (generalizations of Lucas' theorem)
- Properties of $\mathcal{L}^\beta$: Hausdorff dimension, Minkowski dimension, Hölder exponent, Lebesgue measure, etc.
- Iterated functions systems (IFS)
- Comparison/Classification of limit sets

Part II: Regularity of sequences counting scattered subwords

- Extension to other numeration systems/languages (conditions)
- b-regularity of $(S_\varphi(n))_{n\geq 0}$ (Cobham-like theorem)
- Extension of the relation between $(S_2(n))_{n\geq 0}$ and the Farey tree

Part III: Asymptotics of summatory functions

- Extension to other numeration systems/languages (conditions)
- Regularity of $(A_\beta(n))_{n\geq 0}$
- Differentiability of periodic fluctuations
- Comparison/Classification of periodic fluctuations

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