Université de Liège
Faculté des Sciences
Unité de Recherche *Mathematics*

# Extensions of the Pascal Triangle to Words, and Related Counting Problems

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Dissertation présentée
en vue de l'obtention du grade académique de
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# Abstract

The Pascal triangle and the corresponding Sierpiński fractal are fairly well-studied mathematical objects, which both exhibit connections with many different scientific areas. The first is made of binomial coefficients of integers that notably appear in combinatorics to tackle counting problems (for instance, they provide the number of possible ways to choose a given amount of elements from a set of elements). There exist multiple generalizations of those binomial coefficients. In this text, we focus on binomial coefficients of words, which count scattered subwords.

The red thread of this thesis is precisely the combination of the Pascal triangle and binomial coefficients of words.

The first part is dedicated to extensions of the Pascal triangle to various sets of words (languages) associated with different numeration systems. We transport the existing link between the Pascal triangle and the Sierpiński gasket to this wider setting.

Le triangle de Pascal et la fractale de Sierpiński correspondante sont des objets mathématiques relativement bien étudiés et ont des liens avec de nombreuses disciplines scientifiques. Le premier est composé de coefficients binomiaux d'entiers qui apparaissent notamment en combinatoire pour s'attaquer à des problèmes de dénombrement (par exemple, ils fournissent le nombre de façons possibles de choisir un certain nombre d'éléments parmi un ensemble d'éléments). Il existe de multiples généralisations de ces coefficients binomiaux. Dans ce texte, nous nous concentrons sur les coefficients binomiaux de mots qui, quant à eux, comptent des sous-mots dits éclatés.

Le fil conducteur de cette thèse est précisément la combinaison du triangle de Pascal et des coefficients binomiaux de mots.

La première partie est dédiée à des extensions du triangle de Pascal à divers ensembles de mots (langages) associés à différents systèmes de numération. Nous transportons le lien

The second part is concerned with particular sequences extracted from generalized Pascal triangles. They count non-zeroes binomial coefficients on each row of a given Pascal-like triangle. We study their regularity and their automaticity with respect to different numeration systems.

In the third and last part, we establish the asymptotics of the summatory functions of the sequences considered previously. The most important feature of this part might not necessarily be the result itself, but the underlying new method to achieve it.

existant entre les triangles de Pascal et de Sierpiński à ces contextes plus généraux.

La seconde partie est consacrée à des suites particulières extraites des triangles de Pascal généralisés. Celles-ci comptent le nombre de coefficients strictement positifs sur chaque ligne d'un triangle de Pascal généralisé donné. Nous étudions leur régularité et leur automaticité par rapport à différents systèmes de numération.

Dans la troisième et dernière partie, nous établissons le comportement asymptotique des fonctions sommatoires associées aux suites considérées précédemment. Ici, l'aspect le plus intéressant n'est pas nécessairement le résultat en lui-même, mais plutôt la nouvelle méthode sous-jacente pour y parvenir.

2010 *Mathematics Subject Classification*: 05A05, 05A10, 05A15, 11A63, 11A67, 11B57, 11B65, 11B85, 11K16, 28A78, 28A80, 41A60, 68Q70, 68R15.

Keywords: combinatorics on words, automata theory, number theory, graph theory, fractal theory, binomial coefficients, scattered subwords, generalized Pascal triangles, generalized Sierpiński gaskets, regular sequences, synchronized sequences, automatic sequences, numeration systems, Parry numbers, Perron numbers, Bertrand numeration systems, summatory function, digital sequences, asymptotic behavior.

# Remerciements

Tout d'abord, je souhaite chaleureusement remercier mes promoteurs, Émilie Charlier et Michel Rigo, d'avoir accepté de me superviser pendant ces quatre années de doctorat, pour leur intérêt constant, leur soutien, leur aide précieuse, leur patience infinie, leurs encouragements, les nombreuses discussions que j'ai eues avec l'un et l'autre, leurs conseils avisés, les sujets de recherche qu'ils m'ont chacun proposés et les questions scientifiques qui en ont découlé, ainsi que les savoirs qu'ils m'ont transmis.

L'apport de Julien Leroy à ma formation durant ces quatre années est non négligeable. Je le remercie pour les heures (et les heures) passées à travailler avec moi et à me guider dans les méandres mathématiques.

Je remercie Robbert Fokkink, Peter Grabner, Raphaël Jungers, Pierre Mathonet et Jean–Éric Pin de me faire l'honneur d'être membres de mon jury de thèse.

Le Département de Mathématique de l'Université de Liège a été pour moi comme une seconde maison pendant mes neuf années d'études. Je remercie mes professeurs de m'avoir montré à quel point les mathématiques sont belles et d'avoir éveillé en moi une insatiable curiosité à leur égard. Je remercie aussi mes camarades de classe et mes collègues d'avoir partagé avec moi un bout de chemin et de m'avoir offert, pour certains, leur amitié. De belles relations, d'équivalence ou non, sont nées au B37. J'en profite pour remercier Élise, ma grande sœur de thèse. Elle a été un modèle et une oreille attentive. Je remercie également mes petites sœurs de thèse, Adeline, Célia et Marie, d'avoir relu des parties de ce texte. Finalement, je remercie les secrétaires qui ont toujours répondu à mes nombreuses questions.

Je remercie le Fonds de la Recherche Scientifique – FNRS de m'avoir octroyé une bourse de doctorat FRIA. Il est clair que, sans lui, mon parcours doctoral aurait été tout à fait différent. Il m'a notamment permis de voyager, de me former dans le monde entier et, par conséquent, de rencontrer de nombreux chercheurs internationaux et de grandir tant sur le plan personnel que scientifique. En particulier, je remercie Narad Rampersad de m'avoir si gentiment accueillie à Winnipeg (Canada) pendant six mois en 2018. J'ai énormément appris auprès de lui et de ses collègues de l'Université de Winnipeg. Mes hôtes canadiens, Augusta et James, ainsi que ma colocataire Kinga, m'ont beaucoup apporté durant cette période. D'autres voyages m'ont permis de collaborer avec Sébastien Labbé qui m'a proposé d'approcher une autre partie des mathématiques discrètes et qui m'a initiée au logiciel *SageMath*. J'en profite pour remercier Eric Rowland pour ses magnifiques codes en *Mathematica* et d'avoir toujours répondu à mes questions sur ceux-ci; les codes sur *SageMath* de Daniel Krenn m'ont aussi aidée durant mes recherches. Je remercie tous les chercheurs qui se sont intéressés à mon travail, dont Thibault Godin, Emily Gunawan et l'épique Florian Lietard, ainsi que les référés qui ont relu mes travaux et les ont incontestablement améliorés. Finalement, je remercie toutes les personnes que j'ai rencontrées à travers le monde et qui ont contribué à façonner la personne que je suis aujourd'hui.

Je remercie le Réseau des Doctorants de l'Université de Liège au sein duquel j'ai développé des compétences transversales et lié de bonnes amitiés.

Je remercie Mary Richards de m'avoir non seulement enseigné la langue anglaise, mais aussi soutenue durant des périodes plus sombres de ma vie. Au fur et à mesure, Mary est devenue une amie chère.

Cette aventure n'aurait jamais pu être possible ni aboutir sans le soutien constant et la stabilité que m'ont offerts ma famille et mes amis. En particulier, je remercie mon papa pour les valeurs qu'il m'a transmises. Il m'a toujours poussée à donner le meilleur de moi-même en toutes circonstances et m'a permis de mener, comme cela me semblait bon, les études qui me plaisaient. Pour ces raisons, je lui en serai éternellement reconnaissante. Je remercie mes grands-parents et ma petite sœur d'avoir toujours cru en moi. Mon compagnon, Honoré, a lui aussi marché à mes côtés dès le début de cette aventure mathématique.

Mon autre passion, la musique, a toujours été un moteur important. Je remercie les artistes qui me permettent de décompresser et de me vider la tête après de longues journées de recherche.

Enfin, je remercie Michaël pour son soutien quotidien, son écoute attentive, ses encouragements permanents, sa sérénité et l'amour qu'il me porte. Je n'aurais jamais pu achever cette thèse sans lui.

*À mon grand-père,*
*À Honoré*
*Et à toutes les pièces manquantes*
*De mon puzzle*

*To my grandfather,*
*To Honoré,*
*And all my lost jigsaw pieces*

# Contents

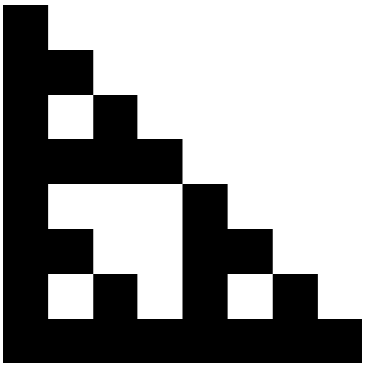
# Introduction

This doctoral dissertation is a contribution to combinatorics on words, which is a relatively new branch of mathematics and theoretical computer science [BP07], since it dates back to the beginning of the 20th century with the work of A. Thue [Thu06]. Nevertheless, the research problems considered in this text often cross the border of other mathematical fields such as fractal theory, ergodic theory, number theory, and automata theory.

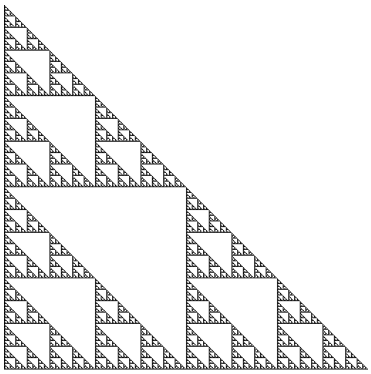
The starting point of this four-year work is the relation between the Pascal triangle P, which is made of binomial coefficients of integers, and the corresponding Sierpiński gasket. The former is named after the French mathematician B. Pascal [Pas65] who lived during the 17th century, though it appeared centuries before in different parts of the world [Coo49]. On its side, the Sierpiński gasket takes its name from the Polish mathematician W. Sierpiński who lived during the 20th century [Ste95]. Those two mathematical objects have been extensively studied through the ages and worldwide, and the related literature is huge, to say the least. They have various connections with the topics of this thesis. They notably exhibit self-similarity, dynamical and fractal features [vHPS92, KL18, Ste95] and they can be obtained via iterated function systems (IFS's) [vHPS92, Ste95]. They can be studied with automata-theoretic techniques [AB97, AB11] or be expressed using first order formulas in an extension of the Presburger arithmetic [CLR15]. Finally, they are linked to simple arithmetic, especially through the celebrated binomial theorem and more generally the multinomial theorem, to enumerative combinatorics [Sta97, BFST18] in order to tackle counting problems, and also to $p$-adic topology and $p$-adic analysis [BCP89, PS14].

Let us briefly explain the link between the Pascal triangle and the Sierpiński gasket. For all $n \in \mathbb{N}$, the first $2^n$ rows and columns $(\binom{i}{j})_{0\leq i,j<2^n}$ of the Pascal triangle can be represented as a grid, at the intersection of $\mathbb{N}^2$ and the square region $[0, 2^n]^2$. In the plane whose $x$-axis (resp., $y$-axis) points rightwards (resp., downwards), each binomial coefficient $\binom{i}{j}$ corresponds to a unit square whose upper-left corner has coordinates $(j, i)$. For the case $n = 3$, the situation is depicted in the figure below, on the left. If we consider the sequence $(\binom{i}{j})_{0 \le i,j < 2^n}$ modulo 2, we can color each unit square in black or white depending on the parity of the corresponding binomial coefficient. We thus obtain a region in $\mathbb{N}^2$ made of black and white squares. Below, the figure on the right illustrates what happens for $n = 3$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 2 | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 3 | 3 | 1 | 0 | 0 | 0 | 0 |
| 1 | 4 | 6 | 4 | 1 | 0 | 0 | 0 |
| 1 | 5 | 10 | 10 | 5 | 1 | 0 | 0 |
| 1 | 6 | 15 | 20 | 15 | 6 | 1 | 0 |
| 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 |

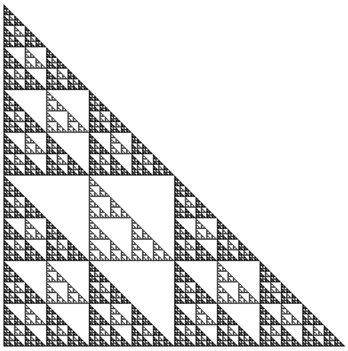


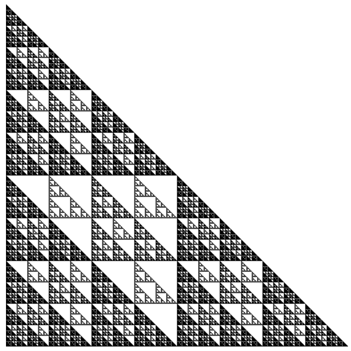
If we normalize this region by a homothety of ratio $1/2^n$, we obtain a sequence of sets in $[0, 1]^2$. It is a folklore fact that it converges, with respect to the Hausdorff distance, to the Sierpiński gasket (see below) when $n$ tends to infinity.



In a similar fashion, when the sequence $(\binom{i}{j})_{0 \le i,j < p^n}$ is considered modulo $p^s$ where $p$ is a prime number and $s$ is a positive integer, then it also converges, with respect to the Hausdorff distance, to some limit object [vHPS92]. More precisely, each unit square is colored in white or black depending on whether the corresponding binomial coefficient is congruent to 0 modulo $p^s$ or not.

For instance, the limit object obtained for $p = 2$ and $s = 2$ is depicted below on the left, and the one for $p = 2$ and $s = 3$ is drawn on the right. Also note that $p = 2$ and $s = 1$ yield the Sierpiński gasket. In [vHPS92], one can find several geometrical and dynamical properties of the studied limit sets such as their Hausdorff dimension.





Several generalizations and variations of the Pascal triangle do already exist and for instance, they are studied with arithmetical and combinatorial viewpoints [BNS16, BS14, DDGS18, Ném18, NP16], dynamical ones [JdlRV05, vHPS92] or analytical ones [HKP18].

In this text, we define new extensions [LRS16, LRS17a, LRS17b, LRS18, Sti19] by means of binomial coefficients of words, which expand the classical notion of binomial coefficients of integers as explained below.

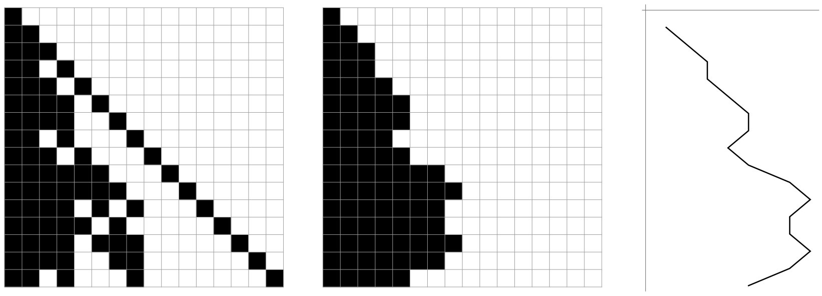
Let $A$ be a finite alphabet, *i.e.*, a finite set of characters or letters. A word over $A$ is simply a sequence of letters belonging to $A$, which can be either finite or infinite. The binomial coefficient $\binom{u}{v}$ of two finite words $u$ and $v$ over $A$ is the number of subsequences of $u$ that exactly match $v$. Observe that if $a$ is a letter, then the binomial coefficient of $a^n$ and $a^k$, which respectively represent $n$ and $k$ letters $a$ glued together, is the number of ways to select $k$ letters $a$ among $n$ available letters $a$, which is exactly $\binom{n}{k}$. Further information on binomial coefficients of words can be found in [Lot97, Chapter 6]. There is a vast literature on the subject with applications in formal language theory [Eil76, FK18, KKS15, KNS16], $p$-adic topology and $p$-adic analysis [BCP89, PS14], combinatorics on words [DE04, RRS15], and model-checking and verification [ABRS05].

This thesis is centered at the Pascal triangle, binomial coefficients, numeration systems and related questions, and is articulated as follows.

The first chapter presents the necessary background to grasp the substance of this text. We first define basics in combinatorics on words, which are taken from [Lot97, Lot02, Rig14a], and we summarily discuss numeration systems based on [BR10, Fra85, Lot02, Rig14b]. In particular, we define numeration systems associated with Parry numbers, *i.e.*, real numbers $\beta > 1$ for which the $\beta$-expansion of 1 is ultimately periodic (the precise definition is given later on). Then we devote an entire section to the key notion of binomial coefficients of words. Using them, we define a new analogue $\mathrm{P}_L$ of the Pascal triangle based on any genealogically ordered language $L$. Therefore, it is natural to consider the case of languages occurring in the theory of numeration systems. As already mentioned, the Pascal triangle P can be seen as an infinite table whose rows and columns are indexed by non-negative integers. This is also the case for its extended version $\mathrm{P}_L$. In the classical version, the sequence $(S(n))_{n\geq 0}$, which counts the number of positive integers on each row of P, satisfies $S(n) = n + 1$ for all $n \geq 0$. However, in the case of $\mathrm{P}_L$, the analogous sequence $S_L = (S_L(n))_{n\geq 0}$, which counts the number of positive integers on each row of $\mathrm{P}_L$, has a much more complicated and irregular behavior, reflecting some combinatorial properties of the language $L$. As we will see further on, one of our goals is to study the properties and the core structure of the sequences $S_L$ for some given languages $L$. Therefore, we give a short introduction to automatic, synchronized and regular sequences [AS92, AS03a, AST00, BR11, CM01]. We finish up the first chapter with notions related to metrics and more precisely, the Hausdorff metric, which turn out to be necessary in the second chapter of this thesis.

The second chapter aims at extending the bond between the Pascal triangle and the Sierpiński gasket, which was explained in the beginning of this introduction. The so-called Parry numeration systems, based on a Parry number $\beta > 1$, form a well-known and widely studied class of numeration systems containing the integer base numeration systems and the Zeckendorf numeration system based on Fibonacci numbers. Within this rather general setting, we study the corresponding Sierpiński gasket. The latter limit object is precisely described in terms of a combinatorial condition on words belonging to the considered numeration language and two maps, one being a homethety of ratio $1/\beta$ as in the classical case of the Pascal triangle and the other mapping $(x, y)$ to $(x, \beta y)$, *i.e.*, multiplying the second component by $\beta$. We show that the Sierpiński-like gasket is the closure of a union of segments whose endpoints are well understood thanks to our combinatorial condition. To simplify the discussion at first, we consider the language of binary expansions of integers, namely words made of 0's and 1's that begin with 1. Some of our reasonings make use of Lucas' theorem, and we can therefore handle the case of a prime number of colors. The results presented in this chapter are published in [LRS16, Sti19]. We close it with some open questions and problems.

In the third chapter, we study the regularity of the sequences $S_L$, which were roughly defined above. If we want to compress the data found in the generalized Pascal triangle $\mathrm{P}_L$, the sequence $S_L$ codes the amount of information we have on each of its row. Below, the figure on the left displays the positive values in the first rows of the generalized Pascal triangle associated with the binary language, *i.e.*, a black (resp., white) square corresponds to a pair of binary words having a positive (resp., zero) binomial coefficient. In the middle, one can find its compressed version, and on the right, the corresponding sequence $S_2 = (S_2(n))_{n\geq 0}$ is plotted.



In fact, the regularity we look at highly depends on the numeration system that is considered, *i.e.*, the definition of the regularity constantly involves the specific numeration system whose numeration language is precisely $L$. In rough words, if $U = (U(n))_{n\geq 0}$ is a strictly increasing sequence of integers starting with 1, then the $U$-kernel of a sequence $s = (s(n))_{n\geq 0}$ of integers is the set of all subsequences of $s$ of the form $(s(i_q(n)))_{n\geq 0}$, where $i_q\colon \mathbb{N} \to \mathbb{N}$ selects all the integers whose representations in the numeration system based on $(U(n))_{n\geq 0}$ end with the suffix $q$. Then a sequence $(s(n))_{n\geq 0}$ is said to be $U$-regular if its $U$-kernel is a finitely-generated $\mathbb{Z}$-module [AST00, RM02, Sha88] (the precise definitions are given in the first and third chapters). As in the second chapter, for pedagogical reasons, we first handle base-2 expansions [LRS17b] (in this case, $U(n) = 2^n$) and the corresponding sequence $S_2 = (S_2(n))_{n\geq 0}$. We present a new method based on trees to show that $S_2$ is 2-regular and also related to the Stern–Brocot sequence, which is a typical example of a 2-regular sequence. The definition of these trees allows us to easily enumerate, and thus count, all subsequences occurring in a given word. Indeed, it is a challenging problem to determine what are the "best" data structures for reasoning with subsequences [BDS16, KNS16]. Even if we are not aware of any relations to extensions of the Stern–Brocot sequence, our general method allows us to tackle the case of all integer base numeration systems [LRS18], and more exotically, the Zeckendorf numeration system [LRS17b] (up to our knowledge, our sequences are not related to already known regular sequences, so the regularity property needs to be properly proved). It is worth noticing that regular sequences in the Fibonacci framework are not so easy to find in the literature, which endows a certain bonus to this work. As a matter of fact, many questions remain unsolved in the theory of $U$-regular sequences, *e.g.*, a Cobham-like theorem, and it is therefore interesting to provide some new natural instances of this type of sequences. Compared to the previous chapter, replacing the Zeckendorf numeration system with an arbitrary Parry numeration system is not obvious. For instance, the Tribonacci numeration system, which naturally generalizes the Zeckendorf numeration system, is not yet fully understood. This observation permits us to end the chapter with some open questions and problems.

In the fourth chapter, we establish the asymptotic behaviors of the summatory functions $A_L = (A_L(n))_{n\geq 0}$ of the sequences $S_L$ for different languages $L$. Otherwise stated, $A_L(n)$ is the total amount of information found on the first $n$ rows in the generalized Pascal triangle $\mathrm{P}_L$. As in the previous chapters, we first take care of the base 2 case [LRS17a], which helps us to investigate the general integer case [LRS18]. As already mentioned above, a nice property of the sequence $S_b = (S_b(n))_{n\geq 0}$ associated with the base-$b$ numeration system is its $b$-regularity. Traditional methods to deal with summatory functions of $b$-regular sequences are on an algebraic or analytic side. They provide general asymptotic formulas usually involving an error term. Our contribution to the study of asymptotic behaviors is to develop a new systematic method to obtain such asymptotic estimates. Our method is based on the construction of a convenient and exotic numeration system, as roughly explained below. In particular, for integer base numeration systems, it gives exact formulas with no error term. The idea is more or less to find two sequences $(R(n))_{n\geq 0}$ and $(T(n))_{n\geq 0}$, each satisfying a linear recurrence relation, such that $A_L(R(n)) = T(n)$ for all $n \geq 0$. Then the asymptotic behavior of $A_L$ depends on the dominant root of the characteristic polynomial of the linear recurrence relation that defines $(T(n))_{n\geq 0}$. In the context of the theory of numeration systems, one interesting feature of our method is to elaborate non-standard representations of $A_L(n)$ in terms of the sequence $(T(n))_{n\geq 0}$. In the aftermath of this original technique, we treat the case of the Zeckendorf numeration system and the Fibonacci numbers [LRS17a]. In this setting, we obtain a formula with an error term for the corresponding summatory function. As in the third chapter, we naturally open the door to applications to other numeration systems and related problems, so we again conclude with some open questions.

As a final comment to this introduction, I would like to mention that the present work is concerned with the sequences `A000032`, `A000045`, `A000788`, `A001590`, `A002487`, `A004601`, `A006356`, `A007306`, `A014417`, `A282714`, `A282715`, `A282716`, `A282717`, `A282718`, `A282719`, `A282720`, `A282728`, `A282729`, `A282730`, `A282731`, `A282732`, `A284441` and `A284442` in [Slo]. In fact, some of the previous sequences were created from scratch in [Slo] after our work was published; we are grateful to N. J. A. Sloane for uploading them in his encyclopedia. To gather data and formulate conjectures prior to this work, mathematical computations were done using the *Mathematica* and *SageMath* programs. For the interested reader, I keep notebooks at their disposal.

During these four years of doctoral studies, I was also able to consider other problems in combinatorics on words giving me the opportunity to write four more papers: Nyldon words [CPS18], palindromic Ziv-Lempel and Crochemore factorizations [JMnRS], Cobham's theorem and automaticity [MRSS18], and formal inverses of sequences [RS18]. For this dissertation, I chose to present the content of the papers [LRS16, LRS17b, LRS17a, LRS18, Sti19] to create a coherent whole.

# Chapter 1

# Preliminaries

This first chapter gives the necessary background to understand the content of this text. In Section 1.2, we start with basic definitions from combinatorics on words. The interested reader will find more information in [Lot97, Lot02, Rig14a]. After, we briefly discuss numeration systems in Section 1.3. This summary is built on [BR10, Chapter 2], [Fra85], [Lot02, Chapter 7] and [Rig14b]. Then Section 1.4 is devoted to the core notion of binomial coefficients of words; see [Lot97, Chapter 6] for more details. Using those binomial coefficients, we are able to define analogues of the well-known Pascal triangle in Section 1.5. Emerging from them, we also define sequences of interest in Section 1.6. In Section 1.7, we write a short introduction to automatic, synchronized and regular sequences based on different texts [AS92, AS03a, AST00, BR11, CM01]. We finish up with notions related to metrics in Section 1.8, which turn out to be useful in Chapter 2.

**Contents**

## 1.1 Basic Notation

In this text, we let $\mathbb{N}$ (resp., $\mathbb{N}_0$) be the set of non-negative (resp., positive) integers. Similarly, $\mathbb{Z}$ is the set of all integers, while $\mathbb{Z}_0$ contains all of them but 0. We let $\mathbb{R}$ (resp., $\mathbb{C}$) denote the set of all real (resp., complex) numbers. For any $\mathbb{K} \in \{\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}\}$, any $x \in \mathbb{K}$, and any $\diamond \in \{<, \leq, >, \geq\}$, we let $\mathbb{K}_{\diamond x}$ denote the set $\{y \in \mathbb{K} \mid y \diamond x\}$. If $\mathbb{K} \in \{\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}\}$, we let $\mathbb{K}[X]$ denote the set of polynomials of indeterminate $X$ and with coefficients in $\mathbb{K}$.

We let $\lceil \cdot \rceil$ denote the *ceiling function* defined by $\lceil x \rceil = \inf\{z \in \mathbb{Z} \mid z \geq x\}$, and $\lfloor \cdot \rfloor$ stands for the *floor function* defined by $\lfloor x \rfloor = \sup\{z \in \mathbb{Z} \mid z \leq x\}$ for all $x \in \mathbb{R}$. The *fractional function* $\{\cdot\}$ is defined for all $x \in \mathbb{R}$ by $\{x\} = x - \lfloor x \rfloor$ [GKP94]. For a real number $x$, $\lceil x \rceil$ (resp., $\lfloor x \rfloor$; resp., $\{x\}$) is also called the *ceiling* (resp., *floor*; resp., *fractional*) *part* of $x$.

## 1.2 The Flexibility of CoW Tails in Curves

For more on combinatorics on words[1], we refer the reader to [Lot97, Lot02, Rig14a].

**Definition 1.1.** An *alphabet* is a non-empty finite set, whose elements are called *letters* or *characters*. In our context, the alphabets are often finite subsets of $\mathbb{N}$. It is worth noticing that infinite alphabets do exist, but we will not consider them unless otherwise specified.

A *word* over an alphabet $A$ is a finite or infinite sequence of letters in $A$. We let $\varepsilon$ denote the *empty word*, *i.e.*, the empty sequence. The *length* of a finite word $w$, denoted by $|w|$, is the number of letters contained in $w$. If $w$ is a non-empty finite (resp., infinite) word, then we let $w_n$ denote its letters for all $n \in \{0, 1, \ldots, |w| - 1\}$ (resp., $n \in \mathbb{N}$). In the finite case, we write $w = w_{|w|-1} w_{|w|-2} \cdots w_0$ or $w = w_0 w_1 \cdots w_{|w|-1}$ depending on the context. If $w$ is a word, we let $w^R$ denote its *reversal* or *mirror* obtained by writing $w$ from right to left, instead of from left to right.

The *set of finite* (resp., *infinite*) words over an alphabet $A$ is denoted by $A^*$ (resp., $A^\omega$ or $A^\mathbb{N}$). Note that $A^*$ is countable whereas $A^\omega$ and $A^\mathbb{N}$ are uncountable. For a unary alphabet $\{a\}$, we usually write $a^*$ instead of $\{a\}^*$.

[1] The title of this section is inspired by one of the sentences that my math teacher in third grade of high school used to repeat a lot. He was incredibly funny and was one of the first to arouse my mathematical curiosity.

A *language* over an alphabet $A$ is a subset of $A^*$.

**Example 1.2.** Let $A = \{0, 1\}$ be the alphabet with the two letters 0 and 1. Consider the finite word $u = 101001$ over $A$. The length of $u$ is $|u| = 6$, and, we write $u = u_0u_1u_2u_3u_4u_5$ with $u_0 = u_2 = u_5 = 1$ and $u_1 = u_3 = u_4 = 0$. We also have $u^R = 100101$. The language of words over $A$ starting with 1 is written $1\{0, 1\}^*$.

**Definition 1.3.** If $u$ and $v$ are two finite words over an alphabet $A$, then the *concatenation of* $u$ *and* $v$, denoted by $u \cdot v$ (or simply $uv$ if there is no need to emphasize), is the finite word $w$ of length $|u| + |v|$ defined by

$$w_n = \begin{cases} u_n, & \text{if } n \in \{0, 1, \ldots, |u| - 1\}; \\ v_{n-|u|}, & \text{if } n \in \{|u|, |u| + 1, \ldots, |u| + |v| - 1\}. \end{cases}$$

In a similar way, we can concatenate a finite word $u$ with an infinite word $v$.

For a finite word $w$ over an alphabet $A$ and a non-negative integer $n$, we let $w^n$ denote the concatenation of $n$ copies of $w$, which is defined by induction by $w^0 = \varepsilon$ and $w^{n+1} = w^n w$ for all $n \in \mathbb{N}$. We say that $w^n$ is the $n$th *power* of $w$. In the same way, we let $w^\omega$ denote the concatenation of infinitely many copies of $w$, which is defined by

$$(w^\omega)_{n\cdot|w|}(w^\omega)_{n\cdot|w|+1} \cdots (w^\omega)_{(n+1)\cdot|w|-1} = w$$

for all $n \in \mathbb{N}$.

Similarly, we define fractional powers of words. Let $w = w_0w_1 \cdots w_{|w|-1}$ be a finite word over $A$. Then its *fractional power* of exponent $p/|w|$ is the word $w^{p/|w|} = w^\ell w_0w_1 \cdots w_{q-1}$ where $\ell$ and $q$ satisfy $p = \ell|w| + q$ with $0 < q \leq |w|$.

An infinite word $w \in A^\omega$ is said to be *ultimately periodic* if there exist finite words $u, v \in A^*$ such that $w = uv^\omega$.

**Example 1.4.** Over the classical Latin alphabet, the concatenation of the words *humming* and *bird* is the word *hummingbird*. Over the binary alphabet $\{0, 1\}$, the concatenation of the words 101 and 001 gives the word 101001 from the previous example.

**Definition 1.5.** Let $L$ and $M$ be two languages over the alphabet $A$. The *concatenation of* $L$ *and* $M$ is the language $LM = \{uv \mid u \in L, v \in M\}$.

For all $n \in \mathbb{N}$, we let $L^n$ denote the concatenation of $n$ copies of $L$, which is defined by $L^0 = \{\varepsilon\}$, and for all $n \in \mathbb{N}_0$,

$$L^n = \{u^{(1)} \cdots u^{(n)} \mid u^{(i)} \in L \text{ for all } i \in \{1, 2, \dots, n\}\}.$$

For all $n \in \mathbb{N}$, we define $L^{\leq n} = \bigcup_{i=0}^{n} L^i$. The *Kleene star* of $L$ is the language $L^* = \bigcup_{n \geq 0} L^n$. For a language $L = \{w\}$ containing only one element $w$, we usually write $w^*$ instead of $\{w\}^*$.

If $L \subset A^*$ is a language and $u \in A^*$ is a finite word, we let $u^{-1}L$ (resp., $Lu^{-1}$) denote the set of words $\{v \in A^* \mid uv \in L\}$ (resp., $\{v \in A^* \mid vu \in L\}$), which contains the words over $A$ that can be put after (resp., before) $u$ to build words in $L$.

In the following definition, we want to emphasize the distinction we make between the terms "factor", or "subword", and "scattered subword".

**Definition 1.6.** Let $w$ be a word over an alphabet $A$. A *factor* or *subword* of $w$ is a finite word $u$ such that there exist $x \in A^*$ and $y \in A^* \cup A^\omega$ satisfying $w = xuy$. More generally, a *scattered subword* of $w$ is a subsequence of $w$ whose indices are not necessarily consecutive. In this case, the idea is to delete letters in $w$ to obtain a scattered subword. Thus, a factor of $w$ is a scattered subword of $w$ whose indices are consecutive.

A *prefix* (resp., *suffix*) of $w$ is a word $u$ (resp., $v$) such that there exists $v \in A^* \cup A^\omega$ (resp., $u \in A^*$) verifying $w = uv$. A prefix or a suffix of $w$ is strict if it is not equal to $w$.

**Example 1.7.** Let $u = 101001 = u_0 u_1 \cdots u_5 \in \{0,1\}^*$. Then 0100 is a factor of $u$ corresponding to the subsequence $(1, 2, 3, 4)$, while 111 is a scattered subword of $u$ corresponding to the subsequence $(0, 2, 5)$, but not a factor of $u$. Moreover, 101 (resp., 001) is a prefix (resp., suffix) of $u$.

If an alphabet $A$ is endowed with a total order, then one can extend this order to $A^*$ or to $A^* \cup A^\omega$. In the following, we define two particular orders on words.

**Definition 1.8.** Let $(A, <)$ be a totally ordered alphabet. The order $<$ on $A$ extends to an order on $A^* \cup A^\omega$, called the *lexicographical order*, as follows.

If $u$ and $v$ are two finite words over $A$, then $u$ is said to be *lexicographically* *less* than $v$, and we write $u <_{\text{lex}} v$, either if $u$ is a strict prefix of $v$, or if there exist $p, s, t \in A^*$, and $a, b \in A$ such that $u = pas$, $v = pbt$, and $a < b$.

Similarly, if $u$ and $v$ are two infinite words over $A$, then $u$ is said to be *lexicographically less* than $v$, and we also write $u <_{\text{lex}} v$, if there exist $p \in A^*$, $s, t \in A^\omega$, and $a, b \in A$ such that $u = pas$, $v = pbt$, and $a < b$. This definition extends to $A^* \cup A^\omega$ if every finite word $z \in A^*$ is replaced by $z\spadesuit^\omega \in (A \cup \{\spadesuit\})^\omega$, where the symbol $\spadesuit$ does not belong to $A$ and is assumed to verify $\spadesuit < a$ for all $a \in A$. Note that the first and the second definitions coincide on finite words.

We write $u \leq_{\text{lex}} v$ for two words $u$ and $v$ satisfying either $u <_{\text{lex}} v$ or $u = v$.

Note that the lexicographical order is commonly used in any language dictionary.

**Definition 1.9.** Let $(A, <)$ be a totally ordered alphabet. The order $<$ on $A$ extends to an order on $A^*$, called the *genealogical order*, as follows. If $u$ and $v$ are two finite words over $A$, then $u$ is said to be *genealogically less* than $v$, and we write $u <_{\text{gen}} v$, if they satisfy either $|u| = |v|$ and $u <_{\text{lex}} v$, or $|u| < |v|$. We write $u \leq_{\text{gen}} v$ for two finite words $u$ and $v$ satisfying either $u <_{\text{gen}} v$ or $u = v$.

In the literature some authors call *radix order* what we call genealogical order.

**Example 1.10.** Consider the alphabet $\{0, 1\}$ totally ordered by $0 < 1$. We have $0011 <_{\text{lex}} 010 <_{\text{lex}} 0100$ but $010 <_{\text{gen}} 0011 <_{\text{gen}} 0100$.

We end this section by the concept of convergence of sequences of words.

**Definition 1.11.** Let $x$ and $y$ be two infinite words over the alphabet $A$. We let $\Lambda(x, y)$ denote the *longest common prefix* of $x$ and $y$. Note that $|\Lambda(x, y)|$ is the smallest index where the two words $x$ and $y$ differ, *i.e.*,

$$|\Lambda(x, y)| = \inf\{i \in \mathbb{N} \mid x_i \neq y_i\}.$$

We define the map

$$d' \colon A^\omega \times A^\omega \to \mathbb{R}_{\geq 0}, (x, y) \mapsto d'(x, y) = 2^{-|\Lambda(x,y)|}.$$

We also set $d'(x,x) = 0$ for all $x \in A^\omega$. It is not difficult to show that $d'$ is an *ultrametric distance*.

A sequence $(z_n)_{n\geq 0}$ of infinite words over the alphabet $A$ *converges* to the infinite word $x \in A^\omega$ if $d'(z_n, x)$ tends to 0 whenever $n$ tends to $+\infty$.

Similarly, we define the convergence of sequences of finite words. If the symbol $\spadesuit$ does not belong to the alphabet $A$, then the sets $A^*$ and $(A \cup \{\spadesuit\})^\omega$ are in bijection via the map $u \in A^* \mapsto u\spadesuit^\omega \in (A \cup \{\spadesuit\})^\omega$. We say that a sequence $(z_n)_{n\geq 0}$ of finite words over $A$ *converges* to the infinite word $x \in A^\omega$ if the sequence $(z_n\spadesuit^\omega)_{n\geq 0}$ of infinite words converges to $x$.

**Example 1.12.** The sequence $((101)^n 2^\omega)_{n\geq 0}$ of infinite words converges to $(101)^\omega$. The sequence $(10^n)_{n\geq 0}$ of finite words converges to $10^\omega$.

Let us build a word that is the limit of a sequence of finite words and that is not periodic.

**Definition 1.13.** Let $A, B$ be two alphabets. A *morphism* $f\colon A^* \to B^*$ is a map satisfying $f(uv) = f(u)f(v)$ for all words $u, v \in A^*$. In particular, we get $f(\varepsilon) = \varepsilon$, and $f$ is completely determined by the images of the letters in the alphabet $A$.

**Example 1.14.** Consider the morphism

$$\tau\colon \{0,1\}^* \to \{0,1\}^*, 0 \mapsto 01, 1 \mapsto 10.$$

The first few iterations of $\tau$ on 0 are

$$\begin{aligned}\tau(0) &= 01,\\ \tau^2(0) &= 0110,\\ \tau^3(0) &= 01101001,\\ \tau^4(0) &= 0110100110010110.\end{aligned}$$

Since $|\tau(0)| = |\tau(1)| = 2$, we have $|\tau^n(0)| = 2^n$ for all $n \geq 0$. It is not difficult to show that $\tau^n(0)$ is a proper prefix of $\tau^{n+1}(0)$ for all $n \geq 0$. Consequently, the sequence $(\tau^n(0))_{n\geq 0}$ of finite words converges to the infinite word

$$t = \lim_{n\to+\infty} \tau^n(0) = 0110100110010110100101100110100110010110 0110\cdots,$$

which is called the *Thue–Morse word* or *Thue–Morse sequence* [Mor21, Thu12]. It is known that this word is not periodic (see, for instance, [AS99]).

## 1.3 Numeration Systems

This section concerns various numeration systems, namely positional, real base, linear, Parry, Bertrand and Perron numeration systems. At this stage, we do not wish to go into deep details, so we refer the interested reader to [BR10, Chapter 2], [Lot02, Chapter 7] or [Fra85, Rig14b]. With simple words, a numeration system is a way to represent numbers with the use of digits, or letters in $\mathbb{N}$. A common property that the previous numeration systems all share is that the digits belong to finite alphabets. It is worth noticing that numeration systems with infinite alphabet do also exist, *e.g.*, the factorial numeration system.

We start this section by defining positional numeration systems. As we will see, they include the daily used decimal numeral system.

**Definition 1.15.** A *positional numeration system* is given by a strictly increasing sequence $U = (U(n))_{n\geq 0}$ of integers such that we have $U(0) = 1$ and $C_U = \sup\{\lceil U(n+1)/U(n)\rceil \mid n \in \mathbb{N}\}$ is finite. If $n$ is a positive integer, we let $\mathrm{rep}_U(n)$ denote its *greedy $U$-expansion*, which is the unique finite word $w = w_{|w|-1}w_{|w|-2}\cdots w_0$ over the alphabet $A_U = \{0, 1, \ldots, C_U - 1\}$ not beginning with 0 and satisfying

$$n = \sum_{i=0}^{|w|-1} w_i\, U(i) \quad \text{and} \quad \sum_{i=0}^{t} w_i\, U(i) < U(t+1) \quad \text{for all } t \in \{0, 1, \ldots, |w|-1\}.$$

Moreover, we set $\mathrm{rep}_U(0) = \varepsilon$. When the context is clear, the greedy $U$-expansion is simply called *$U$-expansion*. The elements of $A_U$ are called the *digits*. The set $L_U = \mathrm{rep}_U(\mathbb{N})$ of all $U$-expansions is referred to as the *numeration language*. If $w = w_{|w|-1}w_{|w|-2}\cdots w_0$ is a finite word over some alphabet made of integers, then we let $\mathrm{val}_U(w)$ denote its *$U$-numerical value*, which is given by

$$\mathrm{val}_U(w) = \sum_{i=0}^{|w|-1} w_i\, U(i).$$

If $\mathrm{val}_U(w) = n$, we say that the word $w$ is a *$U$-representation* of $n$. In this case, observe that $w$ is not necessarily the greedy $U$-expansion of $n$.

The next proposition shows that the genealogical order coincides with the classical order in $\mathbb{N}$.

**Proposition 1.16.** *Let $m$ and $n$ be two non-negative integers. Then $m < n$ if and only if* $\mathrm{rep}_U(m) <_{gen} \mathrm{rep}_U(n)$.

The following two examples are very important. The notation they introduce will be used throughout the text. Note that, if there is no need to emphasize, we usually make no distinction between the symbols $0, 1, 2, 3, \ldots$ and the integers they represent.

**Example 1.17.** Let $b \geq 2$ be an integer. The *integer base-$b$ numeration system* is the positional numeration system built on the sequence

$$U_b = (b^n)_{n \geq 0}.$$

In this case, the alphabet is $A_b = A_{U_b} = \{0, 1, \ldots, b-1\}$, and the numeration language is

$$L_b = L_{U_b} = \mathrm{rep}_{U_b}(\mathbb{N}) = \{1, 2, \ldots, b-1\}\{0, 1, \ldots, b-1\}^* \cup \{\varepsilon\}.$$

Observe that, if leading zeroes were allowed, then different words could represent the same integer. Within this particular numeration system, the (greedy) $U_b$-expansion is also called *base-$b$ expansion*. For the sake of simplicity, we also set $\mathrm{rep}_b = \mathrm{rep}_{U_b}$ and $\mathrm{val}_b = \mathrm{val}_{U_b}$.

When $b = 10$, we find back the common base-10 numeration system that is used to represent numbers in everyday life. As another example, the case $b = 2$ is often used in computer science.

**Example 1.18.** Consider the sequence $F = (F(n))_{n \geq 0} = (1, 2, 3, 5, 8, 13, \ldots)$ of *Fibonacci numbers* (`A000045` in [Slo]) defined by

$$F(0) = 1, \; F(1) = 2, \quad \text{and} \quad F(n+2) = F(n+1) + F(n) \quad \text{for all } n \in \mathbb{N}.$$

The *Fibonacci numeration system*, also called the *Zeckendorf numeration system*, is the positional numeration system built on this sequence $F$. It was proved in [Zec72] that we have $A_F = \{0, 1\}$, and that the set of the greedy $F$-expansions of non-negative integers, *i.e.*, the numeration language, is the set

$$L_F = \mathrm{rep}_F(\mathbb{N}) = 1\{0, 01\}^* \cup \{\varepsilon\}$$

of the words over $\{0, 1\}$ not containing the factor 11. The sequence `A014417` in [Slo] gives the words in $L_F$. For instance, we have $\mathrm{rep}_F(15) = 100010$ and $\mathrm{val}_F(101001) = 13 + 5 + 1 = 19$.

In the last part of this section, we introduce the setting of the particular numeration systems that are used later in this text: the Parry–Bertrand numeration systems. First of all, we recall several definitions and results about representations of real numbers.

**Definition 1.19.** Let $\beta \in \mathbb{R}_{>1}$ and let $A_\beta = \{0, 1, \ldots, \lceil \beta \rceil - 1\}$. Every real number $x \in [0, 1)$ can be written as a series

$$x = \sum_{j=1}^{+\infty} c_j \beta^{-j},$$

where $c_j \in A_\beta$ for all $j \geq 1$. The infinite word $c_1 c_2 \cdots$ is called a *$\beta$-representation* or a *representation in (the real) base $\beta$* of $x$. Among all the $\beta$-representations of $x$, we define the *$\beta$-expansion* $d_\beta(x)$ of $x$ obtained in a greedy way, *i.e.*, for all $j \geq 1$, we have $c_j \beta^{-j} + c_{j+1} \beta^{-j-1} + \cdots < \beta^{-j+1}$. Also note that, if a representation ends with infinitely many zeroes, then it is sometimes convenient to omit the trailing zeroes, and the representation is said to be *finite*.

We also make use of the following convention: if $w = w_n \cdots w_0$ is a finite word (resp., $w = w_1 w_2 \cdots$ is an infinite word) over $A_\beta$, the notation $0.w$ has to be understood as the real number

$$\sum_{j=0}^{n} w_j \beta^{j-n-1} \quad \text{(resp., } \sum_{j=1}^{+\infty} w_j \beta^{-j}\text{)},$$

which actually corresponds to the value of the word $w$ in base $\beta$.

In an analogous way, the *$\beta$-expansion* $d_\beta(1)$ of 1 is the following infinite word over $A_\beta$

$$d_\beta(1) = \begin{cases} (\beta - 1)^\omega, & \text{if } \beta \in \mathbb{N}; \\ (\lceil \beta \rceil - 1)\, d_\beta\left(1 - \frac{\lceil \beta \rceil - 1}{\beta}\right), & \text{otherwise.} \end{cases}$$

In other words, if $\beta$ is not an integer, the first digit of the $\beta$-expansion of 1 is $\lceil \beta \rceil - 1$, and the other digits are derived from the $\beta$-expansion of $1 - (\lceil \beta \rceil - 1)/\beta$.

Let $d_\beta(1) = (t_n)_{n \geq 1}$ be the $\beta$-expansion of 1. Observe that $t_1 = \lceil \beta \rceil - 1$. The *quasi-greedy $\beta$-expansion* $d^*_\beta(1)$ of 1 is an infinite word defined as follows. If $d_\beta(1) = t_1 \cdots t_m$ is finite, *i.e.*, $t_m \neq 0$ and $t_j = 0$ for all $j > m$, then $d^*_\beta(1) = (t_1 \cdots t_{m-1}(t_m - 1))^\omega$. If $d_\beta(1)$ is infinite, then $d^*_\beta(1) = d_\beta(1)$.

This way of representing the real numbers in $[0,1]$ is called the *numeration system in (real) base* $\beta$.

As in Proposition 1.16, the order between real numbers is given by the lexicographic order between their greedy $\beta$-expansions.

**Proposition 1.20.** *Let $x$ and $y$ be two real numbers in $[0,1)$. Then $x < y$ if and only if $d_\beta(x) <_{lex} d_\beta(y)$.*

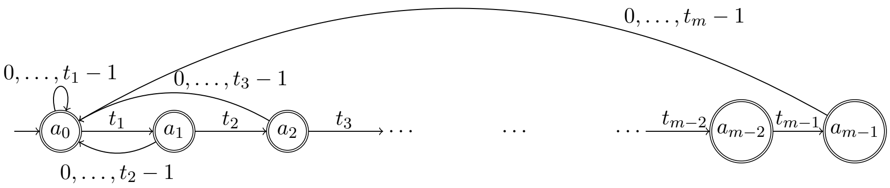
**Example 1.21.** Let $\varphi = \frac{1+\sqrt{5}}{2}$ be the golden ratio. The greedy $\varphi$-expansion of $x = 3 - \sqrt{5}$ is equal to $10010^\omega$ for we have $x = 1/\varphi + 1/\varphi^4$, but other $\varphi$-representations of $x$ are given by $01110^\omega$, or $100(01)^\omega$. Using the equality $1 = 1/\varphi + 1/\varphi^2$, one can prove that the $\varphi$-expansion of 1 is $d_\varphi(1) = 110^\omega$, while its quasi-greedy $\varphi$-expansion is $d^*_\varphi(1) = (10)^\omega$.

We now define a class of numeration systems based on specific real numbers for which the expansion of 1 is ultimately periodic.

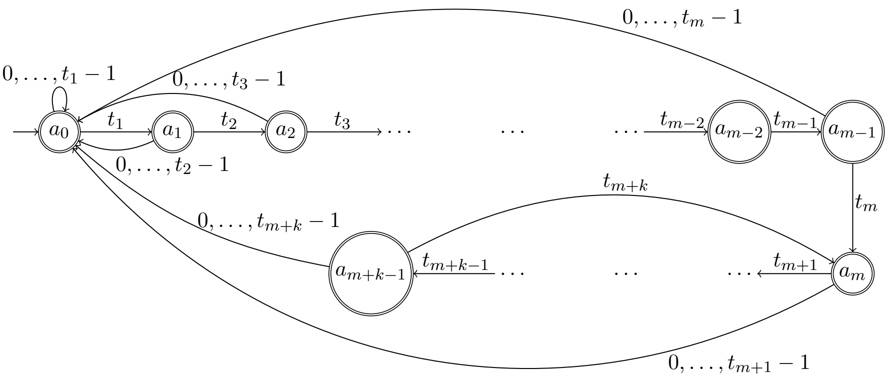
**Definition 1.22.** A real number $\beta > 1$ is a *Parry number* if $d_\beta(1)$ is ultimately periodic. If $d_\beta(1)$ is finite, *i.e.*, $d_\beta(1)$ ends with $0^\omega$, then $\beta$ is called a *simple Parry number*.

Examples of such numbers will be given later on, after Proposition 1.24. In the special case of Parry numbers, this result gives an easy way to decide with the use of automata if an infinite word is the $\beta$-expansion of a real number. This proposition is a reformulation of the well-known Parry's theorem [Par60], which describes the admissible $\beta$-expansions.

**Definition 1.23.** A *deterministic finite automaton (DFA)* over an alphabet $A$ is given by a 5-tuple $\mathcal{A} = (Q, q_0, A, \delta, F)$ where $Q$ is a finite set of *states*, $q_0 \in Q$ is the *initial state*, $\delta \colon Q \times A \to Q$ is the *transition function*, and $F \subset Q$ is the set of *final states* (graphically represented by two concentric circles). The map $\delta$ can be extended to $Q \times A^*$ by setting $\delta(q, \varepsilon) = q$, and $\delta(q, wa) = \delta(\delta(q, w), a)$ for all $q \in Q$, $a \in A$ and $w \in A^*$. We also say that a word $w$ is *accepted* by the automaton if $\delta(q_0, w) \in F$.




(a) The case where $d_\beta(1)$ is finite.




(b) The case where $d_\beta(1)$ is ultimately periodic but not finite.

Figure 1.1: The automaton $\mathcal{A}_\beta$ as a function of the ultimately periodic word $d_\beta(1)$.

**Proposition 1.24.** *Let $\beta \in \mathbb{R}_{>1}$ be a Parry number.*

- *Suppose that $d_\beta(1) = t_1 \cdots t_m$ is finite,* i.e., *$t_m \neq 0$ and $t_j = 0$ for all $j > m$. Then an infinite word is the $\beta$-expansion of a real number in $[0, 1)$ if and only if it is the label of a path in the automaton $\mathcal{A}_\beta = (\{a_0, \dots, a_{m-1}\}, a_0, A_\beta, \delta, \{a_0, \dots, a_{m-1}\})$ depicted in Figure 1.1a, where the transition function $\delta$ is defined as follows: for each $i \in \{1, \dots, m\}$, $\delta(a_{i-1}, t) = a_0$ for all $t \in \{0, \dots, t_i - 1\}$; and for every $i \in \{1, \dots, m-1\}$, $\delta(a_{i-1}, t_i) = a_i$.*

- *Suppose that $d_\beta(1) = t_1 \cdots t_m (t_{m+1} \cdots t_{m+k})^\omega$ where $m, k$ are taken to be minimal. Then an infinite word is the $\beta$-expansion of a real num-*

*ber in* $[0,1)$ *if and only if it is the label of a path in the automaton* $\mathcal{A}_\beta = (\{a_0, \ldots, a_{m+k-1}\}, a_0, A_\beta, \delta, \{a_0, \ldots, a_{m+k-1}\})$ *depicted in Figure 1.1b, where the transition function* $\delta$ *is defined as follows: for each* $i \in \{1, \ldots, m+k\}$, $\delta(a_{i-1}, t) = a_0$ *for all* $t \in \{0, \ldots, t_i - 1\}$*; for every* $i \in \{1, \ldots, m+k-1\}$, $\delta(a_{i-1}, t_i) = a_i$, *and* $\delta(a_{m+k-1}, t_{m+k}) = a_m$.

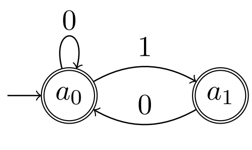
It is worth observing that from any state in the automaton $\mathcal{A}_\beta$, one can reach the initial state by reading a suitable sequence of zeroes, acting as a reset sequence. Note that if $t_i = 0$, then the set $\{0, \ldots, t_i - 1\}$ is empty, so several zeroes might actually be required to reach the initial state. Now let us illustrate the previous proposition. For other examples, see, for instance, [CRRW11].

**Example 1.25.** If $\beta \in \mathbb{R}_{>1}$ is an integer, then $d_\beta(1) = d^*_\beta(1) = (\beta - 1)^\omega$ by definition, and $\beta$ is a Parry number. The automaton $\mathcal{A}_\beta$ consists of a single initial and final state $a_0$ with a loop of labels $0, 1, \ldots, \beta - 1$.

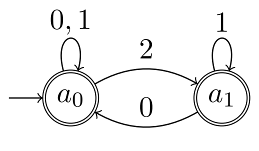
Consider the golden ratio $\varphi$. From Example 1.21, we already know that $\varphi$ is a (simple) Parry number. The automaton $\mathcal{A}_\varphi$ is depicted in Figure 1.2a. The square $\varphi^2$ of the golden ratio is again a Parry number, but a non-simple one. Using the equality

$$1 = \frac{2}{\varphi^2} + \sum_{n=2}^{+\infty} \frac{1}{(\varphi^2)^n},$$

we can show that $d_{\varphi^2}(1) = d^*_{\varphi^2}(1) = 21^\omega$. The automaton $\mathcal{A}_{\varphi^2}$ is depicted in Figure 1.2b.




(a) $\mathcal{A}_\varphi$




(b) $\mathcal{A}_{\varphi^2}$

Figure 1.2: The automata respectively associated with the golden ratio and its square.

With every Parry number is canonically associated a linear numeration system. Let us recall the definition of such numeration systems.

**Definition 1.26.** Let $U = (U(n))_{n\ge 0}$ be a positional numeration system. We say that $U$ is a *linear numeration system* if $U$ satisfies a linear recurrence relation, *i.e.*, there exist $k \ge 1$ and $b_0, \dots, b_{k-1} \in \mathbb{Z}$ such that

$$U(n+k) = b_{k-1}\, U(n+k-1) + \cdots + b_0\, U(n) \quad \text{for all } n \ge 0. \tag{1.1}$$

**Remark 1.27.** Note that if two linear numeration systems are associated with the same recurrence relation, then they only differ by the choice of the initial values $U(0), \dots, U(k-1)$. This choice is sometimes crucial. See, for instance, Example 1.34 below.

**Example 1.28.** It is not difficult to see that the integer base numeration system from Example 1.17 and the Fibonacci numeration system from Example 1.18 are linear.

**Definition 1.29.** Let $\beta \in \mathbb{R}_{>1}$ be a Parry number with $d^*_\beta(1) = (t'_i)_{i\ge 1}$. We define a particular linear numeration system $U_\beta = (U_\beta(n))_{n\ge 0}$ associated with $\beta$ by $U_\beta(n) = t'_1 U_\beta(n-1) + \cdots + t'_n U_\beta(0) + 1$ for all $n \ge 0$. We call it the *Parry numeration system associated with* $\beta$. In particular, if $d_\beta(1) = t_1 \cdots t_m$ is finite ($t_m \neq 0$), then $U_\beta(0) = 1$,

$$U_\beta(i) = t_1 U_\beta(i-1) + \cdots + t_i U_\beta(0) + 1 \quad \text{for all } i \in \{1, \dots, m-1\},$$

and for all $n \ge m$,

$$U_\beta(n) = t_1 U_\beta(n-1) + \cdots + t_m U_\beta(n-m).$$

If $d_\beta(1) = t_1 \cdots t_m (t_{m+1} \cdots t_{m+k})^\omega$ (where $m, k$ are minimal), then we have $U_\beta(0) = 1$,

$$U_\beta(i) = t_1 U_\beta(i-1) + \cdots + t_i U_\beta(0) + 1 \quad \text{for all } i \in \{1, \dots, m+k-1\},$$

and for all $n \ge m+k$,

$$\begin{aligned} U_\beta(n) = {} & t_1 U_\beta(n-1) + \cdots + t_{m+k} U_\beta(n-m-k) + U_\beta(n-k) \\ & - t_1 U_\beta(n-k-1) - \cdots - t_m U_\beta(n-k-m). \end{aligned}$$

**Example 1.30.** When $\beta \in \mathbb{R}_{>1}$ is an integer, then we know from Example 1.25 that $d_\beta(1) = (\beta-1)^\omega$. We are thus in the second case of Definition 1.29, and we have $m = 0$, $k = 1$, $U_\beta(0) = 1$ and, for all $n \ge 1$,

$$U_\beta(n) = (\beta-1) U_\beta(n-1) + U_\beta(n-1) = \beta\, U_\beta(n-1).$$

This yields $U_\beta(n) = \beta^n$ for all $n \geq 0$. The Parry numeration system $U_\beta$ is thus the usual integer base numeration system from Example 1.17.

For the golden ratio $\varphi$, we have $d_\varphi(1) = 11$ from Example 1.25, so we fall into the first case of Definition 1.29. We find $m = 2$, $U_\varphi(0) = 1$,

$$U_\varphi(1) = 1 \cdot U_\varphi(0) + 1 = 2,$$

and, for all $n \geq 2$,

$$U_\varphi(n) = 1 \cdot U_\varphi(n-1) + 1 \cdot U_\varphi(n-2).$$

Consequently, the sequences $(F(n))_{n\geq 0}$ and $(U_\varphi(n))_{n\geq 0}$ are equal, and the Parry numeration system $U_\varphi$ is the Fibonacci numeration system from Example 1.18. In particular, $L_{U_\varphi} = L_F = 1\{0, 01\}^* \cup \{\varepsilon\}$, $\mathrm{rep}_{U_\varphi} = \mathrm{rep}_F$ and $\mathrm{val}_{U_\varphi} = \mathrm{val}_F$.

The linear numeration system $U_\beta$ from Definition 1.29 has an interesting property: we can add or delete trailing zeroes and still keep words in the numeration language.

**Definition 1.31.** A linear numeration system $U = (U(n))_{n\geq 0}$ is a *Bertrand numeration system* if, for all $w \in A_U^+$, $w \in L_U$ if and only if $w0 \in L_U$.

A. Bertrand-Mathis proved in particular that the Parry numeration system $U_\beta$ associated with the Parry number $\beta$ from Definition 1.29 is also a Bertrand numeration system; see [BM89] or [BR10, Chapter 2]. In that case, the alphabet $A_{U_\beta}$ is the set $\{0, 1, \ldots, \lceil\beta\rceil - 1\}$, and any word $w$ in the set $0^* L_{U_\beta}$ of all $U_\beta$-expansions where leading zeroes are allowed is the label of a path in the automaton $\mathcal{A}_\beta$ from Proposition 1.24. For instance, one can easily be convinced that the Parry numeration systems highlighted in Example 1.30 are indeed Bertrand. In the case of the golden ratio $\varphi$, the valid $U_\varphi$-expansions with leading zeroes can be deduced from the automaton drawn in Figure 1.2a.

We end this section by defining other classes of real numbers, giving additional properties to the numeration systems emerging from them.

**Definition 1.32.** A real number $\beta > 1$ is a *Pisot number* (resp., *Perron number*) if it is an algebraic integer, *i.e.*, a root of a monic polynomial in $\mathbb{Z}[X]$, whose conjugates have modulus less than 1 (resp., $\beta$).

Observe that any Pisot number is a Perron number, but the contrary is false in general. For instance, $(5+\sqrt{5})/2$ and the dominant root of the polynomial $X^4 - 3X^3 - 2X^2 - 3$ are Perron, but not Pisot; see [Lot02, Chapter 7]. Note that a Pisot number is a Parry number [Lot02, Chapter 7].

Numeration systems based on Perron numbers are defined below. They have the property (1.2), which allows us to understand their growth rate. Since every Parry number $\beta \in \mathbb{R}_{>1}$ is a Perron number [Lot02, Chapter 7], the Parry numeration system $U_\beta$ also has this property, which will be of interest later on.

**Definition 1.33.** Let $U = (U(n))_{n\geq 0}$ be a linear numeration system. Consider the *characteristic polynomial* of the recurrence (1.1) given by

$$P(X) = X^k - b_{k-1}X^{k-1} - \cdots - b_1 X - b_0.$$

If $P$ is the minimal polynomial of a Perron number $\beta \in \mathbb{R}_{>1}$, we say that $U$ is a *Perron numeration system*. In this case, the polynomial $P$ can be factorized as

$$P(X) = (X - \beta)(X - \alpha_2)\cdots(X - \alpha_k),$$

where the complex numbers $\alpha_2, \ldots, \alpha_k$ are the conjugates of $\beta$, and, for all $j \geq 2$, we have $|\alpha_j| < \beta$. Using a well-known fact regarding recurrence relations, we have

$$U(n) = c_1\beta^n + c_2\alpha_2^n + \cdots + c_k\alpha_k^n \text{ for all } n \geq 0,$$

where $c_1, \ldots, c_k$ are complex numbers depending on the initial values of $U$. Since $|\alpha_j| < \beta$ for all $j \geq 2$, we have

$$\lim_{n\to+\infty} \frac{U(n)}{\beta^n} = c_1. \tag{1.2}$$

**Example 1.34.** It is not difficult to see that the usual integer base numeration system from Example 1.17 is a Perron numeration system having the Bertrand property of Definition 1.31.

The golden ratio $\varphi$ is a Perron, even Pisot, number whose minimal polynomial is $P(X) = X^2 - X - 1$. A Perron and Bertrand numeration system associated with $\varphi$ is the Fibonacci numeration system from Example 1.18 for which there exists $c \in \mathbb{C}$ such that

$$\lim_{n\to+\infty} \frac{F(n)}{\varphi^n} = c.$$

Note that it is possible to determine the exact value of the constant $c$, but it is not important at this step. Now, if we change the initial conditions and set $F'(0) = 1$, $F'(1) = 3$, and $F'(n+2) = F'(n+1) + F'(n)$ for all $n \in \mathbb{N}$, then we again get a Perron numeration associated with $\varphi$, which is not a Bertrand numeration system. Indeed, 2 is a greedy $F'$-expansion, but not 20 because $\text{rep}_{F'}(\text{val}_{F'}(20)) = 102$. In fact, the latter numeration system is built on the Lucas numbers (`A000032` in [Slo]).

## 1.4 Binomial Coefficients of Words

In this section, we introduce one of the main concepts used in this text: the binomial coefficients of words. As we will see, they are a natural extension of the well-known binomial coefficients of integers, which are even used in high schools for multiple purposes (*e.g.*, probabilistic and statistical problems). For more on these binomial coefficients of words, see, for instance, [Lot97, Chapter 6]. In the following definition, recall the difference between "factor" (or "subword") and "scattered subword" highlighted in Definition 1.6.

**Definition 1.35.** The *binomial coefficient* $\binom{u}{v}$ of two finite words $u$ and $v$ over the alphabet $A$ is the number of times $v$ occurs as a scattered subword of $u$. More formally, if $u = u_0 \cdots u_m$ and $v = v_0 \cdots v_n$ where $u_i, v_j$ are letters in $A$ for all $i$ and $j$, then

$$\binom{u}{v} = \#\{(i_0, \ldots, i_n) \mid 0 \leq i_0 < \cdots < i_n \leq m \text{ and } u_{i_0} \cdots u_{i_n} = v\}.$$

It is worth noticing that for any finite word $u$, then $\binom{u}{\varepsilon} = 1$ for the only occurrence of the empty word $\varepsilon$ in $u$ corresponds to the empty sequence.

There is a vast literature on binomial coefficients of words with applications in formal language theory (*e.g.*, Parikh matrices, $p$-group languages, or piecewise testable languages [Eil76, FK18, KKS15, KNS16]), $p$-adic topology and $p$-adic analysis [BCP89, PS14], combinatorics on words (*e.g.*, avoiding binomial repetitions [RRS15]), and model-checking and verification [ABRS05]. For instance, one combinatorial question that can naturally be asked about this topic is to determine when it is possible to uniquely reconstruct a word from some of its binomial coefficients; see, for instance, [DE04].

**Example 1.36.** Take the binary alphabet $A = \{0, 1\}$, and consider the finite words $u = 101001$ and $v = 101$ over $A$. Their binomial coefficient is

$$\binom{101001}{101} = 6.$$

Indeed, if we write $u = u_0 u_1 \cdots u_5 = 101001$, we have

$$u_0u_1u_2 = u_0u_1u_5 = u_0u_3u_5 = u_0u_4u_5 = u_2u_3u_5 = u_2u_4u_5 = 101 = v.$$

**Remark 1.37.** Let $A$ be any alphabet, and take two words $u, v \in A^*$. If $|u| < |v|$, then clearly there is no subsequence of $u$ that matches $v$, so $\binom{u}{v} = 0$. If $|u| = |v|$, then

$$\binom{u}{v} = \begin{cases} 1, & \text{if } u = v; \\ 0, & \text{otherwise.} \end{cases}$$

As mentioned before, the concept of binomial coefficients of words is a natural generalization of the binomial coefficients of integers. For any letter $a$ in the alphabet $A$, we have

$$\binom{a^m}{a^n} = \binom{m}{n} \quad \text{for all } m, n \in \mathbb{N}, \tag{1.3}$$

where $a^m$ denotes the concatenation of $m$ letters $a$ (see Definition 1.3).

The following lemma helps us to compute the binomial coefficient of a pair of words thanks to the binomial coefficients of pairs of shorter words. For a proof, we refer the reader to [Lot97, Chapter 6].

**Lemma 1.38.** *Let $A$ be an alphabet. For any words $u, v$ in $A^*$, and any letters $a, b$ in $A$, we have*

$$\binom{ua}{vb} = \binom{u}{vb} + \delta_{a,b}\binom{u}{v},$$

*where $\delta_{a,b}$ is the Kronecker symbol that is equal to 1 if $a = b$, 0 otherwise. For any three words $s, t, w$ in $A^*$, we have*

$$\binom{sw}{t} = \sum_{\substack{u,v \in A^* \\ uv = t}} \binom{s}{u}\binom{w}{v}.$$

Implied by the previous result, the next useful lemma deals with binomial coefficients of words ending with blocks of a given letter.

**Lemma 1.39.** *Let $A$ be an alphabet containing the letter $a$. For all non-empty words $u, v \in A^*$ and all $k \in \mathbb{N}$, we have*

$$\binom{ua^k}{va^k} = \sum_{j=0}^{k} \binom{k}{k-j} \binom{u}{va^j} = \sum_{j=0}^{k} \binom{k}{j} \binom{u}{va^j}.$$

Let us also recall Lucas' theorem linking classical binomial coefficients modulo a prime $p$ with base-$p$ expansions. See [Luc78, p. 230] or [Fin47]. Note that in the following statement, if the base-$p$ expansions of $m$ and $n$ are not of the same length, then we pad the shortest with leading zeroes.

**Theorem 1.40.** *Let $m$ and $n$ be two non-negative integers, and let $p$ be a prime. If*

$$m = m_k p^k + m_{k-1} p^{k-1} + \cdots + m_1 p + m_0$$

*and*

$$n = n_k p^k + n_{k-1} p^{k-1} + \cdots + n_1 p + n_0$$

*with $m_i, n_i \in \{0, 1, \ldots, p-1\}$ for all $i$, then the following congruence relation holds*

$$\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{m_i}{n_i} \pmod{p},$$

*using the convention that $\binom{m}{n} = 0$ if $m < n$.*

## 1.5 Generalized Pascal Triangles

The Pascal triangle and the corresponding[2] Sierpiński gasket are well-studied objects. They have connections with various topics in mathematics. They notably exhibit self-similarity, dynamical and fractal features [vHPS92, KL18, Ste95] and they can be obtained via iterated function systems (IFS's) [vHPS92, Ste95]. They can be studied with automata-theoretic techniques [AB97, AB11] or be expressed using first order formulas in an extension of the Presburger arithmetic [CLR15]. Finally, they are linked to simple arithmetic, especially through the celebrated binomial theorem and more generally the multinomial theorem, to enumerative combinatorics [Sta97, BFST18] in order to tackle counting problems, and also to $p$-adic topology and $p$-adic analysis [BCP89, PS14].

[2]See Chapter 2 for more details.

**Definition 1.41.** The *(classical) Pascal triangle* $\mathrm{P}\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is represented as an infinite table and defined as follows. The entry $\mathrm{P}(m, n)$ on the $m$th row and $n$th column of P is the integer $\binom{m}{n}$. The first few values in the Pascal triangle P are given in Table 1.3. The construction of the Pascal triangle is directed by the following relation

$$\binom{m}{n} = \binom{m-1}{n-1} + \binom{m-1}{n},$$

known as the *Pascal rule*.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1 | 2 | 1 | 0 | 0 | 0 | 0 | 0 |
| 3 | 1 | 3 | 3 | 1 | 0 | 0 | 0 | 0 |
| 4 | 1 | 4 | 6 | 4 | 1 | 0 | 0 | 0 |
| 5 | 1 | 5 | 10 | 10 | 5 | 1 | 0 | 0 |
| 6 | 1 | 6 | 15 | 20 | 15 | 6 | 1 | 0 |
| 7 | 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 |

Table 1.3: The first few values in the classical Pascal triangle P.

**Warning.** In this text, we choose to draw the Pascal triangle and related pictures in the Cartesian coordinate system in two dimensions whose $x$-axis points rightward, and the $y$-axis points downward. For instance, in the Pascal triangle case, the rows (resp., columns) are depicted on the $y$-axis (resp., $x$-axis). This convention will be valid throughout this text, especially in Chapter 2.

Several generalizations and variations of the Pascal triangle do already exist and are studied with arithmetical and combinatorial viewpoints [BNS16, BS14, DDGS18, Ném18, NP16], dynamical ones [JdlRV05, vHPS92] or analytical ones [HKP18]. On a combinatorics on words side, if $A$ is a finite alphabet, then one can find an analogue of the Pascal triangle indexed by all the words in $A^*$; see [Lot97, Problem 6.3.3 in Chapter 6]. In this section, we will consider an extension of the classical Pascal triangle to binomial coefficients of words [LRS16]. To define such a triangular array, we will consider all the words in an infinite language that is genealogically ordered.

**Definition 1.42.** Let $(A,<)$ be a totally ordered alphabet, and let $L \subset A^*$ be an infinite language over $A$. We order the words in $L$ by increasing genealogical order, and we write $L = \{w_0 <_{\text{gen}} w_1 <_{\text{gen}} w_2 <_{\text{gen}} \cdots\}$ (this corresponds to representations within an abstract numeration system based on $L$ [LR01], [BR10, Chapter 3]). The *generalized Pascal triangle* or *Pascal-like triangle* $\mathrm{P}_L\colon \mathbb{N}\times\mathbb{N}\to\mathbb{N}$ associated with the language $L$ is represented as an infinite table and defined as follows. The entry[3] $\mathrm{P}_L(m,n)$ on the $m$th row and $n$th column of $\mathrm{P}_L$ is the integer $\binom{w_m}{w_n}$.

Using the first relation from Lemma 1.38, one can also derive a rule to build $\mathrm{P}_L$, similar to the Pascal rule. Whereas the latter rule is local in the sense that a specific coefficient can be obtained by adding two coefficients located on the previous row, the gaps between the coefficients to be added in Lemma 1.38 can become bigger and bigger.

For the sake of simplicity, when $\beta > 1$ is a Parry number and $L = L_{U_\beta}$ is the numeration language of the Parry–Bertrand numeration associated with $\beta$ from Definition 1.29, we write $\mathrm{P}_\beta$ instead of $\mathrm{P}_{L_{U_\beta}}$. By Proposition 1.16, $w_n = \mathrm{rep}_{U_\beta}(n)$ for all $n \in \mathbb{N}$.

**Example 1.43.** We consider the language $L_2$ of the base-2 expansions of integers. In $L_2$, we have

$$\varepsilon <_{\text{gen}} 1 <_{\text{gen}} 10 <_{\text{gen}} 11 <_{\text{gen}} 100 <_{\text{gen}} 101 <_{\text{gen}} 110 <_{\text{gen}} 111 <_{\text{gen}} \cdots,$$

hence the first few values in the generalized Pascal triangle $\mathrm{P}_2$ are given in Table 1.4. The sequence `A282714` in [Slo] stores those values. For the base-3 case, see the sequence `A284441` in [Slo].

**Remark 1.44.** Note that, from (1.3) on page 17, the usual Pascal triangle is a "sub-array" of the extended triangle $\mathrm{P}_2$, and more generally of $\mathrm{P}_b$ for any $b \geq 2$. Indeed, $\mathrm{P}_b$ contains at least $b-1$ copies of the classical Pascal triangle by only considering words in the language $a^*$ with $a \in A_b \setminus \{0\}$. In Table 1.4, the elements of the classical Pascal triangle P are written in bold.

[3] Using the notation $\binom{u}{v}$, the rows (resp., columns) of $\mathrm{P}_L$ are indexed by the words $u$ (resp., $v$).

| | $\varepsilon$ | 1 | 10 | 11 | 100 | 101 | 110 | 111 |
|---|---|---|---|---|---|---|---|---|
| $\varepsilon$ | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | **1** | **1** | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 11 | **1** | **2** | 0 | **1** | 0 | 0 | 0 | 0 |
| 100 | 1 | 1 | 2 | 0 | 1 | 0 | 0 | 0 |
| 101 | 1 | 2 | 1 | 1 | 0 | 1 | 0 | 0 |
| 110 | 1 | 2 | 2 | 1 | 0 | 0 | 1 | 0 |
| 111 | **1** | **3** | 0 | **3** | 0 | 0 | 0 | **1** |

Table 1.4: The first few values in the generalized Pascal triangle $\mathrm{P}_2$.

This is even the case for any Parry number $\beta > 1$ since a copy of the classical Pascal triangle can be seen inside $\mathrm{P}_\beta$ by limiting ourselves to words in the language $a0^*$ for $a \in A_{U_\beta} \setminus \{0\}$. For an example, see below.

We can also observe that the second column $(\binom{\mathrm{rep}_2(n)}{1})_{n\geq 0}$ of $\mathrm{P}_2$ is exactly the sum-of-digits function $(s_2(n))_{n\geq 0}$ for base-2 expansions of integers [Del75]. Indeed, for a given integer $n$, $s_2(n)$ counts the number of 1's in the base-2 expansion of $n$. Thus, considering these values modulo 2, the second column of $\mathrm{P}_2$ is exactly the well-known Thue–Morse word [AS03a].

Proceeding as in Example 1.43, we give the generalized Pascal triangle $\mathrm{P}_\varphi$ associated with the golden ratio $\varphi$.

**Example 1.45.** When $\beta$ is the golden ratio $\varphi$, we know from Example 1.30 that the numeration language $L_{U_\varphi}$ is $1\{0, 01\}^* \cup \{\varepsilon\}$. The first values in the generalized Pascal triangle $\mathrm{P}_\varphi$ are given in Table 1.5 (note that the sequence of those values is the sequence `A282716` in [Slo]). In this table, the elements of the classical Pascal triangle P are again written in bold.

The following funny result was observed by J. Raskin during a comprehensible seminar, and states that the sum of the entries on the $n$th row of $\mathrm{P}_2$ is exactly $n + 1$. Notice that it does not seem to be true for any other integer base. A natural question would be to investigate the general case of $\beta$-numeration systems.

| | $\varepsilon$ | 1 | 10 | 100 | 101 | 1000 | 1001 | 1010 |
|---|---|---|---|---|---|---|---|---|
| $\varepsilon$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | **1** | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 1 | **1** | **1** | 0 | 0 | 0 | 0 | 0 |
| 100 | 1 | **1** | **2** | **1** | 0 | 0 | 0 | 0 |
| 101 | 1 | 2 | 1 | 0 | 1 | 0 | 0 | 0 |
| 1000 | 1 | **1** | **3** | **3** | 0 | **1** | 0 | 0 |
| 1001 | 1 | 2 | 2 | 1 | 2 | 0 | 1 | 0 |
| 1010 | 1 | 2 | 3 | 1 | 1 | 0 | 0 | 1 |

Table 1.5: The first few values in the generalized Pascal triangle $\mathrm{P}_{\varphi}$.

**Proposition 1.46.** *For all $n \geq 0$, we have*

$$\sum_{m\in\mathbb{N}} \binom{\mathrm{rep}_2(n)}{\mathrm{rep}_2(m)} = n+1.$$

*Proof*. For the sake of clarity, let us define $S(n) = \sum_{m\in\mathbb{N}} \binom{\mathrm{rep}_2(n)}{\mathrm{rep}_2(m)}$. To prove the claim, we proceed by induction on $n \geq 0$. Using Table 1.4, the result is trivially true for $n \in \{0,\ldots,7\}$. Now assume that $n \geq 8$ and write $\mathrm{rep}_2(n) = ua$ with $u \in L_2$, $|u| \geq 3$ and $a \in \{0,1\}$. We only take care of the case where $a = 0$ for the other is similar. Let $L_{2,0} = L_2 \cap \{0,1\}^*0$ (resp., $L_{2,1} = L_2 \cap \{0,1\}^*1$) be the set of base-2 expansions ending with 0 (resp., 1). We have

$$\begin{aligned} S(n) &= \binom{u0}{\varepsilon} + \sum_{v\in L_{2,0}} \binom{u0}{v} + \sum_{v\in L_{2,1}} \binom{u0}{v} \\ &= 1 + \sum_{v\in L_2\setminus\{\varepsilon\}} \binom{u0}{v0} + \sum_{v\in L_2} \binom{u0}{v1}, \end{aligned}$$

and by Lemma 1.38, we find

$$S(n) = 1 + \sum_{v\in L_2\setminus\{\varepsilon\}} \binom{u}{v0} + \sum_{v\in L_2\setminus\{\varepsilon\}} \binom{u}{v} + \sum_{v\in L_2} \binom{u}{v1}.$$

Now observe that

$$\{v0 \mid v \in L_2 \setminus \{\varepsilon\}\} \cup \{v1 \mid v \in L_2\} = L_{2,0} \cup L_{2,1} = L_2 \setminus \{\varepsilon\},$$

so

$$S(n) = \sum_{v \in L_2} \binom{u}{v} + \sum_{v \in L_2 \setminus \{\varepsilon\}} \binom{u}{v} = 2 \sum_{v \in L_2} \binom{u}{v} - 1$$

since $1 = \binom{u}{\varepsilon}$. Now $\text{rep}_2(n/2) = u$, so we get

$$S(n) = 2S(n/2) - 1,$$

and the result follows by induction hypothesis. □

## 1.6 Counting Non-Zero Binomial Coefficients

In this short section, with any language $L$ and any row of the generalized Pascal triangle $\text{P}_L$, is associated a sequence that counts the number of non-zero elements in $\text{P}_L$. In Chapter 3, we study this particular sequence.

**Definition 1.47.** Let $(A, <)$ be a totally ordered alphabet, and let $L \subset A^*$ be an infinite language over $A$. Write $L = \{w_0 <_{\text{gen}} w_1 <_{\text{gen}} w_2 <_{\text{gen}} \cdots\}$. We let $S_L = (S_L(n))_{n \geq 0}$ denote the sequence whose $n$th term, for $n \geq 0$, is the number of non-zero elements in the $n$th row of $\text{P}_L$. Otherwise stated, for $n \geq 0$, we define

$$S_L(n) = \# \left\{ v \in L \mid \binom{w_n}{v} > 0 \right\} = \# \left\{ m \in \mathbb{N} \mid \binom{w_n}{w_m} > 0 \right\}.$$

As before, for the sake of simplicity, when $\beta > 1$ is a Parry number and $L = L_{U_\beta}$ is the numeration language of the Parry–Bertrand numeration system associated with $\beta$ from Definition 1.29, we write $S_\beta$ instead of $S_{L_{U_\beta}}$. In that case, for $n \geq 0$, we have

$$\begin{aligned} S_\beta(n) &= \# \left\{ v \in L_{U_\beta} \mid \binom{\text{rep}_{U_\beta}(n)}{v} > 0 \right\} \qquad (1.4) \\ &= \# \left\{ m \in \mathbb{N} \mid \binom{\text{rep}_{U_\beta}(n)}{\text{rep}_{U_\beta}(m)} > 0 \right\}. \end{aligned}$$

**Remark 1.48.** Note that we can relate $S_L$ to Simon's congruence for which two finite words are equivalent if they share exactly the same set of scattered subwords [Sim75]. More precisely, two words are $\sim_k$-equivalent if they have the same set of scattered subwords of length at most $k$. Observe that two words of distinct length can be equivalent even if one of them has more scattered subwords than the other (simply take, for example, $11 \sim_2 111$). For instance, in $L_2$, 101 and 110 share the same scattered subwords of length at most 2 ($101 \sim_2 110$ if words are restricted to $L_2$), and

$$S_2(\mathrm{val}_2(101)) = 5 = S_2(\mathrm{val}_2(110))$$

since there is only one length-3 word that is a scattered subword of 101 or 110 respectively.

**Example 1.49.** Let us work with the generalized Pascal triangle $\mathrm{P}_2$ from Example 1.43. Its first few values are stored in Table 1.6 and correspond to the words $\varepsilon$, 1, 10, 11, 100, 101, 110 and 111 in $L_2$. Compared to Table 1.4, we add two new columns: the non-negative integers on the left side of the table, and the rightmost column is the sequence $S_2$. Its first few terms are

$$1, 2, 3, 3, 4, 5, 5, 4, 5, 7, 8, 7, 7, 8, 7, 5, 6, 9, 11, 10, 11, 13, 12, 9, 9, 12, 13, 11, \ldots.$$

| $n$ | $\mathrm{rep}_2(n)$ | $\varepsilon$ | 1 | 10 | 11 | 100 | 101 | 110 | 111 | $S_2(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $\varepsilon$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 2 |
| 2 | 10 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 3 |
| 3 | 11 | 1 | 2 | 0 | 1 | 0 | 0 | 0 | 0 | 3 |
| 4 | 100 | 1 | 1 | 2 | 0 | 1 | 0 | 0 | 0 | 4 |
| 5 | 101 | 1 | 2 | 1 | 1 | 0 | 1 | 0 | 0 | 5 |
| 6 | 110 | 1 | 2 | 2 | 1 | 0 | 0 | 1 | 0 | 5 |
| 7 | 111 | 1 | 3 | 0 | 3 | 0 | 0 | 0 | 1 | 4 |

Table 1.6: The first few values of $S_2$ in the generalized Pascal triangle $\mathrm{P}_2$.

A visual representation is given in Figure 1.7 where we have represented in black the positive values in $\mathrm{P}_2$ and a compressed version of the same figure (the compressed representation is inspired by P. Dumas [Dum]). In fact, the sequence obtained by adding an extra 1 as a prefix of $S_2$ exactly matches the sequence of denominators of the Farey tree (`A007306` in [Slo]). This is a key observation for later on; see Chapter 3.

Note that the case of the generalized Pascal triangle $\mathrm{P}_3$ can be handled similarly, and the corresponding sequence $S_3$ is tagged `A282715` in [Slo].

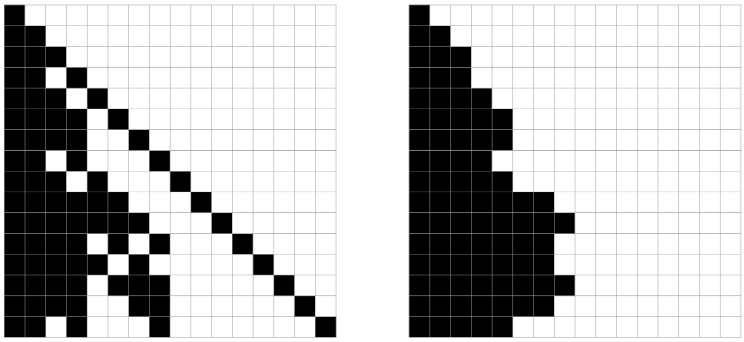


Figure 1.7: Positive values in the generalized Pascal triangle $P_2$ (on the left) and the compressed version (on the right).

**Example 1.50.** Let us consider the generalized Pascal triangle $P_\varphi$ from Example 1.45. Corresponding to the words $\varepsilon$, 1, 10, 100, 101, 1000, 1001, 1010 and 10000 in $L_F = L_{U_\varphi}$ (see Example 1.30), the first few values of $P_\varphi$ are given in Table 1.8. Compared to Table 1.5, as in the previous example, we add new columns, and the rightmost is $S_\varphi$ whose first few terms are

$$1, 2, 3, 4, 4, 5, 6, 6, 6, 8, 9, 8, 8, 7, 10, 12, 12, 12, 10, 12, 12, 8, 12, 15, 16, 16, \ldots.$$

The full version is labeled `A282717` in [Slo].

| $n$ | $\mathrm{rep}_F(n)$ | $\varepsilon$ | 1 | 10 | 100 | 101 | 1000 | 1001 | 1010 | 10000 | $S_\varphi(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $\varepsilon$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 |
| 2 | 10 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 3 |
| 3 | 100 | 1 | 1 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 4 |
| 4 | 101 | 1 | 2 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 4 |
| 5 | 1000 | 1 | 1 | 3 | 3 | 0 | 1 | 0 | 0 | 0 | 5 |
| 6 | 1001 | 1 | 2 | 2 | 1 | 2 | 0 | 1 | 0 | 0 | 6 |
| 7 | 1010 | 1 | 2 | 3 | 1 | 1 | 0 | 0 | 1 | 0 | 6 |
| 8 | 10000 | 1 | 1 | 4 | 6 | 0 | 4 | 0 | 0 | 1 | 6 |

Table 1.8: The first few values of $S_\varphi$ in the generalized Pascal triangle $P_\varphi$ with words in $L_F$.

## 1.7 Automaticity, Synchronicity and Regularity

In this section, we introduce the notions of automatic, synchronized and regular sequences. Here, the sequences we work with are all made of integers. In some sense, the previous properties depend on the numeration system that is considered to represent integers. The literature is vast, so we pick some references to write this short summary [AS92, AS03a, AST00, BR11, CM01, Eil74, RM02, Sha88].

There are different equivalent ways to define automatic sequences; see, for instance, [AS03a]. As an example, they can be defined through automata after which they are named. However, in this text, we choose to define them relatively to their kernel [AS03a, Eil74] because we have in mind a larger class of sequences, called regular (see Definition 1.54).

**Definition 1.51.** Let $b \geq 2$ be an integer. The *b-kernel* of a sequence $s = (s(n))_{n\geq 0}$ is the set of (sub)sequences

$$\mathcal{K}_b(s) = \{(s(b^i n + j))_{n\geq 0} |\ i \geq 0 \text{ and } 0 \leq j < b^i\}.$$

One characterization of $b$-automatic sequences is that their $b$-kernels are finite; see [AS03a, Eil74].

**Definition 1.52.** Let $b \geq 2$ be an integer. A sequence $s = (s(n))_{n\geq 0}$ of integers is *b-automatic* if its $b$-kernel $\mathcal{K}_b(s)$ is finite.

Lots of examples of automatic sequences can be found in [AS03a]. We only give the following famous one.

**Example 1.53.** Let $t = (t_n)_{n\geq 0} = 01101001\cdots$ be the Thue–Morse word introduced in Example 1.14. Note that there are different equivalent definitions of $t$; see, for instance, [AS03a]. For example, we have the following recursive way to define it: $t_0 = 0$, $t_{2n} = t_n$ and $t_{2n+1} = (t_n + 1) \bmod 2$ for all $n \geq 0$. Using this definition, its 2-kernel $\mathcal{K}_2(t)$ contains exactly two elements, namely $t$ and $\underline{t}$ where the map $\underline{\cdot}\colon 0 \mapsto 1, 1 \mapsto 0$ exchanges letters. So $t$ is 2-automatic.

Unbounded sequences, *i.e.*, taking infinitely many integer values, are also of interest but their $b$-kernels are clearly infinite. One way to try to handle such sequences is to introduce the definition of $b$-regularity [AS92, AS03a, BR11]. In fact, the $b$-regularity of a sequence provides interesting structural information about it. For instance, we get matrices to compute its $n$th term in a number of steps proportional to $\log_b(n)$.

**Definition 1.54.** Let $b \geq 2$ be an integer. A sequence $s = (s(n))_{n\geq 0}$ of integers is *$b$-regular* if $\langle \mathcal{K}_b(s) \rangle$ is a finitely-generated $\mathbb{Z}$-module, *i.e.*, there exists a finite number of sequences $t_1 = (t_1(n))_{n\geq 0}, \ldots, t_\ell = (t_\ell(n))_{n\geq 0}$ such that every sequence in the $\mathbb{Z}$-module generated by the $b$-kernel $\mathcal{K}_b(s)$ is a $\mathbb{Z}$-linear combination of the sequences $t_1, \ldots, t_\ell$. Equivalently, for all $i \geq 0$ and for all $j \in \{0, \ldots, b^i - 1\}$, there exist integers $c_1, \ldots, c_\ell$ such that

$$s(b^i n + j) = \sum_{r=1}^{\ell} c_r \, t_r(n) \quad \text{for all } n \geq 0.$$

Another useful characterization [AS03a, Theorem 16.2.3] is the following one: a sequence $s = (s(n))_{n\geq 0}$ of integers is $b$-regular if and only if it admits a *linear representation*, *i.e.*, there exist an integer $k \geq 1$, a row vector $r$, a column vector $c$ and square matrices $\Gamma_0, \ldots, \Gamma_{b-1}$ of size $k$ such that, if $\mathrm{rep}_b(n) = n_j \cdots n_0$, then

$$s(n) = r \, \Gamma_{n_0} \Gamma_{n_1} \cdots \Gamma_{n_j} \, c.$$

Note that by transposing the previous product, one can get a linear representation by reading $\mathrm{rep}_b(n)$ from left to right, *i.e.*, starting with the most significant digit.

Observe that, by definition, any $b$-automatic sequence is $b$-regular, but the converse clearly does not hold. In fact, a sequence is $b$-regular and takes only finitely many values if and only if it is $b$-automatic [AS03a]. In the following, we give two 2-regular sequences that will be useful in Chapters 3 and 4. Many examples of $b$-regular sequences may be found in [AS92, AS03a, AS03b].

A method to show that a sequence $s$ is $b$-regular can be done in two steps. First, the idea is to express the sequences $(s(b^i n + j))_{n\geq 0}$ for a given $i \geq 0$ and all residues $0 \leq j < b^i$ as linear combinations of sequences of the form $(s(b^{i'} n + j'))_{n\geq 0}$ with $0 \leq i' < i$ and $0 \leq j' < b^{i'}$. Secondly, one can use those combinations to express the sequences $(s(b^{i''} n + j''))_{n\geq 0}$ with $i'' > i$ and $0 \leq j'' < b^{i''}$ as linear combinations of sequences of the form $(s(b^{i'} n + j'))_{n\geq 0}$ with $0 \leq i' < i$ and $0 \leq j' < b^{i'}$.

**Example 1.55.** The *Stern–Brocot sequence* $(SB(n))_{n\geq 0}$, which takes its name from the *Stern–Brocot tree* (see Chapter 3), is defined by $SB(0) = 0$, $SB(1) = 1$, $SB(2n) = SB(n)$ and $SB(2n+1) = SB(n) + SB(n+1)$ for all $n \geq 1$. This sequence is unbounded and tagged `A002487` in [Slo].

The $\mathbb{Z}$-module generated by its 2-kernel is simply generated by the sequence itself and the shifted sequence $(SB(n+1))_{n\geq 0}$; see [AS92, Example 7]. So the Stern–Brocot sequence is 2-regular. Using the relations in the definition of the Stern–Brocot sequence, one has

$$\begin{aligned} SB(4n) &= SB(2(2n)) = SB(2n) = SB(n); \\ SB(4n+1) &= SB(2(2n)+1) = SB(2n) + SB(2n+1) \\ &= 2SB(n) + SB(n+1); \\ SB(4n+2) &= SB(2(2n+1)) = SB(2n+1) = SB(n) + SB(n+1); \\ SB(4n+3) &= SB(2(2n+1)+1) = SB(2n+1) + SB(2n+2) \\ &= SB(n) + 2SB(n+1). \end{aligned}$$

**Example 1.56.** Let $s_2 = (s_2(n))_{n\geq 0}$ denote the *sum-of-digits function* for base-2 expansions of integers. For instance, the binary expansion of 6 is $\mathrm{rep}_2(6) = 101$, so $s_2(6) = 1 + 0 + 1 = 2$. The sequence $s_2$ readily satisfies

$$s_2(2n) = s_2(n) \quad \text{and} \quad s_2(2n+1) = 1 + s_2(n)$$

for all $n \geq 0$, which means that $s_2$ is 2-regular. It also admits the linear representation

$$r = \begin{pmatrix} 1 & 0 \end{pmatrix},\ \Gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \Gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ c = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let us also mention a result regarding asymptotic estimates for summatory functions of $b$-regular sequences, which avoids error terms and will be useful in Chapter 4. For instance, this result can be applied to study the behavior of summatory functions of sum-of-digit functions; see [AS03a]. In this result, if $v$ belongs to $\mathbb{C}^k$, then the notation $||v||$ stands for the *Euclidean norm* of $v$, defined by $\left(\sum_{i=1}^{k} |v_i|^2\right)^{\frac{1}{2}}$. Moreover, if $M$ is a square matrix of size $k$ with entries in $\mathbb{C}$, then we let $||M||$ denote the $L^2$ *norm* of $M$, which is the matrix norm associated with the usual Euclidean norm on $\mathbb{C}^k$ by the formula $||M|| = \sup_{||x||=1} ||Mx||$.

**Theorem 1.57.** *[AS03a, Theorem 3.5.1] Let $b \geq 2$ be an integer. Suppose there exist an integer $k \geq 1$, a sequence $(V(n))_{n\geq 0}$ of vectors in $\mathbb{C}^k$ defined by*

$$V(n) = \begin{pmatrix} V_1(n) \\ V_2(n) \\ \vdots \\ V_k(n) \end{pmatrix},$$

*and $b$ square matrices $\Gamma_0, \Gamma_1, \ldots, \Gamma_{b-1}$ of size $k$ such that*

- *$V(bn + r) = \Gamma_r V(n)$ for all $n \geq 0$ and all $0 \leq r < b$;*
- *$||V(n)|| = O(\log n)$; and*
- *there exist a $k \times k$ matrix $\Lambda$ and a constant $c > 0$ such that either $||\Lambda|| < c$, or $\Lambda$ is nilpotent with $\Gamma = \Gamma_0 + \Gamma_1 + \cdots + \Gamma_{k-1} = cI + \Lambda$.*

*The matrix $\Gamma$ being clearly invertible, if $||\Gamma^{-1}|| < 1$, then there exists a continuous function $\mathcal{G} \colon \mathbb{R} \to \mathbb{C}^k$ of period 1 such that*

$$\sum_{0\leq n<N} V(n) = N^{\log_k c} \left(I + c^{-1}\Lambda\right)^{\log_k N} \mathcal{G}\left(\log_k N\right).$$

First introduced in [CM01], the class of $b$-synchronized sequences is a strict intermediate (with respect to the inclusion) between the classes of $b$-automatic sequences and $b$-regular sequences. In [CM01], the authors show that every $b$-synchronized sequence is $b$-regular, but the converse is not true (an example is given in the latter paper). Moreover, they also prove that a sequence is $b$-synchronized and takes on only finitely many values if and only if it is $b$-automatic. Roughly speaking, a sequence $(s(n))_{n\geq 0}$ is $b$-synchronized if there exists a finite automaton accepting the pairs of base-$b$ expansions of $n$ and $s(n)$, as stated in Definition 1.58 below.

**Definition 1.58.** Let $b \geq 2$ be an integer. For the purpose of this definition, the map $\operatorname{rep}_b$ from Example 1.17 is extended to $\mathbb{N} \times \mathbb{N}$ as follows. For all $m, n \in \mathbb{N}$, we set

$$\operatorname{rep}_b(m, n) = \left(0^{M-|\operatorname{rep}_b(m)|} \operatorname{rep}_b(m), 0^{M-|\operatorname{rep}_b(n)|} \operatorname{rep}_b(n)\right),$$

where $M = \max\{|\operatorname{rep}_b(m)|, |\operatorname{rep}_b(n)|\}$. The idea is that the shortest word is padded with leading zeroes to get two words of the same length.

A sequence $(s(n))_{n\geq 0}$ of integers is said to be *b-synchronized* if the language $\{\text{rep}_b(n, s(n)) \mid n \in \mathbb{N}\}$ is accepted by some finite automaton reading pairs of digits.

As an example, it is proved in [GSS13] that if an infinite word is $b$-automatic, then its factor complexity function, *i.e.*, the map counting the number of distinct factors of a given length, is $b$-synchronized.

As a final comment to this section, we will timely see that the notions of automaticity and regularity may be extended to other numeration systems.

## 1.8 Metrics

The aim of this section is to introduce the notation about the metrics used in this text, and more specifically the Hausdorff metric. For more on the subject, see, for instance, [Fal86, Fal97].

In the following, we let $d$ denote the *Euclidean distance* on $\mathbb{R}^2$. If $S$ and $S'$ are non-empty subsets of $\mathbb{R}^2$, we let $d(S, S')$ denote the quantity

$$d(S, S') = \inf\{d(x, y) \mid x \in S, y \in S'\}.$$

When $S = \{x\}$ is reduced to a single point $x$ and $S'$ is a non-empty subset of $\mathbb{R}^2$, then we write $d(x, S')$ instead of $d(\{x\}, S')$.

Let us insist on the fact that $d$ does not define a proper distance between non-empty subsets of $\mathbb{R}^2$. First, $d(S, S')$ may be equal to 0 even if the non-empty subsets $S, S' \subset \mathbb{R}^2$ are not equal (it suffices to consider non-disjoint subsets). Moreover, the triangle inequality is not fulfilled. Also note that if $S$ and $S'$ are non-empty subsets of $\mathbb{R}^2$, there might not exist $x \in S$ and $y \in S'$ such that $d(S, S') = d(x, y)$. However, when $S$ is a non-empty closed set and $S'$ is a non-empty compact set, there always exist $x \in S$ and $y \in S'$ such that $d(S, S') = d(x, y)$. By abuse of terminology, the quantity $d(S, S')$ is sometimes referred to as the *distance* between $S$ and $S'$.

To remedy this situation, one can consider the Hausdorff distance that defines a proper distance between subsets. Thanks to this notion, we obtain a way to measure how far two subsets of a metric space are from each other. Informally, if two subsets are close with respect to the Hausdorff distance, they should look alike. This distance is used in various mathematical fields such as geometry or fractal theory, and even finds applications in digital image processing and computer vision.

**Definition 1.59.** If $\epsilon \in \mathbb{R}_{>0}$ is a positive real number[4] and $x$ is a point in $\mathbb{R}^2$, the *open ball of radius $\epsilon$ centered at $x$* is the set of points in $\mathbb{R}^2$ at a distance at most $\epsilon$ of $x$, *i.e.*,

$$B(x, \epsilon) = \{y \in \mathbb{R}^2 \mid d(x, y) < \epsilon\}.$$

If $S$ is a subset of $\mathbb{R}^2$, we let

$$[S]_\epsilon = \bigcup_{x \in S} B(x, \epsilon)$$

denote the *$\epsilon$-fattening* of $S$, that is the set of all points within $\epsilon$ of the set $S$. The *Hausdorff metric* or *Hausdorff distance* $d_h$ induced by $d$ is defined by

$$d_h(S, S') = \inf\{\epsilon \in \mathbb{R}_{>0} \mid S \subset [S']_\epsilon \text{ and } S' \subset [S]_\epsilon\} \text{ for all } S, S' \subset \mathbb{R}^2.$$

Equivalently, we have

$$d_h(S, S') = \max\{\sup_{x \in S} \inf_{y \in S'} d(x, y), \sup_{y \in S'} \inf_{x \in S} d(x, y)\}.$$

We let $(\mathcal{H}(\mathbb{R}^2), d_h)$ denote the space of the non-empty compact subsets of $\mathbb{R}^2$ equipped with the Hausdorff metric $d_h$ induced by $d$. It is well known that $(\mathcal{H}(\mathbb{R}^2), d_h)$ is complete [Fal86].

As an illustration, take $S$ and $S'$ non-empty compact subsets of $\mathbb{R}^2$, and assume that $d_h(S, S') = \eta \in \mathbb{R}_{\geq 0}$. In particular, $S \subset [S']_{\eta'}$ for all $\eta' > \eta$. Then, for all $x \in S$, there exists $y \in S'$ such that $d(x, y) < \eta'$. Consequently, $d(S, S') \leq d(x, y) < \eta'$.

As a final point to this section, we show that an increasing nested sequence of compact sets whose union is bounded always converges with respect to the Hausdorff distance. This result will turn out to be useful in Chapter 2.

**Proposition 1.60.** *Let $(K_n)_{n \geq 0}$ be a sequence of compact subsets of $\mathbb{R}^2$ such that $K_n \subset K_{n+1}$ for all $n \geq 0$, and their union $\cup_{n \geq 0} K_n$ is bounded. Then $(K_n)_{n \geq 0}$ converges to*

$$\overline{\bigcup_{n \geq 0} K_n}$$

*with respect to the Hausdorff distance.*

---

[4] In this text, there are two different kinds of epsilon: the rounded epsilon $\varepsilon$ designates the empty word whereas the moon-shaped epsilon $\epsilon$ stands for a real number.

*Proof*. First, observe that $\overline{\bigcup_{n\geq 0} K_n}$ is a compact set by hypothesis. To prove the claim, we need to show that, for all $\epsilon \in \mathbb{R}_{>0}$, there exists $N \in \mathbb{N}$ such that, for all $m \geq N$,

$$d_h\left(K_m, \overline{\bigcup_{n\geq 0} K_n}\right) < \epsilon.$$

By definition, we thus have to prove that for all $\epsilon \in \mathbb{R}_{>0}$, there exists $N \in \mathbb{N}$ such that, for all $m \geq N$,

$$K_m \subset \left[\overline{\bigcup_{n\geq 0} K_n}\right]_\epsilon \quad \text{and} \quad \overline{\bigcup_{n\geq 0} K_n} \subset [K_m]_\epsilon.$$

The first inclusion is always satisfied, and since the compact sets are increasingly nested by hypothesis, we must equivalently show that for all $\epsilon \in \mathbb{R}_{>0}$, there exists $N \in \mathbb{N}$ such that

$$\overline{\bigcup_{n\geq 0} K_n} \subset [K_N]_\epsilon.$$

Let $\epsilon \in \mathbb{R}_{>0}$. For every point $x \in \overline{\bigcup_{n\geq 0} K_n}$, the definition of the closure implies that there exists $y \in \bigcup_{n\geq 0} K_n$ such that $d(x, y) < \epsilon/2$. We let $N(x)$ denote the smallest integer $N \geq 0$ such that there exists $y \in K_N$ with $d(x, y) < \epsilon/2$, which exists by hypothesis.

By compactness, there exist points $x_1, \ldots, x_k \in \overline{\bigcup_{n\geq 0} K_n}$ such that

$$\overline{\bigcup_{n\geq 0} K_n} \subset \bigcup_{i=1}^{k} B\left(x_i, \frac{\epsilon}{2}\right).$$

Let $N = \max\{N(x_1), \ldots, N(x_k)\}$. For all $i \in \{1, \ldots, k\}$, there exists a point $y_i \in K_{N(x_i)} \subset K_N$ such that $d(x_i, y_i) < \epsilon/2$.

To conclude, we show that, for all $i \in \{1, \ldots, k\}$, $B(x_i, \epsilon/2) \subset [K_N]_\epsilon$, which suffices. Let $i \in \{1, \ldots, k\}$, and pick $x \in B(x_i, \epsilon/2)$. Then

$$d(x, y_i) \leq d(x, x_i) + d(x_i, y_i) < \epsilon.$$

Since $y_i \in K_N$, then $x \in [K_N]_\epsilon$. □

**Remark 1.61.** It is worth mentioning that the previous result holds for any metric space $(X, \delta)$ if the Hausdorff distance is analogously defined in this context. However, in this text, we focus on $X = \mathbb{R}^2$ and $\delta = d$.

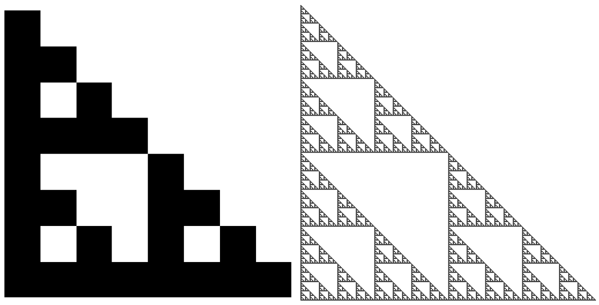
# Chapter 2

# Convergence of Generalized Pascal Triangles

As already mentioned in the introduction of this dissertation and also in Chapter 1, there is a connection between the Pascal triangle (see Definition 1.41) and the Sierpiński gasket. In what follows, we explain how from the first we can obtain the second. Let us consider the intersection of the

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 2 | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 3 | 3 | 1 | 0 | 0 | 0 | 0 |
| 1 | 4 | 6 | 4 | 1 | 0 | 0 | 0 |
| 1 | 5 | 10 | 10 | 5 | 1 | 0 | 0 |
| 1 | 6 | 15 | 20 | 15 | 6 | 1 | 0 |
| 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 |

(a) Portion of P.



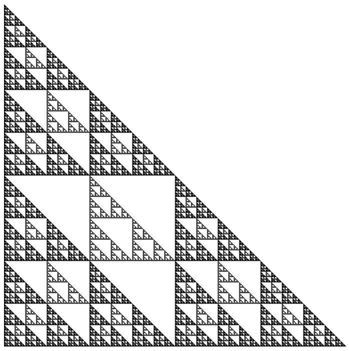
(b) Colored portion of P. (c) The Sierpiński gasket.

Figure 2.1: Relation between the classical Pascal triangle P and the Sierpiński gasket.

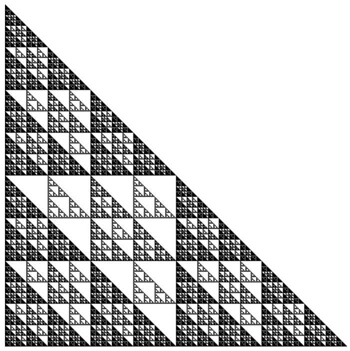
lattice $\mathbb{N}^2$ with the region $[0, 2^n]^2$ for $n \in \mathbb{N}$. Then the first $2^n$ rows and columns $(\binom{i}{j} \bmod 2)_{0 \leq i,j < 2^n}$ of the Pascal triangle modulo 2 provide a coloring of this lattice. If we normalize this region by a homothety of ratio $1/2^n$, it is a folklore fact that we get a sequence in $[0, 1]^2$ converging, with respect to the Hausdorff distance, to the Sierpiński gasket when $n$ tends to infinity.

In Figure 2.1a are depicted the first $2^3$ rows and columns of the Pascal triangle P (note that we have already applied the adequate homothety so that roughly, all the objects have the same size). Then in Figure 2.1b, we color in black (resp., white) the squares corresponding to odd (resp., even) binomial coefficients. When powers of 2 grow, the corresponding limit object is the Sierpiński gasket in Figure 2.1c.

In a similar fashion, when the sequence $(\binom{i}{j})_{0\le i,j<p^n}$ is considered modulo $p^s$ where $p$ is a prime number and $s$ is a positive integer, then it also converges, with respect to the Hausdorff distance, to some well-defined limit object [vHPS92]. More precisely, in an analogous construction, each unit square is colored in white or black depending on whether the corresponding binomial coefficient is congruent to 0 modulo $p^s$ or not. For instance, the limit object obtained for $p = 2$ and $s = 2$ (resp., $p = 2$ and $s = 3$) is drawn in Figure 2.2a (resp., Figure 2.2b). Also note that $p = 2$ and $s = 1$ yield the Sierpiński gasket (see Figure 2.1c). In [vHPS92], one can find several geometrical and dynamical properties of the studied limit sets such as their Hausdorff dimension.



(a) $p = 2$ and $s = 2$



(b) $p = 2$ and $s = 3$

Figure 2.2: Generalized Sierpiński gaskets.

Given an infinite language $L$ over an alphabet $A$, one can wonder whether a similar phenomenon occurs in the context of the generalized Pascal triangle $\mathrm{P}_L$ from Definition 1.42. One of the objectives of this second chapter is to understand when such phenomena happen and to obtain a description of the limit objects.

In a first (pedagogical) approach, we discuss in detail the base-2 case, *i.e.*, when $L = L_2$ and $\mathrm{P}_L = \mathrm{P}_2$ as in Example 1.43, and binomial coefficients modulo a prime number. In Section 2.2, we jump to any Parry–Bertrand numeration system. We finish up with some open questions. The material of this chapter is taken from [LRS16, Sti19].

**Contents**



## 2.1 Results in Base 2

In this section, we focus on the base-2 case. For all $n \in \mathbb{N}$, the generalized Pascal triangle $\mathrm{P}_2$ limited to words in $L_2^{\leq n}$ has $2^n$ rows and columns. They can be seen as a region of $\mathbb{R}^2$. Let us consider a grid of unit squares at the intersection of $\mathbb{N}^2$ and the region $[0, 2^n]^2$. The first $2^n$ rows and columns

$$\left( \binom{\mathrm{rep}_2(i)}{\mathrm{rep}_2(j)} \bmod 2 \right)_{0 \leq i,j < 2^n}$$

of the generalized Pascal triangle $\mathrm{P}_2$ modulo 2 yield a coloring of this grid. This construction leads to the definition of a sequence $(\mathcal{T}_n^2)_{n \geq 0}$ of subsets of $\mathbb{R}^2$ (see Definition 2.2 and Figure 2.4 for a picture of the cases where $n \in \{3, 4\}$). If we normalize each $\mathcal{T}_n^2$ by a homothety of ratio $1/2^n$, we will define a sequence $(\mathcal{U}_n^2)_{n \geq 0}$ in $[0, 1]^2$ (see Definition 2.6 and Figure 2.7 for a picture of the cases where $n \in \{3, 4, 9\}$). Further, we show that this particular sequence converges to an analogue of the Sierpiński gasket, denoted by $\mathcal{L}^2$, with respect to the Hausdorff distance. We also describe the limit object $\mathcal{L}^2$ (see Figure 2.15) as the topological closure of a union of segments that satisfy a simple combinatorial property. As we will see, those segments are well understood since we precisely know their endpoints (see Definition 2.11). The limit set $\mathcal{L}^2$ convinces us that these extended Pascal triangles contain many interesting combinatorial and dynamical questions to consider.

For the sake of simplicity, we will mostly describe the coloring modulo 2. In Section 2.1.5, we shortly discuss colorings modulo a prime number $p$ (see Figure 2.19 for coefficients congruent to 2 modulo 3).

**Remark 2.1.** In our construction, at each step, we exactly take $2^n$ words and a scaling (or normalization) factor of $1/2^n$. For instance, in [BvH03], the authors discussed which sequences can be used as scaling factors for objects related to automatic sequences. In particular, the Pascal triangle P modulo $p^d$ is shown to be $p$-automatic in [AB97], where $p$ is prime, $d$ is an integer, and the scaling sequence has to be of the form $(p^{kn+j})_{n\geq 0}$ with $j = 0, \ldots, p-1$. See the first question in Section 2.3.

### 2.1.1 Black & White Special Compact Sets

In this section, we define a subset $\mathcal{T}_n^2$ of $[0, 2^n]^2$ associated with the parity of the first $2^n \times 2^n$ binomial coefficients of $\mathrm{P}_2$. Afterwards, we prove in Proposition 2.5 that there are exactly $3^n$ pairs of words in $L_2^{\leq n}$ having a non-zero binomial coefficient. Each set $\mathcal{T}_n^2$ is then normalized by a factor $1/2^n$ to give birth to the sequence $(\mathcal{U}_n^2)_{n\geq 0}$ of subsets in $[0, 1]^2$ (see Definition 2.6). One of our goals is to show that the sequence $(\mathcal{U}_n^2)_{n\geq 0}$ converges with respect to the Hausdorff distance.

For the purpose of this chapter, the numerical value $\mathrm{val}_U$ of Definition 1.15 is extended to pairs of words in $A_U^*$: we let $\mathrm{val}_U(w, w')$ denote the pair $(\mathrm{val}_U(w), \mathrm{val}_U(w'))$ for all $w, w' \in A_U^*$.

**Definition 2.2.** Consider the sequence $(\mathcal{T}_n^2)_{n\geq 0}$ of sets in $\mathbb{R}^2$ defined for all $n \geq 0$ by

$$\mathcal{T}_n^2 = \bigcup_{\substack{u,v \in L_2^{\leq n} \\ \binom{u}{v} \equiv 1 \pmod 2}} \mathrm{val}_2(v, u) + [0, 1]^2.$$

Observe that each $\mathcal{T}_n^2$ is a compact subset of $[0, 2^n]^2$.

**Warning.** As already mentioned in Chapter 1, we choose to draw our pictures in the Cartesian coordinate system in two dimensions whose $x$-axis points rightward, and the $y$-axis points downward. This choice corresponds to the indexation of tables associated with Pascal triangles; see Definitions 1.41 and 1.42.

With this convention, for all $n \geq 0$ and all $u, v \in L_2^{\leq n}$ having an odd binomial coefficient, the upper-left corner of the square region $\mathrm{val}_2(v,u) + [0,1]^2$ in $\mathcal{T}_n^2$ associated with the pair $(u,v)$ has coordinates $(\mathrm{val}_2(v), \mathrm{val}_2(u))$ as shown in Figure 2.3.

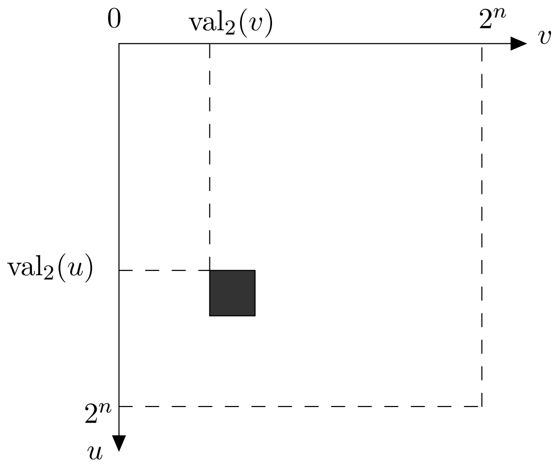



Figure 2.3: Visualization of a square region in $\mathcal{T}_n^2$ indexed by $u, v$.

In the following example, we depict some sets $\mathcal{T}_n^2$.

**Example 2.3.** Let us draw $\mathcal{T}_n^2$ for $n \in \{3,4\}$. First, we have

$$L_2^{\leq 3} = \{\varepsilon, 1, 10, 11, 100, 101, 110, 111\}.$$

Using Table 1.4 from Chapter 1, one can identify the odd binomial coefficients among all displayed values. They correspond to square units in $\mathcal{T}_3^2$ that are colored in black on the left side of Figure 2.4. In the same spirit, one can do the same for $n = 4$. In this case, one has to consider words of length up to 4 in $L_2$. Both sets $\mathcal{T}_3^2$ and $\mathcal{T}_4^2$ are drawn in Figure 2.4. By definition, observe that $\mathcal{T}_4^2$ is four times bigger than $\mathcal{T}_3^2$, and $\mathcal{T}_3^2$ is in fact the left top portion of $\mathcal{T}_4^2$ of size $8 \times 8$.

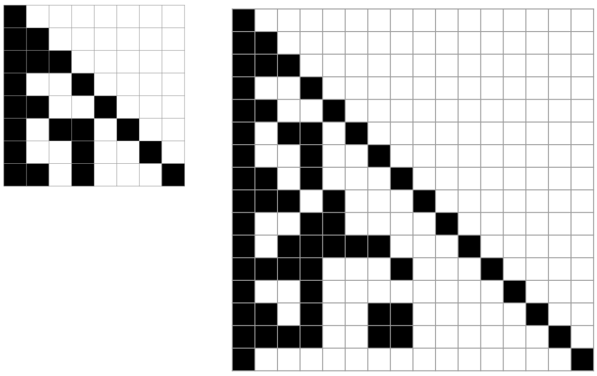


Figure 2.4: The sets $\mathcal{T}_3^2$ and $\mathcal{T}_4^2$.

In Table 2.5, we count the number of unit squares in $\mathcal{T}_n^2$ for the first few values of $n$, and we compare this quantity to the number of positive binomial coefficients of pairs of words in $L_2^{\leq n}$. For instance, using Table 1.4 or Figure 2.4, the first number for $n = 3$ is 22 and the second is 27. In Proposition 2.5, we show that the number of positive binomial coefficients of pairs of words in $L_2^{\leq n}$ is in fact $3^n$. Before proving it, a lemma is needed.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| # unit squares | 1 | 3 | 8 | 22 | 62 | 166 | 458 | 1258 | 3510 | 9838 |
| # positive coefficients | 1 | 3 | 9 | 27 | 81 | 243 | 729 | 2187 | 6561 | 19683 |

Table 2.5: Number of unit squares in $\mathcal{T}_n^2$ compared to the number of positive binomial coefficients of pairs of words in $L_2^{\leq n}$, for $n = 0, \dots, 9$.

**Lemma 2.4.** *For all $n \in \mathbb{N}_0$, the number of pairs $(u, v) \in L_2^n \times L_2^{\leq n}$ of words having a positive binomial coefficient is equal to $2 \cdot 3^{n-1}$.*

*Proof*. For each positive integer $n$, define

$$V_n = \left\{ (x, y) \in \mathbb{N} \times \mathbb{N} \mid 2^{n-1} \leq y < 2^n, 0 \leq x \leq y \text{ and } \binom{\mathrm{rep}_2(y)}{\mathrm{rep}_2(x)} > 0 \right\}.$$

If $(x, y) \in V_n$, then $\mathrm{rep}_2(y)$ is a word of length exactly $n$, *i.e.*, belongs to $(L_2^{\leq n} \setminus L_2^{\leq n-1}) = L_2^n$, and $\mathrm{rep}_2(x)$ is a word in $L_2^{\leq n}$. Thus, $\#V_n$ corresponds to the number of pairs of words in $L_2^n \times L_2^{\leq n}$ having a positive binomial coefficient. We prove the claim by induction on $n \geq 1$. To that aim, we first obtain a partition (2.2) of the set $V_n$, and then we show how to write $V_{n+1}$ in terms of the images of $V_n$ under four maps (see (2.5)).

For all integers $n \geq 1$ and $m \geq 0$, consider the set

$$X_{m,n} = \begin{cases} \emptyset, & \text{if } m > n; \\ V_n \cap (\{0\} \times \mathbb{N}), & \text{if } m = 0; \\ V_n \cap ([2^{m-1}, 2^m) \times \mathbb{N}), & \text{otherwise.} \end{cases}$$

Notice that for $m \geq 1$,

$$X_{m,n} = \left\{ \mathrm{val}_2(v, u) \mid u \in L_2^n, v \in L_2^m \text{ and } \binom{u}{v} > 0 \right\}. \tag{2.1}$$

Indeed, if $m > n$, then $X_{m,n} = \emptyset$, and the left-hand side of (2.1) is also the empty set since $u \in L_2^n$ and $v \in L_2^m$ imply that $|u| = n < m = |v|$ and then $\binom{u}{v} = 0$. If $m \leq n$, then the result follows from the fact that $x \in [2^{m-1}, 2^m)$ implies that $\mathrm{rep}_2(x) \in L_2^m$. We thus have the following partition

$$V_n = \bigcup_{m=0}^{n} X_{m,n}. \tag{2.2}$$

For all $n \geq 1$ and $m \geq 0$, the set $X_{m+1,n+1}$ can be obtained under transformations of the sets $X_{m,n}, X_{m+1,n}$ as follows (see (2.3)). Let us define the functions $f_1$, $f_2$, $f_3$ and $f_4$ by

$$\begin{aligned} f_1 &\colon (x,y) \in \mathbb{N} \times \mathbb{N} \mapsto (2x, 2y) \in \mathbb{N} \times \mathbb{N}, \\ f_2 &\colon (x,y) \in \mathbb{N} \times \mathbb{N} \mapsto (2x+1, 2y+1) \in \mathbb{N} \times \mathbb{N}, \\ f_3 &\colon (x,y) \in \mathbb{N} \times \mathbb{N} \mapsto (x, 2y) \in \mathbb{N} \times \mathbb{N}, \\ f_4 &\colon (x,y) \in \mathbb{N} \times \mathbb{N} \mapsto (x, 2y+1) \in \mathbb{N} \times \mathbb{N}. \end{aligned}$$

Using Lemma 1.38, we show that for all $n \geq 1$ and $m \geq 0$,

$$X_{m+1,n+1} = f_1(X_{m,n}) \cup f_2(X_{m,n}) \cup f_3(X_{m+1,n}) \cup f_4(X_{m+1,n}). \tag{2.3}$$

We show the first inclusion in (2.3) and suppose that $(x, y) \in X_{m+1,n+1}$. Then we can write $\mathrm{rep}_2(x) = vv_0$ and $\mathrm{rep}_2(y) = uu_0$ with $v \in L_2^m$, $u \in L_2^n$ and $v_0, u_0 \in \{0, 1\}$. By Lemma 1.38, we have

$$\binom{\mathrm{rep}_2(y)}{\mathrm{rep}_2(x)} = \binom{uu_0}{vv_0} = \binom{u}{vv_0} + \delta_{u_0,v_0} \binom{u}{v}.$$

Since $\binom{\text{rep}_2(y)}{\text{rep}_2(x)} > 0$, we must have either $\binom{u}{vv_0} > 0$ or $\binom{u}{v} > 0$. In the first case, $\text{val}_2(vv_0, u) \in X_{m+1,n}$, so

$$(x, y) = \text{val}_2(vv_0, uu_0) \in f_3(X_{m+1,n}) \cup f_4(X_{m+1,n}).$$

Proceeding similarly, if $\binom{u}{v} > 0$, then $\text{val}_2(v, u) \in X_{m,n}$, so

$$(x, y) = \text{val}_2(vv_0, uu_0) \in f_1(X_{m,n}) \cup f_2(X_{m,n}).$$

We show the other inclusion in (2.3). Assume that $(x, y) \in X_{m,n}$. In particular, $\text{rep}_2(x) \in L_2^m$ and $\text{rep}_2(y) \in L_2^n$. We have $\binom{\text{rep}_2(y)}{\text{rep}_2(x)} > 0$, and by Lemma 1.38, we deduce that for $a \in \{0, 1\}$

$$\binom{\text{rep}_2(2y + a)}{\text{rep}_2(2x + a)} = \binom{\text{rep}_2(y)a}{\text{rep}_2(y)a} = \binom{\text{rep}_2(y)}{\text{rep}_2(x)a} + \delta_{a,a}\binom{\text{rep}_2(y)}{\text{rep}_2(x)} > 0,$$

implying that $(2x, 2y) = f_1(x, y)$ and $(2x+1, 2y+1) = f_2(x, y)$ both belong to $X_{m+1,n+1}$. Finally, suppose that $(x, y) \in X_{m+1,n}$. Let us write $\text{rep}_2(x) = vb$ with $v \in L_2^m$ and $b \in \{0, 1\}$, and $\text{rep}_2(y) = v \in L_2^n$. We have $\binom{u}{vb} > 0$, and by Lemma 1.38, we get for $a \in \{0, 1\}$

$$\binom{\text{rep}_2(2y + a)}{\text{rep}_2(x)} = \binom{ua}{vb} = \binom{u}{vb} + \delta_{a,b}\binom{u}{v} > 0,$$

showing that $(x, 2y) = f_3(x, y)$ and $(x, 2y+1) = f_4(x, y)$ are both in $X_{m+1,n+1}$. In a similar fashion, one can prove that for all $n \geq 0$

$$X_{0,n+1} = \{0\} \times ([2^n, 2^{n+1}) \cap \mathbb{N}) = f_3(X_{0,n}) \cup f_4(X_{0,n}). \tag{2.4}$$

In the following, we establish that the set $V_{n+1}$ is the union of the sets $f_i(V_n)$ for $i \in \{1, 2, 3, 4\}$. From the partition highlighted in (2.2), we get for all $n \geq 1$

$$\begin{aligned} V_{n+1} = \bigcup_{m=0}^{n+1} X_{m,n+1} &= X_{0,n+1} \cup \bigcup_{m=1}^{n+1} X_{m,n+1} \\ &= X_{0,n+1} \cup \bigcup_{m=0}^{n} X_{m+1,n+1}. \end{aligned}$$

From (2.3) and (2.4), we thus obtain

$$\begin{aligned}
V_{n+1} &= f_3(X_{0,n}) \cup f_4(X_{0,n}) \\
&\quad \cup \left( \bigcup_{m=0}^{n} f_1(X_{m,n}) \cup f_2(X_{m,n}) \cup f_3(X_{m+1,n}) \cup f_4(X_{m+1,n}) \right) \\
&= f_3(X_{0,n}) \cup f_4(X_{0,n}) \cup \left( \bigcup_{m=0}^{n} f_1(X_{m,n}) \right) \cup \left( \bigcup_{m=0}^{n} f_2(X_{m,n}) \right) \\
&\quad \cup \left( \bigcup_{m=0}^{n} f_3(X_{m+1,n}) \right) \cup \left( \bigcup_{m=0}^{n} f_4(X_{m+1,n}) \right).
\end{aligned}$$

From (2.2) and for $i \in \{1,2\}$ , we know that

$$\bigcup_{m=0}^{n} f_i(X_{m,n}) = f_i \left( \bigcup_{m=0}^{n} X_{m,n} \right) = f_i(V_n).$$

Recall that $X_{n+1,n} = \emptyset$, so for $i \in \{3,4\}$, we find

$$\bigcup_{m=0}^{n} f_i(X_{m+1,n}) = f_i \left( \bigcup_{m=0}^{n} X_{m+1,n} \right) = f_i \left( \bigcup_{m=1}^{n+1} X_{m,n} \right) = f_i \left( \bigcup_{m=1}^{n} X_{m,n} \right).$$

Putting everything together, we finally get

$$\begin{aligned}
V_{n+1} =& f_3(X_{0,n}) \cup f_4(X_{0,n}) \cup f_1(V_n) \cup f_2(V_n) \\
& \cup f_3 \left( \bigcup_{m=1}^{n} X_{m,n} \right) \cup f_4 \left( \bigcup_{m=1}^{n} X_{m,n} \right) \\
=& f_1(V_n) \cup f_2(V_n) \cup f_3 \left( \bigcup_{m=0}^{n} X_{m,n} \right) \cup f_4 \left( \bigcup_{m=0}^{n} X_{m,n} \right) \\
=& f_1(V_n) \cup f_2(V_n) \cup f_3(V_n) \cup f_4(V_n),
\end{aligned} \tag{2.5}$$

where (2.2) is used in the last equality.

We now prove that $\#V_n = 2 \cdot 3^{n-1}$ by induction on $n \geq 1$. The result is clear for $n = 1$ since $V_1 = \{(0,1),(1,1)\}$. Let us suppose it holds up to $n \geq 1$, and let us prove it for $n+1$. First, observe that $f_1(V_n) \cap f_2(V_n) = \emptyset$ (resp., $f_3(V_n) \cap f_4(V_n) = \emptyset$) for the second component of pairs in $f_1(V_n)$ (resp., $f_3(V_n)$) is even, and the second component of pairs in $f_2(V_n)$ (resp., $f_4(V_n)$) is odd. Furthermore, if $(x,y) \in V_n$, then exactly one of the two elements $f_3(x,y)$ and $f_4(x,y)$ belongs to $f_1(V_n) \cup f_2(V_n)$. Indeed, suppose that $(x,y)$

belongs to $V_n$. If $x \in \{0,1\}$, then $(\lfloor x/2 \rfloor, y) = (0,y)$ also belongs to $V_n$. Hence, $f_1(0,y) = (0,2y)$ and $f_2(0,y) = (1,2y+1)$ are in $f_1(V_n) \cup f_2(V_n)$. If $x > 1$, write $\text{rep}_2(y) = u$, $\text{rep}_2(x) = vb$ with $u,v \in L_2$ and $b \in \{0,1\}$. We have $\text{rep}_2(\lfloor x/2 \rfloor) = v$, thus if $vb$ is a scattered subword of $u$, so is $v$. This shows that $(\lfloor x/2 \rfloor, y)$ also belongs to $V_n$. Finally, exactly one of the following two equalities is satisfied (depending on the parity of $x$):

$$f_3(x,y) = f_1(\lfloor x/2 \rfloor, y) \quad \text{or} \quad f_4(x,y) = f_2(\lfloor x/2 \rfloor, y).$$

From (2.5), we thus get

$$\begin{aligned}\#V_{n+1} =& \#(f_1(V_n) \cup f_2(V_n)) + \#(f_3(V_n) \cup f_4(V_n)) \\ & - \#((f_1(V_n) \cup f_2(V_n)) \cap (f_3(V_n) \cup f_4(V_n))) \\ =& \sum_{i=0}^{4} \#f_i(V_n) - \#V_n = \sum_{i=0}^{4} \#V_n - \#V_n \\ =& 3\#V_n = 2 \cdot 3^n,\end{aligned}$$

where the last equality comes from the induction hypothesis. □

**Proposition 2.5.** *For all $n \in \mathbb{N}$, the number of pairs of words in $L_2^{\leq n}$ having a positive binomial coefficient is equal to $3^n$.*

*Proof*. For $n = 0$, we have $L_2^{\leq n} = \{\varepsilon\}$, and since $\binom{\varepsilon}{\varepsilon} = 1$, the result holds. As in the proof of Lemma 2.4, define for all $i \geq 1$

$$V_i = \left\{ (x,y) \in \mathbb{N} \times \mathbb{N} \mid 2^{i-1} \leq y < 2^i, 0 \leq x \leq y \text{ and } \binom{\text{rep}_2(y)}{\text{rep}_2(x)} > 0 \right\},$$

and set $V_0 = \{(0,0)\}$. The number of positive binomial coefficients of pairs of words in $L_2^{\leq n}$ is then

$$\sum_{i=0}^{n} \#V_i = \#V_0 + \sum_{i=1}^{n} \#V_i.$$

By Lemma 2.4, this gives

$$\sum_{i=0}^{n} \#V_i = 1 + \sum_{i=1}^{n} 2 \cdot 3^{i-1} = 1 + 2 \cdot \sum_{i=0}^{n-1} 3^i = 1 + 2 \cdot \frac{3^n - 1}{3 - 1} = 3^n,$$

as expected. □

In the following definition, we normalize each set $\mathcal{T}_n^2$ by a factor $1/2^n$ to obtain a sequence of compact subsets in $[0,1]^2$.

**Definition 2.6.** Let $(\mathcal{U}_n^2)_{n\geq 0}$ be the sequence of compact sets in $[0,1]^2$ defined for all $n \geq 0$ by

$$\mathcal{U}_n^2 = \frac{1}{2^n}\mathcal{T}_n^2.$$

By Definition 2.2, each pair $(u,v)$ of words of length at most $n$ with an odd binomial coefficient gives rise to a square region in $\mathcal{T}_n^2$, so it does in $\mathcal{U}_n^2$ too. More accurately, we have the following situation. Let $n \geq 0$ and $u, v \in L_2^{\leq n}$ such that $\binom{u}{v} \equiv 1 \pmod 2$. We have

$$\mathrm{val}_2(v,u) + [0,1]^2 \subset \mathcal{T}_n^2$$

implying

$$\frac{1}{2^n}\,\mathrm{val}_2(v,u) + \left[0, \frac{1}{2^n}\right]^2 = (0.0^{n-|v|}v, 0.0^{n-|u|}u) + \left[0, \frac{1}{2^n}\right]^2 \subset \mathcal{U}_n^2$$

as depicted in Figure 2.6. Recall that if $w = w_n \cdots w_0$ is a finite word over $\{0,1\}$, the notation $0.w$ has to be understood as the real number $\sum_{j=0}^{n} w_j 2^{j-n-1}$ (see Definition 1.19 with $\beta = 2$).

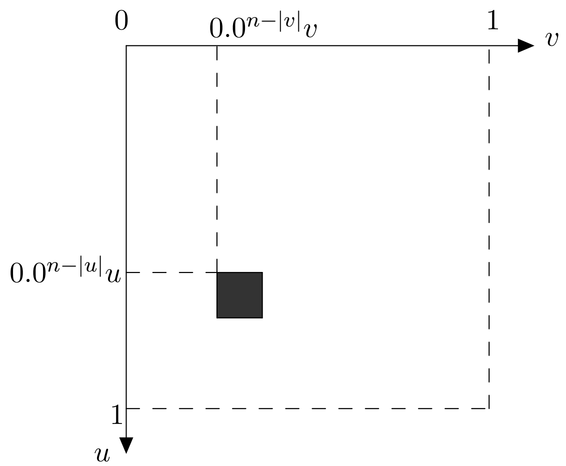



Figure 2.6: Visualization of a square region in $\mathcal{U}_n^2$.

In Figure 2.7, we have depicted the sets $\mathcal{U}_3^2$, $\mathcal{U}_4^2$ and $\mathcal{U}_9^2$. Notice that segments of slopes $1, 2, 2^2, \ldots$ seem to appear in $\mathcal{U}_9^2$. This is a key observation for our discussion.

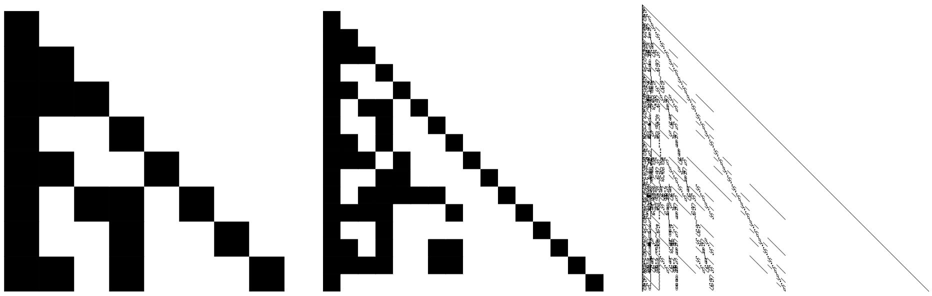


Figure 2.7: The sets $\mathcal{U}_3^2$, $\mathcal{U}_4^2$ and $\mathcal{U}_9^2$.

Our aim is to prove that the sequence $(\mathcal{U}_n^2)_{n\geq 0}$ of compact subsets of $[0,1]^2$ converges with respect to the Hausdorff distance, and to provide a description of the limit set $\mathcal{L}^2$. This description will be referred to as the $(\star)$ condition, which is at the heart of our reasonings. Notice that it will be generalized in Section 2.1.5 to take into account the situation modulo $p$.

### 2.1.2 Twinkle, Twinkle Little $(\star)$

Let us roughly describe the basic idea behind the $(\star)$ condition. Some pairs of words $(u,v) \in L_2 \times L_2$ have the property that not only $\binom{u}{v} \equiv 1 \pmod 2$ but also $\binom{uw}{vw} \equiv 1 \pmod 2$ for all words $w \in \{0,1\}^*$. As an obvious example, take $u = v \in L_2$; less trivial examples can be found in Example 2.8. Such a property creates a particular pattern occurring in $\mathcal{U}_n^2$ for all $n \geq |u|$, as we will see further on.

**Definition 2.7.** Let $(u,v) \in L_2 \times L_2$. We say that $(u,v)$ satisfies the $(\star)$ *condition* or simply $(\star)$ if $(u,v) \neq (\varepsilon,\varepsilon)$,

$$\binom{u}{v} \equiv 1 \pmod 2, \ \binom{u}{v0} = 0 \text{ and } \binom{u}{v1} = 0.$$

Note that if $(u,v)$ satisfies $(\star)$, then $|v| \leq |u|$, and

$$\binom{u}{vw} = 0$$

for all non-empty words $w$.

**Example 2.8.** In Table 2.8, one can find some pairs $(u,v) \in L_2 \times L_2$ satisfying $(\star)$.

| $u$ | 1 | 101 | 1001 | 1101 | 1110 |
|---|---|---|---|---|---|
| $v$ | 1 | 11 | 11 | 111 | 10 |

Table 2.8: Some pairs of words in $L_2$ satisfying $(\star)$.

The following lemma shows that if a pair of words satisfies $(\star)$, then adding the same letter at the end of both words creates a pair that also satisfies $(\star)$. To the contrary, adding distinct letters gives a zero binomial coefficient.

**Lemma 2.9.** *If $(u,v) \in L_2 \times L_2$ satisfies $(\star)$, then both $(u0,v0)$ and $(u1,v1)$ satisfy $(\star)$. Furthermore, the binomial coefficients of the pairs $(u0,v1)$ and $(u0,v1)$ are equal to $0$.*

*Proof.* For the first part of the statement, we only treat $(u0,v0) \in L_2 \times L_2$ since the other case is similar. First, the fact that $\binom{u0}{v0} \equiv 1 \pmod 2$ directly follows from Lemma 1.38 and from the hypothesis. Now we proceed by contradiction. If $\binom{u0}{v00} > 0$ or $\binom{u0}{v01} > 0$, then $v00$ or $v01$ is a scattered subword of $u0$. In both cases, we conclude that $v0$ must appear as a scattered subword of $u$, contradicting the assumption.

The second part of the statement follows by Lemma 1.38 and by the fact that $(u,v)$ satisfies $(\star)$. □

The previous lemma implies that a particular pattern occurs in $\mathcal{U}_n^2$ for all $n$ sufficiently large.

**Remark 2.10.** Let $(u,v)$ be a pair of words in $L_2$ satisfying $(\star)$ such that $0 \leq |v| \leq |u| = \ell$. Then $(0.0^{\ell-|v|}v, 0.u) + [0, 1/2^\ell]^2 \subset \mathcal{U}_\ell^2$. As a consequence of Lemma 2.9, $(u,v)$, $(u0,v0)$ and $(u1,v1)$ have an odd binomial coefficient and thus correspond to square regions in $\mathcal{U}_{\ell+1}^2$, *i.e.*,

$$\{(0.0^{\ell+1-|v|}v, 0.0u), (0.0^{\ell-|v|}v0, 0.u0), (0.0^{\ell-|v|}v1, 0.u1)\} + \left[0, \frac{1}{2^{\ell+1}}\right]^2 \subset \mathcal{U}_{\ell+1}^2.$$

Note that $(u0,v1)$ and $(u1,v0)$ do not give any square region in $\mathcal{U}_{\ell+1}^2$ since their binomial coefficients are equal to 0 by Lemma 2.9. Iterating this argument yields, for all $n \geq 0$,

$$\bigcup_{w \in \{0,1\}^{\leq n}} (0.0^{\ell+n-|w|-|v|}vw, 0.0^{n-|w|}uw) + \left[0, \frac{1}{2^{\ell+n}}\right]^2 \subset \mathcal{U}^2_{\ell+n}.$$

Again, observe that the pair $(uw, vw')$ with $w, w' \in \{0,1\}^{\leq n}$, $w \neq w'$ and $|w| = |w'|$ has a zero binomial coefficient. Indeed, let us proceed by induction on $|w| = |w'|$ to prove the claim. If $w$ and $w'$ are distinct letters, then it is true by Lemma 2.9. If $w = ta$ and $w' = t'b$ with $a \neq b$ in $\{0,1\}$ and if $\binom{uw}{vw'} = \binom{ut}{vt'b} > 0$, then there would exist a letter $c \in \{0,1\}$ such that $vc$ is a scattered subword of $u$, contradicting the $(\star)$ condition. If $w = ta$ and $w' = t'a$ with $t, t' \in \{0,1\}^*$, $t \neq t'$ and $|t| = |t'|$ and $a \in \{0,1\}$, then $\binom{uw}{vw'} = \binom{ut}{vt'a} + \binom{ut}{vt'} > 0$. The first binomial coefficient is 0 otherwise it violates the $(\star)$ condition, and the second is 0 by induction hypothesis. As a consequence, the considered pair $(uw, vw')$ does not induce a square region in $\mathcal{U}^2_{\ell+n}$.

As an example, consider the pair $(101, 11)$ satisfying $(\star)$ (see Table 2.8). The associated square region in $\mathcal{U}^2_3$ is of size $1/8$ and its upper-left corner has coordinates $(\mathrm{val}_2(11)/2^3, \mathrm{val}_2(101)/2^3) = (3/8, 5/8)$. This one black square is divided into two black squares of size $1/16$ in $\mathcal{U}^2_4$, and then four black squares of size $1/32$ in $\mathcal{U}^2_5$. The corresponding squares in $\mathcal{U}^2_3$, $\mathcal{U}^2_4$ and $\mathcal{U}^2_5$ are showed in Figure 2.9.

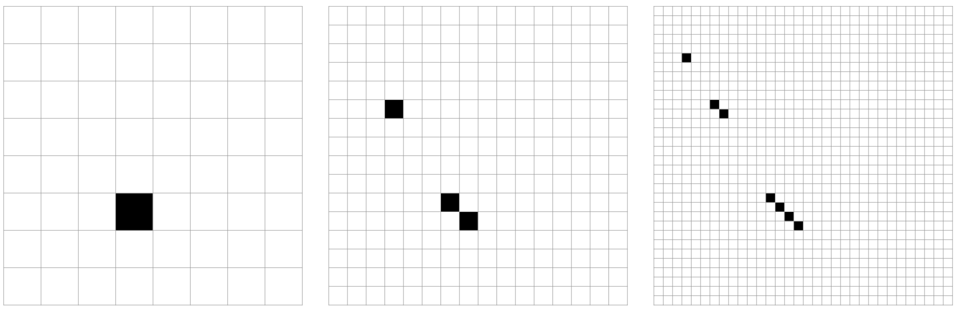


Figure 2.9: The pair $(101, 11)$ satisfying $(\star)$ in $\mathcal{U}^2_3, \mathcal{U}^2_4, \mathcal{U}^2_5$.

At this stage, observe that the sequence of squares

$$\bigcup_{w \in \{0,1\}^n} (0.0^{\ell-|v|}vw, 0.uw) + \left[0, \frac{1}{2^{\ell+n}}\right]^2 \subset \mathcal{U}^2_{\ell+n}$$

roughly tends to the diagonal of the initial square $(0.0^{\ell-|v|}v, 0.u) + [0, 1/2^\ell]^2$ with respect to the Hausdorff distance (see, for instance, what happens in Figure 2.9). This particular pattern will be used in the next section, where we build another sequence of compact sets.

### 2.1.3 New Compact Sets

Inspired by the previous remark, we first define a closed segment associated with a pair of words, and then an initial compact set $\mathcal{A}_0^2$; see Definitions 2.11 and 2.12. Roughly, this compact set contains segments of slope 1, some of which notably appear in $\mathcal{U}_9^2$ in Figure 2.7. Modifying $\mathcal{A}_0^2$ with the help of two maps leads to the definition of a sequence $(\mathcal{A}_n^2)_{n\geq 0}$ of compact sets; see Definition 2.15. Again, the idea behind this construction is that $\mathcal{A}_n^2$ contains segments of slopes $1, 2, 2^2, \ldots, 2^n$, some of which are particularly depicted in $\mathcal{U}_9^2$ in Figure 2.7.

**Definition 2.11.** Let $(u, v)$ in $L_2 \times L_2$ be such that $1 \leq |v| \leq |u|$. We define a closed segment $S_{u,v}$ of slope 1 and of length $\sqrt{2} \cdot 2^{-|u|}$ in $[0, 1] \times [1/2, 1]$. The endpoints of $S_{u,v}$ are given by

$$A_{u,v} = (0.0^{|u|-|v|}v, 0.u) \text{ and } B_{u,v} = (0.0^{|u|-|v|}v + 2^{-|u|}, 0.u + 2^{-|u|}).$$

Note that if we allow infinite binary expansions ending with ones, we have

$$B_{u,v} = (0.0^{|u|-|v|}v111\cdots, 0.u111\cdots).$$

Observe that $S_{u,v}$ is included in $[1/2^{|u|-|v|+1}, 1/2^{|u|-|v|}] \times [1/2, 1]$.

**Definition 2.12.** We let $\mathcal{A}_0^2$ be the following compact set

$$\overline{\bigcup_{\substack{(u,v) \\ \text{satisfying}(\star)}} S_{u,v}},$$

which is the closure of a countable union of segments of slope 1.

Notice that Definition 2.11 implies that $\mathcal{A}_0^2 \subset [0, 1] \times [1/2, 1]$. Also observe that the union is not disjoint as some segments are included in others. For instance, $S_{10,10} \subset S_{1,1}$ since $A_{1,1} = (1/2, 1/2)$, $B_{1,1} = (1, 1)$, $A_{10,10} = (1/2, 1/2)$ and $B_{10,10} = (3/4, 3/4)$.

**Example 2.13.** In Figure 2.10, each segment $S_{u,v}$ is represented for all pairs $(u, v)$ of words satisfying $(\star)$ with $|u| \leq 6$. For instance, the segment of origin $(1/2, 1/2)$ and length $\sqrt{2}/2$ comes from the pair of words $(1, 1)$ satisfying $(\star)$. Now consider the pair $(1101, 111)$ satisfying $(\star)$. Its associated segment $S_{1101,111}$ is depicted in red: it has origin $(7/16, 13/16) \simeq (0.4375, 0.8125)$ and length $\sqrt{2}/16$.

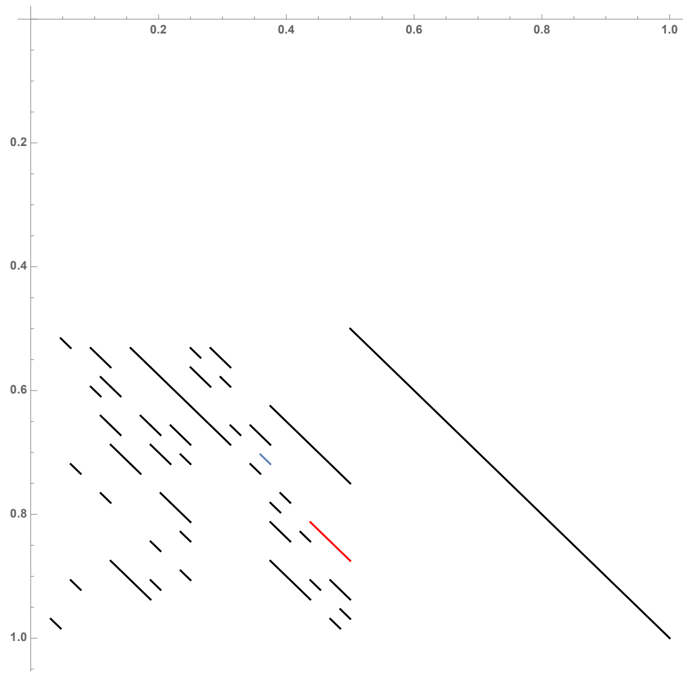



Figure 2.10: An approximation of $\mathcal{A}_0^2$ computed with words of length $\leq 6$.

**Remark 2.14.** In the definition of $\mathcal{A}_0^2$, we take the closure of a union to ensure the compactness of the set. In the following, we build a limit point that does not belong to the union of segments but to the closure $\mathcal{A}_0^2$.

First, the point $(1/32, 1/2)$ does not belong to the union of segments. Let us proceed by contradiction, and suppose there exist words $u, v, w$ in $\{0, 1\}^*$ such that $(u, v)$ satisfies $(\star)$, $1/2 = 0.uw$, and $1/32 = 0.0^{|u|-|v|}vw$. Then $u, v$ both belong to $10^*$, but the $(\star)$ condition implies $u = v$. This is impossible.

For all $n \geq 0$ and all $r \in \{0, 1, \ldots, 7\}$, the pair $(10^{8n+4+r}1, 10^{8n+r}1)$

satisfies ($\star$) if and only if $0 \le r \le 3$. Indeed, observe that

$$\binom{10^{8n+4+r}1}{10^{8n+r}1} = \binom{8n+4+r}{8n+r}.$$

First assume that $0 \le r \le 3$, and let $\mathrm{rep}_2(n) = n_k \cdots n_0$ and $\mathrm{val}_2(r_1 r_0) = r$ with $r_0, r_1 \in \{0,1\}$. Then we have $\mathrm{rep}_2(8n+4+r) = n_k \cdots n_0 1 r_1 r_0$ and $\mathrm{rep}_2(8n+r) = n_k \cdots n_0 0 r_1 r_0$. From Theorem 1.40, we find

$$\binom{8n+4+r}{8n+r} \equiv \binom{n_k}{n_k} \cdots \binom{n_0}{n_0}\binom{1}{0}\binom{r_1}{r_1}\binom{r_0}{r_0} \pmod 2 \equiv 1 \pmod 2.$$

It is also easy to see that

$$\binom{10^{8n+4+r}1}{10^{8n+r}10} = 0 \text{ and } \binom{10^{8n+4+r}1}{10^{8n+r}11} = 0.$$

Now, if $4 \le r \le 7$, then

$$\mathrm{rep}_2(r) \in \{100, 101, 110, 111\} \text{ and } \mathrm{rep}_2(4+r) \in \{1000, 1001, 1010, 1011\}.$$

When applying Theorem 1.40, the corresponding product contains a factor $\binom{0}{1}$, which is equal to 0, and the result is thus even.

For all $n \ge 0$ and all $r \in \{0,1,2,3\}$, define $m = 8n+6+r \ge 6$, and set $u_m = 10^{m-2}1$ and $v_m = 10^{m-6}1$. We know that the pair $(u_m, v_m)$ satisfies ($\star$). The origin $A_{u_m,v_m}$ of the associated segment $S_{u_m,v_m}$ is equal to

$$\begin{aligned} A_{u_m,v_m} &= (0.0^4 v_m, 0.u_m) = (0.0^4 10^{m-6}1, 0.10^{m-2}1) \\ &= (1/32 + 1/2^m, 1/2 + 1/2^m). \end{aligned}$$

We have at our hand a sequence of segments $S_{u_m,v_m}$ in $\mathcal{A}_0^2$ with one endpoint being of the form $(1/32 + 1/2^m, 1/2 + 1/2^m, )$ with $m \ge 6$. Thus, the point $(1/32, 1/2)$ is an accumulation point of $\mathcal{A}_0^2$, as desired.

In what follows, we illustrate the previous reasoning. For $n \in \{0,1\}$ and $r \in \{0,1,2,3\}$, we have $m \in \{6,7,8,9,14,15,16,17\}$, and the coordinates of the origin $A_{u_m,v_m}$ of $S_{u_m,v_m}$ are displayed in Table 2.11. In Figure 2.12, we have represented the segments corresponding to those values. Note that, as $m$ increases, the length of $u_m$ also increases and the segments become shorter.

In the following, we transform $\mathcal{A}_0^2$ under iterations of two maps to create a new sequence of nested compact sets.

| $m$ | 6 | 7 | 8 | 9 |
|---|---|---|---|---|
| $A_{u_m,v_m}$ | $(\frac{3}{64}, \frac{33}{64})$ | $(\frac{5}{128}, \frac{65}{128})$ | $(\frac{9}{256}, \frac{129}{256})$ | $(\frac{17}{512}, \frac{257}{512})$ |
| $m$ | 14 | 15 | 16 | 17 |
| $A_{u_m,v_m}$ | $(\frac{513}{16384}, \frac{8193}{16384})$ | $(\frac{1025}{32768}, \frac{16385}{32768})$ | $(\frac{2049}{65536}, \frac{32769}{65536})$ | $(\frac{4097}{131072}, \frac{65537}{131072})$ |

Table 2.11: Origins of the segments $S_{u_m,v_m}$ for $m \in \{6, 7, 8, 9, 14, 15, 16, 17\}$.

**Definition 2.15.** We let $c$ denote the homothety of center $(0, 0)$ and ratio $1/2$, and we consider the map $h\colon (x, y) \mapsto (x, 2y)$. For all $n \geq 0$, we define the compact set

$$\mathcal{A}_n^2 = \bigcup_{\substack{0\leq i\leq n\\0\leq j\leq i}} h^j(c^i(\mathcal{A}_0^2)).$$

Observe that the application of the map $c$ to a segment does not change its slope whereas $h$ multiplies it by 2. Consequently, since $\mathcal{A}_0^2$ contains segments of slope 1, $\mathcal{A}_n^2$ then contains segments of slopes $1, 2, 2^2, \ldots, 2^n$ for all $n \geq 0$. Also note that, by definition, the sequence $(\mathcal{A}_n^2)_{n\geq 0}$ is increasingly nested, *i.e.*, it satisfies

$$\mathcal{A}_0^2 \subset \mathcal{A}_1^2 \subset \mathcal{A}_2^2 \subset \cdots .$$

**Example 2.16.** Recall that $\mathcal{A}_0^2 \subset [0, 1] \times [1/2, 1]$. In Figure 2.13, the region $R = [0, 1] \times [1/2, 1]$ containing $\mathcal{A}_0^2$ is depicted in gray. Then we apply $c$ and $h \circ c$ to $R$, respectively giving the two regions $[0, 1/2] \times [1/4, 1/2]$ and $[0, 1/2] \times [1/2, 1]$ in red. The union of the gray and red regions contains $\mathcal{A}_1^2$. Finally, we apply $c^2$, $h \circ c^2$ and $h^2 \circ c^2$ to $R$ to draw the three blue regions. The compact set $\mathcal{A}_2^2$ lies into the union of the gray, red and blue regions.

Let us now take a more precise example. In Figure 2.14, we have depicted two original segments in $\mathcal{A}_0^2$ (in black), then one application of $c$ possibly followed by $h$ (in red), then a second application of $c$ followed by at most 2 applications of $h$ (in blue).

With the help of Figure 2.13, it is not too difficult to see that

$$\mathcal{A}_m^2 \cap ([1/2^{m+1}, 1] \times [0, 1]) = \mathcal{A}_n^2 \cap ([1/2^{m+1}, 1] \times [0, 1]) \qquad (2.6)$$

for all $m, n \in \mathbb{N}$ with $m \leq n$. Roughly, the region $[1/2^{m+1}, 1] \times [0, 1]$ gets

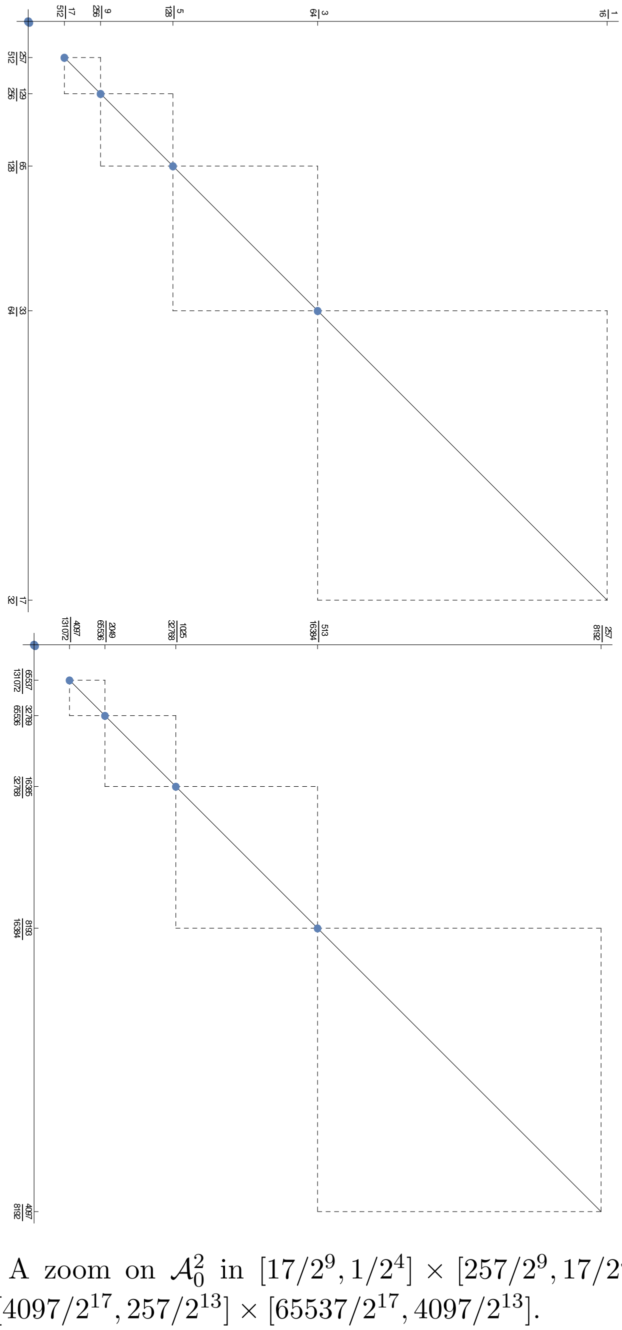


Figure 2.12: A zoom on $\mathcal{A}_0^2$ in $[17/2^9, 1/2^4] \times [257/2^9, 17/2^5]$ and in the smaller area $[4097/2^{17}, 257/2^{13}] \times [65537/2^{17}, 4097/2^{13}]$.

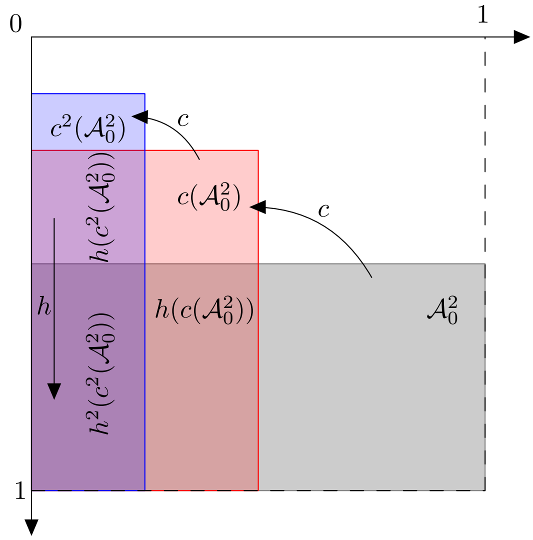



Figure 2.13: Two applications of $c$ and $h$ from $\mathcal{A}_0^2$.

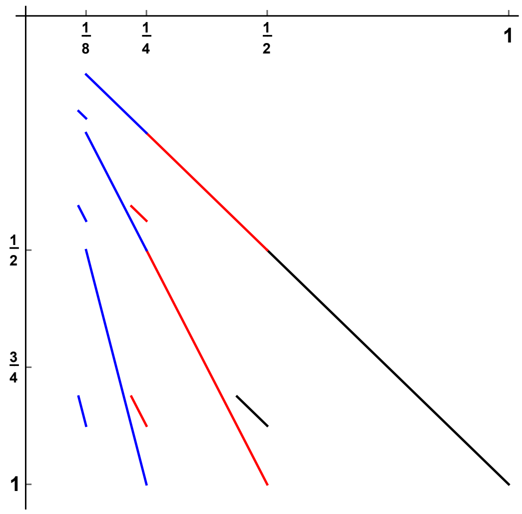



Figure 2.14: A subset of $\mathcal{A}_2^2$.

stabilized in $\mathcal{A}_n^2$ as soon as $n \geq m$. For our needs, we show that a particular segment is in the sequence $(\mathcal{A}_n^2)_{n\geq 0}$.

**Lemma 2.17.** *For all $n \geq 0$, the segment with endpoints $(1/2^{2n+1}, 1/2^{n+1})$ and $(1/2^n, 1)$ is included in $\mathcal{A}_{2n}^2$.*

*Proof*. Since the pair $(1,1) \in L_2 \times L_2$ satisfies $(\star)$, the segment $S_{1,1}$ with endpoints $A_{1,1} = (1/2, 1/2)$ and $B_{1,1} = (1,1)$ is included in $\mathcal{A}_0^2$. By definition, the segment $h^n(c^{n+i}(S_{1,1}))$ with endpoints $(1/2^{n+i+1}, 1/2^{i+1})$ and $(1/2^{n+i}, 1/2^i)$ is in $\mathcal{A}_{n+i}^2$ for all $n \geq 0$ and all $i \geq 0$ (note that applying $h^n$ after $c^{n+i}$ annihilates the division by $2^n$ on the second component). Since $\mathcal{A}_n^2 \subset \cdots \subset \mathcal{A}_{2n}^2$, the union of segments

$$S = \bigcup_{i=0}^{n} h^n(c^{n+i}(S_{1,1}))$$

is included in $\mathcal{A}_{2n}^2$. Observe that

$$h^n(c^{n+i}(A_{1,1})) = (1/2^{n+i+1}, 1/2^{i+1}) = h^n(c^{n+i+1}(B_{1,1}))$$

for all $i \in \{0, \ldots, n-1\}$, so $S$ is in fact a continuous segment with endpoints $(1/2^{2n+1}, 1/2^{n+1})$ and $(1/2^n, 1)$ that is inside $\mathcal{A}_{2n}^2$. $\square$

Applying Proposition 1.60 to the sequence $(\mathcal{A}_n^2)_{n\geq 0}$ gives the following definition[1].

**Definition 2.18.** We let $\mathcal{L}^2 = \overline{\bigcup_{n\geq 0} \mathcal{A}_n^2}$ denote the compact limit of the sequence $(\mathcal{A}_n^2)_{n\geq 0}$ of compact sets.

In particular, observe that each $\mathcal{A}_n^2$ is a subset of $\mathcal{L}^2$. In the following example, we draw an approximation of $\mathcal{L}^2$.

**Example 2.19.** We take all the pairs of words in $L_2$ of length at most 8. Among them, those satisfying $(\star)$ create 1369 different segments in $\mathcal{A}_0^2$. By definition, their length is at least $\sqrt{2}/2^8$, so we are missing segments of length $\leq \sqrt{2}/2^9$. Afterwards, we apply the maps $h^j(c^i(\cdot))$ to those segments for $0 \leq j \leq i \leq 4$ in order to obtain an approximation of $\mathcal{A}_4^2$ in Figure 2.15. Consequently, except the segments of length $\leq \sqrt{2}/2^9$, their images and accumulation points, we have an exact image of $\mathcal{L}^2$ inside $[1/32, 1] \times [0, 1]$ (recall the stabilization from (2.6)).

[1] In [LRS16], we showed in a different way that this definition makes sense: we proved that $(\mathcal{A}_n^2)_{n\geq 0}$ is a Cauchy sequence, which always converges in a complete metric space.

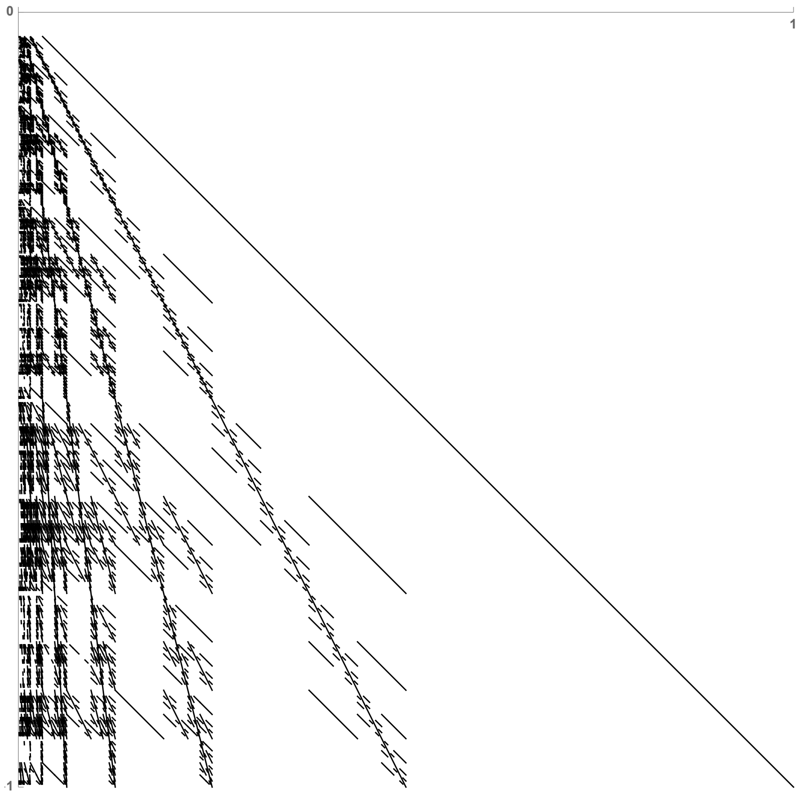



Figure 2.15: An approximation of the limit set $\mathcal{L}^2$.

### 2.1.4 The Sierpiński Counterpart

Now we will show that the sequence $(\mathcal{U}_n^2)_{n\geq 0}$ of compact subsets of $[0,1]^2$ converges to $\mathcal{L}^2$ with respect to the Hausdorff distance, *i.e.*, the Hausdorff distance between $\mathcal{U}_n^2$ and $\mathcal{L}^2$ tends to 0 as $n$ goes to infinity. This is done in two parts. The first is to show that, when $\epsilon$ is a positive real number, then $\mathcal{U}_n^2 \subset [\mathcal{L}^2]_\epsilon$ for all sufficiently large $n \in \mathbb{N}$. Secondly, we will need to prove that $\mathcal{L}^2 \subset [\mathcal{U}_n^2]_\epsilon$ for all sufficiently big $n \in \mathbb{N}$. Putting these two arguments together gives $d_h(\mathcal{U}_n^2, \mathcal{L}^2) < \epsilon$ for all large enough $n \in \mathbb{N}$.

**Lemma 2.20.** *Let $\epsilon > 0$. For all big enough $n \in \mathbb{N}$, we have $\mathcal{U}_n^2 \subset [\mathcal{L}^2]_\epsilon$.*

*Proof*. Let $\epsilon > 0$, choose $n \in \mathbb{N}$, and pick $(x, y) \in \mathcal{U}_n^2$. To prove the claim, we exhibit a point $B \in \mathcal{L}^2$ such that $d((x, y), B) < \epsilon$ if $n$ is large enough.

By definition, there exists $(u,v) \in L_2 \times L_2$ such that $\binom{u}{v} \equiv 1 \pmod 2$, $|u| \leq n$ and $(x,y) \in A + [0,1/2^n]^2$, where $A = (0.0^{n-|v|}v, 0.0^{n-|u|}u)$ is the upper-left corner of the previous square region in $\mathcal{U}_n^2$. In particular, observe that $d((x,y),A) \leq \sqrt{2} \cdot 2^{-n}$.

Assume first that $(u,v)$ satisfies $(\star)$. By Definition 2.12, the segment $S_{u,v}$ is in $\mathcal{A}_0^2$. Now apply $n-|u|$ times the homothety $c$ to it. By Definition 2.15, the segment $c^{n-|u|}(S_{u,v})$ is in $\mathcal{A}^2_{n-|u|}$, thus also in $\mathcal{L}^2$ by definition. In particular, $A = c^{n-|u|}(A_{u,v}) \in \mathcal{L}^2$ and

$$d((x,y),A) \leq \sqrt{2} \cdot 2^{-n} < \epsilon,$$

if $n$ is big enough. We can choose $B = A$.

Now assume that $(u,v)$ does not satisfy $(\star)$. Since $\binom{u}{v}$ is odd, either $u$ and $v$ are non-empty words, or $u$ is non-empty and $v = \varepsilon$, or they are both empty.

First, assume that $u$ is non-empty and $v = \varepsilon$. The point $A$ is on the vertical line of equation $x = 0$ and its $y$-coordinate varies in $[1/2^n, 1]$. By Lemma 2.17, the segment $S$ with endpoints $(1/2^{2n+1}, 1/2^{n+1})$ and $(1/2^n, 1)$ is inside $\mathcal{A}^2_{2n}$, and also in $\mathcal{L}^2$. Since the segment $S$ passes through the square $A + [0,1/2^n]^2$, there exists a point $B \in S \subset \mathcal{L}^2$ that also belongs to the square $A + [0,1/2^n]^2$ such that $d((x,y),B) \leq \sqrt{2} \cdot 2^{-n}$. Thus, we can choose $n$ sufficiently big such that $d((x,y),B) < \epsilon$.

As a second case, if $u = \varepsilon = v$, then $A = (0,0)$, and a reasoning similar to the one developed above allows us to conclude.

Finally, suppose that $u$ and $v$ are non-empty. The idea is to find $k \in \mathbb{N}$ such that the pair $(u0^{2^k}1, v0^{2^k}1)$ of words satisfies $(\star)$, and then apply the argument of the first part of the proof. Let $\binom{u}{v} = r$. For each occurrence of $v$ in $u$, we count the total number of zeroes after it. We thus define a sequence of non-negative integer indices

$$|u| - |v| \geq i_1 \geq i_2 \geq \cdots \geq i_r \geq 0$$

corresponding to the number of zeroes following the first, the second, ..., the $r$th occurrence of $v$ in $u$. In Table 2.16, we illustrate the argument with $u = 100010$ and $v = 10$ for which $r = 5$ (note that on the first row of the table, the occurrence of $v$ in $u$ that is considered is written in bold).

Now let $k$ be a non-negative integer such that $2^k > |u|$. We get

$$\binom{u0^{2^k}1}{v0^{2^k}1} = \sum_{\ell=1}^{r} \binom{2^k + i_\ell}{2^k}.$$

| | **10**0010 | **1**0**0**010 | **1**00**0**10 | **1**0001**0** | 1000**10** |
|---|---|---|---|---|---|
| $\ell$ | 1 | 2 | 3 | 4 | 5 |
| $i_\ell$ | 3 | 2 | 1 | 0 | 0 |

Table 2.16: Number of zeroes after each occurrence of $v = 10$ in $u = 100010$.

Indeed, for each $\ell \in \{1, \ldots, r\}$, write $u = pw$ where the last letter of $p$ is the last letter of $v$ and $|w|_0 = i_\ell$. With the $\ell$th occurrence of $v$, we obtain occurrences of $v0^{2^k}1$ in $u0^{2^k}1$ by choosing $2^k$ zeroes among the $2^k + i_\ell$ zeroes available in $w0^{2^k}1$. Moreover, since $2^k > |u|$, it is not possible to have any other occurrence of $v0^{2^k}1$ in $u0^{2^k}1$. From Theorem 1.40,

$$\binom{2^k + i_\ell}{2^k} \equiv 1 \pmod 2$$

for all $\ell \in \{1, \ldots, r\}$, so since $r$ is odd, we get

$$\binom{u0^{2^k}1}{v0^{2^k}1} \equiv 1 \pmod 2.$$

It is easy to check that the pair $(u_k, v_k) = (u0^{2^k}1, v0^{2^k}1)$ of words satisfies $(\star)$: the first two conditions of $(\star)$ are already fulfilled, while the last two follow from the fact that the block of zeroes is of length $2^k > |u|$. As in the first part of the proof, the segment $S_{u_k,v_k}$ is inside $\mathcal{A}_0^2$, so the segment $c^{n-|u|}(S_{u_k,v_k})$ of origin $A'_{u_k,v_k} = c^{n-|u|}(A_{u_k,v_k})$ is in $\mathcal{A}^2_{n-|u|} \subset \mathcal{L}^2$. Hence

$$d((x,y), A'_{u_k,v_k}) \le d((x,y), A) + d(A, A'_{u_k,v_k}) \le \sqrt{2} \cdot 2^{-n} + \frac{d(A_{u,v}, A_{u_k,v_k})}{2^{n-|u|}}.$$

Since $d(A_{u,v}, A_{u_k,v_k}) = \sqrt{2} \cdot 2^{-|u|-2^k-1}$, we find

$$d((x,y), A'_{u_k,v_k}) \le \sqrt{2} \cdot 2^{-n+1} < \epsilon,$$

if $n$ is large enough. In this case, we may choose $B = A'_{u_k,v_k}$. □

Given $\epsilon > 0$, it remains to show that $\mathcal{L}^2 \subset [\mathcal{U}_n^2]_\epsilon$ for all sufficiently large $n \in \mathbb{N}$. Before getting to this result, an extra lemma is needed, whose main idea is that if $\binom{u}{vb}$ is odd for a letter $b$, then we can find a letter $a$ such that $\binom{ua}{vb}$ is also odd (this observation turns out to be useful in the proof of Lemma 2.22).

| | $v$ | $v0$ | $v1$ |
|---|---|---|---|
| $u$ | 0 | 0 | 0 |
| $u0$ | | 0 | 0 |
| $u1$ | | 0 | 0 |

| | $v$ | $v0$ | $v1$ |
|---|---|---|---|
| $u$ | 0 | 0 | 1 |
| $u0$ | | 0 | 1 |
| $u1$ | | 0 | 1 |

| | $v$ | $v0$ | $v1$ |
|---|---|---|---|
| $u$ | 0 | 1 | 0 |
| $u0$ | | 1 | 0 |
| $u1$ | | 1 | 0 |

| | $v$ | $v0$ | $v1$ |
|---|---|---|---|
| $u$ | 0 | 1 | 1 |
| $u0$ | | 1 | 1 |
| $u1$ | | 1 | 1 |

| | $v$ | $v0$ | $v1$ |
|---|---|---|---|
| $u$ | 1 | 0 | 0 |
| $u0$ | | 1 | 0 |
| $u1$ | | 0 | 1 |

| | $v$ | $v0$ | $v1$ |
|---|---|---|---|
| $u$ | 1 | 0 | 1 |
| $u0$ | | 1 | 1 |
| $u1$ | | 0 | 0 |

| | $v$ | $v0$ | $v1$ |
|---|---|---|---|
| $u$ | 1 | 1 | 0 |
| $u0$ | | 0 | 0 |
| $u1$ | | 1 | 1 |

| | $v$ | $v0$ | $v1$ |
|---|---|---|---|
| $u$ | 1 | 1 | 1 |
| $u0$ | | 0 | 1 |
| $u1$ | | 1 | 0 |

Table 2.17: Residues modulo 2 of $\binom{ua}{vb}$ as a function of the residues modulo 2 of $\binom{u}{v}$ and $\binom{u}{vb}$.

**Lemma 2.21.** *Let $u, v$ be words in $L_2$. If $\binom{u}{vb} \equiv 1 \pmod 2$ for a letter $b \in \{0,1\}$, then there exists a letter $a \in \{0,1\}$ such that $\binom{ua}{vb} \equiv 1 \pmod 2$.*

*Proof.* This result follows from Lemma 1.38 and Table 2.17 that displays the values modulo 2 of the binomial coefficients $\binom{ua}{vb}$ for all $a, b \in \{0,1\}$ when the values modulo 2 of the binomial coefficients $\binom{u}{v}$ and $\binom{u}{vb}$ with $b \in \{0,1\}$ are known. □

In the following lemma, we show that the distance between a given point of $\mathcal{L}^2$ and terms of the sequence $(\mathcal{U}_n^2)_{n\geq 0}$ of large indices can get as small as one wants. Recall that our goal is in fact to prove that $\mathcal{L}^2 \subset [\mathcal{U}_n^2]_\epsilon$ for all sufficiently large $n \in \mathbb{N}$. Thus, afterwards, we will need to permute the quantifiers to show that the Hausdorff distance between $\mathcal{L}^2$ and $\mathcal{U}_n^2$ is small when $n$ gets big. This will be possible by using the compactness of $\mathcal{L}^2$; see the proof of Theorem 2.24.

**Lemma 2.22.** *Let $\epsilon > 0$. For all $(x, y) \in \mathcal{L}^2$, there exists $N$ such that for all $n \geq N$, $d((x, y), \mathcal{U}_n^2) < \epsilon$.*

*Proof.* Let $\epsilon > 0$ and let $(x, y) \in \mathcal{L}^2$. Since $(\mathcal{A}_n^2)_{n\geq 0}$ converges to $\mathcal{L}^2$ with respect to the Hausdorff distance, there exist $N_1 \in \mathbb{N}$ and $(x', y') \in \mathcal{A}_{N_1}^2$ such that,

$$d((x, y), (x', y')) < \epsilon/4.$$

By definition of $\mathcal{A}^2_{N_1}$, there exist integers $i, j$ such that $0 \le j \le i \le N_1$ and $(x_0', y_0') \in \mathcal{A}^2_0$ such that

$$h^j(c^i((x_0', y_0'))) = (x', y').$$

By definition of $\mathcal{A}^2_0$, there exist a pair $(u, v) \in L_2 \times L_2$ satisfying $(\star)$ and $(x_0'', y_0'') \in S_{u,v}$ such that

$$d((x_0', y_0'), (x_0'', y_0'')) < \epsilon/4.$$

Since $j \le i$, we have

$$\begin{aligned} d((x', y'), h^j(c^i((x_0'', y_0'')))) &= d(h^j(c^i((x_0', y_0'))), h^j(c^i((x_0'', y_0'')))) \\ &\le d((x_0', y_0'), (x_0'', y_0'')) \\ &< \epsilon/4. \end{aligned}$$

Consequently, we get that

$$\begin{aligned} d((x, y), h^j(c^i((x_0'', y_0'')))) &\le d((x, y), (x', y')) + d((x', y'), h^j(c^i((x_0'', y_0'')))) \\ &< \epsilon/2. \end{aligned} \tag{2.7}$$

In the second part of the proof, we will show that

$$d(h^j(c^i((x_0'', y_0''))), \mathcal{U}^2_n) < \epsilon/2$$

for all sufficiently large $n$. We will make use of the constants $i, j$ and words $u, v$ given above.

Let $n \ge 0$. Since $(u, v) \in L_2 \times L_2$ satisfies $(\star)$, iteratively applying Lemma 2.9 shows that the pair $(uw, vw)$ satisfies $(\star)$ for all words $w \in \{0, 1\}^*$ of length $n, \ldots, n+i$. Those pairs correspond to square regions in $\mathcal{U}^2_{n+i+|u|}$ located on the segments $c^\ell(S_{u,v})$ for $0 \le \ell \le i$; this can be seen in Figure 2.18 (to draw this picture, we choose $i = 2$). In particular, for a word $w$ of length $n$, then $\binom{uw}{vw} \equiv 1 \pmod 2$, and by Lemma 2.21, at least one of the two binomial coefficients $\binom{uw0}{vw}$, $\binom{uw1}{vw}$ is odd (roughly, in Table 2.17, under a value 1, there is always at least a value 1). Iterating this argument $j$ times, we conclude that at least one of the $2^j$ binomial coefficients of the form $\binom{uwz}{vw}$, with $z \in \{0, 1\}^j$, is odd for all $0 \le j \le i$. In other words, at least one of the square regions

$$(0.0^{i+|u|-|v|}vw, 0.0^{i-j}uwz) + \left[0, \frac{1}{2^{n+i+|u|}}\right]^2, 0 \le j \le i \text{ and } z \in \{0, 1\}^j, \tag{2.8}$$

is a subset of $\mathcal{U}^2_{n+i+|u|}$ (recall $|vw| \leq |uwz| = |u| + n + j \leq |u| + n + i$). We observe that each of the square regions of the form (2.8) is intersected by $h^j(c^i(S_{u,v}))$. Indeed, the latter segment has slope $2^j$ and

$$(0.0^{i+|u|-|v|}v, 0.0^{i-j}u) \text{ and } (0.0^{i+|u|-|v|}v111\cdots, 0.0^{i-j}u111\cdots)$$

as endpoints. This can be visualized in Figure 2.18 where each rectangular gray region contains at least one square region from $\mathcal{U}^2_{n+i+|u|}$. Consequently, every point of $h^j(c^i(S_{u,v}))$ is at distance at most $2^j/2^{n+i+|u|}$ from a point in $\mathcal{U}^2_{n+i+|u|}$. In particular, this holds for $h^j(c^i((x''_0, y''_0)))$. We now choose $N_2$ such that $2^j/2^{N_2+i+|u|} < \epsilon/2$. Hence, for all $n \geq N_2 + i + |u|$,

$$d(h^j(c^i((x''_0, y''_0))), \mathcal{U}^2_n) < \epsilon/2. \tag{2.9}$$

To conclude the proof, for all $n \geq N_2 + i + |u|$, we have $d((x, y), \mathcal{U}^2_n) < \epsilon$ from (2.7) and (2.9). □

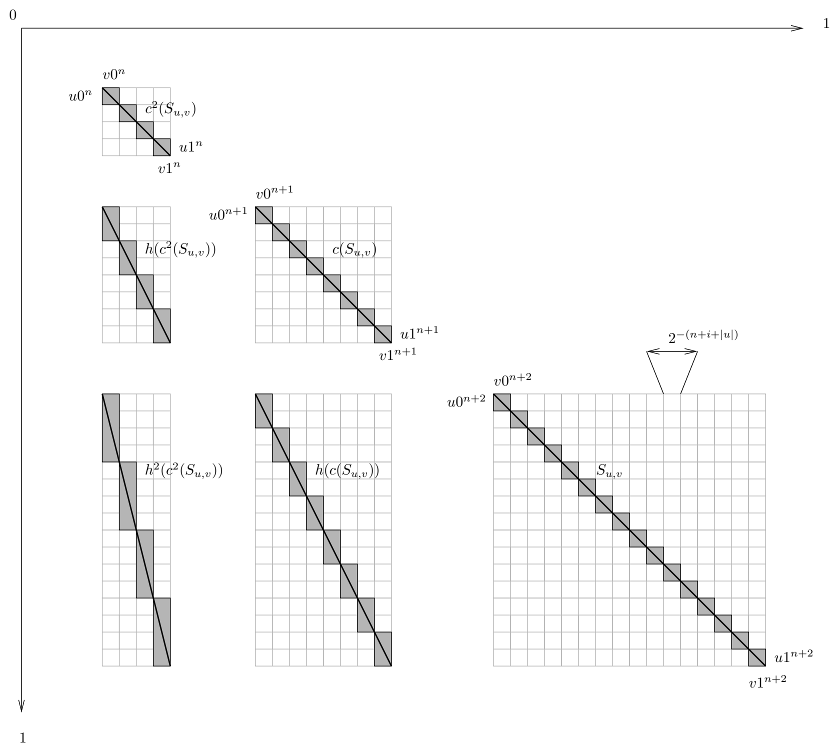



Figure 2.18: Situation occurring in the proof of Lemma 2.22.

The next result follows with no difficulty.

**Corollary 2.23.** *Let* $(u,v) \in L_2 \times L_2$ *satisfying* $(\star)$ *and let* $0 \le j \le i$. *For every point* $(f,g)$ *of the segment* $h^j(c^i(S_{u,v}))$, *there exists a sequence* $((f_n,g_n))_{n\ge 0}$ *converging to* $(f,g)$ *such that* $(f_n,g_n) \in \mathcal{U}_n^2$ *for all* $n \ge 0$.

*Proof*. Let $(f,g)$ be a point of the segment $h^j(c^i(S_{u,v}))$. By definition, this point belongs to $\mathcal{L}^2$. Let $\epsilon > 0$. From Lemma 2.22, we have

$$d((f,g),\mathcal{U}_m^2) < \epsilon$$

for all sufficiently large $m$. When $m$ is big enough, pick $(f_m,g_m) \in \mathcal{U}_m^2$ such that $d((f,g),(f_m,g_m)) < \epsilon$. Consequently, we can build a sequence $((f_n,g_n))_{n\ge 0}$ converging to $(f,g)$ such that $(f_n,g_n) \in \mathcal{U}_n^2$ for all $n \ge 0$. □

We are ready to prove the main result of this section.

**Theorem 2.24.** *The sequence* $(\mathcal{U}_n^2)_{n\ge 0}$ *converges to* $\mathcal{L}^2$ *with respect to the Hausdorff distance.*

*Proof*. Let $\epsilon > 0$. From Lemma 2.20, it suffices to show that $\mathcal{L}^2 \subset [\mathcal{U}_n^2]_\epsilon$ for all sufficiently large $n \ge 0$. For all $(x,y) \in \mathcal{L}^2$, using Corollary 2.23, there exists a (Cauchy) sequence $((f_i(x,y),g_i(x,y))_{i\ge 0}$ such that $(f_i(x,y),g_i(x,y))$ belongs to $\mathcal{U}_i^2$ for all $i$, and there exists $N_{(x,y)}$ such that, for all $i,j \ge N_{(x,y)}$,

$$d((f_i(x,y),g_i(x,y)),(f_j(x,y),g_j(x,y))) < \epsilon/2 \tag{2.10}$$

and

$$d((f_i(x,y),g_i(x,y)),(x,y)) < \epsilon/2. \tag{2.11}$$

From (2.11), we trivially have

$$\mathcal{L}^2 \subset \bigcup_{(x,y)\in\mathcal{L}^2} B((f_{N_{(x,y)}}(x,y),g_{N_{(x,y)}}(x,y)),\epsilon/2).$$

Since $\mathcal{L}^2$ is compact, we can extract a finite covering: there exist a positive integer $k$ and $(x_1,y_1),\dots,(x_k,y_k)$ in $\mathcal{L}^2$ such that

$$\mathcal{L}^2 \subset \bigcup_{j=1}^{k} B((f_{N_{(x_j,y_j)}}(x_j,y_j),g_{N_{(x_j,y_j)}}(x_j,y_j)),\epsilon/2).$$

Let $N = \max_{j=1,\dots,k} N_{(x_j,y_j)}$. For all $j \in \{1,\dots,k\}$ and all $n \ge N$, we deduce from (2.10) that

$$B((f_{N_{(x_j,y_j)}}(x_j,y_j),g_{N_{(x_j,y_j)}}(x_j,y_j)),\epsilon/2) \subset B((f_n(x_j,y_j),g_n(x_j,y_j),\epsilon),$$

and therefore

$$\mathcal{L}^2 \subset \bigcup_{j=1}^{k} B((f_n(x_j, y_j), g_n(x_j, y_j)), \epsilon) \subset [\mathcal{U}_n^2]_\epsilon,$$

as expected. □

### 2.1.5 Extension Modulo $p$

In Section 2.1.4, similarly to the construction of the Sierpiński gasket, we proved that colored sub-blocks $(\mathcal{U}_n^2)_{n\geq 0}$ of the generalized Pascal triangle $\mathrm{P}_2$ tends to $\mathcal{L}^2$ with respect to the Hausdorff distance (see Theorem 2.24). For the sake of simplicity, the presentation was restricted to the case of odd binomial coefficients. Nevertheless, the reasonings, constructions and results can be adapted to the more general setting of congruences modulo a prime $p$. Note that since we make use of Lucas' theorem (that is, Theorem 1.40), we limit ourselves to congruences modulo a prime. In this section, we briefly sketch the main differences with the case $p = 2$.

First, we can extend Definition 2.2, and introduce the corresponding sets $\mathcal{U}_{n,p,r}^2$ as in Definition 2.6.

**Definition 2.25.** Let $p$ be a fixed prime number, and let $r \in \{1, \ldots, p-1\}$ be a positive residue. Consider the sequence $(\mathcal{T}_{n,p,r}^2)_{n\geq 0}$ of sets in $\mathbb{R}^2$ defined for all $n \geq 0$ by

$$\mathcal{T}_{n,p,r}^2 = \bigcup_{\substack{u,v \in L_2^{\leq n} \\ \binom{u}{v} \equiv r \pmod{p}}} \mathrm{val}_2(v, u) + [0, 1]^2.$$

As before, each $\mathcal{T}_{n,p,r}^2$ is a compact subset of $[0, 2^n]^2$. Let $(\mathcal{U}_{n,p,r}^2)_{n\geq 0}$ be the sequence of compact sets in $[0, 1]^2$ defined for all $n \geq 0$ by

$$\mathcal{U}_{n,p,r}^2 = \frac{1}{2^n} \mathcal{T}_{n,p,r}^2.$$

In Figure 2.19, we consider the case $p = 3$ and $r = 2$, and the set $\mathcal{U}_{7,3,2}^2$ is depicted on the left (note that the right side of this figure will become clear in a moment). Then the $(\star)$ condition of Definition 2.7 becomes $(\star)_{p,r}$.

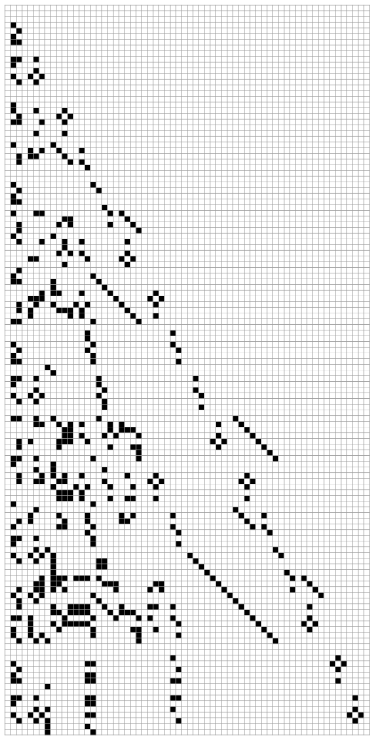


Figure 2.19: The set $\mathcal{U}^2_{7,3,2}$ and an approximation of the corresponding set $\mathcal{L}^2_{3,2}$.

**Definition 2.26.** Let $(u, v) \in L_2 \times L_2$. We say that $(u, v)$ satisfies the $(\star)_{p,r}$ *condition* or simply $(\star)_{p,r}$ if $(u, v) \neq (\varepsilon, \varepsilon)$,

$$\binom{u}{v} \equiv r \pmod{p}, \ \binom{u}{v0} = 0 \text{ and } \binom{u}{v1} = 0.$$

Note that the pairs $(u, v)$ satisfying this condition depend on the choice of $p$ and $r$. For example, the pairs $(110, 10)$ and $(11010110, 11110)$ both satisfy $(\star)_{3,2}$ but not $(\star)_{3,1}$.

Notice that Lemma 2.9 still holds, so does Remark 2.10. The sequence $(\mathcal{A}^2_{n,p,r})_{n\geq 0}$ of sets is defined similarly as in Definitions 2.12 and 2.15, and (2.6) is still valid.

**Remark 2.27.** The pair $(1, 1)$ satisfies $(\star)_{p,r}$ if and only if $r = 1$. Thus, Lemma 2.17 is true only if $r = 1$.

We may again apply Proposition 1.60 to the sequence $(\mathcal{A}^2_{n,p,r})_{n\geq 0}$, and we let $\mathcal{L}^2_{p,r} = \overline{\bigcup_{n\geq 0} \mathcal{A}^2_{n,p,r}}$ denote its compact limit.

**Example 2.28.** In Figure 2.19, we have represented the set $\mathcal{U}^2_{7,3,2}$ when considering binomial coefficients congruent to 2 modulo 3 and an approximation of the limit set $\mathcal{L}^2_{3,2}$ proceeding as in Example 2.19. Similarly, in Figure 2.20, we have depicted the superimposition of approximations of the limit sets $\mathcal{L}^2_{3,1}$ in orange and $\mathcal{L}^2_{3,2}$ in black. Note that the sets $\mathcal{L}^2_{3,1} \cup \mathcal{L}^2_{3,2}$ and $\mathcal{L}^2_{2,1} = \mathcal{L}^2$ are different.

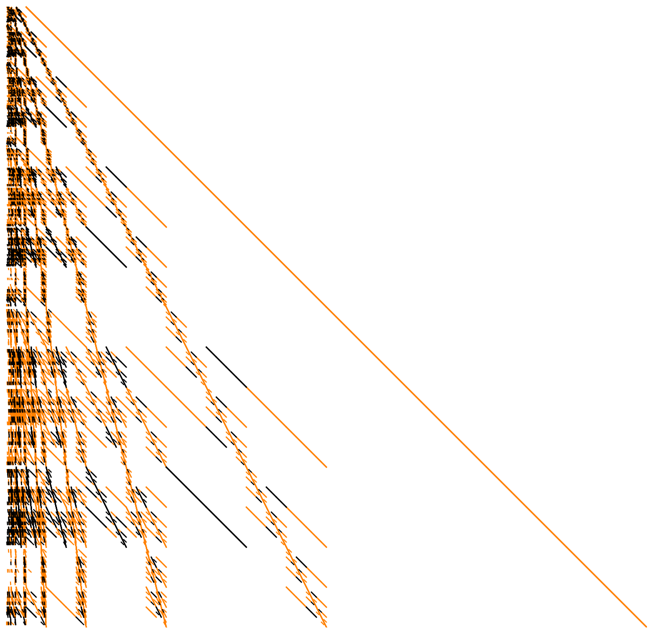


Figure 2.20: The superimposition of approximations of the limit set $\mathcal{L}^2_{3,1}$ in orange and the limit set $\mathcal{L}^2_{3,2}$ in black.

The proof of the analogue of Lemma 2.20 follows the same lines. We simply have to replace the word $u0^{2^k}1$ (resp., $v0^{2^k}1$) with $u0^{p^k}1$ (resp., $v0^{p^k}1$), and then we apply Lucas' theorem with base-$p$ expansions. Also notice that the some cases of that proof can be forgotten if $r \neq 1$.

Analogously to Lemma 2.21, one can observe that if $\binom{u}{vb} \equiv r \pmod p$ for $b \in \{0,1\}$, then there exists $a \in \{0,1\}$ such that $\binom{ua}{vb} \equiv r \pmod p$. This observation is useful to adapt the proof of Lemma 2.22.

Finally, gathering all these extended results allows us to obtain the following theorem.

**Theorem 2.29.** *Let $p$ be a prime and $r \in \{1, \dots, p-1\}$ be a positive residue. When considering binomial coefficients congruent to $r \pmod p$, the sequence $(\mathcal{U}^2_{n,p,r})_{n\geq 0}$ converges to $\mathcal{L}^2_{p,r}$ with respect to the Hausdorff distance.*

## 2.2 Results for Parry–Bertrand Numeration Systems

As mentioned in the introduction of this chapter, the idea is now to adapt the results from Section 2.1 to the more general framework of Parry–Bertrand numeration systems. Compared to the base-2 case (and more generally to the integer base case), new technicalities have to be taken into account to generalize the convergence of Pascal-like triangles to this larger class of numeration systems. However, we will sometimes omit details that are similar to both cases. A noteworthy difference with the base-2 case is that in this section, empty words are allowed in the corresponding combinatorial ($\star$) condition.

The particular setting of this section is the following one: we let $\beta \in \mathbb{R}_{>1}$ be a Parry number, and we constantly use the special Parry–Bertrand numeration $U_\beta$ from Definition 1.29. Recall from Section 1.3 that the alphabet $A_{U_\beta}$ of the system of numeration $U_\beta$ is the set $\{0, 1, \dots, \lceil \beta \rceil - 1\}$, and its numeration language $L_{U_\beta} \subset A^*_{U_\beta}$ can be derived from the automaton $\mathcal{A}_\beta$ in Proposition 1.24. Another essential property of this particular numeration system is the Bertrand condition, which allows us to delete or add trailing zeroes to valid representations. The object we study in this section is the generalized Pascal triangle $\mathrm{P}_\beta$ from Definition 1.42.

Let us consider a grid of unit squares at the intersection of $\mathbb{N}^2$ and $[0, U_\beta(n)]^2$ for all $n \in \mathbb{N}$. The first $U_\beta(n)$ rows and columns

$$\left( \binom{\mathrm{rep}_{U_\beta}(i)}{\mathrm{rep}_{U_\beta}(j)} \bmod 2 \right)_{0 \leq i,j < U_\beta(n)}$$

of the generalized Pascal triangle $\mathrm{P}_\beta$ modulo 2 give a coloring of this grid, regarding the parity of the corresponding binomial coefficients. As before, this construction defines a sequence of compact subsets of $\mathbb{R}^2$. If we normalize these sets respectively by a homothety of ratio $1/U_\beta(n)$, we define a sequence

$(\mathcal{U}_n^\beta)_{n\geq 0}$ of subsets of $[0,1]^2$ (see Definition 2.30). We show that it converges, with respect to the Hausdorff distance, to a limit set described using a simple combinatorial property extending the one from Definition 2.7.

### 2.2.1 The Prettiest $(\star)$

As for the base-2 case with Definition 2.6, we consider a sequence of compact sets that are built on sub-blocks of the generalized Pascal triangle $\mathrm{P}_\beta$. Recall that $\mathrm{val}_{U_\beta}$ was extended to take into account pairs of words at the beginning of Section 2.1.1.

**Definition 2.30.** We consider the sequence $(\mathcal{U}_n^\beta)_{n\geq 0}$ of compact subsets of $[0,1]^2$ defined for all $n \geq 0$ by

$$\mathcal{U}_n^\beta = \frac{1}{U_\beta(n)} \left( \bigcup_{\substack{u,v \in L_{U_\beta}^{\leq n} \\ \binom{u}{v} \equiv 1 \pmod 2}} \mathrm{val}_{U_\beta}(v,u) + [0,1]^2 \right).$$

As in the base-2 case, each pair $(u,v) \in L_{U_\beta} \times L_{U_\beta}$ of words of length at most $n$ with an odd binomial coefficient gives rise to a square region in $\mathcal{U}_n^\beta$ as depicted in Figure 2.21.

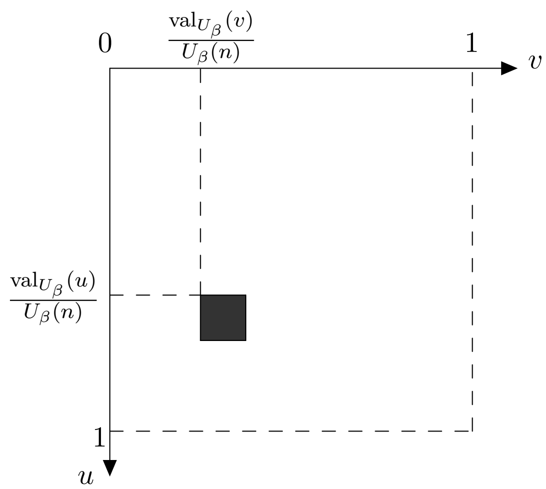
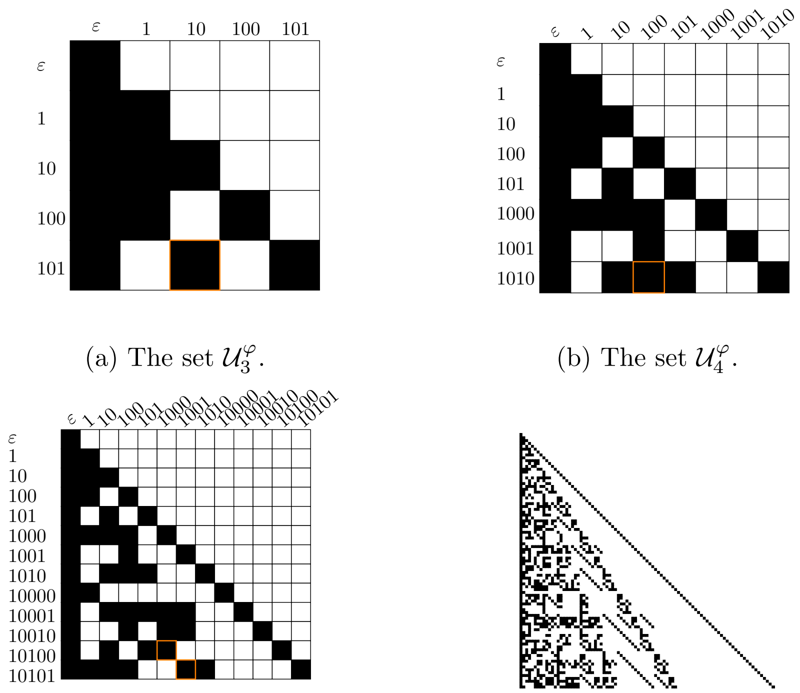



Figure 2.21: Visualization of a square region in $\mathcal{U}_n^\beta$.

**Example 2.31.** When $\beta$ is the golden ratio $\varphi$, the first values in the generalized Pascal triangle $\mathrm{P}_\varphi$ were given in Table 1.5. In this table, one can identify the odd binomial coefficients among all displayed values, which correspond to square units in some sets $\mathcal{U}_n^\varphi$. For instance, the sets $\mathcal{U}_3^\varphi$, $\mathcal{U}_4^\varphi$, $\mathcal{U}_5^\varphi$ and $\mathcal{U}_9^\varphi$ are depicted in Figure 2.22. Note that, for the sake of clarity, in $\mathcal{U}_9^\varphi$, we do not draw the grid nor the corresponding words. By definition, also observe that we find $\mathcal{U}_3^\varphi$ as a smaller left top portion of $\mathcal{U}_4^\varphi$. This observation is general.




(a) The set $\mathcal{U}_3^\varphi$. (b) The set $\mathcal{U}_4^\varphi$.

(c) The set $\mathcal{U}_5^\varphi$. (d) The set $\mathcal{U}_9^\varphi$.

Figure 2.22: The sets $\mathcal{U}_3^\varphi$, $\mathcal{U}_4^\varphi$, $\mathcal{U}_5^\varphi$ and $\mathcal{U}_9^\varphi$ when $\beta$ is the golden ratio $\varphi$.

Our aim is to show that the sequence $(\mathcal{U}_n^\beta)_{n\geq 0}$ of compact subsets of $[0,1]^2$ is converging and to provide a description of its limit set. The idea behind this description is the following one.

Let $(u,v)$ be a pair of words in $L_{U_\beta}$ having an odd binomial coefficient. Some of those pairs are such that $\binom{ua}{va} \equiv 0 \pmod 2$ for all letters $a$ such that

$ua, va \in L_{U_\beta}$. In fact, they create a black square region in $\mathcal{U}^\beta_{|u|}$ while the corresponding square region in $\mathcal{U}^\beta_{|u|+1}$ is white. As an example, take $\beta = \varphi$, $u = 1010$ and $v = 101$. The only authorized letter is $a = 0$, and we have $\binom{u0}{v0} = 2$ (see Figure 2.22).

To the contrary, some of those pairs create a more stable pattern, *i.e.*, $\binom{uw}{vw} \equiv 1 \pmod 2$ for all words $w$ such that $uw, vw \in L_{U_\beta}$. Roughly, they create a diagonal of square regions in $(\mathcal{U}^\beta_n)_{n\geq 0}$. For instance, take $\beta = \varphi$, $u = 101$ and $v = 10$. In this case, $\binom{uw}{vw} \equiv 1 \pmod 2$ for all admissible words $w$. In particular, the pairs $(u, v)$, $(u0, v0)$,$(u00, v00)$ and $(u01, v01)$ have odd binomial coefficients (in Figure 2.22, they are highlighted in orange), and create a diagonal of square regions. This is exactly what happened in the base-2 case; recall Remark 2.10.

With the second type of pairs of words, we define a new sequence $(\mathcal{A}^\beta_n)_{n\geq 0}$ of compact subsets of $[0, 1]^2$, which converges to a well-understood limit set $\mathcal{L}^\beta$ with respect to the Hausdorff distance (see Definition 2.49). Then we show that the first sequence $(\mathcal{U}^\beta_n)_{n\geq 0}$ of compact sets also converges to this limit set with respect to the Hausdorff distance (see Theorem 2.59). The remaining of this chapter is dedicated to formalize and prove those statements.

To reach this goal, for all non-empty words $u, v \in L_{U_\beta}$, we first define the least integer $p$ such that $u0^pw, v0^pw$ belong to $L_{U_\beta}$ for all words $w$ in $0^*L_{U_\beta}$. In other terms, any word $w$ can be read after $u0^p$ and $v0^p$ in the automaton $\mathcal{A}_\beta$ from Proposition 1.24. Then we prove that some pairs of words $(u, v) \in L_{U_\beta} \times L_{U_\beta}$ have the property that not only $\binom{u}{v} \equiv 1 \pmod 2$ but also $\binom{u0^pw}{v0^pw} \equiv 1 \pmod 2$ for all words $w \in 0^*L_{U_\beta}$; see Corollary 2.41. As shown in Remark 2.43, such a property creates a particular pattern occurring in $\mathcal{U}^\beta_n$ for all sufficiently large $n$.

**Proposition 2.32.** *For all non-empty words $u, v \in L_{U_\beta}$, there exists $p \geq 0$ such that*

$$(u0^p)^{-1}L_{U_\beta} = (v0^p)^{-1}L_{U_\beta} = 0^*L_{U_\beta}. \tag{2.12}$$

*Proof*. Using Proposition 1.24, take $p$ such that $\delta(a_0, u0^p) = a_0 = \delta(a_0, v0^p)$. For each $w \in \{u, v\}$, $(w0^p)^{-1}L_{U_\beta}$ is the set of the words accepted by $\mathcal{A}_\beta$ (from the initial state), *i.e.*, $(w0^p)^{-1}L_{U_\beta} = 0^*L_{U_\beta}$, as desired. □

**Definition 2.33.** For all non-empty words $u, v \in L_{U_\beta}$, we let $p(u,v)$ denote the least non-negative integer $p(u,v)$ such that (2.12) holds.

When $u = v = \varepsilon$, Proposition 1.24 shows that $\delta(a_0, \varepsilon) = a_0$. Thus, we have $p(\varepsilon, \varepsilon) = 0$ and $(\varepsilon 0^{p(\varepsilon,\varepsilon)})^{-1} L_{U_\beta} = L_{U_\beta}$.

**Example 2.34.** We make use of Example 1.25. If $\beta > 1$ is an integer, then $p(u,v) = 0$ for all $u, v \in L_{U_\beta}$. If $\beta$ is the golden ratio $\varphi$, then $p(u,v) = 0$ if and only if $u$ and $v$ end with 0 or $u = v = \varepsilon$, otherwise $p(u,v) = 1$.

The integer of Definition 2.33 can be greater than 1 as illustrated in the following example.

**Example 2.35.** Let $\beta$ be the dominant root of the polynomial $P(X) = X^4 - 2X^3 - X^2 - 1$. Then $\beta \approx 2.47098$ is a Parry number with $d_\beta(1) = 2101$ and $d^*_\beta(1) = (2100)^\omega$. The automaton $\mathcal{A}_\beta$ is depicted in Figure 2.23. For instance, $p(101, 21) = 2$. Observe that $p(u,v) \leq 2$ for all words $u, v \in L_{U_\beta}$.

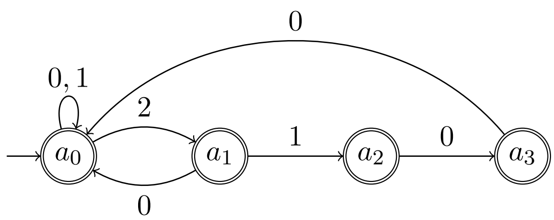



Figure 2.23: The automaton $\mathcal{A}_\beta$ for the dominant root $\beta$ of the polynomial $P(X) = X^4 - 2X^3 - X^2 - 1$.

**Definition 2.36.** Let $(u,v) \in L_{U_\beta} \times L_{U_\beta}$. We say that $(u,v)$ satisfies the $(\star)$ *condition* or simply $(\star)$ if either $u = v = \varepsilon$, or $|u| \geq |v| > 0$ and

$$\binom{u0^{p(u,v)}}{v0^{p(u,v)}} \equiv 1 \pmod 2 \quad \text{and} \quad \binom{u0^{p(u,v)}}{v0^{p(u,v)}a} = 0 \text{ for all } a \in A_{U_\beta},$$

where $p(u,v)$ comes from Definition 2.33.

**Remark 2.37.** When $\beta = 2$, the $(\star)$ condition from Definition 2.7 is slightly different from the $(\star)$ condition defined above. Indeed, in the previous definition, we allow $u$ and $v$ to be empty words at the same time[2]. In this

[2] The reader might be puzzled by this slight difference. In a first attempt to understand

particular case, $p(u,v)=0$ and $v0^{p(u,v)}a \in L_{U_\beta}$ for all $a \in A_{U_\beta} \setminus \{0\}$. Now if $v \neq \varepsilon$, then $v0^{p(u,v)}a \in L_{U_\beta}$ for all $a \in A_{U_\beta}$. It is also worth noticing that if only one of the two words $u$ or $v$ is empty, then the pair $(u,v)$ never satisfies $(\star)$.

The next easy lemma shows that all diagonal elements of $\mathcal{U}_n^\beta$ satisfy $(\star)$.

**Lemma 2.38.** *For any word $u \in L_{U_\beta}$, the pair $(u,u)$ satisfies $(\star)$.*

*Proof.* If $u=\varepsilon$, the result is clear using Definition 2.36. Suppose $u$ is non-empty, and let $p=p(u,u)$. Then we get $\binom{u0^p}{u0^p}=1\equiv 1 \pmod 2$, and for all $a \in A_{U_\beta}$, $\binom{u0^p}{u0^p a}=0$ for we have $|u0^p a| > |u0^p|$. □

If a pair of words satisfies $(\star)$, it has the following two properties. First, as stated in Proposition 2.39, its binomial coefficient is odd. Secondly, it creates a special pattern in $\mathcal{U}_n^\beta$ for all large enough $n$; see Proposition 2.40, Corollary 2.41 and Remark 2.43. These are the extended versions of Lemma 2.9 and Remark 2.10.

**Proposition 2.39.** *Let $(u,v) \in L_{U_\beta} \times L_{U_\beta}$ be a pair of words satisfying $(\star)$. Then $\binom{u}{v} \equiv 1 \pmod 2$.*

*Proof.* If $u=v=\varepsilon$, the result is clear by definition. Suppose that $u$ and $v$ are non-empty. Let us proceed by contradiction and suppose that $\binom{u}{v}$ is even. For the sake of clarity, let us set $p=p(u,v)$. On the one hand, by Definition 2.36, we know that $\binom{u0^p}{v0^p} \equiv 1 \pmod 2$, and on the other hand, Lemma 1.39 states that

$$\binom{u0^p}{v0^p} = \sum_{j=0}^{p} \binom{p}{j}\binom{u}{v0^j} = \sum_{j=1}^{p} \binom{p}{j}\binom{u}{v0^j} + \binom{u}{v}.$$

Consequently, we have

$$\sum_{j=1}^{p} \binom{p}{j}\binom{u}{v0^j} \equiv 1 \pmod 2 > 0,$$

the convergence in the base-2 case in 2015 [LRS16], we restricted ourselves to non-empty words because it was easier to compare associated segments. Later, when the question of generalizations to $\beta$-numeration systems arose in 2018 [Sti19], we realized that this restriction was superfluous. In this text, I chose to stay faithful to both papers, and thus stick to both definitions.

and there must exist $i \in \{1, \ldots, p\}$ such that $\binom{u}{v0^i} > 0$. Using Lemma 1.39 again, we also have

$$\binom{u0^p}{v0^p0} = \sum_{j=0}^{p} \binom{p}{j} \binom{u}{v00^j} = \sum_{j=1}^{p+1} \binom{p}{j-1} \binom{u}{v0^j} \geq \binom{p}{i-1} \binom{u}{v0^i} > 0,$$

which contradicts Definition 2.36. □

**Proposition 2.40.** *Let $(u, v)$ be a pair of non-empty words in $L_{U_\beta}$ satisfying $(\star)$. For any letter $a \in A_{U_\beta}$, the pair $(u0^{p(u,v)}a, v0^{p(u,v)}a)$ of words in $L_{U_\beta}$ satisfies $(\star)$. Furthermore, for any distinct letters $a, b \in A_{U_\beta}$, the binomial coefficient of the pair $(u0^{p(u,v)}a, v0^{p(u,v)}b)$ of words in $L_{U_\beta}$ is equal to 0.*

*Proof.* Set $p = p(u, v)$. By definition of $p$, observe that the words $u0^pa, v0^pa$ belong to $L_{U_\beta}$ for any letter $a \in A_{U_\beta}$. Let $a$ be a letter in $A_{U_\beta}$, and also set $p' = p(u0^pa, v0^pa)$. By combining Lemmas 1.38 and 1.39, we find

$$\begin{aligned}
\binom{u0^pa0^{p'}}{v0^pa0^{p'}} &= \sum_{j=0}^{p'} \binom{p'}{j} \binom{u0^pa}{v0^pa0^j} \\
&= \sum_{j=1}^{p'} \binom{p'}{j} \binom{u0^pa}{v0^pa0^j} + \binom{u0^pa}{v0^pa} \\
&= \sum_{j=1}^{p'} \binom{p'}{j} \binom{u0^pa}{v0^pa0^j} + \binom{u0^p}{v0^pa} + \binom{u0^p}{v0^p}.
\end{aligned}$$

Since $(u, v)$ satisfies $(\star)$, $\binom{u0^p}{v0^pa} = 0$. We now show that all the coefficients $\binom{u0^pa}{v0^pa0^j}$, for $j = 1, \ldots, p'$, are also 0. Let $1 \leq j \leq p'$. From Lemma 1.38, we know that

$$\binom{u0^pa}{v0^pa0^j} = \binom{u0^p}{v0^pa0^j} + \delta_{a,0} \binom{u0^p}{v0^pa0^{j-1}}.$$

Clearly, the first term $\binom{u0^p}{v0^pa0^j}$ must be 0. Indeed, otherwise it means that the word $v0^pa$ appears as a scattered subword of the word $u0^p$, which contradicts $(\star)$. The second term $\binom{u0^p}{v0^pa0^{j-1}}$ only appears if $a = 0$. In that case, this term becomes $\binom{u0^p}{v0^p0^j} = 0$, for otherwise there is an occurrence of the word $v0^p0$ in $u0^p$, contradicting $(\star)$. Consequently, using Definition 2.36, we get

$$\binom{u0^pa0^{p'}}{v0^pa0^{p'}} = \binom{u0^p}{v0^p} \equiv 1 \pmod{2}.$$

Using the same type of argument, for any letter $b \in A_{U_\beta}$, we also have

$$\binom{u0^pa0^{p'}}{v0^pa0^{p'}b} = 0.$$

Thus, $(u0^pa, v0^pa)$ satisfies $(\star)$, as claimed.

The second part of the statement follows by Lemma 1.38 and by the fact that $(u,v)$ satisfies $(\star)$. $\square$

**Corollary 2.41.** *Let $u, v \in L_{U_\beta}$ be two non-empty words such that $(u,v)$ satisfies $(\star)$. Then*

$$\binom{u0^{p(u,v)}w}{v0^{p(u,v)}w} \equiv 1 \pmod 2 \quad \textit{and} \quad \binom{u0^{p(u,v)}w}{v0^{p(u,v)}w'} = 0$$

*for all $w, w' \in 0^*L_{U_\beta}$ with $|w| = |w'|$ and $w' \neq w$.*

*Proof.* Set $p = p(u,v)$. From Proposition 2.32, $u0^pw, v0^pw$ belong to $L_{U_\beta}$ for any word $w \in 0^*L_{U_\beta}$.

Let us prove the first part by induction on the length of $w \in 0^*L_{U_\beta}$. If $|w| = 0$, then $w = \varepsilon$ is the empty word, and the statement is true using Definition 2.36. If $|w| = 1$, then $w = a$ is a letter belonging to $A_{U_\beta}$. By Proposition 2.40, we know that $(u0^pa, v0^pa)$ satisfies $(\star)$, and Proposition 2.39 implies that $\binom{u0^pa}{v0^pa} \equiv 1 \pmod 2$. Now suppose that $|w| \geq 2$ and write $w = atb$ where $a, b$ are letters. From Lemma 1.38, we deduce that

$$\binom{u0^pw}{v0^pw} = \binom{u0^pat}{v0^patb} + \binom{u0^pat}{v0^pat}.$$

By induction hypothesis, $\binom{u0^pat}{v0^pat} \equiv 1 \pmod 2$ since $at \in 0^*L_{U_\beta}$ and also $|at| < |w|$. Furthermore, $\binom{u0^pat}{v0^patb}$ must be 0, otherwise it means that the word $v0^pa$ occurs as a scattered subword of the word $u0^p$, which contradicts the fact that $(u,v)$ satisfies $(\star)$. In conclusion, $(u0^pw, v0^pw)$ has an odd binomial coefficient, as desired.

Let us now prove the second part of the statement by induction on the length of $w, w' \in 0^*L_{U_\beta}$. If $|w| = |w'| = 1$, then $w = a$ and $w' = b$ are distinct letters belonging to $A_{U_\beta}$. The result follows from Proposition 2.40. Now suppose that $|w| = |w'| \geq 2$, and assume that the result holds for shorter words taken as in the statement. As a first case, suppose that $w = sa$ and

$w' = s'b$ with $a, b \in A_{U_\beta}$ and $a \neq b$. From Lemma 1.38,

$$\binom{u0^p w}{v0^p w'} = \binom{u0^p sa}{v0^p s'b} = \binom{u0^p s}{v0^p s'b} = 0.$$

Indeed, if the latter coefficient were positive, then it means that $v0^p s'b$ is a scattered subword of $u0^p s$. In this case, if we let $c \in A_{U_\beta}$ denote the first letter of $s'$, then $v0^p c$ is a scattered subword of $u0^p$, which contradicts the fact that $(u, v)$ satisfies $(\star)$. As a second case, suppose that $w = sa$ and $w' = s'a$ with $a \in A_{U_\beta}$, $s, s' \in 0^* L_{U_\beta}$ and $s \neq s'$. From Lemma 1.38,

$$\binom{u0^p w}{v0^p w'} = \binom{u0^p sa}{v0^p s'a} = \binom{u0^p s}{v0^p s'a} + \binom{u0^p s}{v0^p s'}.$$

By induction hypothesis, $\binom{u0^p s}{v0^p s'} = 0$ since $|s| = |s'| < |w| = |w'|$. By a reasoning similar to the one developed above, $\binom{u0^p s}{v0^p s'a} = 0$, otherwise it violates the fact that $(u, v)$ satisfies $(\star)$. All in all, we have just showed that $(u0^p w, v0^p w')$ has a binomial coefficient equal to 0. This ends the proof. □

The next lemma is useful to characterize the pattern created in $\mathcal{U}_n^\beta$, for all sufficiently large $n$, by pairs of words satisfying $(\star)$; see Remark 2.43 below. In the following statement, we make use of the convention given in Definition 1.19. Note that, in the base-2 case, and more generally in the integer base case, this result is easy because dividing by a term of $U_\beta$ roughly shifts the values or the words to the right.

**Lemma 2.42.** *Let $(u, v) \in L_{U_\beta} \times L_{U_\beta}$ satisfying $(\star)$.*

- *The sequence*

$$\left(\frac{1}{U_\beta(|u| + p(u, v) + n)} \ \mathrm{val}_{U_\beta}(v0^{p(u,v)+n}, u0^{p(u,v)+n})\right)_{n \geq 0}$$

  *converges to the pair of real numbers $(0.0^{|u|-|v|}v, 0.u)$.*

- *For all $n \geq 0$, let $d_n$ denote the length-$n$ prefix of $d_\beta^*(1)$. Then the sequence*

$$\left(\frac{1}{U_\beta(|u| + p(u, v) + n)} \ \mathrm{val}_{U_\beta}(v0^{p(u,v)}d_n, u0^{p(u,v)}d_n)\right)_{n \geq 0}$$

  *converges to the pair of real numbers*

$$(0.0^{|u|-|v|}v0^{p(u,v)}d_\beta^*(1), 0.u0^{p(u,v)}d_\beta^*(1)).$$

*Proof*. Let $(u, v) \in L_{U_\beta} \times L_{U_\beta}$ satisfying $(\star)$ and set $p = p(u, v)$. Note that $0 \le |v| \le |u|$. We prove the first item as the proof of the second one is similar. The result is true if $u = v = \varepsilon$. Suppose that $u$ and $v$ are non-empty words. Let us write $u = u_{|u|-1}u_{|u|-2}\cdots u_0$ where $u_i \in A_{U_\beta}$ for all $0 \le i < |u|$. By definition, we have

$$\frac{\mathrm{val}_{U_\beta}(u0^{p+n})}{U_\beta(|u|+p+n)} = \sum_{i=0}^{|u|-1} u_i \frac{U_\beta(i+p+n)}{U_\beta(|u|+p+n)}.$$

Using (1.2) on page 15, $U_\beta(i+p+n)/U_\beta(|u|+p+n)$ tends to $\beta^i/\beta^{|u|}$ when $n$ tends to infinity. Consequently,

$$\lim_{n\to+\infty} \frac{\mathrm{val}_{U_\beta}(u0^{p+n})}{U_\beta(|u|+p+n)} = \sum_{i=0}^{|u|-1} u_i \beta^{i-|u|} = 0.u.$$

Using the same reasoning on the word $v$, we conclude that the sequence

$$\left(\left(\frac{\mathrm{val}_{U_\beta}(v0^{p(u,v)+n})}{U_\beta(|u|+p(u,v)+n)}, \frac{\mathrm{val}_{U_\beta}(u0^{p(u,v)+n})}{U_\beta(|u|+p(u,v)+n)}\right)\right)_{n\ge 0}$$

converges to the pair of real numbers $(0.0^{|u|-|v|}v, 0.u)$. □

**Remark 2.43.** Let $(u, v) \in L_{U_\beta} \times L_{U_\beta}$ satisfying $(\star)$ and set $p = p(u, v)$. Suppose that $u$ and $v$ are non-empty (the case where $u = v = \varepsilon$ is similar: in the following, replace $0^*L_{U_\beta}$ by $L_{U_\beta}$ where needed). Using Corollary 2.41, the pair of words $(u0^p w, v0^p w)$ has an odd binomial coefficient for any word $w \in 0^*L_{U_\beta}$. In particular, the pair of words $(u0^p w, v0^p w)$ corresponds to a square region in $\mathcal{U}^\beta_{|u|+p+n}$ for all $w \in 0^*L_{U_\beta}$ such that $|w| = n \ge 0$. By definition, this region is

$$\frac{1}{U_\beta(|u|+p+n)} \mathrm{val}_{U_\beta}(v0^p w, u0^p w) + \left[0, \frac{1}{U_\beta(|u|+p+n)}\right]^2 \subset \mathcal{U}^\beta_{|u|+p+n}.$$

Using Lemma 2.42, when $w = 0^n$ (the smallest word of length $n$ in $0^*L_{U_\beta}$), the sequence

$$\left(\frac{1}{U_\beta(|u|+p+n)} \mathrm{val}_{U_\beta}(v0^{p+n}, u0^{p+n})\right)_{n\ge 0}$$

converges to the pair of real numbers $(0.0^{|u|-|v|}v, 0.u)$. This point will be the first endpoint of a segment associated with $u$ and $v$. See Definition 2.45.

Analogously, using Lemma 2.42, when $w = d_n$ is the length-$n$ prefix of $d^*_\beta(1)$ (the greatest word of length $n$ in $0^*L_{U_\beta}$), then the sequence

$$\left(\frac{1}{U_\beta(|u|+p+n)}\,\mathrm{val}_{U_\beta}(v0^pd_n, u0^pd_n)\right)_{n\geq 0}$$

converges to the pair of real numbers $(0.0^{|u|-|v|}v0^pd^*_\beta(1), 0.u0^pd^*_\beta(1))$. This point will be the second endpoint of the same segment associated with $u$ and $v$. See again Definition 2.45. As a consequence, the sequence of sets whose $n$th term is defined by

$$\bigcup_{w\in(0^*L_{U_\beta})^n} \frac{1}{U_\beta(|u|+p+n)}\left(\mathrm{val}_{U_\beta}(v0^pw, u0^pw) + [0,1]^2\right) \tag{2.13}$$

(in which we allow all length-$n$ words in $0^*L_{U_\beta}$) converges, with respect to the Hausdorff distance, to the diagonal of the square

$$\mathcal{Q} = (0.0^{|u|-|v|}v, 0.u) + \left[0, \frac{1}{\beta^{|u|+p}}\right]^2.$$

As a final comment, let us mention that Corollary 2.41 also implies that the pair of words $(u0^pw, v0^pw')$ does not correspond to a square region in $\mathcal{U}^\beta_{|u|+p+n}$ for words $w, w' \in 0^*L_{U_\beta}$ such that $|w| = |w'| = n \geq 0$ and $w \neq w'$. In other words, the only square regions of $(\mathcal{U}^\beta_{|u|+p+n})_{n\geq 0}$ in $\mathcal{Q}$ are located on the diagonal.

The reasoning of the previous remark is illustrated in the next example.

**Example 2.44.** As a first example, when $\beta = 2$, we find back the construction in Remark 2.10. As a second example, let us take $\beta$ to be the golden ratio $\varphi$. Let $u = 101$ and $v = 10$ (resp., $u' = 100 = v'$). Then $p(u,v) = 1$ (resp., $p(u',v') = 0$); see Example 2.34. Those pairs of words satisfy $(\star)$. The first few terms of the sequence of sets (2.13) are respectively depicted in Figure 2.24 and Figure 2.25. Observe that when $n$ tends to infinity, the union of those black squares in $\mathcal{U}^\varphi_{n+4}$ (resp., $\mathcal{U}^\varphi_{n+3}$) converges to the diagonal of $(0.0v, 0.u) + [0, 1/\varphi^4]^2$ (resp., $(0.v', 0.u') + [0, 1/\varphi^3]^2$).

### 2.2.2 Compact Sets Again

The observation made in Remark 2.43 leads to the definition of an initial set $\mathcal{A}^\beta_0$. The same technique is applied in Section 2.1.3. At first, let us define a

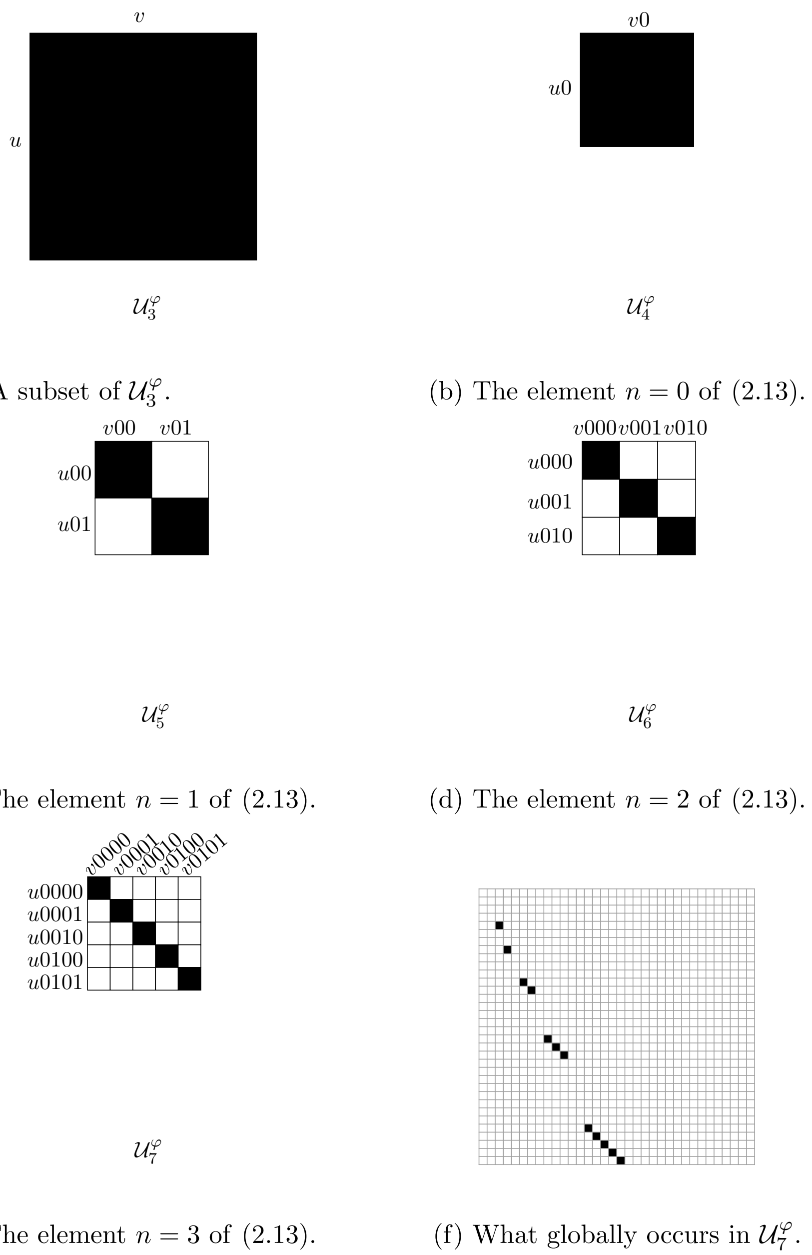
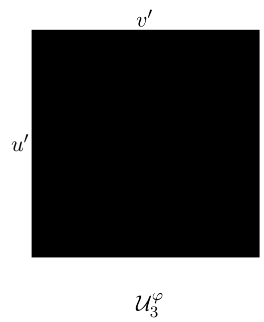
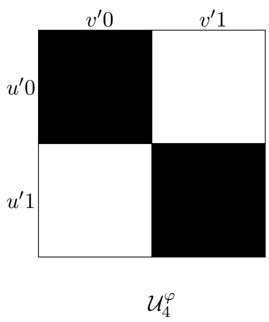


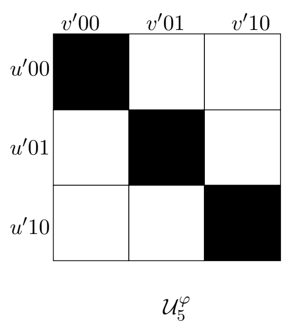

Figure 2.24: The first few terms of the sequence of sets (2.13) converging to the diagonal of the square $(0.0v, 0.u) + [0, 1/\varphi^4]^2$ for $u = 101$ and $v = 10$.



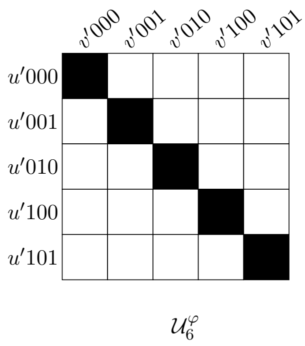

(a) The element $n = 0$ of (2.13).



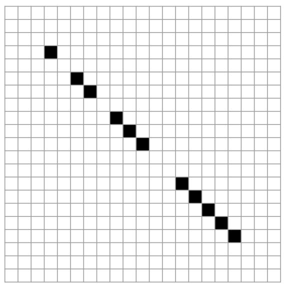

(b) The element $n = 1$ of (2.13).




(c) The element $n = 2$ of (2.13).




(d) The element $n = 3$ of (2.13).



(e) What globally happens in $\mathcal{U}_6^\varphi$.

Figure 2.25: The first few terms of the sequence of sets (2.13) converging to the diagonal of the square $(0.v', 0.u') + [0, 1/\varphi^3]^2$ for $u' = 100$ and $v' = 100$.

segment associated with a pair of words as in Definition 2.11[3].

**Definition 2.45.** Let $(u, v)$ in $L_{U_\beta} \times L_{U_\beta}$ such that $1 \leq |v| \leq |u|$ or $u$ and $v$ are both empty. We define a closed segment $S_{u,v}$ of slope 1 and of length $\sqrt{2} \cdot \beta^{-|u|-p(u,v)}$ in $[0,1]^2$. The endpoints of $S_{u,v}$ are given by $A_{u,v} = (0.0^{|u|-|v|}v, 0.u)$ and

$$\begin{aligned} B_{u,v} &= A_{u,v} + (\beta^{-|u|-p(u,v)}, \beta^{-|u|-p(u,v)}) \\ &= (0.0^{|u|-|v|}v0^{p(u,v)}d_\beta^*(1), 0.u0^{p(u,v)}d_\beta^*(1)). \end{aligned}$$

Observe that, if $u = v = \varepsilon$, the associated segment of slope 1 has endpoints $(0, 0)$ and $(1, 1)$. Otherwise, the segment $S_{u,v}$ lies in $[0, 1] \times [1/\beta, 1]$.

We now give the generalization of Definition 2.12.

**Definition 2.46.** Let us define the following compact set

$$\mathcal{A}_0^\beta = \overline{\bigcup_{\substack{(u,v) \\ \text{satisfying}(\star)}} S_{u,v}},$$

which is the closure of a countable union of segments of slope 1.

Definition 2.45 implies that $\mathcal{A}_0^\beta \subset [0,1]^2$. More accurately, we actually have $\mathcal{A}_0^\beta \setminus S_{\varepsilon,\varepsilon} \subset [0,1] \times [1/\beta, 1]$. Furthermore, observe that we take the closure of a union to ensure the compactness of the set. As for the base-2 case, accumulation points do exist in $\mathcal{A}_0^\beta$. It is not difficult to adapt the reasoning of Remark 2.14. Finally, as it was the case in the base-2 setting, the union of segments is not disjoint since some of them are included in others. For instance, for all $u \in L_{U_\beta}$, the pair $(u, u)$ satisfies $(\star)$ by Lemma 2.38, and $S_{u,u} \subset S_{\varepsilon,\varepsilon}$.

**Example 2.47.** Let $\beta = \varphi$. In Figure 2.26, the segment $S_{u,v}$ is drawn for all pairs $(u, v) \in L_{U_\varphi} \times L_{U_\varphi}$ satisfying $(\star)$ and such that $0 \leq |v| \leq |u| \leq 10$. We thus obtain an approximation of $\mathcal{A}_0^\varphi$.

[3] As in the footnote on page 68, there is a slim difference between Definitions 2.11 and 2.45. As already justified, $u$ and $v$ can simultaneously be empty words in the present section.

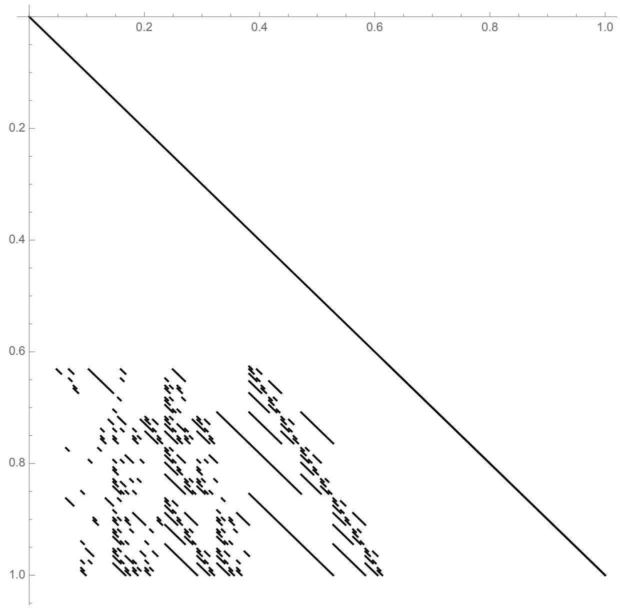



Figure 2.26: An approximation of $\mathcal{A}_0^\varphi$ computed with words of length $\leq 10$.

Analogously to Definition 2.15, we introduce another sequence of compact sets obtained by transforming the initial set $\mathcal{A}_0^\beta$ under iterations of two maps. As we will see, this new sequence allows us to properly define a limit set $\mathcal{L}^\beta$.

**Definition 2.48.** We let $c$ denote the homothety of center $(0, 0)$ and ratio $1/\beta$, and we consider the map $h\colon (x, y) \mapsto (x, \beta y)$. We define a sequence of compact sets by setting, for all $n \geq 0$,

$$\mathcal{A}_n^\beta = \bigcup_{\substack{0 \leq i \leq n \\ 0 \leq j \leq i}} h^j(c^i(\mathcal{A}_0^\beta)).$$

When the map $c$ is applied to a segment, it does not change its slope while $h$ multiplies it by $\beta$. As a consequence, since $\mathcal{A}_0^\beta$ contains segments of slope 1, then $\mathcal{A}_n^\beta$ contains segments of slopes $1, \beta, \beta^2, \ldots, \beta^n$ for all $n \geq 0$. Also note that by definition the sequence $(\mathcal{A}_n^\beta)_{n \geq 0}$ is increasingly nested, *i.e.*,

$$\mathcal{A}_0^\beta \subset \mathcal{A}_1^\beta \subset \mathcal{A}_2^\beta \subset \cdots.$$

As in Figure 2.13 that describes what happens in base 2, we apply $c$ and $h$ at most twice from $\mathcal{A}_0^\beta \setminus S_{\varepsilon,\varepsilon}$ in Figure 2.27. Using this figure, if $m, n \in \mathbb{N}$

satisfy $m \leq n$, observe that

$$\mathcal{A}_m^\beta \cap ([1/\beta^{m+1}, 1] \times [0,1]) = \mathcal{A}_n^\beta \cap ([1/\beta^{m+1}, 1] \times [0,1]). \qquad (2.14)$$

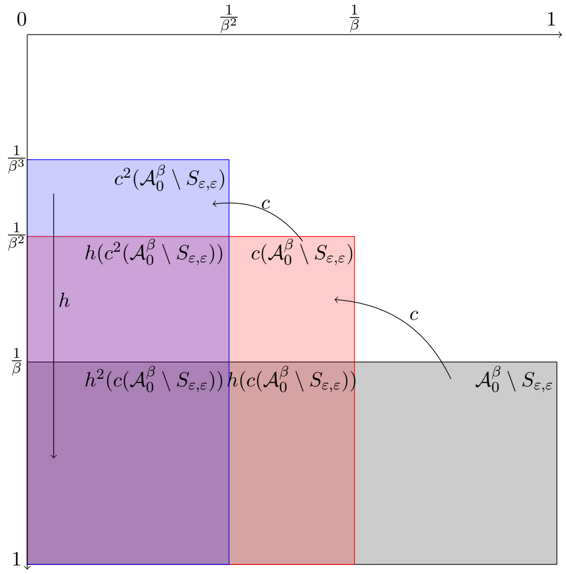



Figure 2.27: Two applications of $c$ and $h$ from $\mathcal{A}_0^\beta \setminus S_{\varepsilon,\varepsilon}$.

Applying Proposition 1.60 to the sequence $(\mathcal{A}_n^\beta)_{n\geq 0}$ gives the following definition[4].

**Definition 2.49.** We let $\mathcal{L}^\beta = \overline{\bigcup_{n\geq 0} \mathcal{A}_n^\beta}$ denote the compact limit set of the sequence $(\mathcal{A}_n^\beta)_{n\geq 0}$.

We proceed as in Example 2.19 to find an approximation of $\mathcal{L}^\varphi$.

**Example 2.50.** Let $\varphi$ be the golden ratio. We have represented in Figure 2.28 all the segments of $\mathcal{A}_0^\varphi$ for words of length at most 10, and we have applied the maps $h^j(c^i(\cdot))$ to this set of segments for $0 \leq j \leq i \leq 4$. Thus, we have an approximation of $\mathcal{A}_4^\varphi$, and even of $\mathcal{L}^\varphi$ (recall the stabilization from (2.14)).

[4]In [Sti19], as it was the case in [LRS16], this definition made sense because we showed that $(\mathcal{A}_n^\beta)_{n\geq 0}$ is a Cauchy sequence.

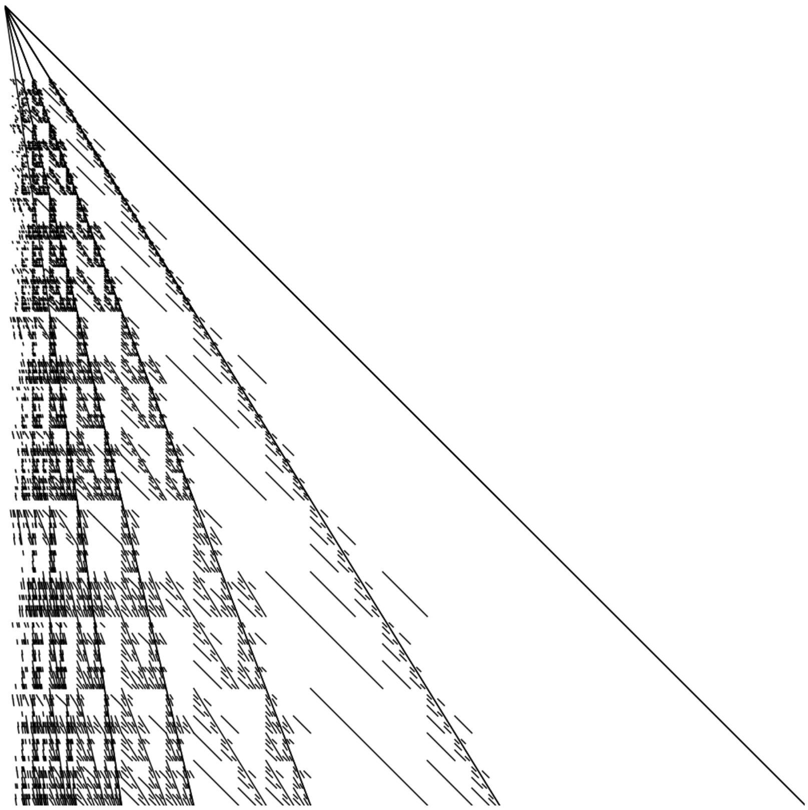


Figure 2.28: An approximation of the limit set $\mathcal{L}^{\varphi}$.

### 2.2.3 The Analogue of the Sierpiński Gasket

In this section, similarly to the sequence $(\mathcal{A}_n^{\beta})_{n\geq 0}$, we show that the sequence $(\mathcal{U}_n^{\beta})_{n\geq 0}$ of compact subsets of $[0,1]^2$ also converges to $\mathcal{L}^{\beta}$ with respect to the Hausdorff distance. The strategy is analogous to the one developed in the base-2 case: for $\epsilon \in \mathbb{R}_{>0}$ and for all sufficiently large $n \in \mathbb{N}$, we first prove that $\mathcal{U}_n^{\beta} \subset [\mathcal{L}^{\beta}]_{\epsilon}$ (that is, Lemma 2.51), and secondly, we show that $\mathcal{L}^{\beta} \subset [\mathcal{U}_n^{\beta}]_{\epsilon}$ (which follows from Lemma 2.57, Corollary 2.58 and the compactness of the set $\mathcal{L}^{\beta}$). The proofs of Lemmas 2.51 and 2.57, which extend Lemmas 2.20 and 2.22 respectively, are essentially the same, so we highlight the main differences.

**Lemma 2.51.** *Let $\epsilon > 0$. For all sufficiently large $n \in \mathbb{N}$, we have*

$$\mathcal{U}_n^{\beta} \subset [\mathcal{L}^{\beta}]_{\epsilon}.$$

*Proof*. The proof is very similar to the one of Lemma 2.20. Let $\epsilon > 0$. Take $n \in \mathbb{N}$, and let $(x, y) \in \mathcal{U}_n^\beta$. In the following, we find a point $B \in \mathcal{L}^\beta$ such that $d((x, y), B) < \epsilon$ if $n$ is sufficiently big, which suffices.

By definition, there exists $(u, v) \in L_{U_\beta} \times L_{U_\beta}$ such that $\binom{u}{v} \equiv 1 \pmod 2$, $0 \le |v| \le |u| \le n$, and the point $(x, y)$ belongs to the square region

$$\frac{1}{U_\beta(n)} \operatorname{val}_{U_\beta}(v, u) + \left[0, \frac{1}{U_\beta(n)}\right]^2 \subset \mathcal{U}_n^\beta. \tag{2.15}$$

Let us set $A = \operatorname{val}_{U_\beta}(v, u)/U_\beta(n)$ to be the upper-left corner of the square region (2.15) in $\mathcal{U}_n^\beta$. In particular, note that $d((x, y), A) \le \sqrt{2}/U_\beta(n)$.

Assume first that $(u, v)$ satisfies $(\star)$. By Definitions 2.46 and 2.48, the segment $S_{u,v}$ is in $\mathcal{A}_0^\beta$ and $c^{n-|u|}(S_{u,v})$ is a segment of origin $A' = c^{n-|u|}(A_{u,v})$ in $\mathcal{A}_{n-|u|}^\beta$. In particular, $A'$ belongs to $\mathcal{L}^\beta$ by definition. Using (1.2) (the reasoning is similar to the one developed in the proof of Lemma 2.42), there exists $N \in \mathbb{N}$ such that, for all $n \ge N$, $d(A, A') < \epsilon/2$. Hence, for all $n \ge N$ such that $\sqrt{2}/U_\beta(n) < \epsilon/2$, we have

$$d((x, y), A') \le d((x, y), A) + d(A, A') < \sqrt{2}/U_\beta(n) + \epsilon/2 < \epsilon.$$

Thus, we may choose $B = A'$.

Now assume that $(u, v)$ does not satisfy $(\star)$. Since $\binom{u}{v} \equiv 1 \pmod 2$, then either $u$ and $v$ are non-empty words, or $u$ is non-empty and $v = \varepsilon$ (recall that, if they are both empty, they satisfy $(\star)$).

First, assume that $u$ is non-empty and $v = \varepsilon$. In this case, the point $A$ is on the vertical line of equation $x = 0$. Since $\mathcal{A}_0^\beta$ contains the segment $S_{\varepsilon,\varepsilon}$ by Definition 2.46, then $\mathcal{A}_n^\beta$ contains the segment $h^n(c^n(S_{\varepsilon,\varepsilon}))$ of slope $\beta^n$ with endpoints $(0, 0)$ and $(1/\beta^n, 1)$. Since $h^n(c^n(S_{\varepsilon,\varepsilon}))$ is in $\mathcal{L}^\beta$ and since this segment passes through the square $A + [0, 1/U_\beta(n)]^2$, we may choose $n$ sufficiently large in order to find a point $B \in h^n(c^n(S_{\varepsilon,\varepsilon})) \subset \mathcal{L}^\beta$ that also belongs to this square and satisfies $d((x, y), B) < \epsilon$.

Finally, suppose that $u$ and $v$ are non-empty. Let $k$ be a non-negative integer such that $2^k > \max\{|u|, p(u, v)\}$. By definition of $p(u, v)$, the words $u0^{2^k}1$ and $v0^{2^k}1$ both belong to $L_{U_\beta}$. As in the proof of Lemma 2.20 that uses Theorem 1.40, we have

$$\binom{u0^{2^k}1}{v0^{2^k}1} \equiv 1 \pmod 2.$$

Using this result and Lemma 1.39 in particular, it is then easy to check that the pair of words $(u0^{2^k}1, v0^{2^k}1)$ satisfies $(\star)$. Finally, proceed as in the first part of the proof (namely replace $u$ by $u0^{2^k}1$ and $v$ by $v0^{2^k}1$, and apply $c^{n-|u|}$ to $S_{u_k,v_k}$). □

In Lemma 2.57, we show that each point of $\mathcal{L}^\beta$ is in $[U_n]_\epsilon$ for $\epsilon > 0$ and all sufficiently large $n \in \mathbb{N}$. To that aim, we need to control the number of consecutive words ending with 0 in $L_{U_\beta}$ (genealogically ordered). In other words, we bound the number of consecutive integers whose $U_\beta$-expansion ends with 0.

**Definition 2.52.** We let $C_\beta = \max\{n \in \mathbb{N} \mid 0^n \text{ is a factor of } d^*_\beta(1)\}$ denote the *maximal number of consecutive zeroes* in $d^*_\beta(1)$.

Before giving examples, the next proposition shows that the maximal number of consecutive words in $L_{U_\beta}$ ending with 0 is $C_\beta + 1$.

**Proposition 2.53.** *There are at most $C_\beta + 1$ consecutive non-negative integers whose $U_\beta$-expansion ends with* 0.

The proof of Proposition 2.53 requires a lemma, so we postpone it just after.

**Lemma 2.54.** *Let $n \geq 0$ be an integer with $\text{rep}_{U_\beta}(n) = c_{\ell-1} \cdots c_0 \in L_{U_\beta}$, and let $i$ denote the length of the longest suffix of $\text{rep}_{U_\beta}(n)$ that is also a prefix of $d^*_\beta(1)$. The following assertions are true.*

- *The word $\text{rep}_{U_\beta}(n+1) \in L_{U_\beta}$ ends with $0^i$.*
- *If $i = 0$, then $\text{rep}_{U_\beta}(n+1) = c_{\ell-1} \cdots c_1(c_0 + 1)$.*
- *If $i = \ell$, then $\text{rep}_{U_\beta}(n+1) = 10^\ell$.*

*Proof*. Recall that the prefixes of $d^*_\beta(1)$ are the maximal words of different lengths in $L_{U_\beta}$ and are also the labels of the maximal paths in the automaton $\mathcal{A}_\beta$. The result now follows from Proposition 1.24. □

*Proof of Proposition 2.53.* Let $n \geq 0$ be an integer such that $\text{rep}_{U_\beta}(n)$ ends with 0, and let $\ell = |\text{rep}_{U_\beta}(n)|$. Observe that $\ell \geq 2$. We can also assume that $\text{rep}_{U_\beta}(n-1)$ does not end with 0.

To prove the claim, we show that there exists an integer $0 < t \leq C_\beta + 1$ such that the word $\mathrm{rep}_{U_\beta}(n+t)$ ends with 1, which suffices. For all $k \geq 0$, we let $i_k$ denote the length of the longest suffix of $\mathrm{rep}_{U_\beta}(n+k)$ that is also a prefix of $d^*_\beta(1)$.

**Step 0.** If $i_0 = 0$, then Lemma 2.54 implies that the word $\mathrm{rep}_{U_\beta}(n+1)$ ends with 1. Thus, we can take $t = 1$, and we are done since $C_\beta \geq 0$.

Suppose that $i_0 > 0$. By Lemma 2.54, we know that $\mathrm{rep}_{U_\beta}(n+1)$ ends with $0^{i_0}$. By hypothesis, 0 is a suffix of $\mathrm{rep}_{U_\beta}(n)$ but cannot be a prefix of $d^*_\beta(1)$. Thus, we must necessarily have $i_0 \geq 2$. Furthermore, $C_\beta \geq 1$ since we have found a prefix of $d^*_\beta(1)$ that ends with 0.

**Step 1.** We examine the word $\mathrm{rep}_{U_\beta}(n+1)$, and we divide the reasoning into two cases as before.

If $i_1 = 0$, then $\mathrm{rep}_{U_\beta}(n+2)$ ends with 1 by Lemma 2.54. Note that we have $2 \leq C_\beta + 1$ (since $C_\beta \geq 1$), so we can take $t = 2$, which ends the procedure.

If $i_1 > 0$, then Lemma 2.54 shows that $\mathrm{rep}_{U_\beta}(n+2)$ ends with $0^{i_1}$. Recall that $\mathrm{rep}_{U_\beta}(n+1)$ ends with the prefix of $d^*_\beta(1)$ of length $i_1$ but also with $0^{i_0}$. In particular, this prefix has the suffix $0^{i_0}$. Consequently, we obtain $i_1 > i_0$ and $C_\beta \geq i_0 \geq 2$.

**Step 2.** We have to consider the word $\mathrm{rep}_{U_\beta}(n+2)$, and we divide the reasoning into two cases as before. On the one hand, if $i_2 = 0$, then we can take $t = 3$ thanks to Lemma 2.54, and the conclusion follows. On the other hand, if $i_2 > 0$, then a reasoning using Lemma 2.54 and similar to what was done in the previous paragraph leads to establish that $\mathrm{rep}_{U_\beta}(n+3)$ ends with $0^{i_2}$, $i_2 > i_1$ and $C_\beta \geq i_1 \geq 3$. Afterwards, we need to consider $\mathrm{rep}_{U_\beta}(n+3)$ and repeat the procedure.

We claim that this process halts after at most $C_\beta + 1$ steps. Indeed, at each new step $j$ with $j \geq 0$, either $i_j = 0$ and we stop (in this case, we can take $t = j + 1$), or $i_j > 0$ and in this case, we have $C_\beta \geq j + 1$. The second case is no longer accessible as soon as $j \geq C_\beta$. □

Let us illustrate the previous proposition.

**Example 2.55.** Let $\varphi$ be the golden ratio. From Example 1.21, $C_\varphi = 1$ since $d^*_\varphi(1) = (10)^\omega$. The first few words in $L_{U_\varphi} = L_F$ are

$$\varepsilon, 1, 10, 100, 101, 1000, 1001, 1010, 10000, 10001, 10010, 10100, 10101, \ldots.$$

The maximal number of consecutive words in $L_{U_\varphi}$ ending with 0 is 2, which corresponds to $C_\varphi + 1$.

**Example 2.56.** Let $\beta \approx 1.38028$ be the dominant root of the polynomial $P(X) = X^4 - X^3 - 1$. Then $\beta$ is a Parry number with $d_\beta(1) = 1001$ and $d^*_\beta(1) = (1000)^\omega$. The automaton $\mathcal{A}_\beta$ is depicted in Figure 2.29. In this example, $C_\beta = 3$. The first few words in $L_{U_\beta}$ are

$\varepsilon, 1, \mathbf{10}, \mathbf{100}, \mathbf{1000}, \mathbf{10000}, 10001, 100000, 100001, 100010, 1000000, 1000001, 1000010, 1000100, 10000000, 10000001, \mathbf{10000010}, \mathbf{10000100}, \mathbf{10001000}, \mathbf{100000000}, 100000001, \mathbf{100000010}, \mathbf{100000100}, \mathbf{100001000}, \mathbf{100010000}, 100010001, 1000000000, \ldots.$

The maximal number of consecutive words in $L_{U_\beta}$ ending with 0 is 4, which is equal to $C_\beta + 1$. Observe that $10, 100, 1000$ are prefixes of $d^*_\beta(1)$, but 10000 is not.

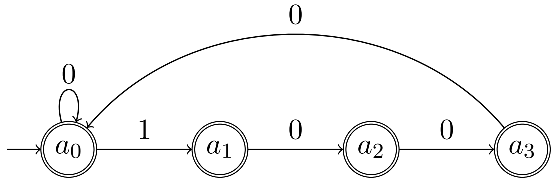



Figure 2.29: The automaton $\mathcal{A}_\beta$ for the dominant root $\beta$ of the polynomial $P(X) = X^4 - X^3 - 1$.

In the view of Definition 1.29 with $d_\beta(1) = 1001 = t_1t_2t_3t_4$, the sequence $U_\beta$ is defined by $U_\beta(0) = 1$,

$$\begin{aligned} U_\beta(1) &= t_1U_\beta(0) + 1 = 2, \\ U_\beta(2) &= t_1U_\beta(1) + t_2U_\beta(0) + 1 = 3, \\ U_\beta(3) &= t_1U_\beta(2) + t_2U_\beta(1) + t_3U_\beta(0) + 1 = 4, \end{aligned}$$

and for all $n \geq 4$,

$$\begin{aligned} U_\beta(n) &= t_1U_\beta(n-1) + t_2U_\beta(n-2) + t_3U_\beta(n-3) + t_4U_\beta(n-4) \\ &= U_\beta(n-1) + U_\beta(n-4). \end{aligned}$$

Thus, its first few terms are $1, 2, 3, 4, 5, 7, 10, 14$. For all $k \in \mathbb{N}$, the number of length-$k$ words in $0^*L_{U_\beta}$ is $U_\beta(k)$. This observation is general and reveals its usefulness in the proof of the next lemma.

As in the base-2 case, the compactness of $\mathcal{L}^\beta$ allows us to permute the quantifiers in the next lemma.

**Lemma 2.57.** *Let $\epsilon > 0$. For all $(x, y) \in \mathcal{L}^\beta$, $d((x, y), \mathcal{U}_n^\beta) < \epsilon$ for all sufficiently large $n$.*

*Proof*. Let $\epsilon > 0$ and let $(x, y) \in \mathcal{L}^\beta$. As in the proof of Lemma 2.22 with (2.7), there exist non-negative integers $N_1, i, j$ with $0 \le j \le i \le N_1$, a pair of words $(u, v) \in L_{U_\beta} \times L_{U_\beta}$ satisfying $(\star)$, and $(x_0, y_0) \in S_{u,v}$ such that

$$d((x, y), h^j(c^i((x_0, y_0)))) < \epsilon/2.$$

Now we will show that

$$d(h^j(c^i((x_0, y_0))), \mathcal{U}_n^\beta) < \epsilon/2$$

for all sufficiently large $n$, which completes the proof when using the triangle inequality. We intensively use the constants $i, j$, the words $u, v$ given above, and the integer $p = p(u, v)$. Set

$$L_{u,v} = \begin{cases} L_{U_\beta}, & \text{if } u = v = \varepsilon; \\ 0^* L_{U_\beta}, & \text{otherwise.} \end{cases}$$

Since $(u, v) \in L_{U_\beta} \times L_{U_\beta}$ satisfies $(\star)$, the pair of words $(u0^p w, v0^p w)$ has an odd binomial coefficient, for all words $w \in L_{u,v}$: if $u = v = \varepsilon$, then $\binom{w}{w} = 1$, otherwise use Corollary 2.41. In particular, this is the case when $w \in L_{u,v}$ is of length $n \ge 0$. We can choose $n$ sufficiently large such that there are at least $2(C_\beta + 1) + 2$ words of length $n$ in $L_{u,v}$. Using Proposition 2.53, there exist at least two words $w \in L_{u,v}$ with $|w| = n$ and not ending with 0. Furthermore, as soon as $w$ does not end with 0, Lemma 1.38 shows that

$$\binom{u0^p w0^k}{v0^p w} = \binom{u0^p w}{v0^p w} \equiv 1 \pmod 2 \text{ for all } k \ge 0.$$

By definition of the sequence $U_\beta$, for all $k \ge 0$, we also have

$$\#\{z \in 0^* L_{U_\beta} \mid u0^p wz \in L_{U_\beta} \text{ and } |z| = k\} \le U_\beta(k).$$

In fact, the number of length-$k$ words in $0^* L_{U_\beta}$ is $U_\beta(k)$. For all $0 \le j \le i$, at least one of the $U_\beta(j)$ binomial coefficients of the form $\binom{u0^p wz}{v0^p w}$ with $w$

not ending with 0, $z \in 0^* L_{U_\beta}$ and $|z| = j$ is odd (indeed, choose $z = 0^j$ for instance). In other terms, at least one of the square regions

$$\frac{1}{U_\beta(n+i+|u|+p)} \operatorname{val}_{U_\beta}(v0^p w, u0^p wz) + \left[0, \frac{1}{U_\beta(n+i+|u|+p)}\right]^2, \tag{2.16}$$

with $0 \leq j \leq i, z \in 0^* L_{U_\beta}$ and $|z| = j$,

is a subset of $\mathcal{U}^\beta_{n+i+|u|+p}$, since $|v0^p w|, |u0^p wz| \leq n + i + |u| + p$. This can be visualized in Figure 2.30. For this, we took the special setting of the golden ratio $\varphi$ and the Zeckendorf numeration system (see Example 1.18).

Now observe that, for any word $w \in L_{u,v}$, each square region of the form (2.16) is intersected by $h^j(c^i(S_{u,v}))$. Indeed, the latter segment has $A = (0.0^{i+|u|-|v|}v, 0.0^{i-j}u)$ and $B = (0.0^{i+|u|-|v|}v0^p d^*_\beta(1), 0.0^{i-j}u0^p d^*_\beta(1))$ as endpoints and slope $\beta^j$. Using (1.2) on page 15, if $n$ is sufficiently large, the points

$$\frac{1}{U_\beta(n+i+|u|+p)} \operatorname{val}_{U_\beta}(v0^p 0^n, u0^p 0^{n+j})$$
$$\left(\text{resp., } \frac{1}{U_\beta(n+i+|u|+p)} \operatorname{val}_{U_\beta}(v0^p d_n, u0^p d_{n+j})\right)$$

and $A$ (resp., $B$) are close for all $0 \leq j \leq i$, where we let $d_n$ denote the length-$n$ prefix of $d^*_\beta(1)$ for all $n \geq 0$. When $u$ and $v$ are non-empty, this can be seen in Figure 2.31 where each rectangular gray region contains at least one square region from $\mathcal{U}^\beta_{n+i+|u|+p}$ (to draw this picture, we take the particular case of the golden ratio $\varphi$, and $i = 2$). When $u = v = \varepsilon$, Figure 2.31 is modified in the following way: simply replace each word of the forms $u0^\ell$, $v0^\ell$ by $\varepsilon$.

As a consequence, every point of $h^j(c^i(S_{u,v}))$ is at distance at most

$$\frac{2 \cdot (C_\beta + 2) \cdot U_\beta(j)}{U_\beta(n+i+|u|+p)}$$

from a point in $\mathcal{U}^\beta_{n+i+|u|+p}$ when $n$ is sufficiently large. Indeed, there are two cases to consider: either the point falls into a gray region from Figure 2.31, or it does not. In the first case, then the point is at distance at most $U_\beta(j)/U_\beta(n+i+|u|+p)$ from a square region in $\mathcal{U}^\beta_{n+i+|u|+p}$; see also Figure 2.30. Recall that this square region is of the form (2.16) where $w$

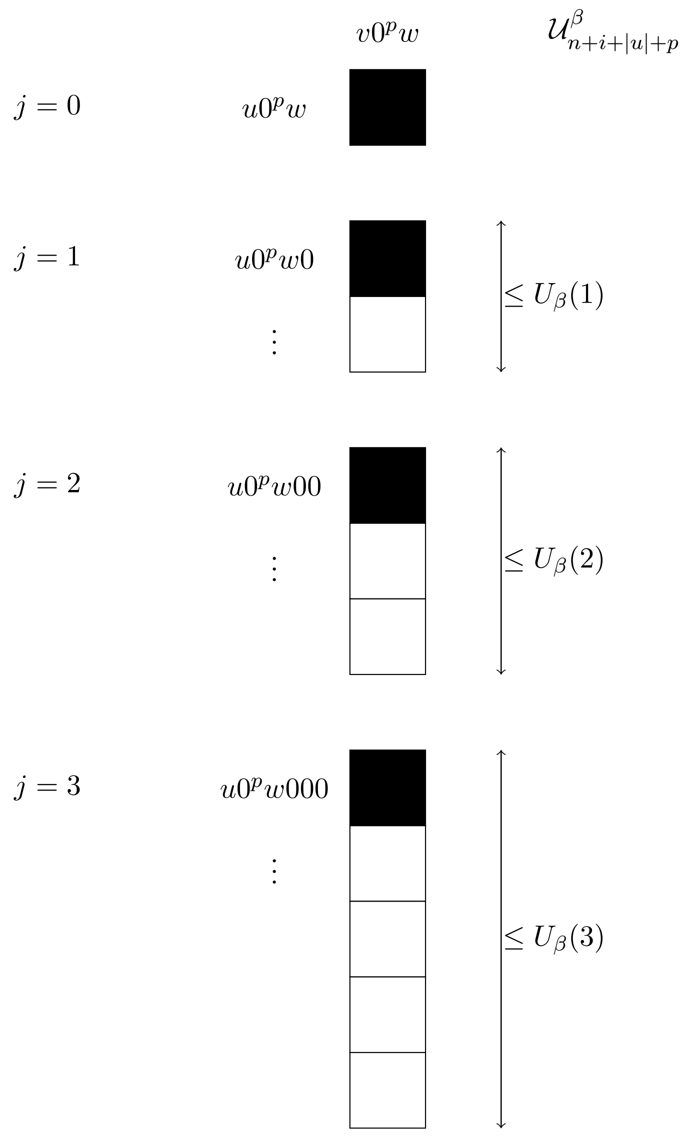



Figure 2.30: If $w$ does not end with 0 and is of length $n$, then $\binom{u0^pw0^j}{v0^pw}$ being odd creates a square region in $\mathcal{U}^{\beta}_{n+i+|u|+p}$.

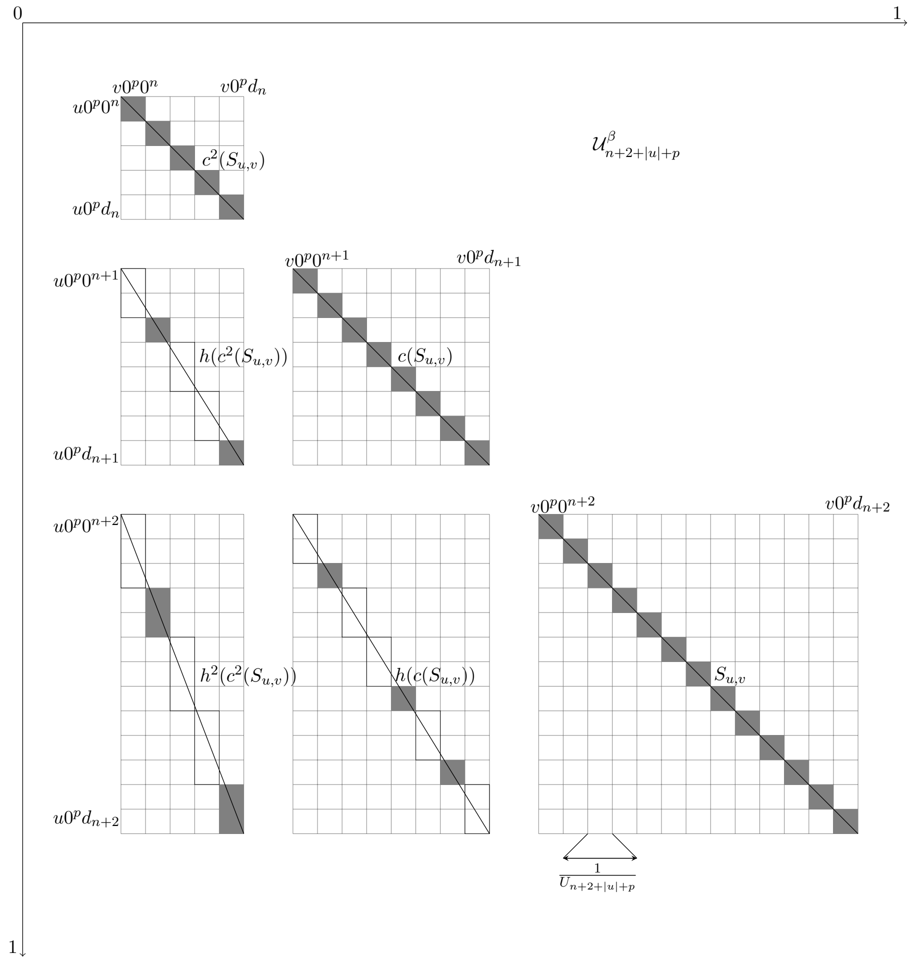



Figure 2.31: The situation occurring in the proof of Lemma 2.57, where we choose $\beta$ to be the golden ratio, and $i = 2$.

does not end with 0. In the second case, the point falls into a (white) square region of the form

$$\frac{1}{U_\beta(n+i+|u|+p)}\,\mathrm{val}_{U_\beta}(v0^pw', u0^pw'z) + \left[0, \frac{1}{U_\beta(n+i+|u|+p)}\right]^2,$$
$$\text{with } |w| = |w'| = n, w' \in L_{u,v}, 0 \le j \le i, z \in 0^*L_{U_\beta} \text{ and } |z| = j.$$

Since $n$ is large enough, there exists a word $w''$ not ending with 0 with $|w''| = n$, which is within a distance of $2(C_\beta + 2)$ of $w'$. Then applying the argument from the previous case proves the statement.

In particular, the result holds for the point $h^j(c^i((x_0, y_0)))$ belonging to $h^j(c^i(S_{u,v}))$. Hence, for all sufficiently large $n$, $d(h^j(c^i((x_0, y_0))), \mathcal{U}_n^\beta) < \epsilon/2$, and the conclusion follows. □

From the previous lemma, we deduce the following result which is the analogue of Corollary 2.23 and whose proof is identical.

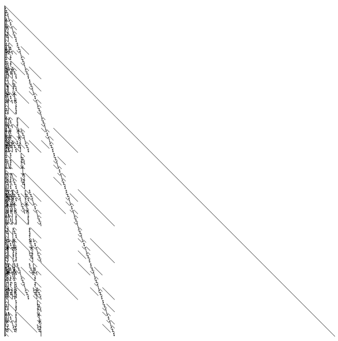
**Corollary 2.58.** *Let $(u, v) \in L_{U_\beta} \times L_{U_\beta}$ satisfying $(\star)$ and let $0 \leq j \leq i$. For every point $(f, g)$ of the segment $h^j(c^i(S_{u,v}))$, there exists a sequence $((f_n, g_n))_{n\geq 0}$ converging to $(f, g)$ and such that $(f_n, g_n) \in \mathcal{U}_n^\beta$ for all $n \geq 0$.*

The proof of the following theorem is the same as the one of Theorem 2.24, so we omit it. It uses the compactness of the set $\mathcal{L}^\beta$, Lemmas 2.51 and 2.57, and Corollary 2.58.

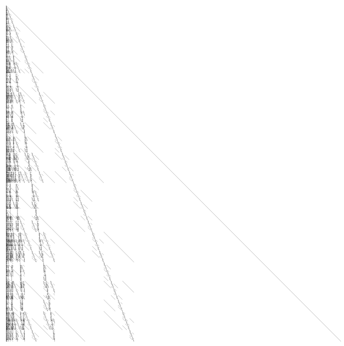
**Theorem 2.59.** *The sequence $(\mathcal{U}_n^\beta)_{n\geq 0}$ converges to $\mathcal{L}^\beta$ with respect to the Hausdorff distance.*

In the next example, we give an approximation of the limit object $\mathcal{L}^\beta$ for different values of $\beta$.

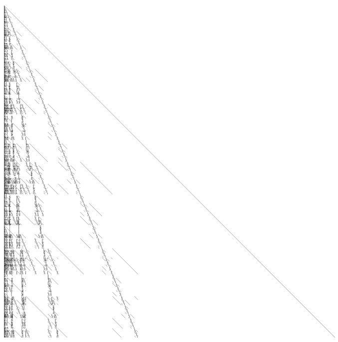
**Example 2.60.** Let us define several Parry numbers. Let $\beta_1 \approx 2.47098$ be the dominant root of the polynomial $P(X) = X^4 - 2X^3 - X^2 - 1$, which is a Parry and Pisot number; see Example 2.35. Let $\beta_2 \approx 1.38028$ be the dominant root of the polynomial $P(X) = X^4 - X^3 - 1$, which is a Parry and Pisot number; see Example 2.56. Let $\beta_3 \approx 2.80399$ be the dominant root of the polynomial $P(X) = X^4 - 2X^3 - 2X^2 - 2$. We can show that $\beta_3$ is a Parry number, but not a Pisot number. Let $\beta_4 \approx 1.32472$ be the dominant root of the polynomial $P(X) = X^5 - X^4 - 1$. We can show that $\beta_4$ is a Parry number and also the smallest Pisot number [BR10, Example 2.3.54]. In Figure 2.32, we depict an approximation of $\mathcal{L}^\beta$ for $\beta$ in $\{3, \varphi^2, \beta_1, \beta_2, \beta_3, \beta_4\}$. For instance, the sets $\mathcal{L}^{\beta_2}$ and $\mathcal{L}^{\beta_4}$ more or less look alike. This might be due to the fact that the associated polynomials are not so different. More generally, a challenging angle of research is to examine the similarities and the differences between limit objects, as stated among the open questions in the next section.



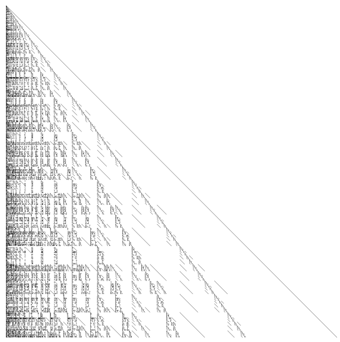
(a) An approximation of $\mathcal{L}^3$.



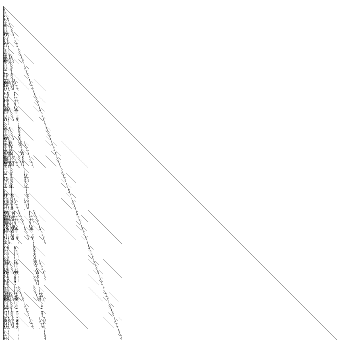
(b) An approximation of $\mathcal{L}^{\varphi^2}$.



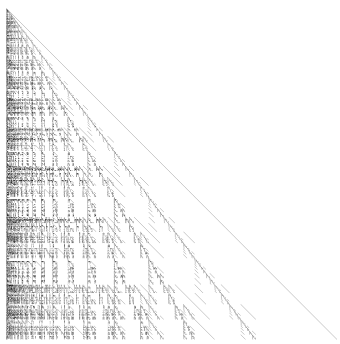
(c) An approximation of $\mathcal{L}^{\beta_1}$.



(d) An approximation of $\mathcal{L}^{\beta_2}$.



(e) An approximation of $\mathcal{L}^{\beta_3}$.



(f) An approximation of $\mathcal{L}^{\beta_4}$.

Figure 2.32: An approximation of the limit object $\mathcal{L}^\beta$ for different values of $\beta$.

As a final comment, let us mention that the extension to any prime number holds: one simply has to adapt all the results, as in Section 2.1.5. The notation used in the following example is taken from that section.

**Example 2.61.** Let us consider the case where $\beta$ is the golden ratio $\varphi$. We have represented $\mathcal{U}^{\varphi}_{9,3,2}$ in Figure 2.33 when considering binomial coefficients congruent to 2 modulo 3 (instead of odd coefficients) and an approximation of the corresponding limit set $\mathcal{L}^{\varphi}_{3,2}$, proceeding as in Example 2.50.

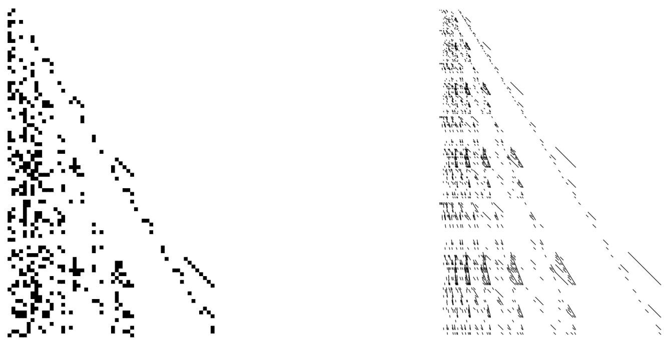


Figure 2.33: The set $\mathcal{U}^{\varphi}_{9,3,2}$ (on the left) and an approximation of the corresponding limit set $\mathcal{L}^{\varphi}_{3,2}$ (on the right).

## 2.3 Open Questions

In the last section of this chapter, we leave some open questions that seem natural to us, or that were asked during different scientific meetings.

For a given numeration system associated with a Parry number $\beta > 1$, we cut the generalized Pascal triangle $\mathrm{P}_\beta$ after terms of the sequence $(U_\beta(n))_{n\geq 0}$, *i.e.*, we consider the first $U_\beta(n)$ rows and columns of $\mathrm{P}_\beta$ at each step (see Definition 2.30). For this reason, $(U_\beta(n))_{n\geq 0}$ is called a *cutting sequence*. In [AB97], authors discuss which cutting sequences lead to a sequence of subblocks of the classical Pascal triangle that converges to some limit object. Inspired by this paper, the next question naturally follows.

**Question 1.** Are there other cutting sequences of interest for our particular matter? What do they look like, *i.e.*, would it be possible to characterize them?

In the same vein, the colorings presented in this chapter are influenced by Lucas' theorem (that is, Theorem 1.40), *i.e.*, we take care of colorings modulo prime numbers. In [vHPS92], authors examine colorings modulo prime powers in the framework of the classical Pascal triangle. Having in mind generalizations of Lucas' theorem (see [AS08, Row11] for instance), we raise the following question.

**Question 2.** Could other colorings be considered for generalized Pascal triangles $\mathrm{P}_\beta$ introduced in Section 1.5?

The construction developed in this chapter highly depends on the manner in which the words are ordered in the considered languages, *i.e.*, the enumeration of the languages. As a consequence, another enumeration would certainly influence the limit object, as well as considering other languages.

**Question 3.** What happens if we change the order of the words in $L_{U_\beta}$, or if we focus on other languages, not specifically derived from numeration systems? Is it still possible to prove a convergence result? Similarly, with other possible extensions of the Pascal triangle (see the list after the definition of the Pascal triangle in Section 1.5), can we adapt the convergence results of the present chapter?

Without diving into technical definitions, we were often asked the following questions that are still open. Note that some bounds for the Hausdorff dimension can be deduced from already known results on the classical Pascal triangle.

**Question 4.** For a fixed Parry number $\beta > 1$, what is the Hausdorff dimension of the limit set $\mathcal{L}^\beta$? And its Minkowski dimension? Could the method in [Neu18] be helpful? What is its Hölder exponent? Could we look at its Lebesgue measure?

A famous method to construct fractals is to use *iterated function systems* (IFS's); see, for instance, [Bar93, Fal97]. Formally, an IFS is a finite set of contracting mappings on a complete metric space. For instance, the Sierpiński gasket can be obtained via IFS's [vHPS92, Ste95].

**Question 5.** For a given Parry number $\beta > 1$, would it be possible to find IFS's that generate $\mathcal{L}^\beta$?

Another interesting direction of investigation is to compare different limit objects.

**Question 6.** Can we compare two limit sets, *e.g.*, $\mathcal{L}^\beta$ and $\mathcal{L}^{\beta'}$ for two distinct Parry numbers $\beta, \beta' > 1$? Can we compare limit objects obtained with different congruences, but for the same numeration system? More generally, can we classify the limit objects?

In [AB97], authors study the *block complexity* of the classical Pascal triangle, *i.e.*, the bidimensionnal factor complexity which counts the number of rectangular blocks of a fixed size. The following question ensues.

**Question 7.** Could we compute the bidimensionnal factor complexity of Pascal-like triangles $\mathrm{P}_\beta$?

# Chapter 3

# Counting Scattered Subwords

In this chapter, we count the number of distinct scattered subwords occurring in a given word. More precisely, we study the sequences $S_\beta$, which were defined in Section 1.6, for different real numbers $\beta > 1$. In particular, these sequences summarize all the information we have on each row of generalized Pascal triangles. When sketching those sequences, some symmetries seem to appear, which makes us think that they are regular in some sense. One of the objectives of this chapter is to establish this regularity by means of specific graph structures which also allow us to easily count scattered subwords.

In Section 3.1, we consider the base-2 case as it was done in the previous chapter, *i.e.*, we work with the generalized Pascal triangle $\mathrm{P}_2$ and the sequence $(S_2(n))_{n\geq 0}$ counting the number of positive entries on each row of $\mathrm{P}_2$; see Chapter 1 for formal definitions. By introducing a convenient and new tree structure, we provide a recurrence relation for $(S_2(n))_{n\geq 0}$. This leads to a connection with the 2-regular Stern–Brocot sequence and the sequence of denominators occurring in the Farey tree, yielding the 2-regularity of $S_2$. Leaving the world of known regular sequences, our method first provides similar results in the general case of integer bases in Section 3.2. Then, in Section 3.3, we extend our construction to the Zeckendorf numeration system, so we deal with $\mathrm{P}_\varphi$ where $\varphi$ is the golden ratio. Again the tree structure permits us to obtain recurrence relations for the corresponding sequence and deduce its Fibonacci-regularity. We conclude the chapter with various remarks and open questions. Its content is taken from [LRS17b, LRS18].

**Contents**



## 3.1 What Happens in Base 2

In this section, we work with the generalized Pascal triangle $\mathrm{P}_2$ from Example 1.43 and the sequence $(S_2(n))_{n\geq 0}$ from Example 1.49. Observe that on a one-letter alphabet, *i.e.*, for the classical Pascal triangle, the analogous sequence $(S(n))_{n\geq 0}$ satisfies $S(n) = n + 1$ for all $n \geq 0$ since $\binom{n}{m} > 0$ for all $m \in \{0, \dots, n\}$ and $\binom{n}{m} = 0$ for all $m \geq n + 1$. It is not difficult to prove that $(S(n))_{n\geq 0}$ is 2-regular but not 2-automatic. As we can see in Figure 3.1, the sequence $(S_2(n))_{n\geq 0}$ has a much more chaotic behavior. However, Figure 3.1 shares some similarities with the sequences studied in [PRRV15] and independently in [Gre15] about the 2-abelian complexity of the Thue–Morse word and other infinite words. For instance, we may observe a palindromic structure over each interval of the form $[2^n, 2^{n+1})$. This particular structure suggests that $(S_2(n))_{n\geq 0}$, as well as the Stern–Brocot sequence of Example 1.55, should be 2-regular. The aim of this section is to study the regularity of $(S_2(n))_{n\geq 0}$; see Section 3.1.4.

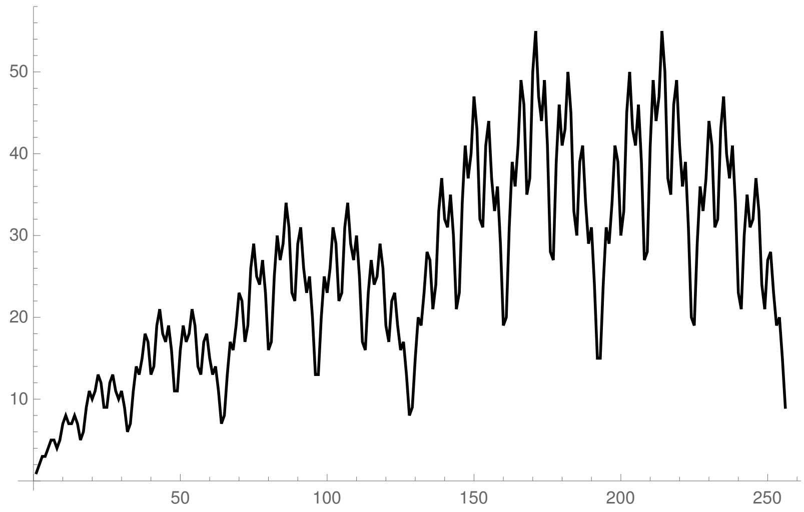



Figure 3.1: The sequence $(S_2(n))_{n\geq 0}$ in the interval $[0, 256]$.

### 3.1.1 Give it a Trie (of Scattered Subwords)

In [FGK+15], an automaton with multiplicities accepting exactly the scattered subwords $v$ of a given word $u$ is presented, whose number of accepting paths is exactly $\binom{u}{v}$. As mentioned in [KNS16], it is an important problem to determine what are the "best" data structures for reasoning with subwords; see also [BDS16].

For our particular needs, we first introduce a convenient tree structure whose nodes correspond to the scattered subwords of a given word and which permits us to easily count them. This tree not only leads to a recurrence relation to compute $S_2(n)$ directly from $\text{rep}_2(n)$, but is also generalized to larger alphabets in Section 3.2 and to the Fibonacci case in Section 3.3.

If $w \in A^*$ is a finite word over the alphabet $A$, then the language of its scattered subwords is *factorial*, *i.e.*, if $xyz$ is a scattered subword of $w$, then so is $y$. Thus the following definition makes sense.

**Definition 3.1.** Let $A$ be an alphabet, and let $w$ be a finite word over $A$. With $w$ is associated the *trie of its scattered subwords* denoted by $\mathcal{T}_A(w)$ and built as follows. The root is $\varepsilon$, and if $u$ and $ua$ are two scattered subwords of $w$ with $a \in A$, then $ua$ is a child of $u$. This trie is also called *prefix tree* or *radix tree* in the sense that all successors of a node have a common prefix.

If we are interested in words and scattered subwords belonging to a specific factorial language $L \subset A^*$, we only consider the part of $\mathcal{T}_A(w)$ that is a subtree of the tree limited to words in $L$. This subtree is denoted by $\mathcal{T}_{A,L}(w)$.

Note that, if the alphabet is clear from the context, we omit it and we respectively write $\mathcal{T}(w)$ and $\mathcal{T}_L(w)$.

**Remark 3.2.** For any word $w \in L \subset A^*$, an easy induction shows that the number of nodes on level $\ell \geq 0$ in $\mathcal{T}_{A,L}(w)$ is the number of distinct scattered subwords of length $\ell$ in $L$ occurring in $w$. In particular, the number of nodes of the trie $\mathcal{T}_{A_{U_\beta},L_{U_\beta}}(\text{rep}_{U_\beta}(n))$ is exactly $S_\beta(n)$ for all $n \geq 0$; see Definition 1.47.

Section 3.1 deals with the base-2 case, so we are interested in words and scattered subwords belonging to $L_2$. We will thus consider the tree $\mathcal{T}_{\{0,1\},L_2}(w)$ for $w \in L_2$. This means that in $\mathcal{T}_{\{0,1\}}(w)$ we will only consider the child 1 of the root $\varepsilon$. For the sake of simplicity, in the following, we write $\mathcal{T}(w)$ (resp., $\mathcal{T}_{L_2}(w)$) instead of $\mathcal{T}_{\{0,1\}}(w)$ (resp., $\mathcal{T}_{\{0,1\},L_2}(w)$).

**Example 3.3.** In Figure 3.2, we have depicted the tree $\mathcal{T}_{L_2}(11001110)$ (the dashed lines and the subtrees $T_\ell$, $\ell \in \{0,\dots,3\}$, will become clear with Definition 3.4 below). The word $w = 11001110$ is highlighted as the leftmost branch. In fact, it is more visual if we do not use the convention that the left child of $u$ is $u0$ and the right child of $u$ is $u1$. The edge between $u$ and its child $u0$ (resp., $u1$) is represented in gray (resp., black).

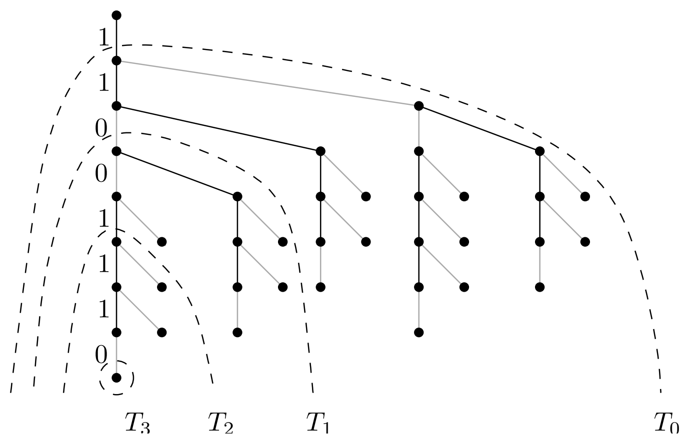



Figure 3.2: The trie $\mathcal{T}_{L_2}(11001110)$.

Since we are dealing with scattered subwords of $w \in L_2$, the tree $\mathcal{T}_{L_2}(w)$ has a particular structure that will be helpful to count the number of distinct scattered subwords occurring in $w$. We now describe this structure, which permits us to construct $\mathcal{T}_{L_2}(w)$ starting with a linear tree and proceeding bottom-up (see Example 3.8).

**Definition 3.4.** Each non-empty word $w$ in $L_2$ is factorized into consecutive maximal blocks of letters 1 and blocks of letters 0 of the form

$$w = \underbrace{1^{n_1}}_{u_1}\underbrace{0^{n_2}}_{u_2}\underbrace{1^{n_3}}_{u_3}\underbrace{0^{n_4}}_{u_4}\cdots\underbrace{1^{n_{2j-1}}}_{u_{2j-1}}\underbrace{0^{n_{2j}}}_{u_{2j}} \tag{3.1}$$

with $j \geq 1$, $n_1, \ldots, n_{2j-1} \geq 1$ and $n_{2j} \geq 0$.

Let $M = M_w$ be such that $w = u_1 u_2 \cdots u_M$ where $u_M$ is the last non-empty block of zeroes or ones. For every $\ell \in \{0, \ldots, M-1\}$, we let $T_\ell$ denote the subtree of $\mathcal{T}_{L_2}(w)$ whose root is the node

$$\begin{cases} u_1 \cdots u_\ell 1, & \text{if } \ell \text{ is even;} \\ u_1 \cdots u_\ell 0, & \text{if } \ell \text{ is odd;} \end{cases}$$

(if $\ell = 0$, then $u_1 \cdots u_\ell = \varepsilon$). Observe that $T_{M-1}$ is a linear tree with $n_M$ nodes. For convenience (Corollary 3.9), we also set $T_M$ to be an empty tree with no node. Roughly speaking, we have a root of a new subtree $T_\ell$ for each new alternation of digits in $w$. Also observe that if $w$ does not contain any letter 0, then the tree $\mathcal{T}_{L_2}(w)$ is linear. In particular, only the trees $T_0$ and $T_M = T_1$ do exist.

**Example 3.5.** For the word $w = 11001110 \in L_2$ of Example 3.3, the factorization of Definition 3.4 is $w = u_1u_2u_3u_4$ with $u_1 = 1^2$, $u_2 = 0^2$, $u_3 = 1^3$ and $u_4 = 0^1$, so we have $M = 4$. In Figure 3.2, we have represented the trees $T_0, \ldots, T_3$. By definition, the root of $T_0$ (resp., $T_1$; resp., $T_2$) is 1 (resp., $u_1 0 = 110$; resp., $u_1u_2 1 = 11001$). Finally, $T_3$ is limited to a single node and its root is $u_1u_2u_3 0 = 11001110 = w$. Indeed, by definition, the number of nodes of $T_{M-1}$ is $n_M$, which is equal to 1 in this example.

In the following result, for a non-empty word $w \in L_2$, we study the structure of the tree $\mathcal{T}_{L_2}(w)$ in relations with the subtrees $T_\ell$ defined previously. Since we are considering scattered subwords of $w$, its proof is not hard.

Note that if $w \in 1^* \cup 10^*$, then $\mathcal{T}_{L_2}(w)$ is linear, and no further interesting information can be brought out.

**Proposition 3.6.** *Let $w$ be a non-empty word in $L_2$. If the tree $\mathcal{T}_{L_2}(w)$ is not linear, it has the following properties.*

- *Assume that $2 \leq 2k < M$. For every $j \in \{0, \dots, n_{2k} - 1\}$, the node of label $x = u_1 \cdots u_{2k-1}0^j$ has two children $x0$ and $x1$. The node $x1$ is the root of a tree isomorphic to $T_{2k}$.*
- *Assume that $3 \leq 2k+1 < M$. For every $j \in \{0, \dots, n_{2k+1} - 1\}$, the node of label $x = u_1 \cdots u_{2k}1^j$ has two children $x0$ and $x1$. The node $x0$ is the root of a tree isomorphic to $T_{2k+1}$.*
- *For every $j \in \{1, \dots, n_1 - 1\}$, the node of label $x = 1^j$ has two children $x0$ and $x1$. The node $x0$ is the root of a tree isomorphic to $T_1$.*

**Example 3.7.** Let us pursue Examples 3.3 and 3.5. Recall that $M = 4$. We illustrate each item of the previous proposition.

First, suppose that $k = 1$. Then $n_{2k} = n_2 = 2$. We observe in Figure 3.2 that the node of label $x = u_1 = 11$ (resp., $x = u_1 0 = 110$) has two children $x0$ and $x1$, and the child $x1$ is the root of a copy of $T_2$.

Now, $n_{2k+1} = n_3 = 3$. We see in Figure 3.2 that the node of label $x = u_1 u_2 = 1100$ (resp., $x = u_1 u_2 1 = 11001$; resp., $x = u_1 u_2 11 = 110011$) has two children $x0$ and $x1$, and the child $x0$ is the root of a copy of $T_3$.

Finally, $n_1 = 2$. From Figure 3.2, we find that the node of label 1 has two children 11 and 10, and the child 10 is the root of a copy of $T_1$. In this case, the result cannot be extended to $1^0 = \varepsilon$ for otherwise we would consider words not in $L_2$.

As depicted in Figure 3.3, Proposition 3.6 permits us to reconstruct the tree $\mathcal{T}_{L_2}(w)$ from $w$. We now explain how.

**Example 3.8.** We continue Examples 3.3 and 3.5. Recall that we have

$$w = 11001110 = \underbrace{1^2}_{u_1}\underbrace{0^2}_{u_2}\underbrace{1^3}_{u_3}\underbrace{0^1}_{u_4}.$$

In Figure 3.3, we show how to build $\mathcal{T}_{L_2}(11001110)$ in four steps. To do so, we start with a linear tree corresponding to the word $w$ (it is depicted on the left in Figure 3.3). We proceed from the bottom of the tree, we progressively add copies of the subtrees $T_\ell$, and we use Proposition 3.6 to guide us at each step.

First, the tree $T_3$ is the linear subtree consisting in the last $n_4 = 1$ node. We add a copy of $T_3$ to each node of the form $u_1 u_2 1^j = 11001^j$ for $j \in \{0, \ldots, n_3 - 1\} = \{0, 1, 2\}$ (second picture).

Then we consider the subtree $T_2$ whose root is the node $u_1 u_2 1 = 11001$. Thanks to Proposition 3.6, we add a copy of it to each node of the form $u_1 0^j = 110^j$ for $j \in \{0, \ldots, n_2 - 1\} = \{0, 1\}$ (third picture).

Finally, we consider the subtree $T_1$ whose root is the node $u_1 0 = 110$. We add a copy of it to the node $1^j$ for $j \in \{1, \ldots, n_1 - 1\} = \{1\}$ (picture at the bottom). Note that if $u_1 = 1^k$, we should add a copy of $T_1$ to each node of the form $1^j$ for $j \in \{1, \ldots, k-1\}$ (if $k = 1$, then no copy of $T_1$ is added).

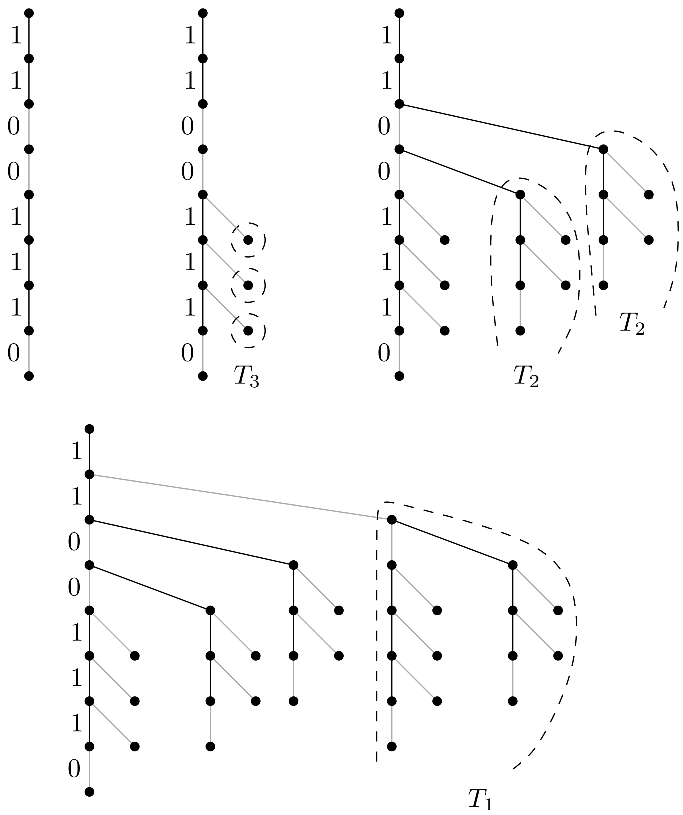



Figure 3.3: Bottom-up construction of $\mathcal{T}_{L_2}(11001110)$.

In the next statement, we count the number of nodes of each subtree $T_\ell$, eventually yielding the total number of nodes of the original tree $\mathcal{T}_{L_2}(w)$. For this result, recall from Definition 3.4 that $T_M$ is an empty tree. If $T$ is a tree, we let $\#T$ denote the *number of nodes* in $T$. Corollary 3.9 will actually give a recurrence relation to compute $S_2(n)$ directly from $\operatorname{rep}_2(n)$.

**Corollary 3.9.** *Let $w$ be a non-empty word in $L_2$. The number of nodes in $T_M$ (resp., $T_{M-1}$) is 0 (resp., $n_M$). For $\ell \in \{0, \dots, M-2\}$, the number of nodes in $T_\ell$ is given by*

$$\#T_\ell = n_{\ell+1}(\#T_{\ell+1} + 1) + \#T_{\ell+2}.$$

*In particular, the number of distinct scattered subwords of $w$ in $L_2$ is given by $1 + \#T_0$.*

*Proof.* Let $w$ be a non-empty word in $L_2$ factorized as in Definition 3.4.

The idea is the same as the one developed in Example 3.8 above and uses Proposition 3.6. Start with a linear tree labeled by $w$ and add, with a bottom-up approach, all the possible subtrees given by Proposition 3.6: first, possible copies of $T_{M-1}$, then copies of $T_{M-2}, \dots, T_1$.

Let $\ell \in \{0, \dots, M-2\}$. By Definition 3.4, $T_\ell$ is the subtree of $\mathcal{T}_{L_2}(w)$ whose root is the node $u_1 \cdots u_\ell a$ with $a \in \{0, 1\}$ depending on the parity of $\ell$ (notice that $u_{\ell+1} \in a^*$). By Proposition 3.6, we know that, for all $j \in \{1, \dots, n_{\ell+1} - 1\}$, $u_1 \cdots u_\ell a^j b$ is the root of a tree isomorphic to $T_{\ell+1}$ with $b = 1 - a \in \{0, 1\}$. This is also the case when $j = n_{\ell+1}$ by Definition 3.4, *i.e.*, $u_1 \cdots u_\ell a^{n_{\ell+1}} b = u_1 \cdots u_\ell u_{\ell+1} b$ is the root of a tree isomorphic to $T_{\ell+1}$. Again by Proposition 3.6, $u_1 \cdots u_\ell u_{\ell+1} b^0 a$ is the root of a tree isomorphic to $T_{\ell+2}$. The formula follows.

By Definition 3.4, $T_0$ is the subtree of $\mathcal{T}_{L_2}(w)$ whose root is the node $\varepsilon \cdot 1 = 1$, so the number of distinct scattered subwords of $w$ in $L_2$ is given by the total number of nodes in $\mathcal{T}_{L_2}(w)$, namely $1 + \#T_0$. □

**Remark 3.10.** If we were interested in the number of distinct scattered subwords of $w$ in $\{0, 1\}^*$ (not only those in $L_2$), we must add the node 0 which will be the root of a subtree isomorphic to $T_1$. Thus, the total number of distinct scattered subwords occurring in $w$ is $1 + \#T_0 + \#T_1$.

We end this section by illustrating the previous corollary.

**Example 3.11.** We carry forward Examples 3.3 and 3.5. For the word $w = 11001110$, we have $\#T_4 = 0$, $n_4 = 1$ and $\#T_3 = 1$. Thus, since $n_3 = 3$, $n_2 = 2$ and $n_1 = 2$, we find

$$\begin{aligned}\#T_2 &= 3(1+1)+0 = 6,\\ \#T_1 &= 2(6+1)+1 = 15,\\ \#T_0 &= 2(15+1)+6 = 38.\end{aligned}$$

The number of distinct scattered subwords of $w$ in $L_2$ is then $1 + \#T_0 = 39$, and since $\mathrm{val}_2(11001110) = 206$, we obtain $S_2(206) = 39$. Moreover, the total number of distinct scattered subwords of 11001110 is $39 + 15 = 54$.

### 3.1.2 A Singular Relation

Thanks to tries of scattered subwords introduced in the previous section, we collect several results about the number of words occurring as scattered subwords of words with a prescribed form; see Lemmas 3.12, 3.13 and 3.14. They lead to a recurrence relation satisfied by $(S_2(n))_{n\geq 0}$ in Proposition 3.15.

For the following result, if $w$ is a finite or infinite word over $\{0,1\}$, we let $\underline{w}$ denote the word obtained by replacing in $w$ every 0 by 1 and every 1 by 0 (see also Example 1.53).

**Lemma 3.12.** *Let $u$ be a word in $\{0,1\}^*$. Then*

$$\#\left\{v \in L_2 \mid \binom{1u}{v} > 0\right\} = \#\left\{v \in L_2 \mid \binom{1\underline{u}}{v} > 0\right\}.$$

*In particular, it means that $S_2(2^\ell + r) = S_2(2^{\ell+1} - r - 1)$ with $0 \leq r < 2^\ell$, i.e., $S_2$ has a local palindromic structure.*

*Proof*. Since the trie of scattered subwords of a word exactly counts the number of distinct scattered subwords of this word, it is enough for the first part to observe that the trees $\mathcal{T}_{L_2}(1u)$ and $\mathcal{T}_{L_2}(1\underline{u})$ are isomorphic. Each node of the form $1x$ in the first tree corresponds to the node $1\underline{x}$ in the second one, and conversely.

For the special case, let $0 \leq r < 2^\ell$, and write $\mathrm{rep}_2(2^\ell + r) = 1z$ with $z \in \{0,1\}^\ell$. Since $\mathrm{val}_2(z) + \mathrm{val}_2(\underline{z}) = \mathrm{val}_2(1^\ell) = 2^\ell - 1$, we have

$$\mathrm{rep}_2(2^{\ell+1} - r - 1) = 1\underline{z}.$$

Using (1.4) on page 23, we obtain

$$\begin{aligned} S_2(2^\ell + r) &= \#\left\{v \in L_2 \mid \binom{1z}{v} > 0\right\} = \#\left\{v \in L_2 \mid \binom{1\underline{z}}{v} > 0\right\} \\ &= S_2(2^{\ell+1} - r - 1). \end{aligned}$$

□

**Lemma 3.13.** *Let $u$ be a word in $\{0,1\}^*$. Then*

$$\begin{aligned} \#\left\{v \in L_2 \mid \binom{100u}{v} > 0\right\} =& 2 \cdot \#\left\{v \in L_2 \mid \binom{10u}{v} > 0\right\} \\ &- \#\left\{v \in L_2 \mid \binom{1u}{v} > 0\right\}. \end{aligned}$$

*Proof*. Our reasoning is again based on the structure of the trees. Recall that the left-hand part of the claimed formula is $\#\mathcal{T}_{L_2}(100u)$ while the first (resp., second) term of its right-hand part is $\#\mathcal{T}_{L_2}(10u)$ (resp., $\#\mathcal{T}_{L_2}(1u)$).

Assume first that $u$ has no 1, then $u = 0^n$ with $n \geq 0$. The tree $\mathcal{T}_{L_2}(1u)$ has $n+2$ nodes, $\mathcal{T}_{L_2}(10u)$ has $n+3$ nodes, and $\mathcal{T}_{L_2}(100u)$ has $n+4$ nodes. Thus the formula is true.

Now assume that $u$ contains at least a letter 1. First, observe that the subtree $S$ of $\mathcal{T}_{L_2}(1u)$ with root 1 is equal to the subtree of $\mathcal{T}_{L_2}(10u)$ with root 10 and also to the subtree of $\mathcal{T}_{L_2}(100u)$ with root 100. Consider the shortest

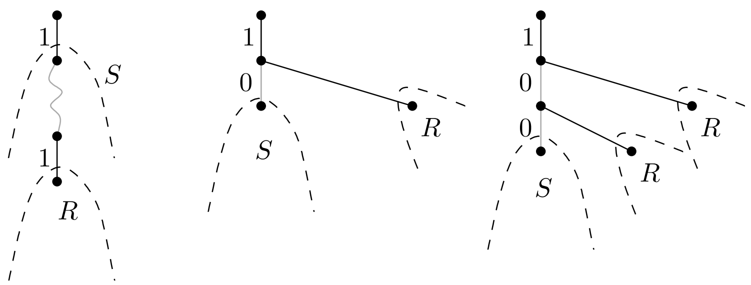



Figure 3.4: Structure of the trees $\mathcal{T}_{L_2}(1u)$, $\mathcal{T}_{L_2}(10u)$ and $\mathcal{T}_{L_2}(100u)$.

prefix of $1u$ of the form $10^r1$ with $r \geq 0$, *i.e.*, we stop after reading the first 1 in $u$. Let $R$ be the subtree of $\mathcal{T}_{L_2}(1u)$ with root $10^r1$. By definition of the prefix $10^r1$, the subtree of $\mathcal{T}_{L_2}(10u)$ with root 11 is $R$. Similarly, $\mathcal{T}_{L_2}(100u)$ contains two copies of $R$: the subtrees of roots 11 and 101. The situation is depicted in Figure 3.4, and the following formula holds:

$$\begin{aligned}\#\mathcal{T}_{L_2}(100u) &= 3 + \#S + 2\#R = 2(2 + \#S + \#R) - (1 + \#S)\\ &= 2\#\mathcal{T}_{L_2}(10u) - \#\mathcal{T}_{L_2}(1u),\end{aligned}$$

as expected. □

**Lemma 3.14.** *Let $u$ be a word in $\{0,1\}^*$. Then*

$$\begin{aligned}\#\left\{v \in L_2 \mid \binom{101u}{v} > 0\right\} =& \#\left\{v \in L_2 \mid \binom{1u}{v} > 0\right\}\\ &+ \#\left\{v \in L_2 \mid \binom{11u}{v} > 0\right\}.\end{aligned}$$

*Proof*. The reasoning is similar to the one of the previous proof. Recall that the left-hand part of the claimed formula is $\#\mathcal{T}_{L_2}(101u)$ while the first (resp., second) term of its right-hand part is $\#\mathcal{T}_{L_2}(1u)$ (resp., $\#\mathcal{T}_{L_2}(11u)$).

If $u = 1^n$ with $n \geq 0$, then $\mathcal{T}_{L_2}(101u)$ has $2n+5$ nodes, and $\mathcal{T}_{L_2}(1u)$ and $\mathcal{T}_{L_2}(11u)$ have respectively $n+2$ and $n+3$ nodes, so the formula holds.

If $u$ has at least a letter 0, then consider the shortest prefix of $1u$ of the form $1^r0$ with $r \geq 1$. Let $S$ be the subtree of $\mathcal{T}_{L_2}(1u)$ with root 1, and $R$ be its subtree with root $1^r0$. The tree $\mathcal{T}_{L_2}(101u)$ (resp., $\mathcal{T}_{L_2}(1u)$; resp., $\mathcal{T}_{L_2}(11u)$) has $3 + 2\#S + \#R$ (resp., $1 + \#S$; resp., $2 + \#S + \#R$) nodes, so

$$\begin{aligned}\#\mathcal{T}_{L_2}(101u) &= 3 + 2\#S + \#R = (1 + \#S) + (2 + \#S + \#R)\\ &= \mathcal{T}_{L_2}(1u) + \mathcal{T}_{L_2}(11u).\end{aligned}$$

□

We now obtain a recurrence relation for $(S_2(n))_{n\geq 0}$. As we will see in Section 3.1.3, it also induces a link between the latter sequence and the sequence of denominators of the Farey tree (`A007306` [Slo]), and also the Stern–Brocot sequence $(SB(n))_{n\geq 0}$ (`A002487` [Slo]).

**Proposition 3.15.** *The sequence $(S_2(n))_{n\geq 0}$ satisfies $S_2(0) = 1$, $S_2(1) = 2$, and for all $\ell \geq 1$ and $0 \leq r < 2^\ell$,*

$$S_2(2^\ell + r) = \begin{cases} S_2(2^{\ell-1} + r) + S_2(r), & \textit{if } 0 \leq r < 2^{\ell-1};\\ S_2(2^{\ell+1} - r - 1), & \textit{if } 2^{\ell-1} \leq r < 2^\ell.\end{cases}$$

*Proof*. Consider some integers $\ell$ and $r$ with $\ell \geq 1$ and $0 \leq r < 2^\ell$.

If $2^{\ell-1} \le r < 2^\ell$, the equality $S_2(2^\ell+r) = S_2(2^{\ell+1}-r-1)$ directly follows from Lemma 3.12. Also note that $0 \le 2^\ell - r - 1 < 2^{\ell-1}$.

So let us suppose $0 \le r < 2^{\ell-1}$. We proceed by induction on $\ell$. The case $\ell = 1$ is easily checked by hand. Let us suppose $\ell \ge 2$. By definition, we have

$$S_2(2^\ell + r) = \#\left\{v \in L_2 \mid \binom{\text{rep}_2(2^\ell + r)}{v} > 0\right\}.$$

Since $r < 2^{\ell-1}$, we have $\text{rep}_2(2^\ell + r) = 10u$ for a word $u \in \{0,1\}^*$ of length $\ell - 1$ verifying $r = \text{val}_2(u)$. We consider two cases depending on the first letter occurring in $u$.

If $u \in 0\{0,1\}^*$, then $r = \text{val}_2(u) \le 2^{\ell-2} - 1$, and we deduce from Lemma 3.13 that $S_2(2^\ell + r) = 2S_2(2^{\ell-1} + r) - S_2(2^{\ell-2} + r)$. The proof is complete after using the induction hypothesis twice

$$\begin{aligned} S_2(2^\ell + r) &= 2(S_2(2^{\ell-2} + r) + S_2(r)) - S_2(2^{\ell-2} + r) \\ &= S_2(2^{\ell-2} + r) + S_2(r) + S_2(r) \\ &= S_2(2^{\ell-1} + r) + S_2(r). \end{aligned}$$

If $u = 1u'$ for $u' \in \{0,1\}^*$, then $\text{val}_2(101u') = \text{val}_2(10u) = 2^\ell + r$, $\text{val}_2(1u') = \text{val}_2(u) = r$ and $\text{val}_2(11u') = 2^{\ell-1} + r$. From Lemma 3.14 applied to $u'$, we deduce that $S_2(2^\ell + r) = S_2(r) + S_2(2^{\ell-1} + r)$, which finishes the proof. □

**Remark 3.16.** If $2^{\ell-1} \le r < 2^\ell$, then $0 \le 2^\ell - r - 1 < 2^{\ell-1}$ and using twice the previous result gives

$$S_2(2^\ell + r) = S_2(2^{\ell+1} - r - 1) = S_2(2^{\ell-1} + 2^\ell - r - 1) + S_2(2^\ell - r - 1).$$

This allows to decrease the involved powers of 2.

We have rough upper bounds for the terms of the sequence $(S_2(n))_{n\ge 0}$, which will turn out to be useful in Chapter 4.

**Corollary 3.17.** *For all $n \ge 1$, we have $S_2(n) \le 2n$.*

*Proof*. We proceed by induction on $n \ge 1$. The case $n = 1$ is easy since $S_2(1) = 2$ (see Example 1.49). Let $n \ge 2$, and let us write $n = 2^\ell + r$ for $\ell \ge 1$ and $0 \le r < 2^\ell$. If $r = 0$, then $S_2(n) = S_2(2^\ell) = \ell + 2$ since the scattered subwords of $\text{rep}_2(n) = \text{rep}_2(2^\ell) = 10^\ell$ are $\varepsilon$ and $10^j$ for all $0 \le j \le \ell$. We have $S_2(n) = S_2(2^\ell) = \ell + 2 \le 2 \cdot 2^\ell = 2n$. If $0 < r < 2^{\ell-1}$, then

$$S_2(n) = S_2(2^\ell + r) = S_2(2^{\ell-1} + r) + S_2(r)$$

by Proposition 3.15. By induction hypothesis,

$$S_2(n) = S_2(2^\ell + r) \le 2 \cdot (2^{\ell-1} + r) + 2r \le 2^{\ell+1} + 2r = 2n.$$

If $2^{\ell-1} \le r < 2^\ell$, then first by Proposition 3.15 and then by induction hypothesis, we find

$$S_2(n) = S_2(2^\ell + r) = S_2(2^{\ell+1} - r - 1) \le 2 \cdot (2^{\ell+1} - r - 1).$$

Observe that $2^{\ell+1} - r - 1 = 2^\ell + 2^\ell - r - 1 < 2^\ell + r$, so we finally have $S_2(n) \le 2n$. □

### 3.1.3 The Farey and Stern–Brocot Trees

Plugging in the first few terms of $(S_2(n))_{n\ge 0}$ in Sloane's On-Line Encyclopedia of Integer Sequences [Slo], it seems to be a shifted version of the sequence `A007306` of the denominators occurring in the Farey tree (the left subtree of the full Stern–Brocot tree, or the Stern–Brocot subtree in the range $[0, 1]$), which contains every (reduced) positive rational less than 1 exactly once. Many papers deal with both trees, but the second has been recently restudied for its link with physical chemistry; see, for instance, [Bat14, BBT10, Gla11]. The *Farey tree* is an infinite binary tree made up of mediants. Given two reduced fractions $\frac{a}{b}$ and $\frac{c}{d}$, with $a, b, c, d \in \mathbb{N}$, their *mediant* is the fraction $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$. This operation is known as the *child's addition*. Observe that for all $\frac{a}{b}$, $\frac{c}{d}$ with $\frac{a}{b} < \frac{c}{d}$, we have $\frac{a}{b} < \frac{a}{b} \oplus \frac{c}{d} < \frac{c}{d}$.

**Definition 3.18.** Starting from the fractions $\frac{0}{1}$ and $\frac{1}{1}$, the *Farey tree* is the infinite tree defined as follows.

- The set of nodes is partitioned into levels indexed by $\mathbb{N}$.
- The level 0 consists in $\{\frac{0}{1}, \frac{1}{1}\}$.
- The level 1 consists in $\{\frac{1}{2}\}$. The node $\frac{1}{2}$ is the only one with two parents, which are $\frac{0}{1}$ and $\frac{1}{1}$.

- For each $n \geq 2$, the level $n$ consists in the children of the nodes of vertices in the level $n-1$. For each node $\frac{a}{b}$ of level $n-1$, we define

$$\text{Left}\left(\frac{a}{b}\right) = \max\left\{\frac{e}{f} \mid \text{level}\left(\frac{e}{f}\right) < n-1 \text{ and } \frac{e}{f} < \frac{a}{b}\right\},$$

and

$$\text{Right}\left(\frac{a}{b}\right) = \min\left\{\frac{e}{f} \mid \text{level}\left(\frac{e}{f}\right) < n-1 \text{ and } \frac{e}{f} > \frac{a}{b}\right\}.$$

It means that Left $\left(\frac{a}{b}\right)$ (resp., Right $\left(\frac{a}{b}\right)$) is the greatest (resp., smallest) fraction that is located in levels before the level of $\frac{a}{b}$ and that is also smaller (resp., greater) than $\frac{a}{b}$. Now, the *left and right children* of $\frac{a}{b}$ are respectively $\frac{a}{b} \oplus \text{Left}\left(\frac{a}{b}\right)$ and $\frac{a}{b} \oplus \text{Right}\left(\frac{a}{b}\right)$.

The first five levels of the Farey tree can be found in Figure 3.5. For instance, the fraction $\frac{a}{b} = \frac{2}{5}$ is located on level 3. The fractions on levels 0, 1 and 2 are $\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}$. We have $\frac{0}{1}, \frac{1}{3} < \frac{2}{5}$ and $\frac{1}{2}, \frac{2}{3}, \frac{1}{1} > \frac{2}{5}$, so

$$\text{Left}\left(\frac{2}{5}\right) = \frac{1}{3} \quad \text{and} \quad \text{Right}\left(\frac{2}{5}\right) = \frac{1}{2}.$$

Consequently, the left (resp., right) child of $\frac{2}{5}$ is $\frac{2}{5} \oplus \text{Left}\left(\frac{2}{5}\right) = \frac{2}{5} \oplus \frac{1}{3} = \frac{3}{8}$ (resp., $\frac{2}{5} \oplus \text{Right}\left(\frac{2}{5}\right) = \frac{2}{5} \oplus \frac{1}{2} = \frac{3}{7}$).

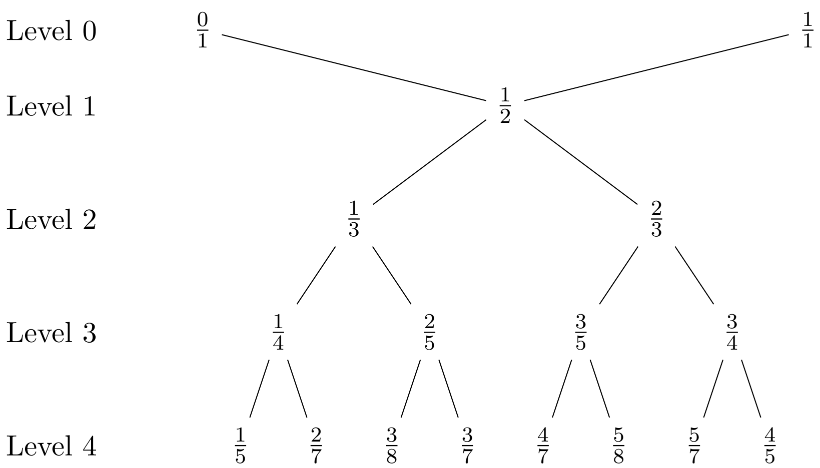



Figure 3.5: The first levels of the Farey tree.

Reading the denominators in Figure 3.5 level-by-level, then from left to right, *i.e.*, conducting a breadth-first traversal of the tree, we obtain the first few terms of the sequence $(S_2(n))_{n\geq 0}$, if we drop the second denominator 1. Have a look at Example 1.49.

If we assume that a branch to the left (resp., right) corresponds to 0 (resp., 1), then with every node $\frac{a}{b}$ (except $\frac{1}{1}$) is associated a unique path from $\frac{0}{1}$ to $\frac{a}{b}$ of label $u \in 1\{0,1\}^* \cup \{\varepsilon\}$. If $\mathrm{val}_2(u) = 2^k + r$ with $k \geq 0$ and $0 \leq r < 2^k$, then we will show that $a = S_2(r)$ and $b = S_2(2^k + r)$ (see Propositions 3.19 and 3.20). For instance, $\frac{3}{7}$ corresponds to the path of label $1011 = \mathrm{rep}_2(11) = \mathrm{rep}_2(2^3 + 3)$ and, $S_2(3) = 3$, $S_2(11) = 7$.

Given an integer $n \geq 0$, we let $D(n)$ (resp., $N(n)$) be the denominator (resp., numerator) of the fraction corresponding to the node reached in the Farey tree from $\frac{0}{1}$ using the path of label $\mathrm{rep}_2(n)$. The sequence $(D(n))_{n\geq 0}$ is called *the sequence of denominators of the Farey tree fractions* and is indexed by `A007306` in [Slo]. Similarly, the sequence $(N(n))_{n\geq 0}$ is *the sequence of numerators of the Farey tree fractions*.

**Proposition 3.19.** *The sequence $(D(n))_{n\geq 0}$ of denominators of the Farey tree fractions coincides with $(S_2(n))_{n\geq 0}$.*

*Proof*. From the definition of the Farey tree, it follows that, for all $\ell \geq 1$ and $u \in \{0,1\}^*$,

$$D(\mathrm{val}_2(u10^\ell)) = D(\mathrm{val}_2(u10^{\ell-1})) + D(\mathrm{val}_2(u))$$

and

$$D(\mathrm{val}_2(u01^\ell)) = D(\mathrm{val}_2(u01^{\ell-1})) + D(\mathrm{val}_2(u)).$$

Otherwise stated, if $n = 2^k + r \geq 1$ with $0 \leq r < 2^k$, we have two possibilities: either there exists $\ell \geq 1$ such that

$$D(n) = \begin{cases} D(n/2) + D(0), & \text{if } \mathrm{rep}_2(n) = 10^\ell; \\ D((n-1)/2) + D(0), & \text{if } \mathrm{rep}_2(n) = 1^\ell; \end{cases}$$

or there exists $\ell \geq 1$ such that

$$D(n) = \begin{cases} D(n/2) + D((n-2^\ell)/2^{\ell+1}), & \text{if } \mathrm{rep}_2(r) \in \{0,1\}^*10^\ell; \\ D((n-1)/2) + D((n-2^\ell+1)/2^{\ell+1}), & \text{if } \mathrm{rep}_2(r) \in \{0,1\}^*01^\ell. \end{cases}$$

We show by induction that the sequences $(D(n))_{n\geq 0}$ and $(S_2(n))_{n\geq 0}$ are equal. A direct inspection shows that $S_2(n) = D(n)$ for $0 \leq n \leq 3$. For the induction step, we assume that $n \geq 4$, and we show that $S_2(n)$ satisfies the same formulas as $D(n)$, *i.e.*, if $n = 2^k + r \geq 1$ with $0 \leq r < 2^k$, we have for some $\ell \geq 1$

$$S_2(n) = \begin{cases} S_2(n/2) + S_2(0), & \text{if } \operatorname{rep}_2(n) = 10^\ell; \\ S_2((n-1)/2) + S_2(0), & \text{if } \operatorname{rep}_2(n) = 1^\ell; \end{cases}$$

or

$$S_2(n) = \begin{cases} S_2(n/2) + S_2((n-2^\ell)/2^{\ell+1}), & \text{if } \operatorname{rep}_2(r) \in \{0,1\}^*10^\ell; \\ S_2((n-1)/2) + S_2((n-2^\ell+1)/2^{\ell+1}), & \text{if } \operatorname{rep}_2(r) \in \{0,1\}^*01^\ell. \end{cases}$$

Let us write $n = 2^k + r$ with $k \geq 2$ and $0 \leq r < 2^k$. We establish the first relations. If $r = 0$ (resp., $r = 2^k - 1$), then $S_2(n) = k + 2$ since the scattered subwords of $\operatorname{rep}_2(n) = 10^k$ (resp., $\operatorname{rep}_2(n) = 1^{k+1}$) are $\varepsilon$ and $10^j$ for $0 \leq j \leq k$ (resp., $\varepsilon$ and $1^j$ for $1 \leq j \leq k+1$). The formula holds since $S_2(n/2) = k+1$ (resp., $S_2((n-1)/2) = k+1$) and $S_2(0) = 1$. Now we may assume that $r \notin \{0, 2^k - 1\}$. In the view of Proposition 3.15, we consider four cases, depending on whether $0 \leq r < 2^{k-1}$ or $2^{k-1} \leq r < 2^k$, and whether $\operatorname{rep}_2(r)$ has a suffix consisting of zeroes or of ones. However, we only give the proof for the case $0 \leq r < 2^{k-1}$ with $\operatorname{rep}_2(r) \in \{0,1\}^*10^\ell$, $\ell \geq 1$, for the other ones are similar. By Proposition 3.15, we first have

$$S_2(n) = S_2(2^{k-1} + r) + S_2(r),$$

and then by induction hypothesis, we find

$$\begin{aligned} S_2(n) =& S_2((2^{k-1}+r)/2) + S_2((2^{k-1}+r-2^\ell)/2^{\ell+1}) \\ &+ S_2(r/2) + S_2((r-2^\ell)/2^{\ell+1}). \end{aligned}$$

Using Proposition 3.15 again, we finally get

$$\begin{aligned} S_2(n) =& S_2(2^{k-1} + r/2) + S_2((2^k + r - 2^\ell)/2^{\ell+1}) \\ =& S_2(n/2) + S_2((n-2^\ell)/2^{\ell+1}). \end{aligned}$$ □

In fact, we have a stronger result.

**Proposition 3.20.** *Let $w \in 1\{0,1\}^*$ be a finite word such that* $\mathrm{val}_2(w) = 2^k + r$ *with $k \geq 0$ and $r \in \{0, \dots, 2^k - 1\}$. Then the fraction in the Farey tree corresponding to the node reached at the end of the path labeled by $w$ is $S_2(r)/S_2(2^k + r)$.*

*Proof*. By definition of the Farey tree, if $n = 2^k + r \geq 1$ with $0 \leq r < 2^k$, we have for some $\ell \geq 1$

$$N(n) = \begin{cases} N(n/2) + N(0) = 1, & \text{if } \mathrm{rep}_2(n) = 10^\ell; \\ N((n-1)/2) + 1 = \ell, & \text{if } \mathrm{rep}_2(n) = 1^\ell; \end{cases}$$

or

$$N(n) = \begin{cases} N(n/2) + N((n-2^\ell)/2^{\ell+1}), & \text{if } \mathrm{rep}_2(r) \in \{0,1\}^* 10^\ell; \\ N((n-1)/2) + N((n-2^\ell+1)/2^{\ell+1}), & \text{if } \mathrm{rep}_2(r) \in \{0,1\}^* 01^\ell; \end{cases}$$

as it was the case for the sequence $(D(n))_{n\geq 0}$ of denominators.

We show that, if $n = 2^k + r \geq 1$ with $0 \leq r < 2^k$, then $N(n) = S_2(r)$. We proceed by induction on $n \geq 1$. The result holds for $1 \leq n \leq 3$ by looking at the Farey tree. Now suppose that $n = 2^k + r \geq 4$. If $r = 0$, then $S_2(r) = 1$ and the formula is true. If $r = 2^k - 1$, then $\mathrm{rep}_2(r) = 1^k$, so $S_2(r) = k + 1$. The formula holds since $\mathrm{rep}_2(n) = 1^{k+1}$ and $N(n) = k + 1$. Now we may suppose that $r \notin \{0, 2^k - 1\}$. Again, we have to consider four cases, depending on whether $0 \leq r < 2^{k-1}$ or $2^{k-1} \leq r < 2^k$, and whether $\mathrm{rep}_2(r)$ has a suffix consisting of zeroes or of ones. However, we only give the proof for the case $0 \leq r < 2^{k-1}$ with $\mathrm{rep}_2(r) \in \{0,1\}^* 10^\ell$, $\ell \geq 1$, since the other ones are similar. Using the previous formulas and the induction hypothesis, we find

$$\begin{aligned} N(n) &= N(n/2) + N((n-2^\ell)/2^{\ell+1}) \\ &= S_2(r/2) + S_2((r-2^\ell)/2^{\ell+1}). \end{aligned}$$

The idea is to use the relations satisfied by $(S_2(n))_{n\geq 0}$ that were highlighted in the proof of the previous proposition. As a first case, suppose that the base-2 expansion of $r$ contains at least a letter 1 before the block $10^\ell$, *i.e.*, $\mathrm{rep}_2(r) \in 1\{0,1\}^* 10^\ell$. Then

$$N(n) = S_2(r/2) + S_2((r-2^\ell)/2^{\ell+1}) = S_2(r),$$

where the last equality comes from the proof of Proposition 3.19. As a second case, suppose that $\mathrm{rep}_2(r) \in 0^*10^\ell$, *i.e.*, $r = 2^\ell$. Then

$$N(n) = S_2(2^{\ell-1}) + S_2(0) = \ell + 2 = S_2(2^\ell) = S_2(r).$$

Now, the main result follows. Indeed, let $w \in 1\{0,1\}^*$ be a finite word such that $\mathrm{val}_2(w) = 2^k + r$ with $k \geq 0$ and $r \in \{0, \ldots, 2^k - 1\}$. By definition, the fraction in the Farey tree corresponding to the node reached at the end of the path labeled by $w$ is $N(2^k + r)/D(2^k + r)$. By the first part of the proof and by Proposition 3.19, this fraction is equal to $S_2(r)/S_2(2^k + r)$, as desired. □

**Remark 3.21.** It is a folklore fact that the sum of the denominators of the fractions on level $k$ (with $k \geq 1$) in the Farey tree is equal to $2 \cdot 3^{k-1}$, *i.e.*,

$$\sum_{r=0}^{2^{k-1}-1} D(2^{k-1} + r) = 2 \cdot 3^{k-1},$$

and is equal to 1 if we only consider the denominator $D(0)$. Thus, the sum $\sum_{i=1}^{2^n-1} D(i)$ of the denominators of the fractions on the levels 1 to $n$ is equal to $3^n - 1$ (or $3^n$ if we add the denominator $D(0)$ on level 0). Using (1.4) on page 23, we observe that $\sum_{i=0}^{2^n-1} S_2(i)$ is the number of pairs of words in $L_2^{\leq n}$ having a positive binomial coefficient. This is yet another proof of Proposition 2.5.

Recall the Stern–Brocot sequence from Example 1.55. Then the equality $D(n) = SB(2n + 1)$ and Proposition 3.19 imply the following.

**Corollary 3.22.** *The sequence* $(S_2(n))_{n\geq 0}$ *satisfies* $S_2(n) = SB(2n+1)$ *for all* $n \in \mathbb{N}$.

**Remark 3.23.** In [CW98], it is shown that the $n$th Stern–Brocot value $SB(n)$ is equal to the number of times words of the form $v \in 1(01)^*$ occur as scattered subwords of the binary expansion of $n$. This result is different from the one obtained here because the form of the scattered subwords is fixed. In [CS11], the authors give a way to build the sequence $(SB(n))_{n\geq 0}$ using occurrences of words appearing in the base-2 expansions of positive integers.

It is known that the Stern–Brocot tree and the Stern–Brocot sequence both have connections with continued fractions (see, for instance, [Nor10]), and continued fractions are deeply bonded with Christoffel words (see, for instance, [BLRS09]). All in all, it implies that the sequence $(S_2(n))_{n\geq 0}$ and Christoffel words are linked together. It is worth mentioning that the Farey tree is also related to the notion of frieze patterns [MGOT15], giving yet another way to think about the sequence $(S_2(n))_{n\geq 0}$. In [ÇS18], it is proved that there exists a bijection between the perfect matchings of snake graphs and the denominators of fractions in the Stern–Brocot tree, so this bijection also extends to the sequence $(S_2(n))_{n\geq 0}$. Numerous discussions with E. Gunawan let us think that the material developed here, namely the sequence $(S_2(n))_{n\geq 0}$ and more especially tries of scattered subwords, could be a way to study cluster algebras and help to count order ideals of a specific poset [BG].

### 3.1.4 2-Regularity

From the 2-regularity of the Stern–Brocot sequence in Example 1.55, Corollary 3.22 and the robustness result [AS03a, Theorem 16.2.2] (which states that if $(s(n))_{n\geq 0}$ is $b$-regular, then so is the sequence $(s(kn+r))_{n\geq 0}$ for $k \geq 1$ and $r \geq 0$), one can immediately deduce that $(S_2(n))_{n\geq 0}$ is also 2-regular. In fact, many properties of $(S_2(n))_{n\geq 0}$ can be deduced from the corresponding properties of the Stern–Brocot sequence. Nevertheless, this section proposes an alternative proof of the 2-regularity property because we have in mind extensions to other numeration systems; see Sections 3.2 and 3.3.

**Theorem 3.24.** *The sequence $(S_2(n))_{n\geq 0}$ satisfies, for all $n \geq 0$,*

$$\begin{aligned} S_2(2n+1) &= 3\,S_2(n) - S_2(2n), \\ S_2(4n) &= 2\,S_2(2n) - S_2(n), \\ S_2(4n+2) &= 4\,S_2(n) - S_2(2n). \end{aligned}$$

*In particular, $(S_2(n))_{n\geq 0}$ is 2-regular.*

*Proof.* To prove those relations, we proceed by induction on $n \geq 0$. It can be checked by hand that the result holds for $n \in \{0, 1\}$. Thus consider $n > 1$, and suppose that the relations hold for all $m < n$. We write $n = 2^\ell + r$ with $\ell \geq 1$ and $0 \leq r < 2^\ell$. For each relation to prove, we divide the proof in two parts according to the position of $r$ inside the interval $[0, 2^\ell)$.

We first prove $S_2(2n + 1) + S_2(2n) = 3S_2(n)$. If $0 \leq r < 2^{\ell-1}$, we get by

Proposition 3.15

$$\begin{aligned}S_2(2n+1)+S_2(2n) &= S_2(2^{\ell+1}+2r+1)+S_2(2^{\ell+1}+2r)\\ &= S_2(2^{\ell}+2r+1)+S_2(2r+1)+S_2(2^{\ell}+2r)+S_2(2r).\end{aligned}$$

By induction hypothesis and then by Proposition 3.15 again, we have

$$S_2(2n+1)+S_2(2n) = 3S_2(2^{\ell-1}+r)+3S_2(r) = 3S_2(n).$$

If $2^{\ell-1} \leq r < 2^{\ell}$, we obtain by Proposition 3.15

$$\begin{aligned}S_2(2n+1)+S_2(2n) &= S_2(2^{\ell+1}+2r+1)+S_2(2^{\ell+1}+2r)\\ &= S_2(2^{\ell+2}-2r-2)+S_2(2^{\ell+2}-2r-1).\end{aligned}$$

By induction hypothesis and then by Proposition 3.15 again, we find

$$S_2(2n+1)+S_2(2n) = 3S_2(2^{\ell+1}-r-1) = 3S_2(n),$$

as desired.

Let us prove $S_2(4n) = 2S_2(2n) - S_2(n)$. If $0 \leq r < 2^{\ell-1}$, we get from Proposition 3.15

$$S_2(4n) = S_2(2^{\ell+2}+4r) = S_2(2^{\ell+1}+4r)+S_2(4r).$$

By induction hypothesis and then by Proposition 3.15 again, we obtain

$$S_2(4n) = 2S_2(2^{\ell}+2r)-S_2(2^{\ell-1}+r)+2S_2(2r)-S_2(r) = 2S_2(2n)-S_2(n).$$

If $2^{\ell-1} \leq r < 2^{\ell}$, we have by Proposition 3.15

$$S_2(4n) = S_2(2^{\ell+2}+4r) = S_2(2^{\ell+3}-4r-1).$$

Observe that the integer $2^{\ell+3}-4r-1$ is odd, so from the first relation, we deduce

$$S_2(4n) = 3S_2(2^{\ell+2}-2r-1)-S_2(2^{\ell+3}-4r-2),$$

and by Proposition 3.15, we in fact have

$$S_2(4n) = 3S_2(2n)-S_2(2^{\ell+3}-4(r+1)+2).$$

By induction hypothesis and then by Proposition 3.15 again, we deduce

$$\begin{aligned}S_2(4n) &= 3S_2(2n)-4S_2(2^{\ell+1}-r-1)+S_2(2^{\ell+2}-2r-2)\\ &= 3S_2(2n)-4S_2(n)+S_2(2n+1).\end{aligned}$$

Lastly, using the first relation, we prove

$$S_2(4n) = 2S_2(2n) - S_2(n),$$

as expected.

Finally, let us show $S_2(4n+2) = 4S_2(n) - S_2(2n)$. If $0 \leq r < 2^{\ell-1}$, we get by Proposition 3.15

$$S_2(4n+2) = S_2(2^{\ell+2} + 4r + 2) = S_2(2^{\ell+1} + 4r + 2) + S_2(4r + 2).$$

By induction hypothesis and then by Proposition 3.15 again, we find

$$S_2(4n+2) = 4S_2(2^{\ell-1}+r) - S_2(2^{\ell}+2r) + 4S_2(r) - S_2(2r) = 4S_2(n) - S_2(2n).$$

If $2^{\ell-1} \leq r < 2^{\ell}$, we get by Proposition 3.15

$$S_2(4n+2) = S_2(2^{\ell+2} + 4r + 2) = S_2(2^{\ell+3} - 4r - 3).$$

Observe that the integer $2^{\ell+3} - 4r - 3$ is odd, so using the first relation leads to

$$S_2(4n+2) = 3S_2(2^{\ell+2} - 2r - 2) - S_2(2^{\ell+3} - 4r - 4),$$

which implies by Proposition 3.15 that

$$S_2(4n+2) = 3S_2(2n+1) - S_2(2^{\ell+3} - 4(r+1)).$$

By induction hypothesis and then by Proposition 3.15 again, we get

$$\begin{aligned} S_2(4n+2) &= 3S_2(2n+1) - 2S_2(2^{\ell+2} - 2r - 2) + S_2(2^{\ell+1} - r - 1) \\ &= 3S_2(2n+1) - 2S_2(2n+1) + S_2(n). \end{aligned}$$

Using the first relation one last time, we have

$$S_2(4n+2) = 4S_2(n) - S_2(2n),$$

as stated.

To finish the proof, observe that the $\mathbb{Z}$-module $\langle \mathcal{K}_2(S_2) \rangle$ is finitely generated: a choice of generators is $(S_2(n))_{n\geq 0}$ and $(S_2(2n))_{n\geq 0}$. □

If a sequence is $b$-regular, then its $n$th term can be obtained by multiplying some matrices, and the length of this product is proportional to $\log_b(n)$; see, for instance, [AS92], [AS03a, Theorem 16.1.3]. In our situation, we will consider products of square matrices of size 2. Observe that due to Corollary 3.22, other matrices can be derived from a linear representation of $(SB(n))_{n\geq 0}$ (see, for instance, [BC18]).

**Corollary 3.25.** *For all* $n \geq 0$, *let*

$$V_2(n) = \begin{pmatrix} S_2(n) \\ S_2(2n) \end{pmatrix}.$$

*Consider the matrix-valued map* $\mu_2 \colon \{0,1\}^* \to \mathbb{Z}_2^2$ *defined by*

$$\mu_2(0) = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, \quad \mu_2(1) = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}.$$

*Then* $V_2(2n+r) = \mu_2(r)V_2(n)$ *for all* $r \in \{0,1\}$ *and* $n \geq 0$. *Consequently, if* $\mathrm{rep}_2(n) = c_k \cdots c_0$, *then*

$$S_2(n) = \begin{pmatrix} 1 & 0 \end{pmatrix} \mu_2(c_0) \cdots \mu_2(c_k) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

*Proof*. Thanks to Theorem 3.24, we have

$$V_2(2n) = \begin{pmatrix} S_2(2n) \\ S_2(4n) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} S_2(n) \\ S_2(2n) \end{pmatrix} = \mu_2(0)V_2(n),$$

and

$$V_2(2n+1) = \begin{pmatrix} S_2(2n+1) \\ S_2(4n+2) \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} S_2(n) \\ S_2(2n) \end{pmatrix} = \mu_2(1)V_2(n)$$

for all $n \geq 0$. Let $r = \sum_{i=0}^{\ell-1} r_i\, 2^i$. Then the word $r_{\ell-1} \cdots r_0$ is a representation of $r$ in base 2 possibly with leading zeroes. By induction, we can show that

$$V_2(2^\ell m + r) = \mu_2(r_0 \cdots r_{\ell-1})V_2(m) \tag{3.2}$$

for all $m \in \mathbb{N}$.

Now let $n \geq 2$. Then there exist $\ell \geq 1$ and $r \in \{0, \dots, 2^\ell - 1\}$ such that $n = 2^\ell + r$. Let $r_{\ell-1} \cdots r_0$ be a representation of $r$ in base 2 possibly with leading zeroes. Using (3.2) and the fact that $V_2(1) = \mu_2(1)V_2(0)$, we get

$$V_2(n) = \mu_2(r_0 \cdots r_{\ell-1})V_2(1) = \mu_2(r_0 \cdots r_{\ell-1}1)V_2(0) = \mu_2((\mathrm{rep}_2(n))^R)V_2(0),$$

where $u^R$ is the reversal of the word $u$. Note that the previous equality also holds for $n \in \{0,1\}$. In particular, we have the following equality

$$S_2(n) = \begin{pmatrix} 1 & 0 \end{pmatrix} \mu_2((\mathrm{rep}_2(n))^R) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

for all $n \in \mathbb{N}$ since the components of $V_2(0)$ are both $S_2(0) = 1$. □

### 3.1.5 Non 2-Synchronicity

Recall that, among the unbounded 2-regular sequences, the 2-synchronized sequences are those that can be "computed" with a finite automaton; see Section 1.7 and Definition 1.58. In the present section, we shortly discuss the fact that $(S_2(n))_{n\geq 0}$ fails to be 2-synchronized for which we give two independent proofs. As a consequence, $(S_2(n))_{n\geq 0}$ is not 2-automatic.

**Proposition 3.26.** *The sequence $(S_2(n))_{n\geq 0}$ is not 2-synchronized.*

*Proof.* Proceed by contradiction, and suppose that there is a deterministic $k$-state automaton that accepts exactly the language $\{\text{rep}_2(n, S_2(n)) \mid n \in \mathbb{N}\}$. Note that for all $\ell \geq 0$, $S_2(2^\ell) = \ell + 2$ using (1.4) on page 23. Consider an integer $\ell$ such that $\ell - \lceil \log_2(\ell + 2)\rceil > k + 1$. Then we have

$$\text{rep}_2(2^\ell, \ell + 2) = (10^\ell, 0^{k+2}u) \tag{3.3}$$

for a word $u \in \{0,1\}^*$ of length $\ell - k - 1$ and such that $\text{val}_2(u) = \ell + 2$. Let $(q_0, q_1, \dots, q_{\ell+2})$ be the path in the automaton starting in the initial state $q_0$ and whose label is $\text{rep}_2(2^\ell, \ell + 2)$. Since $\text{rep}_2(2^\ell, \ell + 2)$ is accepted by the automaton, note that the state $q_{\ell+2}$ is an accepting state. We start by reading $(10^{k+1}, 0^{k+2})$. As the automaton has $k$ states, there exist $1 \leq i < j \leq k + 2$ such that $q_i = q_j$ by the pigeonhole principle. By choice of $\ell$ (also recall (3.3)), there is a path (even a loop) from $q_i$ to $q_j$ whose label is $(0^{j-i}, 0^{j-i})$. Thus, the pair of words $(10^{\ell+j-i}, 0^{k+2+j-i}u)$ is accepted by the automaton. However, we have

$$S_2(\text{val}_2(10^{\ell+j-i})) = S_2(2^{\ell+j-i}) = \ell + j - i + 2 \neq \ell + 2 = \text{val}_2(0^{k+2+j-i}u),$$

which is a contradiction. □

The fact that the sequence $(S_2(n))_{n\geq 0}$ is not 2-synchronized also follows from the next result.

**Lemma 3.27.** *[SS16, Lemma 4] If $s = (s(n))_{n\geq 0}$ is a $b$-synchronized sequence and $s \neq O(1)$, then there exists a constant $C > 0$ such that $s(n) \geq Cn$ infinitely often.*

The joint spectral radius [BKPW08] of a $b$-regular sequence is a good indication of its growth rate [Dum13]. One can numerically estimate that the joint spectral radius $\rho$ of the matrices $\mu_2(0)$ and $\mu_2(1)$ is between 1.61 and 1.71 (for instance, using the *SageMath* program, one can choose the length of the interval in which the joint spectral radius falls, *i.e.*, one can decide the precision of the estimation; in our case, a rough estimate is enough). For bounds, we refer the reader to [The05]. Since, for all $n \geq 0$, $S_2(n)$ can be computed by multiplying those matrices, and the number of multiplications is proportional to $\log_2(n)$ (see Corollary 3.25), it follows that the growth rate of $S_2(n)$ cannot be greater than $\rho^{\log_2(n)} = n^{\log_2(\rho)}$ multiplied by a constant. By Lemma 3.27, the sequence $(S_2(n))_{n\geq 0}$ cannot be 2-synchronized. Note that we can also make use of the growth order of the Stern–Brocot sequence. In fact, its joint spectral radius is exactly the golden ratio [BC18].

## 3.2 The General Case of Integer Bases

In this section, we fix an integer $b \geq 2$, and we investigate the general case of the base-$b$ numeration system of Example 1.17. We consider the generalized Pascal triangle $\mathrm{P}_b$ from Definition 1.42 and the sequence $(S_b(n))_{n\geq 0}$ from Definition 1.47. Recall that the 2-regularity of $(S_2(n))_{n\geq 0}$ particularly follows from its link to the Stern–Brocot sequence (see the beginning of Section 3.1.4). In the general integer base case, $(S_b(n))_{n\geq 0}$ is not known to be related to already existing regular sequences, so the technique developed previously acquires real meaning here. More precisely, we use tries of scattered subwords defined in Section 3.1.1 for this larger context in order to obtain results similar to Proposition 3.6, Corollary 3.9 and more importantly Proposition 3.15. The corresponding results are stated in Propositions 3.29 and 3.32, and Corollary 3.34. Note that the relations in Proposition 3.29 startlingly reduce to three forms. We also show that the sequence $(S_b(n))_{n\geq 0}$ is palindromic over intervals of the form $[(b-1)b^\ell, b^{\ell+1}]$ in Proposition 3.44. Afterwards, as for the case $b = 2$, we show in Section 3.2.2 that $(S_b(n))_{n\geq 0}$ is $b$-regular, but not $b$-synchronized. Moreover, we obtain a linear representation of the sequence with square matrices of size $b$.

**Example 3.28.** If $b = 3$, then we consider $\mathrm{P}_3$ and the sequence $(S_3(n))_{n\geq 0}$ which starts with

$$1, 2, 2, 3, 3, 4, 3, 4, 3, 4, 5, 6, 5, 4, 6, 7, 7, 6, 4, 6, 5, 7, 6, 7, 5, 6, 4, 5, 7, 8, 8, 7, \ldots,$$

and which is depicted in Figure 3.6 (observe the similarity with Figure 3.1).

They are respectively indexed by `A284441` and `A282715` in [Slo]. For instance, the scattered subwords of the word 121 are $\varepsilon, 1, 2, 11, 12, 21, 121$. Thus, $S_3(\mathrm{val}_3(121)) = S_3(16) = 7$.

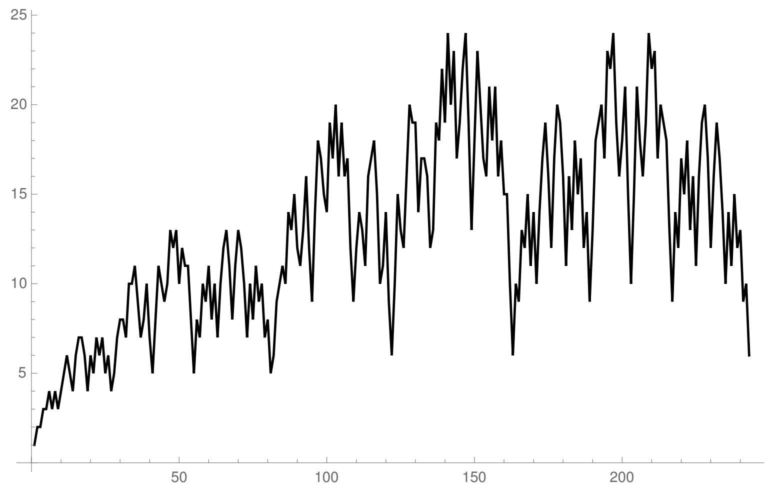



Figure 3.6: The sequence $(S_3(n))_{n\geq 0}$ in the interval $[0, 3^5]$.

Most of the results in this section are proved by induction, and the base case usually takes into account the values of $S_b(n)$ for $0 \leq n < b^2$. They are easily obtained from Definition 1.47 and summarized in Table 3.7.

| $\mathrm{rep}_b(n)$ | $\varepsilon$ | $x$ | $x0$ | $xx$ | $xy$ | $x00$ | $x0x$ | $x0y$ |
|---|---|---|---|---|---|---|---|---|
| $S_b(n)$ | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |

| $\mathrm{rep}_b(n)$ | $xx0$ | $xxx$ | $xxy$ | $xy0$ | $xyx$ | $xyy$ | $xyz$ |
|---|---|---|---|---|---|---|---|
| $S_b(n)$ | 5 | 4 | 6 | 7 | 7 | 6 | 8 |

Table 3.7: The first few values of $S_b(n)$ for $0 \leq n < b^3$, with pairwise distinct $x, y, z \in \{1, \ldots, b-1\}$.

### 3.2.1 A General Recurrence Relation

The aim of this section is to exhibit recurrence relations satisfied by the sequence $(S_b(n))_{n\geq 0}$. They also turn out to be useful to prove results related to summatory functions in Chapter 4. Observe that we divide the following statement into different cases depending on the first letters of the base-$b$ expansion of integers.

**Proposition 3.29.** *The sequence* $(S_b(n))_{n\geq 0}$ *satisfies* $S_b(0) = 1$,

$$S_b(1) = \cdots = S_b(b-1) = 2,$$

*and, for all* $x, y \in \{1, \ldots, b-1\}$ *with* $x$ *and* $y$ *distinct, all* $\ell \geq 1$ *and all* $r \in \{0, \ldots, b^{\ell-1} - 1\}$,

$$S_b(xb^\ell + r) = S_b(xb^{\ell-1} + r) + S_b(r), \tag{3.4}$$

$$S_b(xb^\ell + xb^{\ell-1} + r) = 2S_b(xb^{\ell-1} + r) - S_b(r), \tag{3.5}$$

$$S_b(xb^\ell + yb^{\ell-1} + r) = S_b(xb^{\ell-1} + r) + 2S_b(yb^{\ell-1} + r) - 2S_b(r). \tag{3.6}$$

To prove Proposition 3.29 (and the $b$-regularity of the sequence $(S_b(n))_{n\geq 0}$ as we will see later on), we use tries of scattered subwords, which is a tool introduced in Section 3.1.1, and more precisely the tree $\mathcal{T}_{A_b,L_b}(w) = \mathcal{T}_{L_b}(w)$ for $w \in L_b$. Definition 3.4 is adapted in the following way.

**Definition 3.30.** For each non-empty word $w \in L_b$, we consider a factorization of $w$ into maximal blocks of consecutively distinct letters (*i.e.*, $a_i \neq a_{i+1}$ for all $i$) of the form

$$w = a_1^{n_1} \cdots a_M^{n_M},$$

with $n_\ell \geq 1$ for all $\ell$. Note that for $k - i \geq 2$, one could possibly have $a_k = a_i$.

For each $\ell \in \{0, \ldots, M-1\}$, we consider the subtree $T_\ell$ of $\mathcal{T}_{L_b}(w)$ whose root is the node $a_1^{n_1} \cdots a_\ell^{n_\ell} a_{\ell+1}$. Once again, we conveniently set $T_M$ to be an empty tree with no node. Roughly speaking, we have a root of a new subtree $T_\ell$ for each new alternation of digits in $w$.

For each $\ell \in \{0, \ldots, M-1\}$, we let $\mathrm{Alph}(\ell)$ denote the set of letters occurring in $a_{\ell+1} \cdots a_M$. Then for each letter $a \in \mathrm{Alph}(\ell)$, we let $j(a, \ell)$ denote the smallest index $i$ in $\{\ell + 1, \ldots, M\}$ such that $a_i = a$. Differently put, $j(a, \ell)$ denotes the first occurrence of $a$ in $a_{\ell+1} \cdots a_M$.

**Example 3.31.** In this example, we take $b = 3$ and $w = 22000112 \in L_3$. The tree $\mathcal{T}_{L_3}(22000112)$ is depicted in Figure 3.8. We use three different colors to represent the letters $0, 1, 2$: green for 0, black for 1 and gray for

2. Using the previous notation, we have $M = 4$, $a_1 = 2$, $n_1 = 2$, $a_2 = 0$, $n_2 = 3$, $a_3 = 1$, $n_3 = 2$, $a_4 = 2$ and $n_4 = 1$. The tree $T_0$ (resp., $T_1$; resp., $T_2$; resp., $T_3$) is the subtree of $\mathcal{T}_{L_3}(w)$ with root 2 (resp., $2^2 0$; resp., $2^2 0^3 1$; resp., $2^2 0^3 1^2 2$). These subtrees are represented in Figure 3.8 using dashed lines. The tree $T_3$ is limited to a single node since the number of nodes of $T_{M-1}$ is $n_M$, which is equal to 1 in this example.

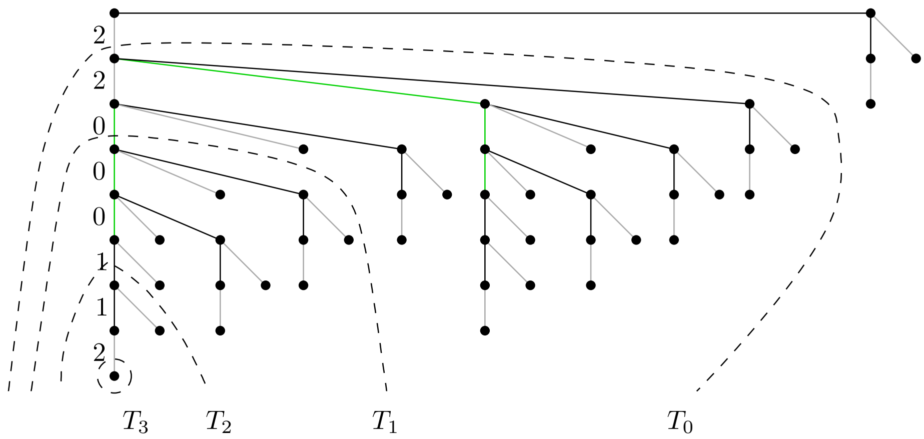



Figure 3.8: The trie $\mathcal{T}_{L_3}(22000112)$.

The sets $\mathrm{Alph}(\ell)$ and the corresponding indices $j(a, \ell)$ are given in Table 3.9. To determine $j(2, 0)$, we have to look at the indices $i \in \{1, 2, 3, 4\}$ such that $a_i = 2$. Since $i = 1$ and $i = 4$ both do the job, we have $j(2, 0) = 1$.

| $\ell$ | $\mathrm{Alph}(\ell)$ | $j(a,\ell)$ with $a \in \mathrm{Alph}(\ell)$ |
|---|---|---|
| 0 | $\{0,1,2\}$ | $j(0,0) = 2, j(1,0) = 3, j(2,0) = 1$ |
| 1 | $\{0,1,2\}$ | $j(0,1) = 2, j(1,1) = 3, j(2,1) = 4$ |
| 2 | $\{1,2\}$ | $j(1,2) = 3, j(2,2) = 4$ |
| 3 | $\{2\}$ | $j(2,3) = 4$ |

Table 3.9: The sets $\mathrm{Alph}(\ell)$ and the corresponding indices $j(a, \ell)$ for the word $w = 22000112 \in L_3$.

The next result describes the structure of the tree $\mathcal{T}_{L_b}(w)$ for $w \in L_b$. It follows from Definition 3.30 and generalizes Proposition 3.6.

**Proposition 3.32.** *Let $w$ be a non-empty word in $L_b$. The tree $\mathcal{T}_{L_b}(w)$ has the following properties.*

- *Every letter $a \in \mathrm{Alph}(0) \setminus \{0\}$ is a child of the node of label $\varepsilon$. This particular node has thus $\#(\mathrm{Alph}(0)) - 1$ children. Each child $a$ is the root of a tree isomorphic $T_{j(a,0)-1}$.*
- *For each $\ell \in \{0, \ldots, M-1\}$ and each $i \in \{0, \ldots, n_{\ell+1} - 1\}$ such that $(\ell, i) \neq (0, 0)$, the node of label $x = a_1^{n_1} \cdots a_\ell^{n_\ell} a_{\ell+1}^i$ has $\#(\mathrm{Alph}(\ell))$ children that are $xa$ for $a \in \mathrm{Alph}(\ell)$. Each child $xa$ with $a \neq a_{\ell+1}$ is the root of a tree isomorphic to $T_{j(a,\ell)-1}$.*

**Example 3.33.** From Example 3.31, we see that the node of label $\varepsilon$ has two children $a \in \mathrm{Alph}(0) \setminus \{0\} = \{1, 2\}$. Its child 1 (resp., 2) is the root of a copy of $T_{j(1,0)-1} = T_2$ (resp., $T_{j(2,0)-1} = T_0$). Let us illustrate the second part of the statement with $\ell = 0$. For $0 < i \leq n_1 - 1 = 1$, the node of label $a_1^i = 2^i$ has $\#(\mathrm{Alph}(0)) = 3$ children, which are $2^i a$ for $a \in \{0, 1, 2\}$. Observe that this is not true if $i = 0$, which corresponds to the node of label $\varepsilon$ that cannot have 0 as a child. For $i = 1$, the child $2^1 0$ (resp., $2^1 1$) is the root of a copy of $T_{j(0,0)-1} = T_1$ (resp., $T_{j(1,0)-1} = T_2$).

The next result, which extends Corollary 3.9, permits us to compute the number of nodes of $T_\ell$ for all $\ell \in \{0, \ldots, M\}$.

**Corollary 3.34.** *Let $w$ be a non-empty word in $L_b$. The number of nodes in $T_{M-1}$ (resp., $T_M$) is $n_M$ (resp., 0). For $\ell \in \{0, \ldots, M-2\}$, the number of nodes in $T_\ell$ is given by*[1]

$$\#T_\ell = n_{\ell+1} \left( 1 + \sum_{\substack{a \in \mathrm{Alph}(\ell+1) \\ a \neq a_{\ell+1}}} \#T_{j(a,\ell+1)-1} \right) + \#T_{j(a_{\ell+1},\ell+1)-1}.$$

*The number of distinct scattered subwords of $w$ in $L_b$ is given by*

$$1 + \#T_0 + \sum_{\substack{a \in \mathrm{Alph}(0) \\ a \neq 0, a_1}} \#T_{j(a,0)-1} = 1 + \sum_{\substack{a \in \mathrm{Alph}(0) \\ a \neq 0}} \#T_{j(a,0)-1}.$$

[1] If $j < 0$, then $T_j$ is set to be an empty tree.

**Remark 3.35.** As for the base-2 case, if we were interested in the total number of distinct scattered subwords of $w$ (not only those in $L_b$), we should add the node 0, which will be the root of a subtree isomorphic to $T_{j(0,0)-1}$. Thus, the total number of distinct scattered subwords occurring in $w$ is

$$1 + \#T_0 + \sum_{\substack{a \in \mathrm{Alph}(0) \\ a \neq a_1}} \#T_{j(a,0)-1} = 1 + \sum_{a \in \mathrm{Alph}(0)} \#T_{j(a,0)-1}.$$

**Example 3.36.** Let us continue Examples 3.31 and 3.33. We have $\#T_4 = 0$ and $\#T_3 = n_4 = 1$. Using Corollary 3.34, since $n_3 = 2$ and $j(2,3) = 4$, we get $\#T_2 = 2(1 + \#T_3) + 0 = 4$. Since $n_2 = 3$, $j(1,2) = 3$ and $j(2,2) = 4$, we have

$$\#T_1 = 3(1 + \#T_2 + \#T_3) + 0 = 18.$$

Similarly, since $n_1 = 2$, $j(0,1) = 2$, $j(1,1) = 3$ and $j(2,1) = 4$, we find

$$\#T_0 = 2(1 + \#T_1 + \#T_2) + \#T_3 = 47.$$

Finally, the only letter $a \in \mathrm{Alph}(0) \setminus \{0, 2\}$ is 1 for which $j(1,0) = 3$. The number of distinct scattered subwords of $w$ in $L_3$ is then

$$1 + \#T_0 + \#T_2 = 52,$$

and since $\mathrm{val}_3(22000112) = 5846$, $S_3(5846) = 52$. Moreover, the total number of distinct scattered subwords of $w$ is

$$(1 + \#T_0 + \#T_2) + \#T_1 = 52 + 18 = 70$$

because we simply have to add $\#T_{j(0,0)-1}$, which is $\#T_1$ in this example.

Using tries of scattered subwords, we prove the following five lemmas, echoing Lemmas 3.13 and 3.14. Their proofs are essentially the same, so we only prove two of them. They particularly lead to a proof of Proposition 3.29.

**Lemma 3.37.** *For each letter $x \in \{1, \dots, b-1\}$ and each finite word $u$ over $\{0, \dots, b-1\}$, we have*

$$\begin{aligned}\#\left\{v \in L_b \mid \binom{x00u}{v} > 0\right\} =& 2 \cdot \#\left\{v \in L_b \mid \binom{x0u}{v} > 0\right\} \\ &- \#\left\{v \in L_b \mid \binom{xu}{v} > 0\right\}.\end{aligned}$$

*Proof*. From Remark 3.2, we need to prove that

$$\#\mathcal{T}_{L_b}(x00u) = 2\#\mathcal{T}_{L_b}(x0u) - \#\mathcal{T}_{L_b}(xu).$$

Assume first that $u$ is of the form $u = 0^n$ with $n \geq 0$. The tree $\mathcal{T}_{L_b}(xu)$ is linear and has $n+2$ nodes, $\mathcal{T}_{L_b}(x0u)$ has $n+3$ nodes, and $\mathcal{T}_{L_b}(x00u)$ has $n+4$ nodes. The formula holds.

Now suppose that $u$ contains letters other than 0. We let $a_1, \ldots, a_m$ denote all the pairwise distinct letters of $u$ different from 0. They are implicitly ordered with respect to their first appearance in $u$. If $x \in \{a_1, \ldots, a_m\}$, we let $i_x \in \{1, \ldots, m\}$ denote the index such that $a_{i_x} = x$. For all $i \in \{1, \ldots, m\}$, we let $u_i a_i$ denote the prefix of $u$ that ends with the first occurrence of the letter $a_i$ in $u$, and we let $R_i$ denote the subtree of $\mathcal{T}_{L_b}(xu)$ with root $xu_i a_i$.

First, observe that the subtree $T$ of $\mathcal{T}_{L_b}(xu)$ with root $x$ is equal to the subtree of $\mathcal{T}_{L_b}(x0u)$ with root $x0$ and also to the subtree of $\mathcal{T}_{L_b}(x00u)$ with root $x00$.

Secondly, for all $i \in \{1, \ldots, m\}$, the subtree of $\mathcal{T}_{L_b}(x0u)$ with root $xa_i$ is $R_i$. Similarly, $\mathcal{T}_{L_b}(x00u)$ contains two copies of $R_i$: the subtrees of roots $xa_i$ and $x0a_i$.

Finally, for all $i \in \{1, \ldots, m\}$ with $i \neq i_x$, the subtree of $\mathcal{T}_{L_b}(x0u)$ with root $a_i$ is $R_i$, and the subtree of $\mathcal{T}_{L_b}(x00u)$ with root $a_i$ is $R_i$.

The situation is depicted in Figure 3.10 where we put a unique edge for several indices when necessary, *e.g.*, the edge labeled by $a_i$ stands for $m$ edges labeled by $a_1, \ldots, a_m$. The claimed formula holds since

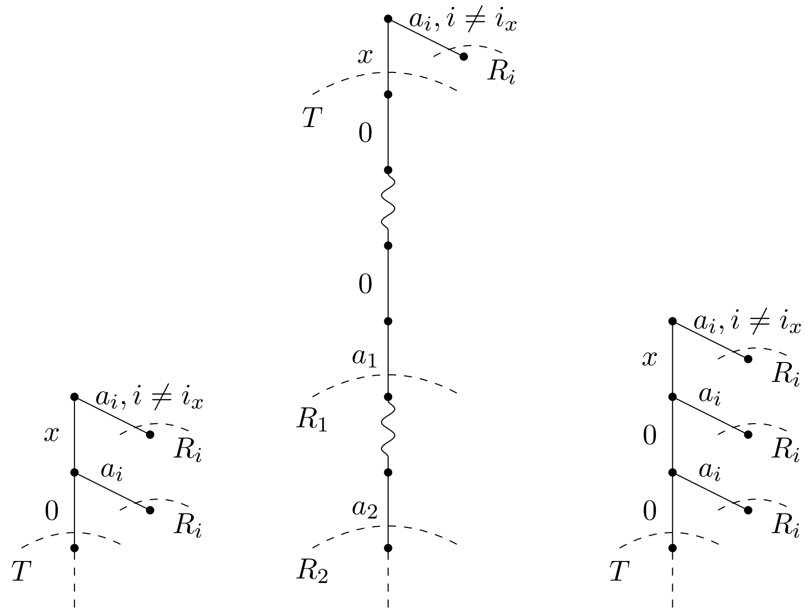
$$\begin{aligned} 2 \cdot &\left( 2 + \#T + 2 \sum_{\substack{1 \leq i \leq m \\ i \neq i_x}} \#R_i + \#R_{i_x} \right) - \left( 1 + \#T + \sum_{\substack{1 \leq i \leq m \\ i \neq i_x}} \#R_i \right) \\ &= 3 + \#T + 3 \sum_{\substack{1 \leq i \leq m \\ i \neq i_x}} \#R_i + 2\#R_{i_x}. \end{aligned}$$

□

**Lemma 3.38.** *For each letter $x \in \{1, \ldots, b-1\}$ and each finite word $u$ over $\{0, \ldots, b-1\}$, we have*

$$\begin{aligned} \#\left\{ v \in L_b \mid \binom{xx0u}{v} > 0 \right\} =& \#\left\{ v \in L_b \mid \binom{x0u}{v} > 0 \right\} \\ &+ \#\left\{ v \in L_b \mid \binom{xu}{v} > 0 \right\}. \end{aligned}$$

*Proof*. The proof is analogous to the proof of Lemma 3.37. □




(a) The tree $\mathcal{T}(x0u)$. (b) The tree $\mathcal{T}(xu)$. (c) The tree $\mathcal{T}(x00u)$.

Figure 3.10: Schematic structure of the trees $\mathcal{T}(x0u)$, $\mathcal{T}(xu)$ and $\mathcal{T}(x00u)$.

**Lemma 3.39.** *For all letters $x, y \in \{1, \ldots, b-1\}$ and each finite word $u$ over $\{0, \ldots, b-1\}$, we have*

$$\#\left\{v \in L_b \mid \binom{x0yu}{v} > 0\right\} = \#\left\{v \in L_b \mid \binom{xyu}{v} > 0\right\} + \#\left\{v \in L_b \mid \binom{yu}{v} > 0\right\}.$$

*Proof*. The proof is similar to the proof of Lemma 3.37. Observe that one needs to divide the proof into two cases according to whether $x$ is equal to $y$ or not. As a first case, also consider $u = y^n$ with $n \geq 0$ instead of $u = 0^n$ with $n \geq 0$. □

**Lemma 3.40.** *For all letters $x, y \in \{1, \ldots, b-1\}$ and each finite word $u$ over $\{0, \ldots, b-1\}$, we have*

$$\#\left\{v \in L_b \mid \binom{xxyu}{v} > 0\right\} = 2 \cdot \#\left\{v \in L_b \mid \binom{xyu}{v} > 0\right\} - \#\left\{v \in L_b \mid \binom{yu}{v} > 0\right\}.$$

*Proof*. The proof is akin to the proof of Lemma 3.39. □

The next lemma has a slightly more technical proof, so we present it.

**Lemma 3.41.** *For all pairwise distinct letters $x, y \in \{1, \ldots, b-1\}$, each letter $z \in \{0, \ldots, b-1\}$ and each finite word $u$ over $\{0, \ldots, b-1\}$, the following equality holds*

$$\begin{aligned}\#\left\{v \in L_b \mid \binom{xyzu}{v} > 0\right\} =& \#\left\{v \in L_b \mid \binom{xzu}{v} > 0\right\} \\ &+ 2 \cdot \#\left\{v \in L_b \mid \binom{yzu}{v} > 0\right\} \\ &- 2 \cdot \#\left\{v \in L_b \mid \binom{\mathrm{rep}_b(\mathrm{val}_b(zu))}{v} > 0\right\}.\end{aligned}$$

*Proof*. Let $x, y \in \{1, \ldots, b-1\}$ with $x \neq y$, $z \in \{0, \ldots, b-1\}$, and let $u \in \{0, \ldots, b-1\}^*$. Our reasoning is again based on the structure of the associated trees $\mathcal{T}_{L_b}(xyzu)$, $\mathcal{T}_{L_b}(xzu)$, $\mathcal{T}_{L_b}(yzu)$ and $\mathcal{T}_{L_b}(\mathrm{rep}_b(\mathrm{val}_b(zu)))$. The proof is divided into two cases depending on whether $z = 0$ or not.

• As a first case, suppose that $z \neq 0$. Then $\mathrm{rep}_b(\mathrm{val}_b(zu)) = zu$. Now we consider two subcases according to whether $u$ is only made of letters $z$ or not.

Assume that $u$ is of the form $u = z^n$ with $n \geq 0$.

If $x \neq z$ and $y \neq z$, the tree $\mathcal{T}_{L_b}(zu)$ is linear and has $n+2$ nodes, $\mathcal{T}_{L_b}(xzu)$ and $\mathcal{T}_{L_b}(yzu)$ have $2(n+2)$ nodes, and $\mathcal{T}_{L_b}(xyzu)$ has $4(n+2)$ nodes. The claimed formula holds.

If $x \neq z$ and $y = z$, the tree $\mathcal{T}_{L_b}(zu)$ is linear and has $n+2$ nodes, $\mathcal{T}_{L_b}(xzu)$ has $2(n+2)$ nodes, $\mathcal{T}_{L_b}(yzu)$ has $n+3$ nodes, and $\mathcal{T}_{L_b}(xyzu)$ has $2(n+3)$ nodes. Again, the announced formula holds.

If $x = z$ and $y \neq z$, the tree $\mathcal{T}_{L_b}(zu)$ is linear and has $n+2$ nodes, $\mathcal{T}_{L_b}(xzu)$ has $n+3$ nodes, $\mathcal{T}_{L_b}(yzu)$ has $2(n+2)$ nodes, and $\mathcal{T}_{L_b}(xyzu)$ has

$3(n+2)+1$ nodes. The desired formula holds too.

Now suppose that $u$ contains letters other than $z$. We let $a_1, \dots, a_m$ denote all the pairwise distinct letters of $u$ different from $z$. They are implicitly ordered with respect to their first appearance in $u$. If $x, y, 0 \in \{a_1, \dots, a_m\}$, we let $i_x, i_y, i_0 \in \{1, \dots, m\}$ respectively denote the indices such that $a_{i_x} = x$, $a_{i_y} = y$ and $a_{i_0} = 0$. For all $i \in \{1, \dots, m\}$, we let $u_i a_i$ denote the prefix of $u$ that ends with the first occurrence of the letter $a_i$ in $u$, and we let $R_i$ denote the subtree of $\mathcal{T}_{L_b}(zu)$ with root $zu_i a_i$.

First, observe that the subtree $T$ of $\mathcal{T}_{L_b}(zu)$ with root $z$ is equal to the subtree of $\mathcal{T}_{L_b}(xzu)$ with root $xz$, to the subtree of $\mathcal{T}_{L_b}(yzu)$ with root $yz$ and also to the subtree of $\mathcal{T}_{L_b}(xyzu)$ with root $xyz$.

Suppose that $x \neq z$ and $y \neq z$. Using the same reasoning as in the proof of Lemma 3.37, the situation is depicted in Figure 3.11. For all $i \in \{1, \dots, m\}$, the subtree of $\mathcal{T}_{L_b}(xzu)$ with root $xa_i$ is $R_i$. Similarly, the subtree of $\mathcal{T}_{L_b}(yzu)$ with root $ya_i$ is $R_i$. The tree $\mathcal{T}_{L_b}(xyzu)$ contains two copies of $R_i$: the subtrees of roots $ya_i$ and $xya_i$. Furthermore, for all $i \in \{1, \dots, m\}$ such that $i \neq i_0$ (resp., $i \notin \{i_x, i_0\}$; resp., $i \notin \{i_y, i_0\}$; resp., $i \notin \{i_x, i_y, i_0\}$), the subtree of $\mathcal{T}_{L_b}(zu)$ (resp., $\mathcal{T}_{L_b}(xzu)$; resp., $\mathcal{T}_{L_b}(yzu)$; resp., $\mathcal{T}_{L_b}(xyzu)$) with root $a_i$ is $R_i$. The expected formula holds since

$$\begin{aligned}
&\left(2 + 2\#T + 2 \sum_{\substack{1 \le i \le m \\ i \neq i_x, i_y, i_0}} \#R_i + \#R_{i_x} + 2\#R_{i_y} + \#R_{i_0}\right) \\
&+ 2 \cdot \left(2 + 2\#T + 2 \sum_{\substack{1 \le i \le m \\ i \neq i_x, i_y, i_0}} \#R_i + 2\#R_{i_x} + \#R_{i_y} + \#R_{i_0}\right) \\
&- 2 \cdot \left(1 + \#T + \sum_{\substack{1 \le i \le m \\ i \neq i_x, i_y, i_0}} \#R_i + \#R_{i_x} + \#R_{i_y}\right) \\
&= 4 + 4\#T + 4 \sum_{\substack{1 \le i \le m \\ i \neq i_x, i_y, i_0}} \#R_i + 3\#R_{i_x} + 2\#R_{i_y} + 3\#R_{i_0}.
\end{aligned}$$

Suppose that $x \neq z$ and $y = z$. The situation is depicted in Figure 3.12.

The requested formula holds since

$$\begin{aligned}
&\left(2+2\#T+2\sum_{\substack{1\le i\le m\\ i\ne i_x,i_0}}\#R_i+\#R_{i_x}+\#R_{i_0}\right)\\
&+2\cdot\left(2+\#T+2\sum_{\substack{1\le i\le m\\ i\ne i_x,i_0}}\#R_i+2\#R_{i_x}+\#R_{i_0}\right)\\
&-2\cdot\left(1+\#T+\sum_{\substack{1\le i\le m\\ i\ne i_x,i_0}}\#R_i+\#R_{i_x}\right)\\
&=4+2\#T+4\sum_{\substack{1\le i\le m\\ i\ne i_x,i_0}}\#R_i+3\#R_{i_x}+3\#R_{i_0}.
\end{aligned}$$

Suppose that $x = z$ and $y \neq z$. The situation is depicted in Figure 3.13. The sought formula holds since

$$\begin{aligned}
&\left(2+\#T+2\sum_{\substack{1\le i\le m\\ i\ne i_y,i_0}}\#R_i+2\#R_{i_y}+\#R_{i_0}\right)\\
&+2\cdot\left(2+2\#T+2\sum_{\substack{1\le i\le m\\ i\ne i_y,i_0}}\#R_i+\#R_{i_y}+\#R_{i_0}\right)\\
&-2\cdot\left(1+\#T+\sum_{\substack{1\le i\le m\\ i\ne i_y,i_0}}\#R_i+\#R_{i_y}\right)\\
&=4+3\#T+4\sum_{\substack{1\le i\le m\\ i\ne i_y,i_0}}\#R_i+2\#R_{i_y}+3\#R_{i_0}.
\end{aligned}$$

• As a second case, suppose that $z = 0$. It is useful to note that $\mathrm{rep}_b(\mathrm{val}_b(\cdot))\colon \{0,\ldots,b-1\}^* \to L_b$ plays a normalization role in the sense that it removes leading zeroes. Consequently, $\mathrm{rep}_b(\mathrm{val}_b(zu)) = \mathrm{rep}_b(\mathrm{val}_b(u))$.

In this case, we must prove that the following formula holds

$$\begin{aligned}\#\left\{v \in L_b \mid \binom{xy0u}{v} > 0\right\} =& \#\left\{v \in L_b \mid \binom{x0u}{v} > 0\right\}\\ &+ 2 \cdot \#\left\{v \in L_b \mid \binom{y0u}{v} > 0\right\}\\ &- 2 \cdot \#\left\{v \in L_b \mid \binom{\mathrm{rep}_b(\mathrm{val}_b(u))}{v} > 0\right\}.\end{aligned}$$

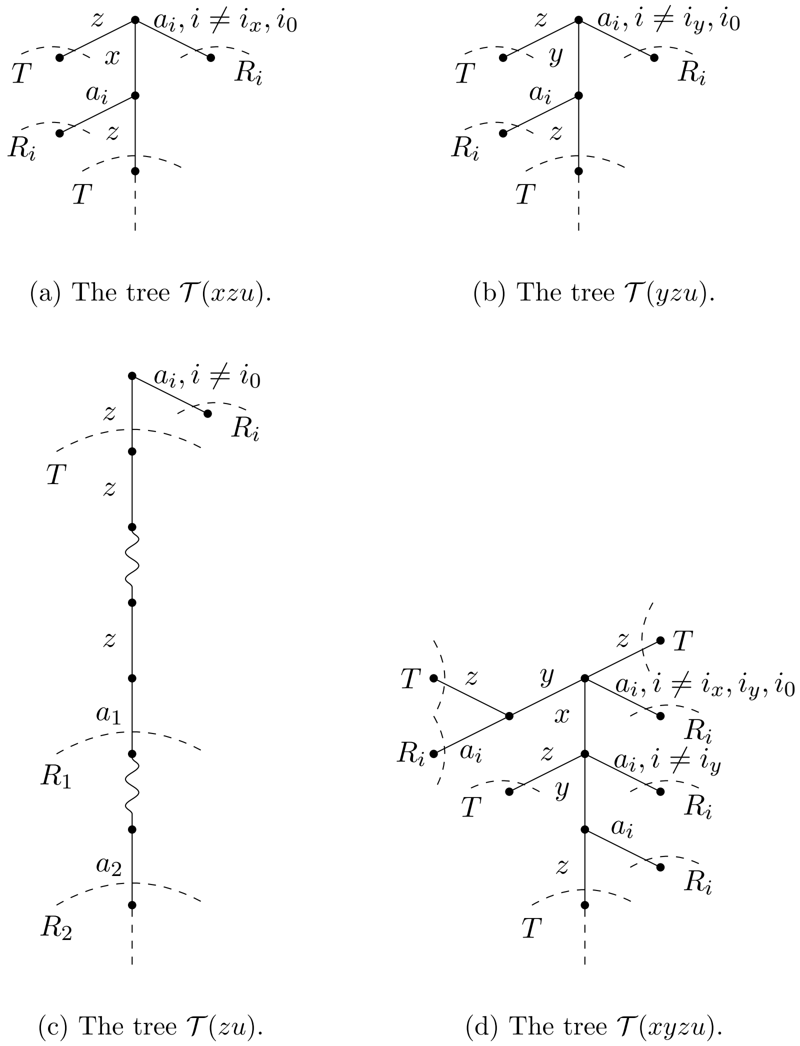
If $u = 0^n$ with $n \geq 0$, $\mathrm{rep}_b(\mathrm{val}_b(u)) = \varepsilon$, and the tree $\mathcal{T}_{L_b}(\mathrm{rep}_b(\mathrm{val}_b(u)))$ has only one node. The trees $\mathcal{T}_{L_b}(x0u)$ and $\mathcal{T}_{L_b}(y0u)$ both have $n+3$ nodes whereas the tree $\mathcal{T}_{L_b}(xy0u)$ has $3(n+2)+1$ nodes. The wished formula holds.

Now suppose that $u$ contains letters other than 0. We let $a_1, \ldots, a_m$ denote all the pairwise distinct letters of $u$ different from 0. They are implicitly ordered with respect to their first appearance in $u$. If $x, y \in \{a_1, \ldots, a_m\}$, we let $i_x, i_y \in \{1, \ldots, m\}$ respectively denote the indices such that $a_{i_x} = x$ and $a_{i_y} = y$. For all $i \in \{1, \ldots, m\}$, we let $u_i' a_i$ denote the prefix of $\mathrm{rep}_b(\mathrm{val}_b(u))$ that ends with the first occurrence of the letter $a_i$ in $\mathrm{rep}_b(\mathrm{val}_b(u))$, and we let $R_i$ denote the subtree of $\mathcal{T}_{L_b}(\mathrm{rep}_b(\mathrm{val}_b(u)))$ with root $u_i' a_i$.

The situation is depicted in Figure 3.14. Observe that the subtree $T$ of $\mathcal{T}_{L_b}(x0u)$ with root $x0$ is equal to the subtree of $\mathcal{T}_{L_b}(y0u)$ with root $y0$ and to the subtree of $\mathcal{T}_{L_b}(xy0u)$ with root $xy0$. The required formula holds since

$$\begin{aligned}&\left(2 + \#T + 2 \sum_{\substack{1 \leq i \leq m\\ i \neq i_x, i_y}} \#R_i + \#R_{i_x} + 2\#R_{i_y}\right)\\ &+ 2 \cdot \left(2 + \#T + 2 \sum_{\substack{1 \leq i \leq m\\ i \neq i_x, i_y}} \#R_i + 2\#R_{i_x} + \#R_{i_y}\right)\\ &- 2 \cdot \left(1 + \sum_{\substack{1 \leq i \leq m\\ i \neq i_x, i_y}} \#R_i + \#R_{i_x} + \#R_{i_y}\right)\\ &= 4 + 3\#T + 4 \sum_{\substack{1 \leq i \leq m\\ i \neq i_x, i_y}} \#R_i + 3\#R_{i_x} + 2\#R_{i_y}.\end{aligned}$$

□



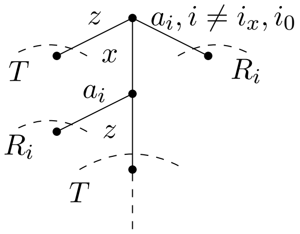

(a) The tree $\mathcal{T}(xzu)$.

(b) The tree $\mathcal{T}(yzu)$.

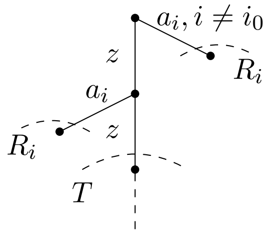
(c) The tree $\mathcal{T}(zu)$.

(d) The tree $\mathcal{T}(xyzu)$.

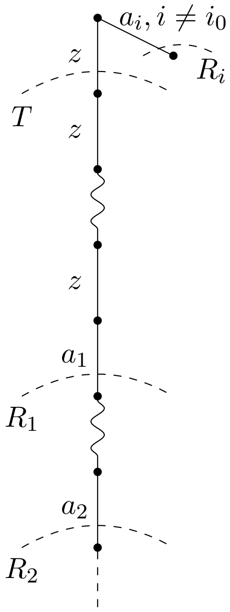
Figure 3.11: Schematic structure of the trees $\mathcal{T}(xzu)$, $\mathcal{T}(yzu)$, $\mathcal{T}(zu)$ and $\mathcal{T}(xyzu)$ when $x \neq z$, $y \neq z$ and $z \neq 0$.



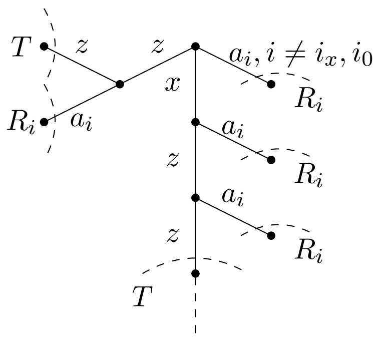

(a) The tree $\mathcal{T}(xzu)$.




(b) The tree $\mathcal{T}(yzu)$.

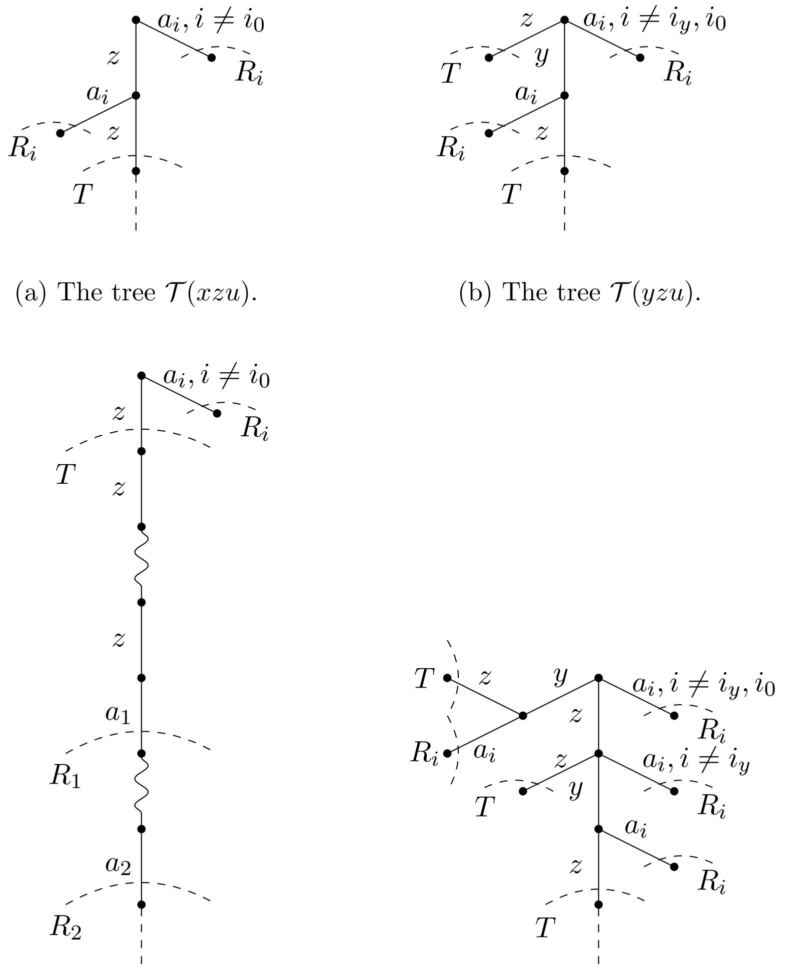



(c) The tree $\mathcal{T}(zu)$.




(d) The tree $\mathcal{T}(xyzu)$.

Figure 3.12: Schematic structure of the trees $\mathcal{T}(xzu)$, $\mathcal{T}(yzu)$, $\mathcal{T}(zu)$ and $\mathcal{T}(xyzu)$ when $x \neq z$, $y = z$ and $z \neq 0$.



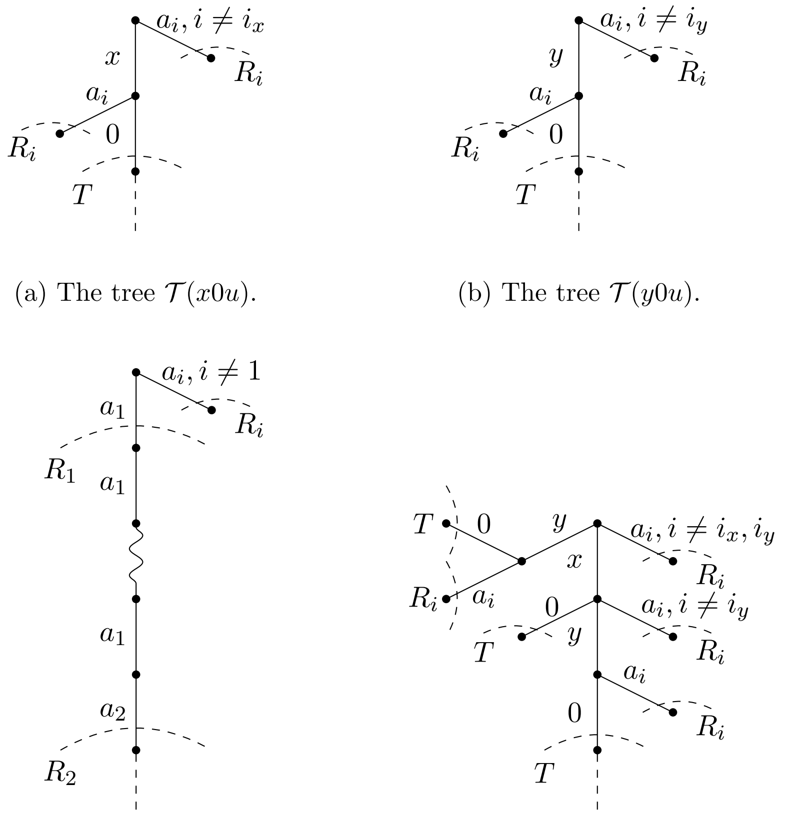

(a) The tree $\mathcal{T}(xzu)$.

(b) The tree $\mathcal{T}(yzu)$.

(c) The tree $\mathcal{T}(zu)$.

(d) The tree $\mathcal{T}(xyzu)$.

Figure 3.13: Schematic structure of the trees $\mathcal{T}(xzu)$, $\mathcal{T}(yzu)$, $\mathcal{T}(zu)$ and $\mathcal{T}(xyzu)$ when $x = z$, $y \neq z$ and $z \neq 0$.




(a) The tree $\mathcal{T}(x0u)$.

(b) The tree $\mathcal{T}(y0u)$.

(c) The tree $\mathcal{T}(\mathrm{rep}_b(\mathrm{val}_b(u)))$.

(d) The tree $\mathcal{T}(xy0u)$.

Figure 3.14: Schematic structure of the trees $\mathcal{T}(x0u)$, $\mathcal{T}(y0u)$, $\mathcal{T}(\mathrm{rep}_b(\mathrm{val}_b(u)))$ and $\mathcal{T}(xy0u)$.

Those five lemmas can be translated into recurrence relations satisfied by the sequence $(S_b(n))_{n\geq 0}$ using Definition 1.47. This proves Proposition 3.29.

*Proof of Proposition 3.29.* The first part is clear using Table 3.7. Let us take $x, y \in \{1, \dots, b-1\}$ with $x \neq y$, and proceed by induction on $\ell \geq 1$.

Let us first prove (3.4). If $\ell = 1$, then $r = 0$, and the result ensues from Table 3.7. Indeed, we have $S_b(xb^1 + 0) = 3$, $S_b(xb^0 + 0) = 2$ and $S_b(0) = 1$. Now suppose that $\ell \geq 2$, and assume that (3.4) holds for all $\ell' < \ell$. Let $r \in \{0, \dots, b^{\ell-1} - 1\}$, and let $u$ be a word in $\{0, \dots, b-1\}^*$ such that $|u| \geq 1$ and $\mathrm{rep}_b(xb^\ell + r) = x0u$. The proof is divided into two parts according to the first letter of $u$. If $u = 0u'$ with $u' \in \{0, \dots, b-1\}^*$, by Lemma 3.37 and using the induction hypothesis twice, we find

$$\begin{aligned}
S_b(xb^\ell + r) &= 2S_b(xb^{\ell-1} + r) - S_b(xb^{\ell-2} + r) \\
&= 2(S_b(xb^{\ell-2} + r) + S_b(r)) - S_b(xb^{\ell-2} + r) \\
&= S_b(xb^{\ell-2} + r) + S_b(r) + S_b(r) \\
&= S_b(xb^{\ell-1} + r) + S_b(r).
\end{aligned}$$

Now if $u = zu'$ with $z \in \{1, \dots, b-1\}$ and $u' \in \{0, \dots, b-1\}^*$, then (3.4) follows from Definition 1.47 and Lemma 3.39.

Let us prove (3.5). If $\ell = 1$, then $r = 0$, and the claimed formula follows from Table 3.7. Indeed, $S_b(xb^1 + xb^0 + 0) = 3$, $S_b(xb^0 + 0) = 2$ and $S_b(0) = 1$. Now suppose that $\ell \geq 2$, and assume that (3.5) holds for all $\ell' < \ell$. Let $r \in \{0, \dots, b^{\ell-1} - 1\}$, and let $u$ be a word in $\{0, \dots, b-1\}^*$ such that $|u| \geq 1$ and $\mathrm{rep}_b(xb^\ell + xb^{\ell-1} + r) = xxu$. The proof is again divided into two parts according to the first letter of $u$. If $u = 0u'$ with $u' \in \{0, \dots, b-1\}^*$, then applying first Lemma 3.38 and then (3.4) twice leads to

$$\begin{aligned}
S_b(xb^\ell + xb^{\ell-1} + r) &= S_b(xb^{\ell-1} + r) + S_b(xb^{\ell-2} + r) \\
&= S_b(xb^{\ell-2} + r) + S_b(r) + S_b(xb^{\ell-2} + r) \\
&= 2(S_b(xb^{\ell-2} + r) + S_b(r)) - S_b(r) \\
&= 2S_b(xb^{\ell-1} + r) - S_b(r).
\end{aligned}$$

Now if $u = zu'$ with $z \in \{1, \dots, b-1\}$ and $u' \in \{0, \dots, b-1\}^*$, then (3.5) follows from Definition 1.47 and Lemma 3.40.

Let us finally prove (3.6). If $\ell = 1$, then $r = 0$, and the result is true by Table 3.7. Indeed, $S_b(xb^1 + yb^0 + 0) = 4$, $S_b(xb^0 + 0) = 2$, $S_b(yb^0 + 0) = 2$ and $S_b(0) = 1$. Now suppose that $\ell \geq 2$, and assume that (3.6) holds for all

$\ell' < \ell$. Let $r \in \{0, \dots, b^{\ell-1} - 1\}$, let $z$ be a letter in $\{0, \dots, b-1\}$, and let $u$ be a word in $\{0, \dots, b-1\}^*$ such that $\mathrm{rep}_b(xb^\ell + yb^{\ell-1} + r) = xyzu$. Using Definition 1.47 and Lemma 3.41, we have

$$S_b(xb^\ell + yb^{\ell-1} + r) = S_b(xb^{\ell-1} + r) + 2S_b(yb^{\ell-1} + r) - 2S_b(r)$$

since $\mathrm{rep}_b(r) = \mathrm{rep}_b(\mathrm{val}_b(zu))$. □

**Example 3.42.** When $b = 2$, Proposition 3.29 gives recurrence relations that are slightly different from those of Proposition 3.15. In fact, the second relation in Proposition 3.15 reflects a palindromic structure in the sequence $(S_2(n))_{n\geq 0}$. As we will see with Proposition 3.44 below, this property is general.

**Example 3.43.** When $b = 3$, Proposition 3.29 yields the following six relations. The sequence $(S_3(n))_{n\geq 0}$ satisfies $S_3(0) = 1$, $S_3(1) = S_3(2) = 2$ and, for all $\ell \geq 1$ and all $0 \leq r < 3^{\ell-1}$,

$$\begin{aligned}
S_3(3^\ell + r) &= S_3(3^{\ell-1} + r) + S_3(r), \\
S_3(2 \cdot 3^\ell + r) &= S_3(2 \cdot 3^{\ell-1} + r) + S_3(r), \\
S_3(3^\ell + 3^{\ell-1} + r) &= 2S_3(3^{\ell-1} + r) - S_3(r), \\
S_3(2 \cdot 3^\ell + 2 \cdot 3^{\ell-1} + r) &= 2S_3(2 \cdot 3^{\ell-1} + r) - S_3(r), \\
S_3(3^\ell + 2 \cdot 3^{\ell-1} + r) &= S_3(3^{\ell-1} + r) + 2S_3(2 \cdot 3^{\ell-1} + r) - 2S_3(r), \\
S_3(2 \cdot 3^\ell + 3^{\ell-1} + r) &= S_3(2 \cdot 3^{\ell-1} + r) + 2S_3(3^{\ell-1} + r) - 2S_3(r).
\end{aligned}$$

To conclude this section, the following result shows that the sequence $(S_b(n))_{n\geq 0}$ has a local palindromic structure as the sequence $(S_2(n))_{n\geq 0}$; see Lemma 3.12 and Proposition 3.15. For instance, the sequence $(S_3(n))_{n\geq 0}$ is depicted in Figure 3.15 inside the interval $[2 \cdot 3^4, 3^5]$. As before, if $w$ is a finite or infinite word over $\{0, 1, \dots, b-1\}$, we let $\underline{w}$ denote the word obtained by replacing in $w$ every letter $a \in \{0, 1, \dots, b-1\}$ by the letter $\underline{a} = (b-1) - a \in \{0, 1, \dots, b-1\}$.

**Proposition 3.44.** *Let $u$ be a word in $\{0, 1, \dots, b-1\}^*$. Then*

$$\#\left\{v \in L_b \mid \binom{(b-1)u}{v} > 0\right\} = \#\left\{v \in L_b \mid \binom{(b-1)\underline{u}}{v} > 0\right\}.$$

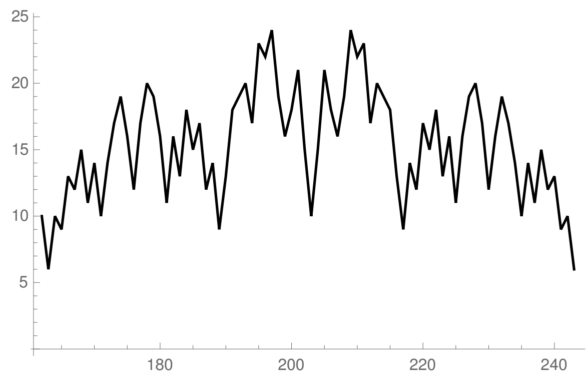



Figure 3.15: The sequence $(S_3(n))_{n\geq 0}$ inside the interval $[2\cdot 3^4, 3^5]$.

*In particular,* $S_b((b-1)\cdot b^\ell + r) = S_b((b-1)\cdot b^\ell + b^\ell - r - 1)$ *for all* $\ell \geq 0$ *and all* $0 \leq r < b^\ell$, i.e., *there exists a palindromic substructure inside the sequence* $(S_b(n))_{n\geq 0}$.

*Proof*. The trees $\mathcal{T}_{L_b}((b-1)u)$ and $\mathcal{T}_{L_b}((b-1)\underline{u})$ are isomorphic. Indeed, on the one hand, each node of the form $(b-1)x$ in the first tree corresponds to the node $(b-1)\underline{x}$ in the second one, and conversely. On the other hand, if there exist letters $a \in \{1,\dots,b-2\}$ in the word $(b-1)u$, the position of the first letter $a$ in this word is equal to the position of the first letter $\underline{a} = (b-1) - a$ in the word $(b-1)\underline{u}$, and conversely. Consequently, the node of the form $ax$ in the first tree corresponds to the node of the form $\underline{ax} = ((b-1)-a)\underline{x}$ in the second tree, and conversely.

For the special case, let $\ell \geq 0$ and $0 \leq r < b^\ell$, and write

$$\text{rep}_b((b-1)\cdot b^\ell + r) = (b-1)z$$

with $z \in \{0,\dots,b-1\}^\ell$. Since $\text{val}_b(z) + \text{val}_b(\underline{z}) = b^\ell - 1$, we have

$$\text{rep}_b((b-1)\cdot b^\ell + b^\ell - 1 - r) = (b-1)\underline{z}.$$

The desired result follows from (1.4) on page 23. □

### 3.2.2 $b$-Regularity and Non-$b$-Synchronicity

In the base-2 case, the sequence $(S_2(n))_{n\geq 0}$ is 2-regular (see Theorem 3.24) but not 2-synchronized (see Proposition 3.26). In this section, we extend those results. As a consequence, we get matrices to compute $S_b(n)$ and the number of matrix multiplications is proportional to $\log_b(n)$. To prove the $b$-regularity, we first need a lemma involving some matrix manipulations.

**Lemma 3.45.** *Let $I$ and $0$ respectively be the identity matrix of size $b^2 \times b^2$ and the zero matrix of size $b^2 \times b^2$. Let $M_b$ be the block-matrix of size $b^3 \times b^3$*

$$M_b = \begin{pmatrix} I & I & 2I & \cdots & \cdots & \cdots & 2I \\ 2I & 3I & 3I & 4I & \cdots & \cdots & 4I \\ \vdots & \vdots & 4I & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & 4I \\ \vdots & \vdots & \vdots & & & \ddots & 3I \\ 2I & 3I & 4I & \cdots & \cdots & \cdots & 4I \end{pmatrix}.$$

*More accurately, the matrix $M_b$ is the block-matrix $(B_{i,j})_{1\le i,j\le b}$, where $B_{i,j}$ is the matrix of size $b^2 \times b^2$ defined by*

$$B_{ij} = \begin{cases} I, & \text{if } i = 1 \text{ and } j \in \{1,2\}; \\ 2I, & \text{if } (i = 1 \text{ and } j \ge 3) \text{ or } (j = 1 \text{ and } i \ge 2); \\ 3I, & \text{if } (j = 2 \text{ and } i \ge 2) \text{ or } (j = i+1 \ge 3); \\ 4I, & \text{otherwise.} \end{cases}$$

*This matrix is invertible, and its inverse is given by*

$$M_b^{-1} = \begin{pmatrix} 3I & 2I & \cdots & \cdots & 2I & -(2b-3)I \\ -2I & 0 & \cdots & \cdots & 0 & I \\ 0 & -I & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & -I & I \end{pmatrix}.$$

*More precisely, the matrix $M_b^{-1}$ is the block-matrix $(C_{i,j})_{1\le i,j\le b}$, where $C_{i,j}$*

*is the matrix of size* $b^2 \times b^2$ *such that*

$$C_{ij} = \begin{cases} 3I, & \text{if } i = j = 1; \\ -(2b-3)I, & \text{if } i = 1 \text{ and } j = b; \\ 2I, & \text{if } i = 1 \text{ and } 2 \le j < b; \\ -2I, & \text{if } i = 2 \text{ and } j = 1; \\ I, & \text{if } j = b \text{ and } i \ge 2; \\ -I, & \text{if } i = j+1 \ge 3; \\ 0, & \text{otherwise.} \end{cases}$$

For the proof of the previous lemma, simply proceed to the multiplication of the matrices $M_b$ and $M_b^{-1}$. Using this lemma, we prove that the sequence $(S_b(n))_{n\ge 0}$ is $b$-regular.

**Theorem 3.46.** *For all* $r \in \{0,\dots,b^2-1\}$, *we have*

$$S_b(nb^2 + r) = a_r S_b(n) + \sum_{s=0}^{b-2} c_{r,s} S_b(nb+s) \quad \text{for all } n \ge 0, \tag{3.7}$$

*where the coefficients* $a_r$ *and* $c_{r,s}$ *are unambiguously determined by* $S_b(0)$, $S_b(1)$, $\dots$, $S_b(b^3-1)$ *and* $s$ *in Tables 3.16, 3.17 and 3.18. In particular, the sequence* $(S_b(n))_{n\ge 0}$ *is* $b$*-regular. Furthermore, a set of generators for* $\langle \mathcal{K}_b(s) \rangle$ *is given by the* $b$ *sequences* $(S_b(n))_{n\ge0}$, $(S_b(bn))_{n\ge0}$, $(S_b(bn+1))_{n\ge0}$, $\dots$, $(S_b(bn+b-2))_{n\ge0}$.

*Proof*. We proceed by induction on $n \ge 0$. For the base case $0 \le n \le b^2-1$, we first compute the coefficients $a_r$ and $c_{r,s}$ using the values of $S_b(nb^2+r)$ for $n \in \{0,\dots,b-1\}$ and $r \in \{0,\dots,b^2-1\}$. Then we show that (3.7) also holds with these coefficients for $n \in \{b,\dots,b^2-1\}$.

**Base case.** Let $I$ denote the identity matrix of size $b^2 \times b^2$. The system of $b^3$ equations (3.7) when $n \in \{0,\dots,b-1\}$ and $r \in \{0,\dots,b^2-1\}$ can be written as $MX = V$ where the matrix $M \in \mathbb{Z}_{b^3}^{b^3}$ is equal to

$$\begin{pmatrix} S_b(0)I & S_b(0)I & S_b(1)I & \cdots & S_b(b-2)I \\ S_b(1)I & S_b(b)I & S_b(b+1)I & \cdots & S_b(b+b-2)I \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ S_b(b-1)I & S_b(b(b-1))I & S_b(b(b-1)+1)I & \cdots & S_b(b(b-1)+b-2)I \end{pmatrix}$$

and the vectors $X, V \in \mathbb{Z}^{b^3}$ are respectively given by

$$\begin{aligned} X^\top &= \begin{pmatrix} a_0 & \cdots & a_{b^2-1} & c_{0,0} & \cdots & c_{b^2-1,0} & \cdots & c_{0,b-2} & \cdots & c_{b^2-1,b-2} \end{pmatrix}, \\ V^\top &= \begin{pmatrix} S_b(0) & S_b(1) & \cdots & S_b(b^3-1) \end{pmatrix}. \end{aligned}$$

Each block-row of $M$ corresponds to one value of $0 \leq n \leq b-1$. For instance, when $n = 0$, then the right-hand side of (3.7) contains the terms $S_b(0)$ and $S_b(s)$ for $s \in \{0, \ldots, b-2\}$. This allows us to build to first block-row of $M$. When $n = 1$, then we find the terms $S_b(1)$ and $S_b(b+s)$ for $s \in \{0, \ldots, b-2\}$ in the right-hand side of (3.7), which permits us to construct the second blow-row of $M$. Also note that in the vector $X$, the coefficients $c_{r,s}$ are first sorted by $s$, then by $r$.

Using Table 3.7, the matrix $M$ is equal to the matrix $M_b$ of Lemma 3.45. Let us explain why. Let $(M_{i,j})_{1 \leq i,j \leq b}$ denote the blocks of the block-matrix $M$. If $i = 1$ and $j \in \{1, 2\}$, then $M_{i,j} = S_b(0)I = I$ by Table 3.7. If $i = 1$ and $j \geq 3$, or $j = 1$ and $i \geq 2$, then $M_{i,j} = S_b(m)I = 2I$ with $m \in \{1, \ldots, b-1\}$. If $j = 2$ and $i \geq 2$, then $M_{i,j} = S_b(m)I = 3I$ since $\text{rep}_b(m)$ is of the form $x0$. If $j = i + 1 \geq 3$, then $M_{i,j} = S_b(m)I = 3I$ since $\text{rep}_b(m)$ is of the form $xx$. In the remaining cases, $M_{i,j} = S_b(m)I = 4I$ since $\text{rep}_b(m)$ is of the form $xy$ with $x \neq y$. By this lemma, the previous system admits the unique solution $X = M_b^{-1}V$. Consequently, for all $r \in \{0, \ldots, b^2-1\}$ and all $s \in \{1, \ldots, b-2\}$, we have

$$\begin{aligned} a_r &= 3S_b(r) + 2\sum_{j=1}^{b-2} S_b(jb^2 + r) - (2b-3)\, S_b((b-1)b^2 + r), \\ c_{r,0} &= -2S_b(r) + S_b((b-1)b^2 + r), \\ c_{r,s} &= -S_b(sb^2 + r) + S_b((b-1)b^2 + r). \end{aligned}$$

The values of those coefficients can be computed without difficulty using Table 3.7, and are stored in Tables 3.16, 3.17 and 3.18. Note that they are unambiguously determined by $S_b(0)$, $S_b(1)$, …, $S_b(b^3-1)$ and the value of $s$, as desired.

For $n \in \{b, \ldots, b^2-1\}$, the values of $S_b(nb^2 + r)$ are given in Tables 3.19, 3.20 and 3.21 according to whether $\text{rep}_b(n)$ is of the form $x0$, $xx$ or $xy$ with $x \neq y$, $x \neq 0$ and $y \neq 0$.

The proof that (3.7) holds for each $n \in \{b, \ldots, b^2-1\}$ only requires easy (but long) computations that are left to the reader. To illustrate the

| $\mathrm{rep}_b(r)$ | $\varepsilon$ | $x$ | $b-1$ | $x0$ | $(b-1)0$ | $xx$ |
|---|---|---|---|---|---|---|
| $a_r$ | $-1$ | $-2$ | $2b-3$ | $-2$ | $4b-4$ | $-1$ |

| $\mathrm{rep}_b(r)$ | $(b-1)(b-1)$ | $xy$ | $(b-1)x$ | $x(b-1)$ |
|---|---|---|---|---|
| $a_r$ | $4b-3$ | $-2$ | $4b-4$ | $2b-3$ |

Table 3.16: Values of $a_r$ for $0 \le r < b^2$ with $x, y \in \{1, \dots, b-2\}$ and $x \neq y$.

| $\mathrm{rep}_b(r)$ | $\varepsilon$ | $x$ | $b-1$ | $x0$ | $(b-1)0$ | $xx$ |
|---|---|---|---|---|---|---|
| $c_{r,0}$ | 2 | 2 | 1 | 1 | $-1$ | 0 |

| $\mathrm{rep}_b(r)$ | $(b-1)(b-1)$ | $xy$ | $(b-1)x$ | $x(b-1)$ |
|---|---|---|---|---|
| $c_{r,0}$ | $-2$ | 0 | $-2$ | $-1$ |

Table 3.17: Values of $c_{r,0}$ for $0 \le r < b^2$ with $x, y \in \{1, \dots, b-2\}$ and $x \neq y$.

| $\mathrm{rep}_b(r)$ | $\varepsilon$ | $x$ | | $b-1$ | $x0$ | | $(b-1)0$ | $xx$ | |
|---|---|---|---|---|---|---|---|---|---|
| $s$ | $z$ | $x$ | $z$ | $z$ | $x$ | $z$ | $z$ | $x$ | $z$ |
| $c_{r,s}$ | 0 | 1 | 0 | $-1$ | 2 | 0 | $-2$ | 2 | 0 |

| $\mathrm{rep}_b(r)$ | $(b-1)(b-1)$ | $xy$ | | | $x(b-1)$ | | $(b-1)x$ | |
|---|---|---|---|---|---|---|---|---|
| $s$ | $z$ | $x$ | $y$ | $z$ | $x$ | $z$ | $x$ | $z$ |
| $c_{r,s}$ | $-2$ | 2 | 1 | 0 | 1 | $-1$ | $-1$ | $-2$ |

Table 3.18: Values of $c_{r,s}$ for $0 \le r < b^2$ and $1 \le s \le b-2$ with $x, y, z$ pairwise distinct letters in $\{1, \dots, b-2\}$.

| $\mathrm{rep}_b(r)$ | $\varepsilon$ | $x$ | $y$ | $x0$ | $y0$ | $xx$ | $yy$ | $xy$ | $yx$ | $yz$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $S_b(nb^2+r)$ | 5 | 7 | 8 | 8 | 10 | 7 | 9 | 10 | 11 | 12 |

Table 3.19: Values of $S_b(nb^2 + r)$ for $b \le n < b^2$ with $\mathrm{rep}_b(n) = x0$ and $x, y, z \in \{1, \dots, b-1\}$ pairwise distinct.

reasoning, we only treat the case where $\mathrm{rep}_b(n) = x0$ with $x \in \{1, \dots, b-1\}$ and $r = 0$ (note that we first need to consider three cases according to the form of $\mathrm{rep}_b(n)$, and inside each case, we divide the argument into three cases

| $\mathrm{rep}_b(r)$ | $\varepsilon$ | $x$ | $y$ | $x0$ | $y0$ | $xx$ | $yy$ | $xy$ | $yx$ | $yz$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $S_b(nb^2+r)$ | 7 | 8 | 10 | 7 | 11 | 5 | 9 | 8 | 10 | 12 |

Table 3.20: Values of $S_b(nb^2+r)$ for $b \le n < b^2$ with $\mathrm{rep}_b(n) = xx$ and $x, y, z \in \{1, \dots, b-1\}$ pairwise distinct.

| $\mathrm{rep}_b(r)$ | $\varepsilon$ | $x$ | $y$ | $z$ | $x0$ | $y0$ | $z0$ | $xx$ | $yy$ |
|---|---|---|---|---|---|---|---|---|---|
| $S_b(nb^2+r)$ | 10 | 13 | 12 | 14 | 13 | 11 | 15 | 10 | 8 |

| $\mathrm{rep}_b(r)$ | $zz$ | $xy$ | $xz$ | $yx$ | $yz$ | $zx$ | $zy$ | $zt$ |
|---|---|---|---|---|---|---|---|---|
| $S_b(nb^2+r)$ | 12 | 12 | 14 | 11 | 12 | 15 | 14 | 16 |

Table 3.21: Values of $S_b(nb^2+r)$ for $b \le n < b^2$ with $\mathrm{rep}_b(n) = xy$ and $x, y, z, t \in \{1, \dots, b-1\}$ pairwise distinct.

according to whether $\mathrm{rep}_b(r)$ is of length 0, 1 or 2). By Table 3.19, we know that $S_b(nb^2+r) = 5$. Let us compute the right-hand side of (3.7). Using Table 3.7, observe that $S_b(n) = 3$ (recall that the scattered subwords of $x0$ are $\varepsilon, x, x0$), and for all $s \in \{0, \dots, b-1\}$,

$$S_b(nb+s) = \begin{cases} 4, & \text{if } s = 0; \\ 5, & \text{if } x = s; \\ 6, & \text{if } x \neq s. \end{cases}$$

Using Tables 3.16, 3.17 and 3.18, the right-hand side of (3.7) is equal to

$$a_0 S_b(n) + c_{0,0} S_b(nb) + \sum_{s=1}^{b-2} c_{0,s} S_b(nb+s) = -1 \cdot 3 + 2 \cdot 4 + 0 = 5,$$

as expected.

**Inductive step.** Consider $n \ge b^2$, and suppose that the relation (3.7) holds for all $m < n$. We show that it still holds for $n$. We have $|\mathrm{rep}_b(n)| \ge 3$. As for the base case, we need to consider several cases according to the form of the base-$b$ expansion of $n$. More precisely, we have to look at the following five forms, where $u \in \{0, \dots, b-1\}^*$, $x, y, z \in \{1, \dots, b-1\}$, $x \neq z$, and $t \in \{0, \dots, b-1\}$:

$$x00u \quad \text{or} \quad x0yu \quad \text{or} \quad xx0u \quad \text{or} \quad xxyu \quad \text{or} \quad xztu.$$

Let us focus on the first one since the same reasoning can be applied for the other ones. Assume that $\mathrm{rep}_b(n) = x00u$ where $x \in \{1, \ldots, b-1\}$ and $u \in \{0, \ldots, b-1\}^*$. For all $r \in \{0, \ldots, b^2-1\}$, there exist $r_1, r_2 \in \{0, \ldots, b-1\}$ such that $\mathrm{val}_b(r_1 r_2) = r$. Using Lemma 3.37, we find

$$\begin{aligned} S_b(nb^2 + r) &= S_b(\mathrm{val}_b(x00ur_1r_2)) \\ &= 2S_b(\mathrm{val}_b(x0ur_1r_2)) - S_b(\mathrm{val}_b(xur_1r_2)). \end{aligned}$$

Then by the induction hypothesis, we obtain

$$\begin{aligned} S_b(nb^2 + r) &= a_r\, 2S_b(\mathrm{val}_b(x0u)) + \sum_{s=0}^{b-2} c_{r,s}\, 2S_b(\mathrm{val}_b(x0us)) \\ &\quad - a_r S_b(\mathrm{val}_b(xu)) - \sum_{s=0}^{b-2} c_{r,s} S_b(\mathrm{val}_b(xus)). \end{aligned}$$

The proof is complete by using Lemma 3.37 again for we have

$$\begin{aligned} S_b(nb^2 + r) &= a_r S_b(\mathrm{val}_b(x00u)) + \sum_{s=0}^{b-2} c_{r,s} S_b(\mathrm{val}_b(x00us)) \\ &= a_r S_b(n) + \sum_{s=0}^{b-2} c_{r,s} S_b(nb + s), \end{aligned}$$

which proves (3.7).

**$b$-regularity.** From the first part of the statement, we deduce that the $\mathbb{Z}$-module $\langle \mathcal{K}_b(S_b) \rangle$ is generated by the $(b+1)$ sequences $(S_b(n))_{n\geq 0}$, $(S_b(bn))_{n\geq 0}$, $(S_b(bn+1))_{n\geq 0}$, …, $(S_b(bn+b-2))_{n\geq 0}$ and $(S_b(bn+b-1))_{n\geq 0}$. We now show that we can reduce the number of generators. To that aim, we prove that for all $n \geq 0$,

$$S_b(nb + b - 1) = (2b-1)S_b(n) - \sum_{s=0}^{b-2} S_b(nb + s). \tag{3.8}$$

Once again, we proceed by induction on $n \geq 0$. As a base case, the proof that (3.8) holds for each $n \in \{0, \ldots, b^2 - 1\}$ only requires easy (but long) computations using Table 3.7 that are left to the reader. In fact, the reasoning is divided into three cases according to the length of $\mathrm{rep}_b(n)$. To illustrate the argument, we show that (3.8) holds for $n = 0$, *i.e.*, $|\mathrm{rep}_b(n)| = 0$. On the one hand, using Table 3.7, we have

$$S_b(nb + b - 1) = S_b(b-1) = 2,$$

and on the other hand, we have

$$\begin{aligned}(2b-1)S_b(n) - \sum_{s=0}^{b-2} S_b(nb+s) &= (2b-1)S_b(0) - \sum_{s=0}^{b-2} S_b(s) \\ &= (2b-1)\cdot 1 - (1+2\cdot(b-2)) = 2\end{aligned}$$

again using Table 3.7.

Now consider $n \geq b^2$ and suppose that the relation (3.8) holds for all integers $m < n$. We prove it is still true for $n$. Note that $|\operatorname{rep}_b(n)| \geq 3$. Mimicking the first induction step of this proof, we need to consider the same five cases according to the form of the base-$b$ expansion of $n$. As previously, let us concentrate on the first form of $\operatorname{rep}_b(n)$ since the same reasoning can be applied for the other ones. Assume that $\operatorname{rep}_b(n) = x00u$ where $x \in \{1,\ldots,b-1\}$ and $u \in \{0,\ldots,b-1\}^*$. Using Lemma 3.37 first, we know that

$$\begin{aligned}S_b(nb+b-1) &= S_b(\operatorname{val}_b(x00u(b-1))) \\ &= 2S_b(\operatorname{val}_b(x0u(b-1))) - S_b(\operatorname{val}_b(xu(b-1))).\end{aligned}$$

The induction hypothesis yields

$$S_b(\operatorname{val}_b(x0u(b-1))) = (2b-1)S_b(\operatorname{val}_b(x0u)) - \sum_{s=0}^{b-2} S_b(\operatorname{val}_b(x0us))$$

and

$$S_b(\operatorname{val}_b(xu(b-1))) = (2b-1)S_b(\operatorname{val}_b(xu)) - \sum_{s=0}^{b-2} S_b(\operatorname{val}_b(xus)),$$

which in turn gives

$$\begin{aligned}S_b(nb+b-1) &= (2b-1)\left(2S_b(\operatorname{val}_b(x0u)) - S_b(\operatorname{val}_b(xu))\right) \\ &\quad - \sum_{s=0}^{b-2} \left(2S_b(\operatorname{val}_b(x0us)) - S_b(\operatorname{val}_b(xus))\right).\end{aligned}$$

The application of Lemma 3.37 then leads to

$$\begin{aligned}S_b(nb+b-1) &= (2b-1)S_b(\operatorname{val}_b(x00u)) - \sum_{s=0}^{b-2} S_b(\operatorname{val}_b(x00us)) \\ &= (2b-1)S_b(n) - \sum_{s=0}^{b-2} S_b(nb+s),\end{aligned}$$

which proves (3.8). Consequently, the $\mathbb{Z}$-module $\langle \mathcal{K}_b(S_b) \rangle$ is generated by the $b$ sequences

$$(S_b(n))_{n\geq 0}, (S_b(bn))_{n\geq 0}, (S_b(bn+1))_{n\geq 0}, \ldots, (S_b(bn+b-2))_{n\geq 0}.$$

For example, if one wants to generate the sequence $(S_b(b^3n+1))_{n\geq 0}$ belonging to $\langle \mathcal{K}_b(S_b) \rangle$, one may use (3.7) twice that gives

$$\begin{aligned} S_b(nb^3+1) &= a_1 S_b(nb) + \sum_{s=0}^{b-2} c_{1,s} S_b(nb^2+s) \\ &= a_1 S_b(nb) + \sum_{s=0}^{b-2} c_{1,s} \left( a_s S_b(n) + \sum_{t=0}^{b-2} c_{s,t} S_b(nb+t) \right) \end{aligned}$$

for all $n \geq 0$. □

Let us illustrate the previous theorem.

**Example 3.47.** Let $b = 2$. Using Tables 3.16, 3.17 and 3.18, we find that $a_0 = -1$, $a_1 = 1$, $a_2 = 4$, $a_3 = 5$, $c_{0,0} = 2$, $c_{1,0} = 1$, $c_{2,0} = -1$, and $c_{3,0} = -2$. In this particular case, there are no $c_{r,s}$ with $s > 0$. Applying Theorem 3.46 and from (3.8), we get

$$\begin{aligned} S_2(2n+1) &= 3S_2(n) - S_2(2n), \\ S_2(4n) &= -S_2(n) + 2S_2(2n), \\ S_2(4n+1) &= S_2(n) + S_2(2n), \\ S_2(4n+2) &= 4S_2(n) - S_2(2n), \\ S_2(4n+3) &= 5S_2(n) - 2S_2(2n) \end{aligned}$$

for all $n \geq 0$. This result is a rewriting of Theorem 3.24. Observe that the third and the fifth identities are superfluous: they follow from the other ones.

**Example 3.48.** Take $b = 3$. Using Tables 3.16, 3.17 and 3.18, the values of the coefficients $a_r$, $c_{r,0}$ and $c_{r,1}$ can be found in Table 3.22. Applying

Theorem 3.46 and from (3.8), we get

$$\begin{aligned}
S_3(3n+2) &= 5S_3(n) - S_3(3n) - S_3(3n+1), \\
S_3(9n) &= -S_3(n) + 2S_3(3n), \\
S_3(9n+1) &= -2S_3(n) + 2S_3(3n) + S_3(3n+1), \\
S_3(9n+2) &= 3S_3(n) + S_3(3n) - S_3(3n+1), \\
S_3(9n+3) &= -2S_3(n) + S_3(3n) + 2S_3(3n+1), \\
S_3(9n+4) &= -S_3(n) + 2S_3(3n+1), \\
S_3(9n+5) &= 3S_3(n) - S_3(3n) + S_3(3n+1), \\
S_3(9n+6) &= 8S_3(n) - S_3(3n) - 2S_3(3n+1), \\
S_3(9n+7) &= 8S_3(n) - 2S_3(3n) - S_3(3n+1), \\
S_3(9n+8) &= 9S_3(n) - 2S_3(3n) - 2S_3(3n+1)
\end{aligned}$$

for all $n \geq 0$. Note that this result proves [LRS17b, Conjecture 26]. Also observe that the fourth, the seventh and the tenth identities are redundant.

| $r$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $a_r$ | $-1$ | $-2$ | 3 | $-2$ | $-1$ | 3 | 8 | 8 | 9 |
| $c_{r,0}$ | 2 | 2 | 1 | 1 | 0 | $-1$ | $-1$ | $-2$ | $-2$ |
| $c_{r,1}$ | 0 | 1 | $-1$ | 2 | 2 | 1 | $-2$ | $-1$ | $-2$ |

Table 3.22: The values of $a_r, c_{r,0}, c_{r,1}$ when $b = 3$ and $r \in \{0, \ldots, 8\}$.

As can be seen in the previous examples, some relations are unnecessary. The following remark establishes this as a general fact.

**Remark 3.49.** Combining (3.7) and (3.8) yields $b^2+1$ identities to generate the $\mathbb{Z}$-module $\langle \mathcal{K}_b(S_b) \rangle$. However, as illustrated in Examples 3.47 and 3.48, only $b^2-b+1$ identities are useful: the relations established for the sequences $(S_b(b^2n + br + b - 1))_{n \geq 0}$, with $r \in \{0, \ldots, b-1\}$, can be deduced from the other identities. For $r \in \{0, \ldots, b-1\}$, (3.7) gives

$$S_b(b^2n + br + b - 1) = a_{br+b-1} S_b(n) + \sum_{s=0}^{b-2} c_{br+b-1,s} S_b(nb + s) \tag{3.9}$$

for all $n \geq 0$. Let us show how we can find back those relations by using the others. Thanks to (3.8) and then (3.7), we can also write

$$\begin{aligned} S_b(b^2n + br + b - 1) =& S_b(b(bn + r) + b - 1) \\ =& (2b-1)S_b(bn+r) - \sum_{s=0}^{b-2} S_b(b^2n + br + s) \\ =& (2b-1)S_b(bn+r) - \sum_{s=0}^{b-2} a_{br+s} S_b(n) \\ & - \sum_{s'=0}^{b-2}\sum_{s=0}^{b-2} c_{br+s,s'} S_b(nb + s') \end{aligned}$$

for all $n \geq 0$. Using Tables 3.16, 3.17 and 3.18, it is not difficult to compute the quantities

$$T_1(r) = \sum_{s=0}^{b-2} a_{br+s} \quad \text{and} \quad T_2(r) = \sum_{s=0}^{b-2} c_{br+s,s'}$$

for all values of $r \in \{0, \ldots, b-1\}$ and $s' \in \{0, \ldots, b-2\}$, which afterwards gives back (3.9).

As in Corollary 3.25, one can build a matrix representation of $(S_b(n))_{n\geq 0}$.

**Remark 3.50.** Using Theorem 3.46 and (3.8), we already know that the $\mathbb{Z}$-module $\langle \mathcal{K}_b(S_b) \rangle$ is generated by the following set of $b$ generators

$$\{(S_b(n))_{n\geq 0}, (S_b(bn))_{n\geq 0}, (S_b(bn+1))_{n\geq 0}, \ldots, (S_b(bn+b-2))_{n\geq 0}\},$$

so we get matrices to compute $S_b(n)$ in a number of steps proportional to $\log_b(n)$. For all $n \geq 0$, let

$$V_b(n) = \begin{pmatrix} S_b(n) \\ S_b(bn) \\ S_b(bn+1) \\ \vdots \\ S_b(bn+b-2) \end{pmatrix} \in \mathbb{Z}^b.$$

Consider the matrix-valued map $\mu_b \colon \{0, 1, \ldots, b-1\}^* \to \mathbb{Z}^{b\times b}$ defined as follows. If $s \in \{0, \ldots, b-2\}$, then we set

$$\mu_b(s) = \begin{pmatrix} A(s) & C_0(s) & \cdots & C_{s-1}(s) & C_s(s) & C_{s+1}(s) & \cdots & C_{b-2}(s) \end{pmatrix},$$

where the vectors $A(s), C_0(s), \ldots, C_{b-2}(s) \in \mathbb{Z}^b$ are given by

$$\begin{aligned}
A(s)^{\mathsf{T}} &= \begin{pmatrix} 0 & a_{bs} & a_{bs+1} & \cdots & a_{bs+b-2} \end{pmatrix}, \\
C_i(s)^{\mathsf{T}} &= \begin{pmatrix} 0 & c_{bs,i} & c_{bs+1,i} & \cdots & c_{bs+b-2,i} \end{pmatrix} \text{ for all } 0 \le i \le b-2, i \neq s, \\
C_s(s)^{\mathsf{T}} &= \begin{pmatrix} 1 & c_{bs,s} & c_{bs+1,s} & \cdots & c_{bs+b-2,s} \end{pmatrix}.
\end{aligned}$$

The matrix $\mu_b(b-1)$ is also set to be

$$\begin{pmatrix}
(2b-1) & -1 & -1 & \cdots & -1 \\
a_{b(b-1)} & c_{b(b-1),0} & c_{b(b-1),1} & \cdots & c_{b(b-1),b-2} \\
a_{b(b-1)+1} & c_{b(b-1)+1,0} & c_{b(b-1)+1,1} & \cdots & c_{b(b-1)+1,b-2} \\
\vdots & \vdots & \vdots & & \vdots \\
a_{b(b-1)+b-2} & c_{b(b-1)+b-2,0} & c_{b(b-1)+b-2,1} & \cdots & c_{b(b-1)+b-2,b-2}
\end{pmatrix}.$$

Observe that the number of generators of $\langle \mathcal{K}_b(S_b) \rangle$ explains the size of the matrices above. For each $s \in \{0, \ldots, b-2\}$, exactly $b-1$ identities from Theorem 3.46 are used to define the matrix $\mu_b(s)$. If $s, s' \in \{0, \ldots, b-2\}$ are such that $s \neq s'$, then the relations used to define the matrices $\mu_b(s)$ and $\mu_b(s')$ are pairwise distinct. The first row of the matrix $\mu_b(b-1)$ is (3.8), and the other rows are $b-1$ identities from Theorem 3.46, which are distinct from the previous relations. Consequently, $(b-1)(b-1) + b = b^2 - b + 1$ identities are used, which corroborates Remark 3.49.

Using the definition of the map $\mu_b$, we can show that

$$V_b(bn+s) = \mu_b(s) V_b(n)$$

for all $s \in \{0, \ldots, b-1\}$ and $n \geq 0$. Consequently, if $\mathrm{rep}_b(n) = n_k \cdots n_0$, then proceeding as in the proof of Corollary 3.25 gives

$$S_b(n) = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \mu_b(n_0) \cdots \mu_b(n_k)\, V_b(0).$$

with

$$V_b(0)^{\mathsf{T}} = \begin{pmatrix} 1 & 1 & 2 & \cdots & 2 \end{pmatrix}.$$

For example, when $b = 2$, the matrices $\mu_2(0)$ and $\mu_2(1)$ are those given in Corollary 3.25. When $b = 3$, we get

$$\mu_3(0) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ -2 & 2 & 1 \end{pmatrix}, \ \mu_3(1) = \begin{pmatrix} 0 & 0 & 1 \\ -2 & 1 & 2 \\ -1 & 0 & 2 \end{pmatrix},$$
$$\mu_3(2) = \begin{pmatrix} 5 & -1 & -1 \\ 8 & -1 & -2 \\ 8 & -2 & -1 \end{pmatrix}.$$

To build the matrix $\mu_3(0)$ (resp., $\mu_3(1)$; resp., $\mu_3(2)$), one may look at the first and second columns (resp., the fourth and fifth columns; resp., the seventh and eighth columns) of Table 3.22.

Finally, the sequence $(S_b(n))_{n\geq 0}$ is not $b$-synchronized, which can be proved in the same fashion as Proposition 3.26, and thus also not $b$-automatic.

**Proposition 3.51.** *The sequence $(S_b(n))_{n\geq 0}$ is neither $b$-synchronized, nor $b$-automatic.*

**Remark 3.52.** Since the sequence $(S_2(n))_{n\geq 0}$ is intimately bonded with the Stern–Brocot sequence (see Section 3.1.3), the present generalization to any integer base gives a motivation to study further extensions of the Farey and Stern–Brocot trees, and associated sequences. Some of them already exist, *e.g.*, [Aiy17, CW98, Gar13, GLR$^+$18, MGOT15], but can one reasonably define a tree structure, or some other combinatorial structure, in which the sequence $(S_b(n))_{n\geq 0}$ naturally appears, extending Propositions 3.19 and 3.20? Observe that Proposition 3.20 gives a natural way to build new trees associated with the sequences $(S_b(n))_{n\geq 0}$, which are worth studying. The multidimensional Farey graphs in [GLR$^+$18] make us think that this question might lead to interesting combinatorial structures that would be intrinsically linked to the sequences $(S_b(n))_{n\geq 0}$.

## 3.3 Dealing with the Fibonacci Numeration System

In this section, we consider the generalized Pascal triangle $\mathrm{P}_\varphi$ from Example 1.45 and the sequence $(S_\varphi(n))_{n\geq 0}$ from Example 1.50 in the Zeckendorf numeration system. In Figure 3.23, the latter sequence is depicted in the interval $[0, F(11)]$; see the resemblance with Figures 3.1 and 3.6. One of the main differences in this section is that we leave the classical setting of $b$-regular sequences to look at Fibonacci-regular sequences, or $F$-regular sequences for short. In Section 3.3.1, we start by precisely defining this new concept. Then we adapt and study the tries of scattered subwords in our particular framework. Using them, we prove that $(S_\varphi(n))_{n\geq 0}$ is $F$-regular in Theorem 3.63. As an easy corollary, $S_\varphi(n)$ can be computed by multiplying square matrices of size 2, and the number of multiplications is proportional to $\log_\varphi(n)$ (see Corollary 3.64). In Section 3.3.2, we get a recurrence relation similar to the case of integer bases with Propositions 3.15 and 3.29. Finally, we build a convenient arrangement of the terms of the sequence $(S_\varphi(n))_{n\geq 0}$ in Proposition 3.68 that might turn out to be useful for further generalizations.

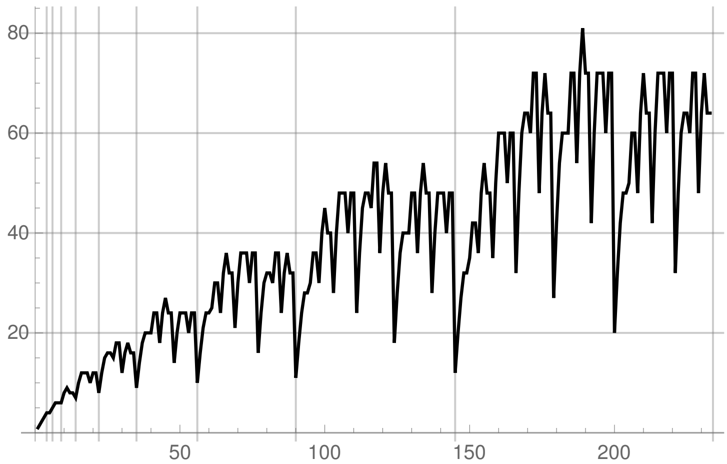



Figure 3.23: The sequence $(S_\varphi(n))_{n\geq 0}$ in the interval $[0, F(11)]$.

### 3.3.1 $F$-Regularity

The notions of automaticity and regularity introduced in Section 1.7 of Chapter 1 can be widened to take into account a larger class of numeration systems [AST00, RM02, Sha88]. Observe that in Definition 1.51, to obtain one specific element of the kernel $\mathcal{K}_b(s)$ of an integer sequence $s = (s(n))_{n\geq 0}$, it is equivalent to consider a word $q \in \{0, \dots, b-1\}^*$ and evaluate $s$ at all the integers whose base-$b$ expansion (possibly with some leading zeroes) ends with the suffix $q$. As an example, $(s(2^3 n + 1))_{n\geq 0}$ corresponds to the word

$q = 001$ (in base 2). It means that, among all the words in $L_2$, we select those (padded with leading zeroes if necessary) that end with $q = 001$, and then we evaluate $s$ at the values in base 2 of those words. For instance, we select the words 1, 1001, 10001 whose values in base 2 are respectively $1 = 2^3 \cdot 0 + 1$, $9 = 2^3 \cdot 1 + 1$, $17 = 2^3 \cdot 2 + 1$. In what follows, we formalize this idea.

**Definition 3.53.** For each $q$ in $\{0,1\}^*$, we define the map

$$i_q \colon X \subset \mathbb{N} \mapsto i_q(X) \subset \mathbb{N},$$

where $i_q(X) = \mathrm{val}_F(0^* \,\mathrm{rep}_F(X) \cap \{0,1\}^* q)$.

In other words, for a given subset $X \subset \mathbb{N}$, $i_q$ selects elements in $X$ whose $F$-expansion (padded with leading zeroes) ends with $q$. In particular, $i_q(X)$ is a subset of $X$, and $i_q(X) = i_q(\mathbb{N}) \cap X$. Observe that $i_q(X)$ might be empty even if $X$ is a non-empty subset of $\mathbb{N}$. For instance, no $F$-expansions (padded with leading zeroes) end with 11, thus $i_{11}(X) = \emptyset$ for any subset $X$ of $\mathbb{N}$. If $i_q(X)$ is non-empty, it is naturally ordered

$$i_q(X) = \{x_{q,0} < x_{q,1} < x_{q,2} < \cdots\},$$

and by abuse of notation, we set $i_q(n) = x_{q,n}$ for all $0 \le n < \# i_q(X)$.

**Example 3.54.** Recall that the first pairs $(n, \mathrm{rep}_F(n))$ for $n \in \mathbb{N}$ are

$$(0,\varepsilon), (1,1), (2,10), (3,100), (4,101), (5,1000), (6,1001),$$
$$(7,1010), (8,10000), (9,10001).$$

The $F$-expansions of $0,2,3,5,7,8$ (resp., $1,4,6,9$) (with leading zeroes) all end with 0 (resp., 1), so the first values in $i_0(\mathbb{N})$ (resp., $i_1(\mathbb{N})$) are $0,2,3,5,7,8$ (resp., $1,4,6,9$). In particular, $i_0(0) = 0$, $i_0(1) = 2$, $\ldots$, $i_0(5) = 8$. The first values in $i_{10}(\mathbb{N})$ are $2,7$.

**Remark 3.55.** One might be tempted to replace the definition of the map $i_q$ by the following: for each $q$ in $\{0,1\}^*$,

$$i'_q \colon \mathbb{N} \to \mathbb{N}, n \mapsto i'_q(n),$$

where $i'_q(n)$ is the $n$th element of the set $\mathrm{val}_F(0^* \,\mathrm{rep}_F(\mathbb{N}) \cap \{0,1\}^* q)$. Then, for a given subset $X \subset \mathbb{N}$, we let $i'_q(X)$ denote the set $\{i'_q(n) \mid n \in X\}$.

However, the two definitions are different. For instance, if $X = \{0, 1, 3, 5\}$, then $i_0(X) = \{0, 3, 5\}$ since

$$\text{rep}_F(0) = \varepsilon,\ \text{rep}_F(1) = 1,\ \text{rep}_F(3) = 100 \text{ and } \text{rep}_F(5) = 1000,$$

and we have to discard 1. To build $i'_0(X)$, recall that the first words in $L_F$ that, when padded with leading zeroes, end with 0 are $\varepsilon$, 10, 100, 1000, 1010 and 10000. Now we have to select the 0th, the first, the third and the fifth among these words, so

$$i'_0(X) = \{\text{val}_F(\varepsilon), \text{val}_F(10), \text{val}_F(1000), \text{val}_F(10000)\} = \{0, 2, 5, 8\}.$$

The second part of the next lemma is particularly important when considering elements of the $F$-kernel (see Theorem 3.63). Note that $\text{rep}_F(i_p(\mathbb{N}))q$ means that we concatenate each word in the language $\text{rep}_F(i_p(\mathbb{N}))$ with the suffix $q$.

**Lemma 3.56.** *We have $i_{pq}(\mathbb{N}) \subseteq i_q(\mathbb{N})$ and $i_{pq}(\mathbb{N}) = i_{pq}(i_q(\mathbb{N}))$. Moreover, if $pq \in 0^*L_F$, then*

$$\text{rep}_F(i_p(\mathbb{N}))q = \text{rep}_F(i_{pq}(\mathbb{N})),$$

i.e., *if $up \in 0^*L_F$ is such that $\text{val}_F(up) = i_p(n)$, then $\text{val}_F(upq) = i_{pq}(n)$.*

*Proof*. The first inclusion is easy because $x \in i_{pq}(\mathbb{N})$ implies that the words in $0^* \text{rep}_F(x)$ end with $pq$, so with $q$ in particular.

The set $i_q(\mathbb{N})$ contains all the integers $x$ such that the words in $0^* \text{rep}_F(x)$ end with $q$. Among those integers, $i_{pq}$ selects those whose $F$-expansion (with leading zeroes) ends with $pq$. Hence the equality.

On the one hand, $\text{rep}_F(i_{pq}(\mathbb{N}))$ contains all the $F$-expansions ending with $pq$, which is an authorized suffix since $pq \in 0^*L_F$. On the other hand, $w \in \text{rep}_F(i_p(\mathbb{N}))q$ if and only if $w = uq$ with $u \in \text{rep}_F(i_p(\mathbb{N}))$. In other words, $w$ is the $F$-expansion of an integer and ends with $pq$. □

Let us illustrate the second part of Lemma 3.56 in Table 3.24 with $p = 0$ and $q = 1$ such that $pq = 01 \in 0^*L_F$. We can directly determine the $n$th word ending with 01 in the language $L_F$ from the $n$th word ending with 0 by simply adding a suffix 1.

**Remark 3.57.** The second part of Lemma 3.56 holds because the words in $L_F$ are defined by avoiding the factor 11 (see Example 1.18). Indeed, since

11 has length 2, we have $0^*L_Fp^{-1} = 0^*L_F(pq)^{-1}$ when $pq \in 0^*L_F$ and $p \neq \varepsilon$ (recall the notation from Chapter 1 in Definition 1.5 on page 3). In this particular case, it is enough to look at the first letter of $p$. For instance, examine Table 3.24: the words in the second column obtained by removing the suffix 0 are the words in the third column obtained by deleting the suffix 01.

| $i_0(\mathbb{N})$ | $\mathrm{rep}_F(i_0(\mathbb{N}))$ | $\mathrm{rep}_F(i_{01}(\mathbb{N}))$ | $i_{01}(\mathbb{N})$ |
|---|---|---|---|
| $0 = i_0(0)$ | $\varepsilon$ | 1 | $i_{01}(0) = 1$ |
| $2 = i_0(1)$ | 10 | 101 | $i_{01}(1) = 4$ |
| $3 = i_0(2)$ | 100 | 1001 | $i_{01}(2) = 6$ |
| $5 = i_0(3)$ | 1000 | 10001 | $i_{01}(3) = 9$ |
| $7 = i_0(4)$ | 1010 | 10101 | $i_{01}(4) = 12$ |
| $8 = i_0(5)$ | 10000 | 100001 | $i_{01}(5) = 14$ |

Table 3.24: Illustration of Lemma 3.56.

This does not always hold, notably when there are longer forbidden factors in the language of the numeration.

Let us take the example of the Tribonacci numeration system. Consider the sequence $T = (T(n))_{n\geq 0} = (1, 2, 4, 7, 13, 24, 44, 81, \ldots)$ of *Tribonacci numbers* (A001590 in [Slo]) defined by $T(0) = 1$, $T(1) = 2$, $T(3) = 4$, and for all $n \geq 0$

$$T(n + 3) = T(n + 2) + T(n + 1) + T(n).$$

The *Tribonacci numeration system* is the positional numeration system built on this sequence $T$ (see also the Fibonacci numeration system in Example 1.18 that is highly similar). In this case, the alphabet is $A_T = \{0, 1\}$, and the numeration language $L_T$ is the set of the words over $\{0, 1\}$ not containing the factor 111. As for the Fibonacci numeration system in Example 1.30, it can be shown that the Tribonacci numeration system is also a Parry–Bertrand numeration system; see, for instance, [Rig14a, Rig14b].

Let us now come back to our matter. For the language $L_T$, we have $1 \in 0^*L_T1^{-1}$ but $1 \notin 0^*L_T(11)^{-1}$ even if $11 \in L_T$. As a consequence, the languages $\mathrm{rep}_T(i_1(\mathbb{N}))1$ and $\mathrm{rep}_T(i_{11}(\mathbb{N}))$ are different: $111 \in \mathrm{rep}_T(i_1(\mathbb{N}))1$ while 111 is not a valid Tribonacci-expansion.

Definitions 1.51, 1.52 and 1.54 are replaced for the Fibonacci numeration system by the following notions where the subsequences of a given sequence are selected by suffixes of $F$-expansions. This extension was first introduced in [AST00, Sha88]. Note that, for each new concept, several equivalent definitions exist, as it is the case for their classical version, but we again focus on the ones emerging from the kernels.

**Definition 3.58.** Let $q$ be a word in $\{0,1\}^*$ such that $i_q(\mathbb{N}) \neq \emptyset$, and let $s = (s(n))_{n\geq 0}$ be a sequence of integers.

The subsequence of $s$ defined by $n \mapsto s(i_q(n))$ is called the *subsequence of $s$ with least significant digits equal to $q$.*

The set of all these subsequences, for $q \in \{0,1\}^*$ such that $i_q(\mathbb{N}) \neq \emptyset$, is called the *Fibonacci-kernel* or *$F$-kernel* of the sequence $s$. We let $\mathcal{K}_F(s)$ denote it.

The sequence $s$ is *Fibonacci-automatic* or *$F$-automatic* if $\mathcal{K}_F(s)$ is finite. We say that $s$ is *Fibonacci-regular* or *$F$-regular* if $\langle \mathcal{K}_F(s) \rangle$ is a finitely-generated $\mathbb{Z}$-module.

For instance, if $s \in \{0,1\}^{\mathbb{N}}$ is the characteristic sequence of Fibonacci numbers, *i.e.*, $s(n) = 1$ if and only if $n$ is a Fibonacci number, then $s$ is Fibonacci-automatic [Sha88]. Without giving a lot of details, one could extend the notion of Fibonacci-automaticity to Tribonacci-automaticy using the numeration system from Remark 3.57. In this case, the abelian complexity of the Tribonacci word is shown to be $T$-automatic [Tur15]. Other generalized automatic or regular sequences may be found in [Sha88, RM02]. The aim is now to show that $(S_\varphi(n))_{n\geq 0}$ is $F$-regular, which is a nice addition to the existing zoology of $F$-regular sequences. First, we establish a formula, depending on the form of the word $u \in L_F$, to count the number of distinct scattered subwords of $u$ in $L_F$.

**Proposition 3.59.** *Let $u$ be a non-empty word in $L_F$ of the form*

$$10^{n_k}10^{n_{k-1}} \cdots 10^{n_2}10^{n_1}$$

*with $n_1 \geq 0$ and $n_2, \ldots, n_k > 0$. Then*

$$\#\left\{ v \in L_F \mid \binom{u}{v} > 0 \right\} = (n_1 + 2) \cdot \prod_{j=2}^{k} (n_j + 1).$$

To prove this result, we reuse the notion of tries of scattered subwords from Definition 3.1 but restricted to $L_F$. For a word $w \in L_F$, the tree $\mathcal{T}_{L_F}(w)$ is given in Definition 3.1. The factorization (3.1) (on page 99) of words in $L_F$ has a very particular form because there is no factor 11. To refer to the same subtrees as in Definition 3.4, we stick to the notation of (3.1), even though the blocks of letters 1 are limited to a single digit

$$w = \underbrace{1}_{u_1}\underbrace{0^{n_2}}_{u_2}\underbrace{1}_{u_3}\underbrace{0^{n_4}}_{u_4}\cdots\underbrace{1}_{u_{2j-1}}\underbrace{0^{n_{2j}}}_{u_{2j}}$$

with $j \geq 1$, $n_2, \ldots, n_{2j-2} \geq 1$ and $n_{2j} \geq 0$. We let $M = M_w$ be such that $w = u_1 u_2 \cdots u_M$, where $u_M$ is the last non-empty block of zeroes or the last one. The trees $T_\ell$ for $\ell \in \{0, \ldots, M\}$ are similar to those of Definition 3.4.

**Example 3.60.** Consider the word $w = 101000100 \in L_F$. With the above notation, the factorization (3.1) of $w$ is $w = u_1u_2u_3u_4u_5u_6$ with $u_1 = 1$, $u_2 = 0^1$, $u_3 = 1$, $u_4 = 0^3$, $u_5 = 1$ and $u_6 = 0^2$, so $M = 6$. In Figure 3.25 (to be compared with Figure 3.2), we have represented the trie $\mathcal{T}_{L_F}(w)$ of scattered subwords in $L_F$ and the subtrees $T_0, \ldots, T_5$. The roots of these subtrees correspond to a prefix of $w$ ending with 1 or 10: for $\ell \in \{0, 2, 4\}$ (resp., $\{1, 3, 5\}$), the tree $T_\ell$ has the root $u_1 \cdots u_\ell 1$ (resp., $u_1 \cdots u_\ell 0 = u_1 \cdots u_{\ell-1}10$).

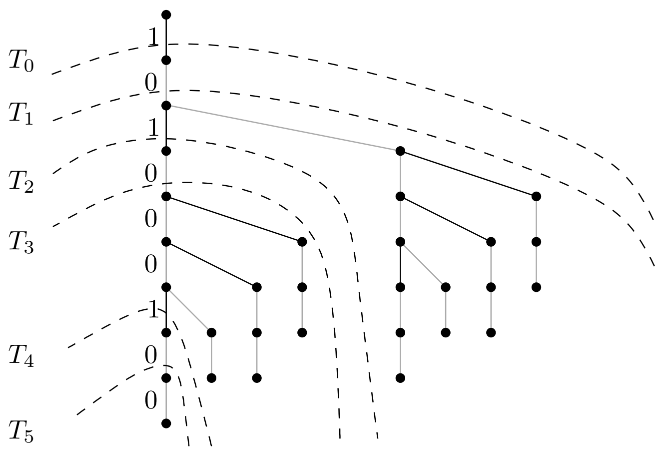



Figure 3.25: The trie $\mathcal{T}_{L_F}(101000100)$.

Since we are considering the language $L_F$, the analogue of Proposition 3.6

becomes the following. The main difference is that the factor 11 is forbidden.

**Proposition 3.61.** *Let $w$ be a non-empty word in $L_F$. If the tree $\mathcal{T}_{L_F}(w)$ is not linear, it has the following properties.*

- *Assume that $2 \leq 2k < M$. For every $j \in \{1, \dots, n_{2k} - 1\}$, the node of label $x = u_1 \cdots u_{2k-1}0^j$ has two children $x0$ and $x1$. The node $x1$ is the root of a tree isomorphic to $T_{2k}$. Moreover, $x = u_1 \cdots u_{2k-1}$ has a single child $x0$.*

- *Assume that $3 \leq 2k+1 < M$. The node of label $x = u_1 \cdots u_{2k}$ has two children $x0$ and $x1$. The node $x0$ is the root of a tree isomorphic to $T_{2k+1}$.*

- *The node of label $x = \varepsilon$ has only one child $x1$, which is the root of a tree isomorphic to $T_0$. The node of label $x = 1$ has only one child, namely $x0$, which is the root of a tree isomorphic to $T_1$.*

**Example 3.62.** Let us continue Example 3.60. As in Example 3.8, Figure 3.26 illustrates the previous proposition in which we see how the subtrees are connected to the "initial" linear subtree labeled by $w$.

First, the tree $T_5$ is the linear subtree consisting in the last $n_6 = 2$ nodes. By Proposition 3.61, we add a copy of $T_5$ to the node $u_1u_2u_3u_40 = 1010000$.

Then we consider the subtree $T_4$ whose root is $u_1u_2u_3u_41 = 1010001$. We add a copy of it to each node of the form $u_1u_2u_30^j1 = 1010^j1$ for all $j \in \{1, \dots, n_4 - 1\} = \{1, 2\}$.

Afterwards, we examine the subtree $T_3$ with the root $u_1u_2u_30 = 1010$. We add a copy of it to the node $u_1u_20 = 100$.

The root of the tree $T_2$ is $u_1u_21$. In this case, $n_2 = 1$, so no copy of $T_2$ is actually added.

Finally, we consider the subtree $T_1$ (resp., $T_0$) with root $u_10 = 10$ (resp., 1). In the present context, no copy of $T_1$ nor $T_0$ is added.

The proof of Proposition 3.59 is similar to the proof of Corollary 3.9.

*Proof of Proposition 3.59.* Let $u = 10^{n_k}10^{n_{k-1}} \cdots 10^{n_2}10^{n_1}$ be a non-empty word in $L_F$ with $n_1 \geq 0$ and $n_2, \dots, n_k > 0$. We proceed by induction on

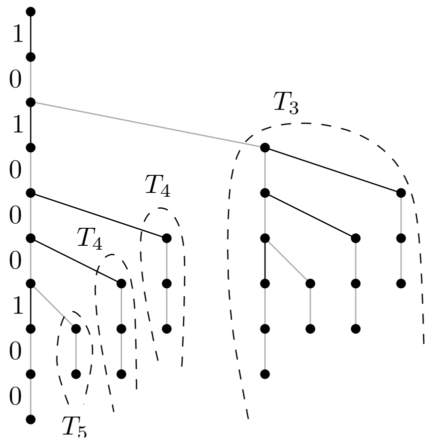



Figure 3.26: The trie $\mathcal{T}_{L_F}(101000100)$ and the connected subtrees $T_0, \ldots, T_5$.

the number $k$ of blocks of zeroes in $u$. If $k = 0$, then $u = 1$, and

$$\#\left\{v \in L_F \mid \binom{u}{v} > 0\right\} = \#\{\varepsilon, 1\} = 0 + 2,$$

as expected. If $k = 1$, then $u = 10^{n_1}$ with $n_1 > 0$, and

$$\#\left\{v \in L_F \mid \binom{u}{v} > 0\right\} = \#\{\varepsilon, 1, 10, 10^2, \ldots, 10^{n_1}\} = n_1 + 2,$$

as desired. Now suppose that $k \geq 2$, and define $u' = 10^{n_{k-1}}10^{n_{k-2}} \cdots 10^{n_2}10^{n_1}$ such that $u = 10^{n_k}u'$. By induction hypothesis, we know that

$$\#\mathcal{T}_{L_F}(u') = \#\left\{v \in L_F \mid \binom{u'}{v} > 0\right\} = (n_1 + 2) \cdot \prod_{j=2}^{k-1}(n_j + 1).$$

We now count the number of nodes of $\mathcal{T}_{L_F}(u)$, giving exactly the quantity $\#\left\{v \in L_F \mid \binom{u}{v} > 0\right\}$. Observe that its subtree of root $10^{n_k}$ is $\mathcal{T}_{L_F}(u')$. By definition, the subtree of $\mathcal{T}_{L_F}(u')$ of root 1 (resp., 10) is $T_0$ (resp., $T_1$). So, the subtree of $\mathcal{T}_{L_F}(u)$ of root $10^{n_k}1$ (resp., $10^{n_k}10$) is $T_0$ (resp., $T_1$). To build $\mathcal{T}_{L_F}(u)$ from $\mathcal{T}_{L_F}(u')$ using Proposition 3.61, we have to add the nodes $\varepsilon$ and $10^i$ for $i \in \{0, \ldots, n_k\}$, then a copy of $T_0$ to each node of the form $10^i1$ with $i \in \{1, \ldots, n_k - 1\}$, and also a copy of $T_1$ to the node $10^{n_k}0$. Thus, we have

$$\#\mathcal{T}_{L_F}(u) = \#\mathcal{T}_{L_F}(u') + \#T_1 + (n_k - 1)(\#T_0 + 1) + 2.$$

Since $\#T_0 + 1 = \#\mathcal{T}_{L_F}(u')$ and $\#T_1 + 2 = \#\mathcal{T}_{L_F}(u') = \#T_0 + 1$, we get

$$\begin{aligned}\#\mathcal{T}_{L_F}(u) = (n_k+1)\#\mathcal{T}_{L_F}(u') &= (n_k+1)\cdot(n_1+2)\cdot\prod_{j=2}^{k-1}(n_j+1)\\ &= (n_1+2)\cdot\prod_{j=2}^{k}(n_j+1),\end{aligned}$$

using the induction hypothesis in the second equality. □

We now prove that $(S_\varphi(n))_{n\geq 0}$ is $F$-regular. Other sequences exhibiting this $F$-regularity can be found in [Ber01, DMR$^+$17, DMSS16].

**Theorem 3.63.** *The sequence $(S_\varphi(n))_{n\geq 0}$ satisfies, for all $n \geq 0$,*

$$\begin{aligned}S_\varphi(i_{00}(n)) &= 2S_\varphi(i_0(n)) - S_\varphi(i_\varepsilon(n)),\\ S_\varphi(i_{01}(n)) &= 2S_\varphi(i_\varepsilon(n)),\\ S_\varphi(i_{10}(n)) &= 3S_\varphi(i_\varepsilon(n)).\end{aligned}$$

*In particular, $(S_\varphi(n))_{n\geq 0}$ is $F$-regular.*

*Proof.* Let $q \in \{0,1\}^*$ be a word such that $i_q(\mathbb{N}) \neq \emptyset$. From (1.4) on page 23, recall that, for all $n \geq 0$,

$$S_\varphi(i_q(n)) = \#\left\{v \in L_F \mid \binom{\mathrm{rep}_F(i_q(n))}{v} > 0\right\}.$$

In order to prove the claim, the idea is to use Proposition 3.59.

Let us show that the first relation holds. Let $u$ be a non-empty word in $L_F$ written as $10^{n_k}10^{n_{k-1}}\cdots 10^{n_1}$ with $n_1 \geq 0$ and $n_2, \ldots, n_k > 0$. By Proposition 3.59 we have

$$\#\left\{v \in L_F \mid \binom{u}{v} > 0\right\} = (n_1+2)\cdot\prod_{j=2}^{k}(n_j+1),$$

$$\#\left\{v \in L_F \mid \binom{u0}{v} > 0\right\} = (n_1+3)\cdot\prod_{j=2}^{k}(n_j+1),$$

and

$$\#\left\{v \in L_F \mid \binom{u00}{v} > 0\right\} = (n_1+4)\cdot\prod_{j=2}^{k}(n_j+1).$$

Hence, we get

$$\begin{aligned}&2 \cdot \# \left\{ v \in L_F \mid \binom{u0}{v} > 0 \right\} - \# \left\{ v \in L_F \mid \binom{u}{v} > 0 \right\} \\ &\quad = \# \left\{ v \in L_F \mid \binom{u00}{v} > 0 \right\}.\end{aligned}$$

This leads to the expected relation. If $n = 0$, then $i_\varepsilon(0) = 0 = i_0(0) = i_{00}(0)$, so

$$S_\varphi(i_{00}(0)) = 2S_\varphi(i_0(0)) - S_\varphi(i_\varepsilon(0))$$

is obviously true. Now if $n > 0$, then $\text{rep}_F(n)$ is a non-empty word in $L_F$, and $\text{val}_F(\text{rep}_F(n)\varepsilon) = n = i_\varepsilon(n)$. Thus, by the second part of Lemma 3.56, we have

$$\text{val}_F(\text{rep}_F(n)0) = i_0(n) \text{ and } \text{val}_F(\text{rep}_F(n)00) = i_{00}(n).$$

The last two relations are obtained using the same reasoning. One has to simply use Proposition 3.59 with words of the form $u$, $u01$ and $u010$ where $u$ is a non-empty word in $L_F$. We derive that

$$\# \left\{ v \in L_F \mid \binom{u01}{v} > 0 \right\} = 2 \cdot \# \left\{ v \in L_F \mid \binom{u}{v} > 0 \right\},$$

and

$$\# \left\{ v \in L_F \mid \binom{u010}{v} > 0 \right\} = 3 \cdot \# \left\{ v \in L_F \mid \binom{u}{v} > 0 \right\}.$$

Note that, to establish the third relation of the statement, we have to use the fact that any word in $0^*L_F$ ending with 10 must end with 010. In other words, we have $i_{010}(n) = i_{10}(n)$ for all $n \geq 0$.

The $F$-regularity of the sequence itself follows: the $\mathbb{Z}$-module generated by the $F$-kernel of $S_\varphi$ is generated by $(S_\varphi(i_\varepsilon(n)))_{n\geq 0} = (S_\varphi(n))_{n\geq 0}$ and $(S_\varphi(i_0(n)))_{n\geq 0}$. As an example, for all $n \geq 0$,

$$S_\varphi(i_{1001}(n)) = 2S_\varphi(i_{10}(n)) = 6S_\varphi(i_\varepsilon(n)) = 6S_\varphi(n). \qquad \square$$

Corollary 3.25 and Remark 3.50 are replaced by the following result stating that any term of $S_\varphi$ can be obtained as a product of matrices. The length of this product is proportional to $\log_\varphi(n)$. Here, we get square matrices of size 2 thanks to Theorem 3.63. However, one of them is defined on a block of letters rather than on one letter; see Remark 3.65 for matrices specifically defined on letters.

**Corollary 3.64.** *Consider the matrix-valued map* $\mu_{U_\varphi}\colon \{0,01\}^* \to \mathbb{Z}_2^2$ *defined by*

$$\mu_{U_\varphi}(0) = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, \quad \mu_{U_\varphi}(01) = \begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix}.$$

*For all* $n \geq 0$*, if the* $F$*-expansion of* $n$ *with a leading* 0 *is factorized into blocks* 0 *and* 01*,* i.e.*,* $0\,\mathrm{rep}_F(n) = u_k \cdots u_1$ *where* $u_i \in \{0, 01\}$ *for all* $i \in \{1, \ldots, k\}$*, then*

$$S_\varphi(n) = \begin{pmatrix} 1 & 0 \end{pmatrix} \mu_{U_\varphi}(u_1) \cdots \mu_{U_\varphi}(u_k) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

*Proof*. For convenience, let

$$V_{U_\varphi}(0) = \begin{pmatrix} S_\varphi(i_\varepsilon(0)) \\ S_\varphi(i_0(0)) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{3.10}$$

We proceed by induction on the number $k$ of different blocks in the factorization $0\,\mathrm{rep}_F(n) = u_k \cdots u_1$ with $u_i \in \{0, 01\}$ for all $i$. One can observe that the result is true for $k \in \{1, 2\}$ and $0\,\mathrm{rep}_F(n) \in \{0, 01, 00, 001, 010, 0101\}$. For example, if $k = 2$ and $0\,\mathrm{rep}_F(n) = 0101$, then $n = 4$ and

$$S_\varphi(4) = 4 = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 6 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \mu_{U_\varphi}(01)\mu_{U_\varphi}(01)V_{U_\varphi}(0).$$

Assume now that $0\,\mathrm{rep}_F(n) = u_k \cdots u_1$ with $k \geq 3$. We only consider the case $u_2 = 0$, the other case where $u_2 = 01$ being similar.

If $u_1 = 0$, then $0\,\mathrm{rep}_F(n)$ ends with 00, so $n = i_{00}(m)$ for an integer $m \geq 0$. By Theorem 3.63, we get

$$S_\varphi(n) = S_\varphi(i_{00}(m)) = 2S_\varphi(i_0(m)) - S_\varphi(i_\varepsilon(m)).$$

By Lemma 3.56, $i_0(m) = \mathrm{val}_F(u_k \cdots u_2)$ and $i_\varepsilon(m) = \mathrm{val}_F(u_k \cdots u_3)$. This, together with the induction hypothesis, leads to

$$\begin{aligned} S_\varphi(n) &= 2S_\varphi(\mathrm{val}_F(u_k \cdots u_2)) - S_\varphi(\mathrm{val}_F(u_k \cdots u_3)) \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} (2\mu_{U_\varphi}(0) - I)\, \mu_{U_\varphi}(u_3) \cdots \mu_{U_\varphi}(u_k)\, V_{U_\varphi}(0). \end{aligned}$$

The desired equality follows by observing that

$$(2\mu_{U_\varphi}(0) - I) = \mu_{U_\varphi}(0)^2 = \mu_{U_\varphi}(u_1)\mu_{U_\varphi}(u_2).$$

Similarly, if $u_1 = 01$, then $n = i_{01}(m)$ for an integer $m \geq 0$. By Theorem 3.63, we find

$$S_\varphi(n) = S_\varphi(i_{01}(m)) = 2S_\varphi(i_\varepsilon(m)).$$

Lemma 3.56 tells us that $i_\varepsilon(m) = \mathrm{val}_F(u_k \cdots u_2)$, so with the induction hypothesis, we have

$$\begin{aligned} S_\varphi(n) &= 2S_\varphi(\mathrm{val}_F(u_k \cdots u_2)) \\ &= 2\begin{pmatrix}1 & 0\end{pmatrix} \mu_{U_\varphi}(u_2) \cdots \mu_{U_\varphi}(u_k)\, V_{U_\varphi}(0). \end{aligned}$$

The expected equality holds after observing that

$$2\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\end{pmatrix} \mu_{U_\varphi}(01) = \begin{pmatrix}1 & 0\end{pmatrix} \mu_{U_\varphi}(u_1). \qquad \square$$

**Remark 3.65.** In the previous corollary, the second matrix is not defined on a letter, but on a block of two letters. Discussions with É. Charlier lead to find other square matrices that not only compute $(S_\varphi(n))_{n\geq 0}$, but are also associated with 0 and 1 (and not 0 and 01). Let us define the matrix-valued map $\mu'_{U_\varphi} : \{0,1\}^* \to \mathbb{Z}_2^2$ by

$$\mu'_{U_\varphi}(0) = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} = \mu_{U_\varphi}(0), \quad \mu'_{U_\varphi}(1) = \begin{pmatrix} 4 & -2 \\ 6 & -3 \end{pmatrix}.$$

Then we have

$$\mu_{U_\varphi}(01) = \mu'_{U_\varphi}(1)\mu'_{U_\varphi}(0).$$

Now from Corollary 3.64 and with (3.10), if $0\,\mathrm{rep}_F(n) = u_k \cdots u_1$ where $u_i \in \{0, 01\}$ for all $i \in \{1, \ldots, k\}$, then

$$\begin{aligned} S_\varphi(n) &= \begin{pmatrix}1 & 0\end{pmatrix} \mu_{U_\varphi}(u_1) \cdots \mu_{U_\varphi}(u_k)\, V_{U_\varphi}(0) \\ &= \begin{pmatrix}1 & 0\end{pmatrix} \mu'_{U_\varphi}((\mathrm{rep}_F(n))^R)\, \mu'_{U_\varphi}(0)\, V_{U_\varphi}(0) \\ &= \begin{pmatrix}1 & 0\end{pmatrix} \mu'_{U_\varphi}((\mathrm{rep}_F(n))^R)\, V_{U_\varphi}(0). \end{aligned}$$

However, it is not clear how to interpret the matrix $\mu'_{U_\varphi}(1)$ in terms of relations between sequences of the $F$-kernel of $(S_\varphi(n))_{n\geq 0}$, while the matrices $\mu_{U_\varphi}(0)$ and $\mu_{U_\varphi}(01)$ symbolize the relations of Theorem 3.63.

### 3.3.2 The Holy Grail

We obtain a recurrence relation satisfied by $(S_\varphi(n))_{n\geq 0}$, which is similar to Propositions 3.15 and 3.29.

**Proposition 3.66.** *We have* $S_\varphi(0) = 1$, $S_\varphi(1) = 2$, *and for all* $\ell \geq 1$ *and all* $0 \leq r < F(\ell-1)$, *we have*

$$S_\varphi(F(\ell)+r) = \begin{cases} S_\varphi(F(\ell-1)+r) + S_\varphi(r), & \text{if } 0 \leq r < F(\ell-2); \\ 2S_\varphi(r), & \text{if } F(\ell-2) \leq r < F(\ell-1). \end{cases}$$

*Proof.* We make use of the previous corollary. Assume that $n = F(\ell) + r$ with $\ell \geq 1$ and $0 \leq r < F(\ell-1)$. We have $0\,\mathrm{rep}_F(n) = u_k \cdots u_1$ for $k \geq 2$, with $u_i \in \{0, 01\}$ for all $i$. In particular, $u_k = 01$.

If $F(\ell-2) \leq r < F(\ell-1)$, then $u_{k-1} = 01$ and $0\,\mathrm{rep}_F(r) = u_{k-1}\cdots u_1$. By Corollary 3.64 and with (3.10), we get

$$\begin{aligned} S_\varphi(F(\ell)+r) &= \begin{pmatrix}1 & 0\end{pmatrix} \mu_{U_\varphi}(u_1)\cdots\mu_{U_\varphi}(u_{k-2})\mu_{U_\varphi}(u_{k-1})\mu_{U_\varphi}(u_k)\, V_{U_\varphi}(0) \\ &= \begin{pmatrix}1 & 0\end{pmatrix} \mu_{U_\varphi}(u_1)\cdots\mu_{U_\varphi}(u_{k-2})\mu_{U_\varphi}(01)\mu_{U_\varphi}(01)\, V_{U_\varphi}(0) \end{aligned}$$

and

$$\begin{aligned} S_\varphi(r) &= \begin{pmatrix}1 & 0\end{pmatrix} \mu_{U_\varphi}(u_1)\cdots\mu_{U_\varphi}(u_{k-2})\mu_{U_\varphi}(u_{k-1})\, V_{U_\varphi}(0) \\ &= \begin{pmatrix}1 & 0\end{pmatrix} \mu_{U_\varphi}(u_1)\cdots\mu_{U_\varphi}(u_{k-2})\mu_{U_\varphi}(01)\, V_{U_\varphi}(0). \end{aligned}$$

Since $(\mu_{U_\varphi}(01))^2 = 2\mu_{U_\varphi}(01)$, the claimed equality $S_\varphi(F(\ell)+r) = 2S_\varphi(r)$ holds.

If $0 \leq r < F(\ell-2)$, then $u_{k-1} = 0$. Let $m < k-1$ be the greatest integer such that $u_m = 01$ (we set $m = 0$ if $u_i = 0$ for all $i \leq k-1$). In this case, we have

$$\begin{aligned} 0\,\mathrm{rep}_F(F(\ell)+r) &= u_k 0^{k-m-1} u_m \cdots u_1, \\ 0\,\mathrm{rep}_F(F(\ell-1)+r) &= 010^{k-m-2} u_m \cdots u_1, \\ 0\,\mathrm{rep}_F(r) &= u_m \cdots u_1. \end{aligned}$$

By Corollary 3.64 with (3.10), we get

$$S_\varphi(F(\ell)+r) = \begin{pmatrix}1 & 0\end{pmatrix} \mu_{U_\varphi}(u_1)\cdots\mu_{U_\varphi}(u_m)\mu_{U_\varphi}(0)^{k-m-1}\mu_{U_\varphi}(01)\, V_{U_\varphi}(0),$$

$$S_\varphi(F(\ell-1)+r) = \begin{pmatrix}1 & 0\end{pmatrix} \mu_{U_\varphi}(u_1)\cdots\mu_{U_\varphi}(u_m)\mu_{U_\varphi}(0)^{k-m-2} \mu_{U_\varphi}(01)\, V_{U_\varphi}(0),$$

and

$$S_\varphi(r) = \begin{pmatrix}1 & 0\end{pmatrix} \mu_{U_\varphi}(u_1)\cdots\mu_{U_\varphi}(u_m)\, V_{U_\varphi}(0).$$

Now, it is not difficult to show by induction that, for all $j \geq 0$,

$$\mu_{U_\varphi}(0)^j \mu_{U_\varphi}(01) = \begin{pmatrix} j+2 & 0 \\ j+3 & 0 \end{pmatrix}.$$

If we let $I$ denote the identity matrix of size 2, the previous result gives

$$(\mu_{U_\varphi}(0)^{k-m-2}\mu_{U_\varphi}(01) + I)\, V_{U_\varphi}(0) = \mu_{U_\varphi}(0)^{k-m-1}\mu_{U_\varphi}(01)\, V_{U_\varphi}(0).$$

This proves that $S_\varphi(F(\ell)+r) = S_\varphi(F(\ell-1)+r)+S_\varphi(r)$ holds, as desired. □

As for Corollary 3.17, one can proceed by induction and use Proposition 3.66 to bound the terms of $(S_\varphi(n))_{n\geq 0}$. This result will prove its utility in Chapter 4.

**Corollary 3.67.** *We have $S_\varphi(1) = S_\varphi(F(0)) \leq 2^1$, and for all $\ell \geq 1$ and all $0 \leq r < F(\ell-1)$, we also have $S_\varphi(F(\ell)+r) \leq 2^{\ell+1}$.*

Using Proposition 3.66, there is a convenient way to arrange the terms of the sequence $(S_\varphi(n))_{n\geq 0}$, which is given in Table 3.27. The 0th (resp., first) row of this table contains the element $S_\varphi(0)$ (resp., $S_\varphi(1)$) of the sequence. Then for all $n \geq 2$, the $n$th row contains the elements

$$(S_\varphi(i))_{F(n-1)\leq i\leq F(n)-1},$$

and thus has $F(n-2)$ elements. For example, the second (resp., third) row contains $S_\varphi(F(1))$ (resp., $S_\varphi(F(2))$ and $S_\varphi(F(2)+1)$). Using Proposition 3.66, for $n \geq 4$, the first $F(n-3)$ elements in the $n$th row are derived from the previous row: in other words, the difference of two consecutive rows is a prefix of $(S_\varphi(n))_{n\geq 0}$. For instance, when $n = 7$, with the help of Table 3.27, the first $F(7-3) = F(4) = 8$ elements in the seventh row are $8, 12, 15, 16, 16, 15, 18, 18$, and their difference with the first eight elements in the sixth row gives $1, 2, 3, 4, 4, 5, 6, 6$, which is a prefix of $(S_\varphi(n))_{n\geq 0}$. The last $F(n-4)$ elements in the $n$th row are twice the elements in the $(n-2)$th

| row | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | | | | |
| 1 | 2 | | | | | | | | | | | | |
| 2 | 3 | | | | | | | | | | | | |
| 3 | 4 | 4 | | | | | | | | | | | |
| 4 | 5 | 6 | 6 | | | | | | | | | | |
| 5 | 6 | 8 | 9 | 8 | 8 | | | | | | | | |
| 6 | 7 | 10 | 12 | 12 | 12 | 10 | 12 | 12 | | | | | |
| 7 | 8 | 12 | 15 | 16 | 16 | 15 | 18 | 18 | 12 | 16 | 18 | 16 | 16 |

Table 3.27: Arrangement of the first few terms of $(S_\varphi(n))_{n\geq 0}$.

row. Continuing the same example, the last $F(7-4) = F(3) = 5$ elements in the seventh row are $12, 16, 18, 16, 16$, which are the five elements in the fifth row multiplied by 2. For $n \geq 1$, also observe that the first element in the $n$th row is equal to $S_\varphi(F(n-1))$, which is easily computed since from (1.4) on page 23, we have

$$\begin{aligned} S_\varphi(F(n-1)) &= \#\left\{v \in L_F \mid \binom{\mathrm{rep}_F(F(n-1))}{v} > 0\right\} \\ &= \#\left\{v \in L_F \mid \binom{10^{n-1}}{v} > 0\right\} = n+1. \end{aligned}$$

**Proposition 3.68** ("Knights of Ni")**.** *The number of occurrences of each integer $i \geq 1$ in $S_\varphi$ is finite. If $n_i$ denotes the position of the last occurrence of $i$ in $S_\varphi$, then $n_i = F(i-2)$ for all $i \geq 5$. In particular, the sequence $(n_i)_{i\geq 1}$ satisfies the same linear relation as the Fibonacci sequence if $i \geq 5$.*

*Proof.* First of all, the definition itself of $S_\varphi$ implies that

$$S_\varphi(n) > 1 = S_\varphi(0) \text{ for all } n \geq 1. \tag{3.11}$$

Then for all $\ell \geq 0$, using (1.4) and the fact that $\mathrm{rep}_F(F(\ell)) = 10^\ell$, we have

$$S_\varphi(F(\ell)) = \ell + 2. \tag{3.12}$$

Now we prove the following result: for all $\ell \geq 3$ and all $0 < r < F(\ell-1)$,

$$S_\varphi(F(\ell)) < S_\varphi(F(\ell) + r). \tag{3.13}$$

We show this by induction on $\ell$. If $\ell \in \{3,4\}$, then the fourth and fifth rows of Table 3.27 give the result. We assume that $\ell \geq 5$, and we suppose the result holds up to $\ell-1$, and we show it also holds for $\ell$. For $0 < r < F(\ell-2)$, we have by Proposition 3.66

$$S_\varphi(F(\ell)+r) = S_\varphi(F(\ell-1)+r) + S_\varphi(r),$$

but by induction hypothesis and by (3.11), we find

$$S_\varphi(F(\ell)+r) > S_\varphi(F(\ell-1)) + S_\varphi(0),$$

which in turn gives $S_\varphi(F(\ell)+r) > S_\varphi(F(\ell))$ by Proposition 3.66. Now, for $F(\ell-2) \leq r < F(\ell-1)$, we have $S_\varphi(F(\ell)+r) = 2S_\varphi(r)$ by Proposition 3.66. There exists $0 \leq r' < F(\ell-3)$ such that $r = F(\ell-2)+r'$. By induction hypothesis (note that $r'=0$ forces the equality), we get

$$S_\varphi(F(\ell)+r) = 2S_\varphi(F(\ell-2)+r') \geq 2S_\varphi(F(\ell-2)).$$

By (3.12) twice and since $\ell \geq 5$, we have

$$S_\varphi(F(\ell)+r) \geq 2\ell > \ell+2 = S_\varphi(F(\ell)).$$

This ends the proof of the intermediate result.

In fact, we can say a little more. If $\ell \geq 3$, we also have

$$S_\varphi(F(\ell)) < S_\varphi(n) \tag{3.14}$$

for all $n > F(\ell)$. Indeed, it suffices to use (3.13) and the fact that for $m \geq 1$,

$$S_\varphi(F(\ell)) = \ell+2 < \ell+m+2 = S_\varphi(F(\ell+m)),$$

which follows from (3.12).

For all $i \geq 1$, it is not difficult to conclude that the number of occurrences of $i$ in $(S_\varphi(n))_{n\geq 0}$ is finite. The sequence $(n_i)_{i\geq 1}$ is thus well defined. We also get that $n_i = F(i-2)$ for all $i \geq 5$. Indeed, from (3.12) first and then (3.14),

$$i = S_\varphi(F(i-2)) < S_\varphi(n)$$

for all $n > F(i-2)$. The last part of the statement easily follows. □

## 3.4 Concluding Remarks

After having considered the numeration language $L_b$ of base-$b$ expansions, then the numeration language $L_F$ of Fibonacci expansions, one can naturally wonder whether similar properties can be observed for an arbitrary initial language $L$ (because Pascal-like triangles may be defined in this general setting; see Definition 1.42). As already observed in Remark 3.57, what seems to be important for further generalizations is that the numeration language $L$ is the set of words not starting with 0 and not containing occurrences of a set of words of length 2. For the Fibonacci numeration system, the language of the numeration is obtained by avoiding the factor 11.

A first generalization of the Fibonacci case would be to consider the $m$-bonacci case where the corresponding numeration language is made of the words over $\{0,1\}$ avoiding the factor $1^m$ (the Fibonacci case is $m=2$, and the Tribonacci case $m=3$ has been considered in Remark 3.57).

**Example 3.69.** For $m=3$, the analogue $(S_{\beta_T}(n))_{n\geq 0}$ (`A282719` in [Slo]) of the sequence $(S_{\varphi}(n))_{n\geq 0}$ counting admissible scattered subwords associated with the Tribonacci numeration system starts with

$$1,2,3,3,4,5,5,5,7,8,6,7,7,6,9,11,9,11,12,10,9,11,11,9,7,11,14,12,\ldots.$$

Due to Remark 3.57 that easily extends to $m$-bonacci numeration languages for $m>3$, Lemma 3.56 does not hold for the $m$-bonacci numeration system as soon as $m\geq 3$. Consequently, it is not clear whether the sequence $(S_{\beta_T}(n))_{n\geq 0}$ is $T$-regular or the analogue sequence for the general $m$-bonacci case is $m$-bonacci-regular. Nevertheless, the sequence $S_{\beta_T}$ seems to partially satisfy a relation similar to the first part of Proposition 3.66. To build a table similar to the arrangement found in Table 3.27, numerical observations lead to the following conjecture: if $n_i$ denotes the position of the last occurrence of $i$ in $S_{\beta_T}$ (assuming that $n_i$ is thus well defined, which is the case for the Fibonacci case, as shown by Proposition 3.68 above), then

$$S_{\beta_T}(n_i+r)=S_{\beta_T}(n_{i-1}+r)+S_{\beta_T}(r)$$

for $0\leq r<n_i-n_{i-1}$ and $i\geq 5$ (see Table 3.28). Observe that combining Propositions 3.66 and 3.68 gives the same result in the Fibonacci case. Moreover, the sequence $(n_i)_{i\geq 1}$ (`A282718` in [Slo])

$$(n_i)_{i\geq 1}=0,1,3,4,7,13,24,44,81,149,274,504,\ldots$$

satisfies the same linear relation as the Tribonacci sequence when $i \geq 4$, which was also the case for the Fibonacci numeration system in Proposition 3.68. However, it is not clear that one can determine a "simple" relation for $S_{\beta_T}(n_i + r)$ when $n_i - n_{i-1} \leq r < n_{i+1} - n_i$ (corresponding to the second part of Proposition 3.66), and thus derive a possible regularity of the sequence $S_{\beta_T}$. Again, one can also try with larger values of the parameter $m$, and imagine partial relations of the same form.

| row | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | | | | | | | |
| 1 | 2 | 3 | | | | | | | | | | | | | | |
| 2 | 3 | | | | | | | | | | | | | | | |
| 3 | 4 | 5 | 5 | | | | | | | | | | | | | |
| 4 | 5 | 7 | 8 | 6 | 7 | 7 | | | | | | | | | | |
| 5 | 6 | 9 | 11 | 9 | 11 | 12 | 10 | 9 | 11 | 11 | 9 | | | | | |
| 6 | 7 | 11 | 14 | 12 | 15 | 17 | 15 | 14 | 18 | 19 | 15 | 14 | 14 | 11 | 15 | ... |
| 7 | 8 | 13 | 17 | 15 | 19 | 22 | ... | | | | | | | | | |

Table 3.28: Arrangement of the first few terms in $(S_{\beta_T}(n))_{n\geq 0}$.

As a second generalization, for a Parry number $\beta > 1$, one could consider the sequence $(S_\beta(n))_{n\geq 0}$ from Definition 1.47 for the Parry–Bertrand numeration system $U_\beta$ associated with $\beta$ from Definition 1.29. In this extended context, can we provide recurrence relations verified by $(S_\beta(n))_{n\geq 0}$, and is the sequence regular with respect to the numeration system $U_\beta$?

**Question 1.** Can we extend the results of Chapter 3 to other numeration systems? What are the precise conditions on the numeration systems to obtain regular sequences? See in particular [Sha88, AST00]. If we consider other possible extensions of the Pascal triangle (see the list after the definition of the Pascal triangle in Section 1.5), would it be possible to deduce similar results?

A famous theorem due to A. Cobham allows us to compare automatic sequences: if $a$ and $b$ are multiplicatively independent integers, *i.e.*, $a^m \neq b^n$ for all $m, n \in \mathbb{N}_{>0}$, then a sequence that is both $a$- and $b$-automatic is ultimately periodic [Cob69]. In the context of regular sequences, J. Bell obtained a similar result [Bel07]: if $a$ and $b$ are multiplicatively independent integers, then a sequence that is both $a$- and $b$-regular satisfies a linear recurrence relation. This leads to the following question.

**Question 2.** Would it be possible to show that the sequence $(S_\varphi(n))_{n\geq 0}$ is not $b$-regular for any integer $b \geq 2$? More generally, would it possible to establish a Cobham-like theorem for sequences that present a regularity with respect to (abstract) numeration systems that are sufficiently different?

Recall that the sequence $(S_2(n))_{n\geq 0}$ is bonded with the Stern–Brocot sequence (see Propositions 3.19 and 3.20), and its 2-regularity, which is one of the main results of this chapter, naturally ensues from the 2-regularity of the Stern–Brocot sequence. As the sequence $(S_\beta(n))_{n\geq 0}$ generalizes the sequence $(S_2(n))_{n\geq 0}$, one can ask the following question, which was already raised in Remark 3.52.

**Question 3.** Echoing Remark 3.52, can one reasonably define some combinatorial structure, *e.g.*, a tree in the base-2 case, in which the sequence $(S_\beta(n))_{n\geq 0}$ naturally appears? What would be the analogue of the Stern–Brocot sequence for $(S_\beta(n))_{n\geq 0}$?

In Remark 3.65, a linear representation of the sequence $(S_\varphi(n))_{n\geq 0}$ was made of matrices associated with 0 and 1. However, the matrix associated with 1 does not reflect any relations between sequences of the $F$-kernel of $(S_\varphi(n))_{n\geq 0}$.

**Question 4.** Would it be possible to find a linear representation of the sequence $(S_\varphi(n))_{n\geq 0}$ made of matrices associated with 0 and 1 such that both can be interpreted in terms of relations between sequences of the $F$-kernel of $(S_\varphi(n))_{n\geq 0}$? More generally, is the series $\sum_{n\geq 0} S_\beta(n)\,\mathrm{rep}_{U_\beta}(n)$ $\mathbb{N}$- or $\mathbb{Z}$-recognizable? What is its rank?

# Chapter 4

# Asymptotics Through Exotic Numerations

In a general sense, digital functions have a definition that depends on the digits in some representation of the integers [BR10, Chapter 9]. Many of them, *e.g.*, the sum of the output labels of a finite transducer reading base-$b$ expansions of integers [HKP15], have been extensively studied in the literature and exhibit an interesting behavior that usually involves some periodic fluctuation [BR10, Del75, Dum13, Dum14, GH05, GR03, GT00, HKP18, HK18]. Such functions are commonly studied using techniques from analytic number theory or linear algebra. For instance, consider the archetypal sum-of-digits function $(s_2(n))_{n\geq 0}$ for base-2 expansions of integers [Tro68]. Its summatory function $A\colon \mathbb{N} \to \mathbb{N}, n \mapsto \sum_{j=0}^{n-1} s_2(j)$ counts the total number of ones occurring in the base-2 expansion of the first $n$ integers, *i.e.*, the sum of the sums of digits of the first $n$ integers in base 2. In [Del75], it is showed that there exists a continuous nowhere differentiable periodic function $\mathcal{G}$ of period 1 such that

$$\frac{A(n)}{n} = \frac{1}{n}\sum_{j=0}^{n-1} s_2(j) = \frac{1}{2}\log_2 n + \mathcal{G}(\log_2 n), \tag{4.1}$$

which gives an exact formula for the summatory function of $s_2$. For an account on this result, see, for instance, [AS03a, Theorem 3.5.4]. As observed in Example 1.56, $s_2$ has a specific internal structure: it is 2-regular. Based on linear algebra techniques, general asymptotic estimates for summatory functions of $b$-regular sequences can be provided [Dum13, Dum14] and are similar to (4.1), although an error term usually appears. Comparable results are also discussed in [BR10, Theorem 9.2.15] and [AS03a].

In this chapter, we expose a new method based on exotic numeration systems to tackle the behavior of the summatory function $A_s = (A_s(n))_{n\geq 0}$ of a digital sequence $s = (s(n))_{n\geq 0}$. Roughly, the idea is to find two sequences $r = (r(n))_{n\geq 0}$ and $t = (t(n))_{n\geq 0}$, each satisfying a linear recurrence relation, such that $A_s(r(n)) = t(n)$ for all $n \geq 0$. From a recurrence relation satisfied by $s$, we deduce a recurrence relation for $A_s$ in which $t$ is involved. This allows us to find relevant representations of $A_s$ in some exotic numeration system associated with the sequence $t$. The adjective "exotic" means that we have a decomposition of particular integers as a linear combination of terms of the sequence $t$ possibly with unbounded coefficients. Then the behavior of $A_s$ depends on the dominant root of the characteristic polynomial of the linear recurrence relation that defines $t$. We present this method on examples inspired by the study of generalized Pascal triangles and binomial coefficients of words, and we obtain behaviors similar to (4.1).

In Section 4.1, the first example is the summatory function of $(S_2(n))_{n\geq 0}$. Since the latter sequence is 2-regular (see Chapter 3), the asymptotics of its summatory function can be studied via classical techniques [Dum13, Dum14]. Anyway, our method provides an exact behavior for this summatory function. Then Section 4.2 extends the results to integer bases. More importantly, the approach also allows us to deal with sequences that do not present any $b$-regular structure (up to our knowledge), as illustrated by the example taken in Section 4.3, which is Fibonacci-regular. In the last section of this chapter, we present some open problems and conjectures in a more general context. The results presented in this chapter come from [LRS17a, LRS18].

## Contents

## 4.1 Combining Base 2 and Base 3

In this section, we work with the sequence $(S_2(n))_{n\geq 0}$ from Example 1.49 and its summatory function defined below.

**Definition 4.1.** We let $A_2 = (A_2(n))_{n\geq 0}$ denote the *summatory function* of the sequence $(S_2(n))_{n\geq 0}$ defined by

$$A_2(n) = \sum_{j=0}^{n-1} S_2(j)$$

for all $n \geq 0$. Its first few terms are

$$0, 1, 3, 6, 9, 13, 18, 23, 27, 32, 39, 47, 54, 61, 69, 76, 81, 87, 96, 107, 117, \ldots$$

(see `A282720` in [Slo]). The quantity $A_2(n)$ counts the total number of base-2 expansions occurring as scattered subwords in the base-2 expansion of integers less than $n$ (the same scattered subword is counted $k$ times if it occurs in the base-$b$ expansion of $k$ distinct integers).

We immediately know that $A_2$ is 2-regular. Note that it is possible to obtain a linear representation of the summatory function $(A_2(n))_{n\geq 0}$ using the one of $(S_2(n))_{n\geq 0}$ stated in Corollary 3.25 (see Remark 4.3 below).

**Proposition 4.2.** *The sequence* $(A_2(n))_{n\geq 0}$ *is* 2*-regular.*

*Proof*. This is a direct consequence of Theorem 3.24 and of the fact that the summatory function of a 2-regular sequence is also 2-regular; see [AS03a, Theorem 16.4.1]. □

**Remark 4.3.** From a linear representation with matrices of size $k$ associated with a $b$-regular sequence, one can derive a linear representation with matrices of size $2k$ associated with its summatory function; see [Dum13, Lemma 1]. Corollary 3.25 yields the following linear representation of $(S_2(n))_{n\geq 0}$

$$r = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad c = V_2(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$\Gamma_0 = \mu_2(0) = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, \quad \Gamma_1 = \mu_2(1) = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}.$$

Thus, a linear representation of $(A_2(n))_{n\geq 0}$ is given by

$$r_{A_2} = \begin{pmatrix} r & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}, \quad c_{A_2} = \begin{pmatrix} c \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

$$\Gamma_{A_2,0} = \begin{pmatrix} \Gamma_0 + \Gamma_1 & -\Gamma_1 \\ 0 & \Gamma_0 \end{pmatrix}, \quad \Gamma_{A_2,1} = \begin{pmatrix} \Gamma_0 + \Gamma_1 & 0 \\ 0 & \Gamma_1 \end{pmatrix}.$$

The sequence $(S_2(n))_{n\geq 0}$ being 2-regular, asymptotic estimates of the sequence $(A_2(n))_{n\geq 0}$ could be deduced from [Dum13]. However, as already mentioned, such estimates usually contain an error term. Applying our method, we get an exact formula for $A_2(n)$ given by Theorem 4.4 below[1]. It is worth noticing that the sequences $(S_2(n))_{n\geq 0}$ and $(A_2(n))_{n\geq 0}$ fail to satisfy the hypotheses of the stronger result [AS03a, Theorem 3.5.1] (see Theorem 1.57 in Chapter 1), which gives yet an exact behavior. Let us show why we cannot directly make use of this result. Using the notation of Theorem 1.57 and Remark 4.3 above, we have

$$V_2(n) = \begin{pmatrix} S_2(n) \\ S_2(2n) \end{pmatrix}, \quad \Gamma = \Gamma_0 + \Gamma_1, \quad \text{and } \Gamma^{-1} = \begin{pmatrix} \frac{1}{3} & 0 \\ -1 & 1 \end{pmatrix}.$$

A direct computation shows that the eigenvalues of $\Gamma^{-1}$ are 1/3 and 1. For any matrix norm $\|\cdot\|_a$ compatible with a vector norm $\|\cdot\|_b$ on $\mathbb{C}^2$, *i.e.*, they verify $\|Mx\|_b \leq \|M\|_a \, \|x\|_b$ for all $x \in \mathbb{C}^2$ and all $M \in \mathbb{C}_2^2$, if $x$ is an eigenvector of $M$ associated with the eigenvalue $\lambda$ of $M$, we have

$$|\lambda| \, \|x\|_b = \|Mx\|_b \leq \|M\|_a \, \|x\|_b.$$

For any compatible norm, we thus obtain $\|\Gamma^{-1}\| \geq 1$, which violates the last hypothesis of Theorem 1.57.

**Theorem 4.4.** *There exists a continuous and periodic function $\mathcal{H}_2$ of period 1 such that, for all large enough $n$,*

$$A_2(n) = \sum_{j=0}^{n-1} S_2(j) = 3^{\log_2 n} \; \mathcal{H}_2(\log_2 n).$$

[1] The classical results on asymptotics of summatory functions of regular sequences are often stated in the same way [AS03a, BR10, Dum13, Dum14]. Theorem 4.4, and later Theorems 4.39 and 4.43, are no exceptions and are stated in this manner.

To derive this result, we make an extensive use of a particular decomposition of $A_2(n)$, with $n \geq 0$, based on powers of 3 that we call 3-decomposition. These occurrences of powers of 3 come from the following lemma, which is in fact a rewriting of Proposition 2.5.

**Lemma 4.5.** *For all $n \in \mathbb{N}$, we have $A_2(2^n) = 3^n$.*

For the sake of presentation, we introduce the *relative position* $\mathrm{relpos}_2(x)$ of a positive real number $x$ inside the interval $[2^{\lfloor \log_2 x \rfloor}, 2^{\lfloor \log_2 x \rfloor + 1})$, *i.e.*,

$$\mathrm{relpos}_2(x) = \frac{x - 2^{\lfloor \log_2 x \rfloor}}{2^{\lfloor \log_2 x \rfloor}} = 2^{\{\log_2 x\}} - 1 \in [0, 1),$$

where $\{\cdot\}$ denotes the fractional function (see Section 1.1). In Figure 4.1, the map $\lfloor \log_2(\cdot) \rfloor + \mathrm{relpos}_2(\cdot)$ (in orange) is compared with $\log_2(\cdot)$ (the dashed line). Observe that both functions take the same value at powers of 2, and the first one is affine between two consecutive powers of 2.

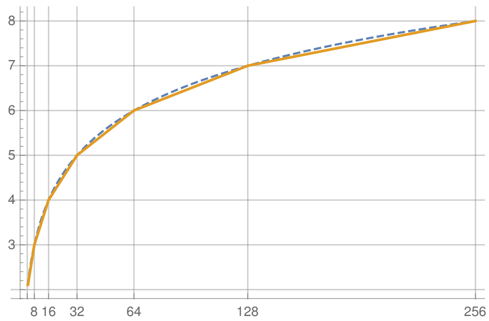



Figure 4.1: The map $\lfloor \log_2(\cdot) \rfloor + \mathrm{relpos}_2(\cdot)$ compared with $\log_2(\cdot)$.

In the remaining of the section, we prove an equivalent version of Theorem 4.4 when considering the function $\mathcal{H}_2$ defined by $\mathcal{H}_2(x) = \Phi_2(\mathrm{relpos}_2(2^x))$.

**Theorem 4.6.** *There exists a continuous function $\Phi_2$ over $[0, 1)$ such that $\Phi_2(0) = 1$, $\lim_{\alpha \to 1^-} \Phi_2(\alpha) = 1$, and the sequence $(A_2(n))_{n \geq 0}$ satisfies, for all $n \geq 1$,*

$$A_2(n) = 3^{\log_2 n}\ \Phi_2(\mathrm{relpos}_2(n)) = n^{\log_2 3}\ \Phi_2(\mathrm{relpos}_2(n)).$$

The graph of $\Phi_2$ is depicted in Figure 4.2. We will show in Lemma 4.24 that $\Phi_2$ can be computed on a dense subset of $[0, 1)$.

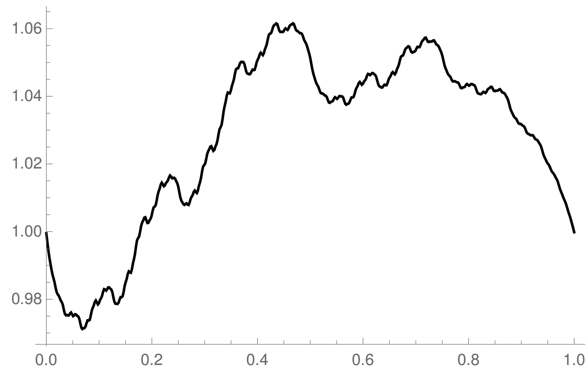



Figure 4.2: The graph of $\Phi_2$.

Particularly giving recurrences for $S_2$, Proposition 3.15 also permits us to derive two convenient relations for $A_2$ where powers of 3 appear. This is the starting point of the 3-decompositions mentioned above.

**Lemma 4.7.** *Let $\ell \geq 1$. If $0 \leq r \leq 2^{\ell-1}$, then*

$$A_2(2^\ell + r) = 2 \cdot 3^{\ell-1} + A_2(2^{\ell-1} + r) + A_2(r).$$

*If $2^{\ell-1} < r < 2^\ell$, then*

$$A_2(2^\ell + r) = 4 \cdot 3^\ell - 2 \cdot 3^{\ell-1} - A_2(2^{\ell-1} + r') - A_2(r') \quad \text{where } r' = 2^\ell - r.$$

*Proof*. Let us start with the first case. If $r = 0$, the result follows from Lemma 4.5. Now assume that $0 < r \leq 2^{\ell-1}$. Applying Proposition 3.15 and Lemma 4.5 twice, we get

$$\begin{aligned}
A_2(2^\ell + r) &= \sum_{j=0}^{2^\ell+r-1} S_2(j) \\
&= A_2(2^\ell) + \sum_{j=0}^{r-1} S_2(2^\ell + j) \\
&= 3^\ell + \sum_{j=0}^{r-1} S_2(2^{\ell-1} + j) + \sum_{j=0}^{r-1} S_2(j) \\
&= 3^\ell + A_2(2^{\ell-1} + r) - A_2(2^{\ell-1}) + A_2(r) \\
&= 2 \cdot 3^{\ell-1} + A_2(2^{\ell-1} + r) + A_2(r).
\end{aligned}$$

Let us proceed to the second case with $2^{\ell-1} < r < 2^\ell$ and $r' = 2^\ell - r$. Notice that $0 < r' < 2^{\ell-1}$. From Lemma 4.5, we obtain

$$\begin{aligned} A_2(2^\ell + r) = A_2(2^{\ell+1} - r') &= A_2(2^{\ell+1}) - \sum_{j=1}^{r'} S_2(2^{\ell+1} - j) \\ &= 3^{\ell+1} - \sum_{j=1}^{r'} S_2(2^\ell + 2^\ell - j). \end{aligned}$$

Applying Proposition 3.15 and then Lemma 4.5 again, we find

$$\begin{aligned} A_2(2^\ell + r) &= 3^{\ell+1} - \sum_{j=1}^{r'} S_2(2^\ell + j - 1) \\ &= 3^{\ell+1} - A_2(2^\ell + r') + A_2(2^\ell) \\ &= 4 \cdot 3^\ell - A_2(2^\ell + r'). \end{aligned}$$

We may apply the first part of this lemma with $r'$ and thus get

$$A_2(2^\ell + r) = 4 \cdot 3^\ell - 2 \cdot 3^{\ell-1} - A_2(2^{\ell-1} + r') - A_2(r'). \qquad \square$$

**Remark 4.8.** If it were authorized, plugging $r = 2^{\ell-1}$ in the second formula of Lemma 4.7 would give $A_2(2^\ell + 2^{\ell-1}) = 2 \cdot 3^\ell$, which is equal to the value given by the first formula of Lemma 4.7.

The values taken by the summatory function $A_2$ at multiples of 2 are multiples of 3, as demonstrated below.

**Corollary 4.9.** *For all $n \geq 0$, $A_2(2n) = 3A_2(n)$.*

*Proof.* Let us proceed by induction on $n \geq 0$. The result holds for $n \in \{0, 1\}$ since $A_2(0) = 0$, $A_2(1) = 1$ and $A_2(2) = 3$. Thus consider $n \geq 2$, and suppose that the result holds for all $m < n$. Let us write $n = 2^\ell + r$ with $\ell \geq 1$ and $0 \leq r < 2^\ell$. Let us first assume that $0 \leq r \leq 2^{\ell-1}$. Then, by Lemma 4.7, we have

$$3A_2(n) - A_2(2n) = 2 \cdot 3^\ell + 3A_2(2^{\ell-1} + r) + 3A_2(r) - 2 \cdot 3^\ell - A_2(2^\ell + 2r) - A_2(2r).$$

We conclude this case by using the induction hypothesis. Now suppose that

$2^{\ell-1} < r < 2^\ell$. Lemma 4.7 leads to

$$\begin{aligned}3A_2(n) - A_2(2n) = &4 \cdot 3^{\ell+1} - 2 \cdot 3^\ell - 3A_2(2^{\ell-1} + r') - 3A_2(r') \\ &- 4 \cdot 3^{\ell+1} + 2 \cdot 3^\ell + A_2(2^\ell + 2r') + A_2(2r'),\end{aligned}$$

where $r' = 2^\ell - r$. The result follows from the induction hypothesis. □

### 4.1.1 Special 3-Decompositions

Let us consider two examples to understand the forthcoming notion of 3-decompositions. The idea is to iteratively apply Lemma 4.7 in order to derive a decomposition of $A_2(n)$, for $n \geq 0$, as a particular linear combination of powers of 3. Indeed, each application of Lemma 4.7 provides a "leading" term of the form $2 \cdot 3^{\ell-1}$ or $4 \cdot 3^\ell - 2 \cdot 3^{\ell-1}$, plus terms where smaller powers of 3 occur. To be precise, the special case of $A_2(2^\ell + 2^{\ell-1})$ gives, when applying the lemma twice, a term $2 \cdot 3^{\ell-1} + 2 \cdot 3^{\ell-1} = 4 \cdot 3^{\ell-1}$, plus terms where smaller powers of 3 occur. Note that Lemma 4.7 only applies to integers greater than 1, so we choose to set $A_2(0) = 0 \cdot 3^0$ and $A_2(1) = 1 \cdot 3^0$ (this last decomposition in terms of 3-powers is coherent with Lemma 4.5).

**Example 4.10.** To compute $A_2(42)$, four applications of Lemma 4.7 yield

$$\begin{aligned}A_2(42) = A_2(2^5 + 10) &= 2 \cdot 3^4 + A_2(2^4 + 10) + A_2(2^3 + 2), \\ A_2(2^4 + 10) &= 4 \cdot 3^4 - 2 \cdot 3^3 - A_2(2^3 + 6) - A_2(2^2 + 2), \\ A_2(2^3 + 2) &= 2 \cdot 3^2 + A_2(2^2 + 2) + A_2(2), \\ A_2(2) = A_2(2^1) &= 2 \cdot 3^0 + A_2(1) + A_2(0) = 3 \cdot 3^0.\end{aligned}$$

We thus get

$$A_2(42) = 6 \cdot 3^4 - 2 \cdot 3^3 - A_2(2^3 + 6) + 2 \cdot 3^2 + 3 \cdot 3^0.$$

At this stage, we already know that, in the next applications of the lemma, no other term in $3^4$ can occur because we are left with the decomposition of $A_2(2^3 + 6)$. Applying again Lemma 4.7 yields

$$\begin{aligned}A_2(2^3 + 6) &= 4 \cdot 3^3 - 2 \cdot 3^2 - A_2(2^2 + 2) - A_2(2), \\ A_2(2^2 + 2) &= 2 \cdot 3 + A_2(2 + 2) + A_2(2), \\ A_2(4) = A_2(2^2) &= 2 \cdot 3 + A_2(2) + A_2(0).\end{aligned}$$

So we have $A_2(4) = 2 \cdot 3 + 3 \cdot 3^0$, $A_2(2^2 + 2) = 4 \cdot 3 + 6 \cdot 3^0$ and thus also $A_2(2^3 + 6) = 4 \cdot 3^3 - 2 \cdot 3^2 - 4 \cdot 3 - 9 \cdot 3^0$. Finally[2],

$$A_2(42) = 6 \cdot 3^4 - 6 \cdot 3^3 + 4 \cdot 3^2 + 4 \cdot 3 + 12 \cdot 3^0. \tag{4.2}$$

Proceeding similarly with $A_2(84)$, we have

$$\begin{aligned} A_2(84) = A_2(2^6 + 20) &= 2 \cdot 3^5 + A_2(2^5 + 20) + A_2(2^4 + 4), \\ A_2(2^5 + 20) &= 4 \cdot 3^5 - 2 \cdot 3^4 - A_2(2^4 + 12) - A_2(2^3 + 4), \\ A_2(2^4 + 4) &= 2 \cdot 3^3 + A_2(2^3 + 4) + A_2(4) \\ &= 2 \cdot 3^3 + A_2(2^3 + 4) + 2 \cdot 3 + 3 \cdot 3^0. \end{aligned}$$

We thus get

$$A_2(84) = 6 \cdot 3^5 - 2 \cdot 3^4 - A_2(2^4 + 12) + 2 \cdot 3^3 + 2 \cdot 3 + 3 \cdot 3^0,$$

and

$$\begin{aligned} A_2(2^4 + 12) &= 4 \cdot 3^4 - 2 \cdot 3^3 - A_2(2^3 + 4) - A_2(4), \\ A_2(2^3 + 4) &= 2 \cdot 3^2 + A_2(2^2 + 4) + A_2(4), \\ A_2(2^2 + 4) &= A_2(2^3) = 2 \cdot 3^2 + A_2(2^2) + A_2(0), \\ A_2(4) &= 2 \cdot 3 + 3 \cdot 3^0. \end{aligned}$$

Since $A_2(2^3 + 4) = 4 \cdot 3^2 + 4 \cdot 3 + 6 \cdot 3^0$, we lastly find

$$A_2(84) = 6 \cdot 3^5 - 6 \cdot 3^4 + 4 \cdot 3^3 + 4 \cdot 3^2 + 8 \cdot 3 + 12 \cdot 3^0. \tag{4.3}$$

If we compare (4.2) and (4.3), we may already notice that the same leading coefficients $6, -6, 4$ and $4$ occur in front of dominating powers of 3.

**Definition 4.11** (3-decomposition)**.** We have $A_2(0) = 0 \cdot 3^0$ (resp., $A_2(1) = 1 \cdot 3^0$), so we say that the single-letter word

$$\mathsf{3dec}(A_2(0)) = 0 \quad \text{(resp., } \mathsf{3dec}(A_2(1)) = 1\text{)}$$

is the 3-*decomposition* of $A_2(0)$ (resp., $A_2(1)$).

For $n \geq 2$, we write $n = 2^{\lfloor \log_2 n \rfloor} + r$, and we define

$$\ell_2(n) = \begin{cases} \lfloor \log_2 n \rfloor - 1, & \text{if } 0 \leq r \leq 2^{\lfloor \log_2 n \rfloor - 1}; \\ \lfloor \log_2 n \rfloor, & \text{if } 2^{\lfloor \log_2 n \rfloor - 1} < r < 2^{\lfloor \log_2 n \rfloor}. \end{cases}$$

[2] *The Ultimate Question of Life, the Universe and Everything.*

Iteratively applying Lemma 4.7 provides a decomposition of the form

$$A_2(n) = \sum_{i=0}^{\ell_2(n)} a_i(n)\, 3^{\ell_2(n)-i},$$

where $a_i(n)$'s are integer coefficients and $a_0(n) \neq 0$. We say that the word

$$\mathsf{3dec}(A_2(n)) = a_0(n) \cdots a_{\ell_2(n)}(n)$$

is the 3-*decomposition* of $A_2(n)$.

When the integer $n$ is clear from the context, we simply write $a_i$ instead of $a_i(n)$. For the sake of clarity, we will also write $(a_0(n), \ldots, a_{\ell_2(n)}(n))$.

As an example, we have $\ell_2(84) = 5$ and, using (4.3), the 3-decomposition of $A_2(84)$ is $(6, -6, 4, 4, 8, 12)$. In Table 4.3, we compute the 3-decomposition of $A_2(n)$ for $0 \leq n \leq 20$ (see also `A282728` in [Slo]). Notice that the notion of 3-decomposition is only valid when the values taken by the sequence $(A_2(n))_{n\geq 0}$ are concerned. For instance, the 3-decomposition of 5 is not defined because 5 is not in $\{A_2(n) \mid n \in \mathbb{N}\}$.

**Remark 4.12.** Assume that we want to develop $A_2(n)$ using Lemma 4.7 only, *i.e.*, to get the 3-decomposition of $A_2(n)$. Several cases may occur.

- If $\operatorname{rep}_2(n) = 10u$, with $u \in \{0,1\}^*$ possibly starting with 0, then we apply the first part of Lemma 4.7, and we are left with evaluations of $A_2$ at integers whose base-2 expansions are shorter and given by $1u$ and $\operatorname{rep}_2(\operatorname{val}_2(u))$. Note that $\operatorname{rep}_2(\operatorname{val}_2(u))$ removes the possible leading zeroes in front of $u$.

- If $\operatorname{rep}_2(n) = 11u$, with $u \in \{0,1\}^* \setminus 0^*$, *i.e.*, $u$ contains at least one letter 1, then we apply the second part of Lemma 4.7. We are thus are left with evaluations of $A_2$ at integers whose base-2 expansions are shorter and given by $1u'$ and $\operatorname{rep}_2(\operatorname{val}_2(u'))$ with

  $$u' \in \{0,1\}^*, |u'| = |u| \text{ and } \operatorname{val}_2(u') = \operatorname{val}_2(\underline{u}) + 1,$$

  where $\underline{\cdot}$ is the involutory morphism exchanging 0 and 1. Indeed, in this case, we have $r = \operatorname{val}_2(1u)$ and thus $\operatorname{val}_2(u') = r'$ is equal to $2^{|u|+1} - r = 2^{|u|} - \operatorname{val}_2(u) = (2^{|u|} - 1 - \operatorname{val}_2(u)) + 1 = \operatorname{val}_2(\underline{u}) + 1$. As an example, if $u = 01011000$, then $\underline{u} = 10100111$ and $u' = 10101000$. In

| $n$ | $a_0(n)$ | $a_1(n)$ | $a_2(n)$ | $a_3(n)$ | $A_2(n)$ |
|---|---|---|---|---|---|
| 0 | 0 | | | | $0 \times 1 = 0$ |
| 1 | 1 | | | | $1 \times 1 = 1$ |
| 2 | 3 | | | | $3 \times 1 = 3$ |
| 3 | 6 | | | | $6 \times 1 = 6$ |
| 4 | 2 | 3 | | | $2 \times 3 + 3 \times 1 = 9$ |
| 5 | 2 | 7 | | | $2 \times 3 + 7 \times 1 = 13$ |
| 6 | 4 | 6 | | | $4 \times 3 + 6 \times 1 = 18$ |
| 7 | 4 | $-2$ | $-7$ | | $4 \times 3^2 - 2 \times 3 - 7 \times 1 = 23$ |
| 8 | 2 | 2 | 3 | | $\vdots$ |
| 9 | 2 | 2 | 8 | | |
| 10 | 2 | 4 | 9 | | |
| 11 | 6 | $-2$ | $-1$ | | |
| 12 | 4 | 4 | 6 | | |
| 13 | 4 | $-6$ | 2 | 1 | $\vdots$ |
| 14 | 4 | $-2$ | $-4$ | $-9$ | |
| 15 | 4 | $-2$ | $-2$ | $-8$ | |
| 16 | 2 | 2 | 2 | 3 | |
| 17 | 2 | 2 | 2 | 9 | |
| 18 | 2 | 2 | 4 | 12 | |
| 19 | 2 | 6 | $-2$ | 5 | |
| 20 | 2 | 4 | 6 | 9 | $\vdots$ |

Table 4.3: The 3-decomposition of $A_2(0), A_2(1), \ldots, A_2(20)$.

fact, if we mark the last occurrence of 1 in $u$ (such an occurrence always exists by assumption), *i.e.*, $u = v10^n$ for some $n \geq 0$, then $u' = \underline{v}10^n$.

- If $\text{rep}_2(n) = 110^k$ with $k \geq 0$, then we will apply the first part of Lemma 4.7, and we are left with evaluations of $A_2$ at integers whose base-2 expansions are given by $10^{k+1}$ and $10^k$. This situation seems not so nice: we are left with the word $10^{k+1}$ of the same length as the original one $110^k$. However, the next application of Lemma 4.7 provides the word $10^k$, and the computation easily ends with a total number of calls to this lemma equal to $k+1$, namely the computations of

$$A_2(2^{k+1}), A_2(2^k), \dots, A_2(2^0), A_2(0)$$

are needed. This situation is not so bad since the numbers of calls to Lemma 4.7 to evaluate $A_2$ at integers with base-2 expansions of the same length can be equal. For instance, the computation of $A_2(12)$ requires the computations of $A_2(8), A_2(4), A_2(2), A_2(1), A_2(0)$ (and we have $\text{rep}_2(12) = 1100$ for which $k = 2$), and the one of $A_2(14)$ needs those of $A_2(6), A_2(4), A_2(2), A_2(1), A_2(0)$.

As already observed with (4.2) and (4.3), the 3-decompositions of $A_2(42)$ and $A_2(84)$ share the same first digits. This is a general fact as stated in the next lemma. Roughly speaking, if the base-2 expansions of two integers $m$ and $n$ have a long common prefix, then the most significant coefficients in the corresponding 3-decompositions of $A_2(m)$ and $A_2(n)$ are the same.

**Lemma 4.13.** *Let $u \in \{0,1\}^*$ be a finite word of length at least 2. For all finite words $v, v' \in \{0,1\}^* \setminus 0^*$, the 3-decompositions of $A_2(\text{val}_2(1uv))$ and $A_2(\text{val}_2(1uv'))$ share the same coefficients $a_0, \dots, a_{|u|-2}$,* i.e.*, their first $|u|-1$ coefficients are equal.*

*Proof*. It is a consequence of Lemma 4.7. Proceed by induction on the length of the words. The word $u$ is of the form $0^{n_1}10^{n_2}1\cdots 10^{n_k}$ with $k \geq 1$ and $n_1, \dots, n_k \geq 0$. If $n_1 > 0$, due to Lemma 4.7, $A_2(\text{val}_2(1uv))$ is decomposed as

$$2 \cdot 3^{|u|+|v|-1} + A_2(\text{val}_2(10^{n_1-1}10^{n_2}1\cdots 10^{n_k}v)) + A_2(\text{val}_2(10^{n_2}1\cdots 10^{n_k}v)).$$

Proceeding similarly, $A_2(\text{val}_2(1uv'))$ is decomposed as

$$2 \cdot 3^{|u|+|v'|-1} + A_2(\text{val}_2(10^{n_1-1}10^{n_2}1\cdots 10^{n_k}v')) + A_2(\text{val}_2(10^{n_2}1\cdots 10^{n_k}v')).$$

The first term in these two expressions will equally contribute to the coefficient $a_0$ in the two 3-decompositions. For the last two terms, we may apply the induction hypothesis. If $n_1 = 0$, then

$$1uv = 110^{n_2}1\cdots 10^{n_k}v = 110^{n_2}1\cdots 10^{n_k}x10^t,$$

where $v = x10^t$ with $x \in \{0,1\}^*$ and $t \geq 0$ by hypothesis. Recall the second case of Remark 4.12. Applying Lemma 4.7 to $A_2(\mathrm{val}_2(1uv))$ gives

$$4 \cdot 3^{|u|+|v|} - 2 \cdot 3^{|u|+|v|-1} + A_2(\mathrm{val}_2(11^{n_2}0 \cdots 01^{n_k}\underline{x}10^t)) \\ + A_2(\mathrm{val}_2(1^{n_i}01^{n_{i+1}}0 \cdots 01^{n_k}\underline{x}10^t)),$$

where $i$ is the smallest index in $\{2, \ldots, k\}$ such that $n_i > 0$. We can conclude in the same way as in the first case. □

**Example 4.14.** Take

$$\mathrm{rep}_2(745) = 1(01110)1001 = 1uv \text{ and } \mathrm{rep}_2(5904) = 1(01110)0010000 = 1uv'$$

with $u = 01110$, $|u| = 5$, $v = 1001$ and $v' = 0010000$. If we compare the 3-decompositions of $A_2(745)$ and $A_2(5904)$ in Table 4.4, they share the same first four coefficients.

| $n$ | $a_0$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $a_8$ | $a_9$ | $a_{10}$ | $a_{11}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 745 | 6 | 2 | −4 | −12 | 12 | −42 | −10 | 32 | 121 | | | |
| 5904 | 6 | 2 | −4 | −12 | −16 | 14 | 14 | 28 | 60 | 60 | 60 | 90 |

Table 4.4: The 3-decomposition of $A_2(745)$ and $A_2(5904)$.

Now we show that the assumption that $v \notin 0^*$ is important. Consider

$$\mathrm{rep}_2(448) = 111000000 \text{ and } \mathrm{rep}_2(449) = 111000001.$$

Even though these two words have the same prefix of length 8, the third coefficients of the 3-decompositions of $A_2(448)$ and $A_2(449)$ differ. See Table 4.5.

| $n$ | $a_0$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $a_8$ |
|---|---|---|---|---|---|---|---|---|---|
| 448 | 4 | −2 | −4 | −6 | −6 | −6 | −6 | −6 | −9 |
| 449 | 4 | −2 | −6 | −6 | 6 | 6 | 6 | 6 | 31 |

Table 4.5: The 3-decomposition of $A_2(448)$ and $A_2(449)$.

The idea in the next three definitions is that to a real number $\alpha$ corresponds the relative position of an integer between two consecutive powers of 2. We also define an infinite word $a(\alpha)$ based on 3-decompositions of specific integers.

**Definition 4.15.** Let $\alpha$ be a real number in $[0, 1)$. Define the sequence $(w_n(\alpha))_{n\geq 1}$ of finite words where for all $n \geq 1$

$$w_n(\alpha) = \mathrm{rep}_2\left(2^n + \lfloor \alpha 2^n \rfloor\right) 1.$$

Roughly, $w_n(\alpha)$ is a word of length $n+2$, and its relative position among the words of length $n+2$ in $1\{0,1\}^*$ is given by an approximation of $\alpha$ (more accurately, it is given by $2\lfloor \alpha 2^n \rfloor$; see Definition 4.17 below).

**Remark 4.16.** As a first observation, we add an extra 1 as least significant digit for convenience (*i.e.*, to avoid the third case of Remark 4.12, and to use Lemma 4.13).

Let $d_2(\alpha) = (d_i)_{i\geq 1}$ be the infinite word over $\{0, 1\}$ that is the 2-expansion of $\alpha$ (see Definition 1.19). In particular, we have

$$\alpha = \sum_{i\geq 1} d_i 2^{-i},$$

and the digits $d_i$ are not all eventually equal to 1. Then for all $n \geq 1$, we find

$$w_n(\alpha) = 1 d_1 \cdots d_n 1$$

since we have $\alpha 2^n = d_1 2^{n-1} + d_2 2^{n-2} + \cdots + d_n 2^0 + \sum_{i\geq n+1} d_i 2^{n-i}$. Recalling Definition 1.11, it is easy to see that the sequence $(w_n(\alpha))_{n\geq 1}$ of finite words converges to the infinite word $1(d_i)_{i\geq 1} = 1d_2(\alpha)$. In particular, we may apply Lemma 4.13 to

$$w_n(\alpha) = 1 d_1 \cdots d_n 1 \text{ and } w_{n+1}(\alpha) = 1 d_1 \cdots d_n d_{n+1} 1$$

with $u = d_1 \cdots d_n$, $|u| = n$, $v = 1$ and $v' = d_{n+1}1$. Consequently, for all $n \geq 2$, the 3-decompositions of $A_2(\mathrm{val}_2(w_n(\alpha)))$ and $A_2(\mathrm{val}_2(w_{n+1}(\alpha)))$ have the same first $n - 1$ coefficients.

**Definition 4.17.** Let $\alpha$ be a real number in $[0, 1)$. Define the sequence $(e_n(\alpha))_{n\geq 1}$ of integers where for all $n \geq 1$

$$e_n(\alpha) = \mathrm{val}_2(w_n(\alpha)) = 2^{n+1} + 2\lfloor \alpha 2^n \rfloor + 1.$$

Note that $e_n(\alpha)$ only takes odd integer values in $[2^{n+1}+1, 2^{n+2}-1]$, and

$$\mathrm{relpos}_2(e_n(\alpha)) = \frac{e_n(\alpha) - 2^{n+1}}{2^{n+1}} = \frac{\lfloor \alpha 2^n \rfloor}{2^n} + \frac{1}{2^{n+1}} \to \alpha$$

as $n$ tends to infinity. This echoes the observation made in Definition 4.15.

**Definition 4.18.** Let $\alpha$ be a real number in $[0,1)$. We consider the sequence $(\mathtt{3dec}(A_2(e_n(\alpha))))_{n\geq 1}$ of finite words. As already mentioned in Remark 4.16, thanks to Lemma 4.13, this sequence of finite words converges to an infinite sequence of integers denoted by

$$a(\alpha) = a_0(\alpha)\, a_1(\alpha)\, \cdots .$$

**Example 4.19.** Take $\alpha = \pi - 3$. The sequence $(w_n(\alpha))_{n\geq 1}$ converges to

$$1d_2(\alpha) = 10010010000111111011\cdots$$

(see `A004601` in [Slo], which is the expansion of $\pi$ in base 2). Computing the sequence $(e_n(\alpha))_{n\geq 1}$ leads to the second column of Table 4.6 (also given by `A282730` in [Slo]). In this table, we also find the 3-decomposition of $A_2(e_n(\alpha))$. By looking at the different rows, we deduce that the first terms of the sequence $a(\alpha)$ are 2, 6, $-6$, 2, 24, $-24$, 6, 30 (see `A282729` in [Slo]). At each step, all coefficients in the 3-decomposition are fixed except for the last two ones (thanks to Lemma 4.13).

### 4.1.2 To Infinity, and Beyond!

In order to prove Theorem 4.4, as already noticed in Theorem 4.6, we introduce an auxiliary function $\Phi_2(\alpha)$, for $\alpha \in [0,1)$, defined as the limit of a converging sequence of step functions built on the 3-decomposition of $A_2(e_n(\alpha))$. For all $n \geq 1$, let $\phi_n$ be the function defined by

$$\phi_n(\alpha) = \frac{A_2(e_n(\alpha))}{3^{\log_2(e_n(\alpha))}} \quad \text{for } \alpha \in [0,1).$$

Note that the proof of the next result will come in due time, after Remark 4.23.

| $n$ | $e_n(\alpha)$ | $a_0$ | $a_1$ | $a_2$ | $a_3$ | $\cdots$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 2 | 7 | | | | | | | | | |
| 2 | 9 | 2 | 2 | 8 | | | | | | | | |
| 3 | 19 | 2 | 6 | $-2$ | 5 | | | | | | | |
| 4 | 37 | 2 | 6 | $-6$ | 6 | 15 | | | | | | |
| 5 | 73 | 2 | 6 | $-6$ | 2 | 8 | 31 | | | | | |
| 6 | 147 | 2 | 6 | $-6$ | 2 | 24 | $-8$ | 14 | | | | |
| 7 | 293 | 2 | 6 | $-6$ | 2 | 24 | $-24$ | 22 | 53 | | | |
| 8 | 585 | 2 | 6 | $-6$ | 2 | 24 | $-24$ | 6 | 30 | 116 | | |
| 9 | 1169 | 2 | 6 | $-6$ | 2 | 24 | $-24$ | 6 | 30 | 30 | 131 | |
| 10 | 2337 | 2 | 6 | $-6$ | 2 | 24 | $-24$ | 6 | 30 | 30 | 30 | 146 |

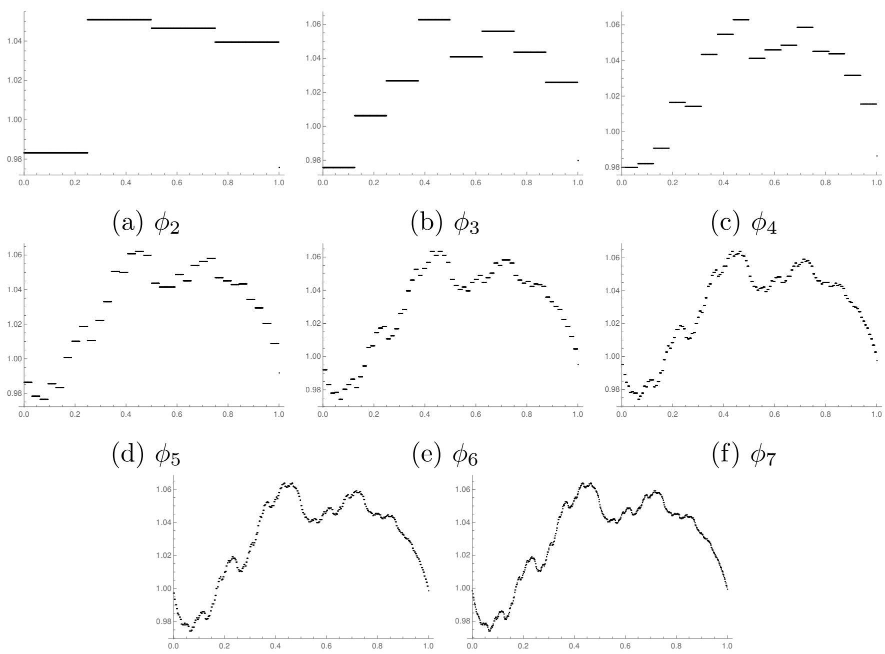
Table 4.6: The 3-decomposition of $A_2(e_n(\alpha))$ for $\alpha = \pi - 3$.

**Proposition 4.20.** *The sequence $(\phi_n)_{n\ge1}$ uniformly converges to the function $\Phi_2$ defined, for $\alpha \in [0,1)$, by*

$$\Phi_2(\alpha) = \begin{cases} \dfrac{1}{3^{1+\log_2(\alpha+1)}} \displaystyle\sum_{i=0}^{+\infty} \frac{a_i(\alpha)}{3^i}, & \text{if } \alpha < 1/2; \\ \dfrac{1}{3^{\log_2(\alpha+1)}} \displaystyle\sum_{i=0}^{+\infty} \frac{a_i(\alpha)}{3^i}, & \text{if } \alpha \ge 1/2. \end{cases}$$

**Remark 4.21.** If the reader is puzzled by the difference between the exponents in the definition of $\Phi_2$, observe that, if $\alpha$ tends to $(1/2)^+$, then one can prove that the 3-decomposition of $A_2(e_n(\alpha))$ converges to the infinite word $(z_n)_{n\ge0} = 4, -6, -2, 4, 4, 4, \ldots$, and $\sum_{i=0}^{+\infty}(z_i/3^i) = 2$. If $\alpha$ tends to $(1/2)^-$, then one can show that the 3-decomposition of $A_2(e_n(\alpha))$ converges to the infinite word $(z'_n)_{n\ge0} = 6, 2, -4, -4, -4, \ldots$, and $\sum_{i=0}^{+\infty}(z'_i/3^i) = 6$. The continuity of $\Phi_2$ will be discussed in the proof of Theorem 4.6.

To visualize the uniform convergence stated in Proposition 4.20, we have depicted the first functions $\phi_2, \ldots, \phi_9$ in Figure 4.7. Observe that if the 2-expansions of $\alpha$ and $\gamma$ share a long common prefix, then the words $w_n(\alpha)$ and $w_n(\gamma)$ are equal by definition (recall Remark 4.16) if $n$ is sufficiently small. Then we have $e_n(\alpha) = e_n(\gamma)$, implying that $\phi_n(\alpha) = \phi_n(\gamma)$ for all sufficiently small $n$. This explains why $\phi_n$ is a step function. One could wonder how many steps there are in a given $\phi_n$. Since $w_n(\alpha)$ is a word of length $n+2$ starting and ending with 1 for all $\alpha$, there are $2^n$ choices left for the remaining $n$ letters in $w_n(\alpha)$, in turn giving $2^n$ different odd integers $e_n(\alpha)$. For instance, we have $w_2(\alpha) \in \{1001, 1011, 1101, 1111\}$ and thus $e_2(\alpha) \in \{9, 11, 13, 15\}$, explaining the four subintervals defining the step function $\phi_2$.



(a) $\phi_2$ (b) $\phi_3$ (c) $\phi_4$

(d) $\phi_5$ (e) $\phi_6$ (f) $\phi_7$

(g) $\phi_8$ (h) $\phi_9$

Figure 4.7: Representation of $\phi_2, \ldots, \phi_9$ in $[0, 1]$.

To ensure the convergence of the series that we will encounter, we need some very rough estimate on the coefficients occurring in 3-decompositions.

**Lemma 4.22.** *For all $n \geq 2$ and for $0 \leq i \leq \ell_2(n)$, we have $|a_i(n)| \leq 10 \cdot 2^i$. In particular, for all $\alpha \in [0, 1)$ and all $i \geq 0$, we have $|a_i(\alpha)| \leq 10 \cdot 2^i$.*

*Proof*. Take $n = 2^\ell + r$ with $\ell \geq 1$ and $0 \leq r < 2^\ell$. Using Definition 4.11, let us write

$$A_2(n) = \sum_{j=0}^{\ell_2(n)} a_j(n)\, 3^{\ell_2(n)-j},$$

where $a_j(n)$'s are integers, $a_0(n) \neq 0$. Observe that we have $\ell_2(n) \in \{\ell, \ell-1\}$. Let us fix $i \in \{0, \ldots, \ell_2(n)\}$. By Lemma 4.7, terms of the form

$$\begin{array}{ll} A_2(2^{\ell_2(n)-i+1} + r') & \text{where } r' \in \{0, \ldots, 2^{\ell_2(n)-i+1} - 1\}, \text{ or} \\ A_2(2^{\ell_2(n)-i} + r'') & \text{where } r'' \in \{2^{\ell_2(n)-i-1} + 1, \ldots, 2^{\ell_2(n)-i} - 1\} \end{array} \tag{4.4}$$

are the only ones possibly contributing to $a_i(n)$. Those of the first (resp., second) form yield $2 \cdot 3^{\ell_2(n)-i}$ (resp., $4 \cdot 3^{\ell_2(n)-i}$) in modulus. Observe that for a term $A_2(2^{\ell_2(n)-i+1}+r')$ of the first form with $2^{\ell_2(n)-i-1} < r' < 2^{\ell_2(n)-i}$, a second application of Lemma 4.7 gives, in addition to $2 \cdot 3^{\ell_2(n)-i}$, the term $A_2(2^{\ell_2(n)-i} + r')$, which is of the second form. Together, these terms give $6 \cdot 3^{\ell_2(n)-i}$. Similarly, if $r' = 2^{\ell_2(n)-i}$, a second application of Lemma 4.7 gives a contribution equal to $4 \cdot 3^{\ell_2(n)-i}$. Finally, if $2^{\ell_2(n)-i} < r' < 2^{\ell_2(n)-i+1}$, then we get a contribution equal to either $2 \cdot 3^{\ell_2(n)-i}$ or $6 \cdot 3^{\ell_2(n)-i}$ in modulus.

Our aim is now to understand, starting from $A_2(2^\ell+r)$, how the successive applications of Lemma 4.7 lead to terms of the form (4.4). Notice that the successive applications of the lemma can give terms of the form $A_2(2^p + r')$ where $r'$ can take several values for a given value of $p$ (see, for instance, Example 4.10 with $A_2(2^3 + 2)$ and $A_2(2^3 + 6)$). This is the reason why we consider a second index $q$ in the sum below.

Let us describe a transformation process starting from a linear combination of the form

$$\sum_{\substack{0 \le p \le k \\ 0 \le q \le s_p}} x_{p,q} A_2(2^p + r_{p,q}),$$

where $k > \ell_2(n) - i + 1$ and, for all $p$ and $q$, $s_p \in \mathbb{N}$ is a counter, $x_{p,q} \in \mathbb{Z}$ and $r_{p,q} \in \{0, 1, \ldots, 2^p - 1\}$.

First, applying Lemma 4.7 to every term of the form $A_2(2^p + r_{p,q})$ with $p < \ell_2(n) - i$ will provide terms of the form $A_2(2^{p'} + r')$ with $p' \le p$ and $r' < 2^{p'}$. Hence, these terms are not of the form (4.4), and will never contribute to $a_i(n)$.

Secondly, applying the lemma to every term of the form $A_2(2^p + r_{p,q})$ with $p > \ell_2(n) - i + 1$ gives a linear combination of $3^p$ and $3^{p-1}$, together with a linear combination of the form $x_1 A_2(2^{p_1} + r_1) + x_2 A_2(2^{p_2} + r_2)$ with $p_1 < p_2 \le p$ and $x_1, x_2 \in \{-1, 1\}$. Observe that $p_2 = p$ if and only if $r_{p,q} = 2^{p-1}$. In this case, we get $p_1 = p - 1$, $r_1 = r_2 = 0$, and the terms $A_2(2^p) = 3^p$ and $A_2(2^{p-1}) = 3^{p-1}$ (by Lemma 4.5).

Therefore, applying Lemma 4.7 to all terms of the form $A_2(2^p+r_{p,q})$ with $p > \ell_2(n) - i + 1$ gives a linear combination of the form

$$\sum_{j=\ell_2(n)-i+1}^{k} y_j 3^j + \sum_{\substack{0\leq p<k \\ 0\leq q\leq t_p}} y_{p,q} A_2(2^p + r'_{p,q}),$$

where for all $j$, we have $y_j \in \mathbb{Z}$, and for all $p$ and $q$, $t_p \in \mathbb{N}$ is a counter, $y_{p,q} \in \mathbb{Z}$, $r'_{p,q} \in \{0, 1, \ldots, 2^p - 1\}$, and where

$$\sum_{\substack{0\leq p<k \\ 0\leq q\leq t_p}} |y_{p,q}| \leq 2 \sum_{\substack{0\leq p\leq k \\ 0\leq q\leq s_p}} |x_{p,q}|.$$

Note that $p < k$ in the right term of the new linear combination because a zero residue gives powers of 3 that are already included in the left term. So we get some information about how behave the coefficients when applying the transformation process once.

Starting from the particular combination $1 \cdot A_2(2^\ell + r)$ and iterating this process $\ell - \ell_2(n) + i - 1$ times (*i.e.*, for $k = \ell$, $k = \ell - 1$, ..., and $k = \ell - (\ell - \ell_2(n) + i - 1) + 1 = \ell_2(n) - i + 2$), we thus obtain a linear combination of the form

$$\sum_{j=\ell_2(n)-i+1}^{\ell_2(n)} y_j 3^j + \sum_{\substack{0\leq p\leq \ell_2(n)-i+1 \\ 0\leq q\leq t_p}} y_{p,q} A_2(2^p + r'_{p,q}),$$

where

$$\sum_{\substack{0\leq p\leq \ell_2(n)-i+1 \\ 0\leq q\leq t_p}} |y_{p,q}| \leq 2^{\ell-\ell_2(n)+i-1} \cdot 1 \leq 2^i.$$

Using what was previously said, we conclude by observing that

$$|a_i(n)| \leq 6 \sum_{0\leq q\leq t_{\ell_2(n)-i+1}} |y_{\ell_2(n)-i+1,q}| + 4 \sum_{0\leq q\leq t_{\ell_2(n)-i}} |y_{\ell_2(n)-i,q}| \leq 10 \cdot 2^i,$$

as desired. The particular case follows from the definition of $a(\alpha)$. □

**Remark 4.23.** With a deeper analysis, one could probably refine the above lemma (even though this is not required for what follows). Computer experiments suggest that $|a_i(\alpha)| \leq 6\, F(i-1)$ for all $\alpha \in [0, 1)$ and all $i \geq 1$, where

$(F(n))_{n\geq 0}$ is the Fibonacci sequence (see Example 1.18). The equality holds for $\alpha = 1/3$. In this particular case, the sequence $(w_n(\alpha))_{n\geq 1}$ converges to $1d_2(1/3) = (10)^\omega$ (observe that $\sum_{j\geq 1} 1/2^{2j} = 1/3$).

We are now ready to prove Proposition 4.20.

*Proof of Proposition 4.20.* Using the 3-decomposition of $A_2(e_n(\alpha))$ from Definition 4.11, we have

$$\phi_n(\alpha) = \frac{1}{3^{\log_2(e_n(\alpha))}} \sum_{i=0}^{\ell_2(e_n(\alpha))} a_i(e_n(\alpha))\, 3^{\ell_2(e_n(\alpha))-i}.$$

We now simplify the previous formula by studying the values of $\log_2(e_n(\alpha))$ and $\ell_2(e_n(\alpha))$.

From Definition 4.17, note that $\log_2(e_n(\alpha)) = n+1+\{\log_2(e_n(\alpha))\}$ since $e_n(\alpha) \in [2^{n+1}, 2^{n+2})$. Moreover, if we write $e_n(\alpha) = 2^{n+1} + r$, then by Definition 4.11, we have

$$\ell_2(e_n(\alpha)) = \begin{cases} n, & \text{if } 0 \leq r \leq 2^n; \\ n+1, & \text{if } 2^n < r < 2^{n+1}. \end{cases}$$

Using Definition 4.17, if $\alpha < 1/2$, then $e_n(\alpha) = 2^{n+1} + r$ with $r \leq 2^n - 1$. If $\alpha \geq 1/2$, then $e_n(\alpha) = 2^{n+1} + r$ with $2^n + 1 \leq r < 2^{n+1}$. Consequently, if $\alpha < 1/2$, we have

$$\phi_n(\alpha) = \frac{1}{3^{1+\{\log_2(e_n(\alpha))\}}} \sum_{i=0}^{n} \frac{a_i(e_n(\alpha))}{3^i}, \tag{4.5}$$

and if $\alpha \geq 1/2$, we get

$$\phi_n(\alpha) = \frac{1}{3^{\{\log_2(e_n(\alpha))\}}} \sum_{i=0}^{n+1} \frac{a_i(e_n(\alpha))}{3^i}. \tag{4.6}$$

First, in both expressions, both sums are converging when $n$ tends to infinity to the series

$$\sum_{i=0}^{+\infty} \frac{a_i(\alpha)}{3^i}.$$

Indeed, thanks to Lemma 4.13, the sequence $(\mathtt{3dec}(A_2(e_n(\alpha))))_{n\geq 1}$ of finite words converges to $a(\alpha)$ (see Definition 4.18). Moreover, due to Lemma 4.22, the sequence of partial sums uniformly converges to the series.

Secondly, by Definition 4.17, we get

$$\left|\frac{e_n(\alpha)}{2^{n+1}} - (\alpha+1)\right| = \left|\frac{2\lfloor \alpha 2^n \rfloor + 1}{2^{n+1}} - \alpha\right| \leq \frac{3}{2^{n+1}}.$$

Thus, the sequence $(e_n(\alpha)/2^{n+1})_{n\geq 1}$ of functions (whose variable is $\alpha$) uniformly converges to $(\alpha+1)$. Since the function $\log_2$ is uniformly continuous on $[1,+\infty[$, the sequence $(\log_2(e_n(\alpha)/2^{n+1}))_{n\geq 1}$ also uniformly converges to $\log_2(\alpha+1)$. Now observe that

$$\begin{aligned}
\log_2\left(\frac{e_n(\alpha)}{2^{n+1}}\right) &= \log_2(e_n(\alpha)) - (n+1) \\
&= \lfloor \log_2(e_n(\alpha)) \rfloor + \{\log_2(e_n(\alpha))\} - n - 1 \\
&= \{\log_2(e_n(\alpha))\}.
\end{aligned}$$

This shows that the sequence $(\{\log_2(e_n(\alpha))\})_{n\geq 1}$ uniformly converges to $\log_2(\alpha+1)$.

To end the proof, let $\epsilon > 0$. For all $\alpha \geq 1/2$, we observe, using (4.6), that

$$\begin{aligned}
|\phi_n(\alpha) - \Phi_2(\alpha)| \leq{}& \left|\sum_{i=0}^{n+1} \frac{a_i(e_n(\alpha))}{3^i}\right| \cdot \left|\frac{1}{3^{\{\log_2(e_n(\alpha))\}}} - \frac{1}{3^{\log_2(\alpha+1)}}\right| \\
&+ \left|\frac{1}{3^{\log_2(\alpha+1)}}\right| \cdot \left|\sum_{i=0}^{n+1} \frac{a_i(e_n(\alpha))}{3^i} - \sum_{i=0}^{+\infty} \frac{a_i(\alpha)}{3^i}\right|.
\end{aligned}$$

We claim that $|\phi_n(\alpha) - \Phi_2(\alpha)| < \epsilon$ for all large enough $n$. Indeed, first

$$\left|\sum_{i=0}^{n+1} \frac{a_i(e_n(\alpha))}{3^i}\right| < C,$$

where $C$ is a positive constant (to see this, use Lemma 4.22). Moreover, the sequence $(\{\log_2(e_n(\alpha))\})_{n\geq 1}$ of functions uniformly converges to $\log_2(\alpha+1)$, so

$$\left|\frac{1}{3^{\{\log_2(e_n(\alpha))\}}} - \frac{1}{3^{\log_2(\alpha+1)}}\right| < \frac{\epsilon}{2C}$$

for large enough $n$. Finally, if $C' = |1/3^{\log_2(\alpha+1)}|$, then

$$\left|\sum_{i=0}^{n+1} \frac{a_i(e_n(\alpha))}{3^i} - \sum_{i=0}^{+\infty} \frac{a_i(\alpha)}{3^i}\right| < \frac{\epsilon}{2C'}$$

for big enough $n$. One proceeds similarly with (4.5) for the case where $\alpha < 1/2$. This finishes the proof. □

The function $\Phi_2$ defined by Proposition 4.20 takes particular values over rational numbers of the form $r/2^k$ with an odd residue $r < 2^k$. This result is the key to get an exact formula in Theorem 4.6.

**Lemma 4.24.** *Let $k \geq 1$ and $0 \leq r < 2^k$ be integers. We have*

$$A_2(2^k + r) = 3^{\log_2(2^k+r)}\, \Phi_2\left(\frac{r}{2^k}\right).$$

*Proof*. From Definitions 4.15 and 4.17, for all $n \geq k$, we have

$$w_n\left(\frac{r}{2^k}\right) = \mathrm{rep}_2(2^k + r)\, 0^{n-k}\, 1 \quad \text{and} \quad e_n\left(\frac{r}{2^k}\right) = 2^{n-k+1}(2^k + r) + 1$$

(recall that multiplying by 2 shifts the base-2 expansions to the left). By Proposition 4.20, we know that

$$\lim_{n\to+\infty} \frac{A_2(2^{n-k+1}(2^k + r) + 1)}{3^{\log_2(2^{n-k+1}(2^k+r)+1)}} = \lim_{n\to+\infty} \phi_n\left(\frac{r}{2^k}\right) = \Phi_2\left(\frac{r}{2^k}\right).$$

Now we claim that

$$\frac{A_2(2^{n-k+1}(2^k + r) + 1)}{A_2(2^{n-k+1}(2^k + r))}\, \frac{3^{\log_2(2^{n-k+1}(2^k+r))}}{3^{\log_2(2^{n-k+1}(2^k+r)+1)}} \to 1$$

when $n$ tends to infinity. The second factor is easily handled since it tends to $3^{\log_2(1)}$ when $n$ tends to infinity. By definition of the summatory function, the first factor is equal to

$$1 + \frac{S_2(2^{n-k+1}(2^k + r))}{A_2(2^{n-k+1}(2^k + r))}. \tag{4.7}$$

Since $(S_2(n))_{n\geq 0}$ is positive, $(A_2(n))_{n\geq 0}$ is increasing, so from Lemma 4.5,

$$A_2(2^{n-k+1}(2^k + r)) \geq 3^n.$$

By Corollary 3.17, $S_2(m) \leq 2m$ for all positive $m$. This implies that (4.7) tends to 1 when $n$ tends to infinity, which in turn proves the intermediate claim.

In particular, it shows that the sequence

$$\left(\frac{A_2(2^{n-k+1}(2^k + r))}{3^{\log_2(2^{n-k+1}(2^k+r))}}\right)_{n\geq k} \tag{4.8}$$

also converges to $\Phi_2(r/2^k)$. Thanks to Corollary 4.9, for all $n \geq k$, we have

$$\frac{A_2(2^{n-k+1}(2^k+r))}{3^{n-k+1}} = A_2(2^k+r).$$

Thus the sequence (4.8) is constant and equal to

$$\frac{A_2(2^{n-k+1}(2^k+r))}{3^{n-k+1}\ 3^{\log_2(2^k+r)}} = \frac{A_2(2^k+r)}{3^{\log_2(2^k+r)}}.$$

Consequently, $A_2(2^k+r)/3^{\log_2(2^k+r)} = \Phi_2(r/2^k)$, as expected. □

We have all the necessary material to prove the main theorems of this section.

*Proof of Theorem 4.6.* This proof is divided into four parts: we obtain the exact formula for the sequence $(A_2(n))_{n\geq 0}$, the fact that $\Phi_2(0)=1$, the limit $\lim_{\alpha\to 1^-}\Phi_2(\alpha)=1$, and the continuity of the function $\Phi_2$.

**Exact formula.** Every integer $n \geq 1$ can be uniquely written as $n = 2^j(2^k+r)$ for $j \geq 0$ maximum, $k \geq 0$ and $r$ in $\{0,\dots,2^k-1\}$. Thanks to Corollary 4.9, $A_2(n) = 3^j\, A_2(2^k+r)$. From Lemma 4.24, we get

$$A_2(n) = 3^j\, A_2(2^k+r) = 3^{j+\log_2(2^k+r)}\, \Phi_2\left(\frac{r}{2^k}\right) = 3^{\log_2 n}\, \Phi_2\left(\frac{r}{2^k}\right).$$

To obtain the relation of the statement, observe that

$$\mathrm{relpos}_2(n) = \frac{n-2^{j+k}}{2^{j+k}} = \frac{r}{2^k}.$$

**Value of $\Phi_2(0)$.** On the one hand, from Lemma 4.24, for any $k \geq 1$, we have

$$A_2(2^k) = 3^{\log_2(2^k)}\Phi_2(0).$$

On the other hand, $A_2(2^k) = 3^k$ thanks to Lemma 4.5. Hence, $\Phi_2(0)=1$.

**Limit for $1^-$.** To show that

$$\lim_{\alpha\to 1^-}\Phi_2(\alpha) = \lim_{\alpha\to 1^-}\lim_{n\to+\infty}\phi_n(\alpha) = 1,$$

we make use of the uniform convergence in Proposition 4.20, and permute the two limits

$$\lim_{\alpha\to 1^-}\Phi_2(\alpha) = \lim_{n\to+\infty}\lim_{\alpha\to 1^-}\phi_n(\alpha) = \lim_{n\to+\infty}\lim_{\alpha\to 1^-}\frac{1}{3^{\{\log_2(e_n(\alpha))\}}}\sum_{i=0}^{n+1}\frac{a_i(e_n(\alpha))}{3^i}$$

(for the last equality, recall (4.6)). Observe that if $\alpha$ is close enough to 1, then the infinite word $d_2(\alpha) = (d_i)_{i\geq 1}$, *i.e.*, the 2-expansion of $\alpha$, has a long prefix containing only letters 1 (since $d_2(1) = 1^\omega$ by Definition 1.19). By definition, we get $w_n(\alpha) = 1^{n+2}$ and $e_n(\alpha) = 2^{n+2} - 1$. Iteratively applying Lemma 4.7 gives

$$\begin{aligned} A_2(e_n(\alpha)) &= A_2(2^{n+1} + 2^{n+1} - 1) \\ &= 4 \cdot 3^{n+1} - 2 \cdot 3^n - A_2(2^n + 1) - A_2(1) \\ &= 4 \cdot 3^{n+1} - 2 \cdot 3^n - 2 \cdot 3^{n-1} - A_2(2^{n-1} + 1) - 2 \cdot A_2(1), \end{aligned}$$

which yields the 3-decomposition of $A_2(e_n(\alpha))$

$$(a_i(e_n(\alpha)))_{0\leq i\leq n+1} = (4, -2, -2, -2, \ldots, -2, a_n(e_n(\alpha)), a_{n+1}(e_n(\alpha))).$$

Recall that $(\{\log_2(e_n(\alpha))\})_{n\geq 1}$ uniformly converges to $\log_2(\alpha + 1)$, so

$$\lim_{\alpha\to 1^-} \Phi_2(\alpha) = \lim_{n\to+\infty} \frac{1}{3} \left( 4 - 2\sum_{i=1}^{n-1} 3^{-i} + \frac{a_n(e_n(\alpha))}{3^n} + \frac{a_{n+1}(e_n(\alpha))}{3^{n+1}} \right).$$

Since, by Lemma 4.22, the last two terms are respectively less than $10 \cdot 2^n$ and $10 \cdot 2^{n+1}$, we find

$$\lim_{\alpha\to 1^-} \Phi_2(\alpha) = \lim_{n\to+\infty} \frac{1}{3}(3 + 3^{1-n}) = 1.$$

**Continuity.** We finally prove that $\Phi_2$ is continuous. Let us take $\alpha$ in $[0, 1)$, and let $d_2(\alpha) = (d_n)_{n\geq 1}$ denote the 2-expansion of $\alpha$. To show that $\Phi_2$ is continuous at $\alpha$, we make use of the uniform convergence of the sequence $(\phi_n)_{n\geq 0}$ to $\Phi_2$ in Proposition 4.20, and we consider

$$\begin{aligned} \lim_{\gamma\to\alpha} |\Phi_2(\gamma) - \Phi_2(\alpha)| &= \lim_{\gamma\to\alpha} \lim_{n\to+\infty} |\phi_n(\gamma) - \phi_n(\alpha)| \\ &= \lim_{n\to+\infty} \lim_{\gamma\to\alpha} |\phi_n(\gamma) - \phi_n(\alpha)|. \end{aligned}$$

First, assume that $\alpha$ is not of the form $r/2^k$ with $k \geq 1$, $0 \leq r < 2^k$, and $r$ odd, *i.e.*, $(d_n)_{n\geq 1}$ does not belong to $\{0, 1\}^*10^\omega$ (note that $\alpha = 0$ is allowed in this case since $r = 0$ is not odd). For any fixed integer $n$, we can choose $\gamma_n$ close enough to $\alpha$ such that $d_2(\gamma_n) \in d_1 d_2 \cdots d_n\{0, 1\}^\omega$. Therefore, we have $w_n(\gamma_n) = w_n(\alpha)$ and $e_n(\gamma_n) = e_n(\alpha)$, hence we also obtain $\phi_n(\gamma_n) = \phi_n(\alpha)$ by definition. This ends the first case.

Now assume that $d_2(\alpha) = d_1 d_2 \cdots d_k 0^\omega$ with $d_k = 1$. For any fixed integer $n > k$, we can chose $\gamma_n$ close enough to $\alpha$ such that

$$d_2(\gamma_n) \in \begin{cases} d_1 d_2 \cdots d_k 0^n \{0,1\}^\omega, & \text{if } \gamma_n \geq \alpha; \\ d_1 d_2 \cdots d_{k-1} 0 1^n \{0,1\}^\omega, & \text{if } \gamma_n < \alpha. \end{cases}$$

If $\gamma_n \geq \alpha$, we get $\phi_n(\gamma_n) = \phi_n(\alpha)$ as in the first case, and we end the proof in a similar way. If $\gamma_n < \alpha$, we get

$$e_n(\alpha) = \mathrm{val}_2(1 d_1 d_2 \cdots d_k 0^{n-k} 1) = 2^{n+1} + 2 \sum_{i=1}^{k} d_i 2^{n-i} + 1,$$

$$e_n(\gamma_n) = \mathrm{val}_2(1 d_1 d_2 \cdots d_{k-1} 0 1^{n-k} 1) = 2^{n+1} + 2 \sum_{i=1}^{k-1} d_i 2^{n-i} + 2 \sum_{j=0}^{n-k-1} 2^j + 1,$$

giving $e_n(\alpha) = e_n(\gamma_n) + 2$. Since $\lfloor \log_2(e_n(\alpha)) \rfloor = \lfloor \log_2(e_n(\gamma_n)) \rfloor = n+1$, we get by definition of $\phi_n$

$$\begin{aligned} |\phi_n(\alpha) - \phi_n(\gamma_n)| &= \left| \frac{A_2(e_n(\alpha))}{3^{\log_2(e_n(\alpha))}} - \frac{A_2(e_n(\gamma_n))}{3^{\log_2(e_n(\gamma_n))}} \right| \\ &\leq \left| \frac{A_2(e_n(\alpha))}{3^{n+1}} \left( \frac{1}{3^{\{\log_2(e_n(\alpha))\}}} - \frac{1}{3^{\{\log_2(e_n(\gamma_n))\}}} \right) \right| \\ &\quad + \left| \frac{1}{3^{\log_2(e_n(\gamma_n))}} \left( A_2(e_n(\alpha) - A_2(e_n(\gamma_n)) \right) \right|. \end{aligned}$$

We now bound each term. For the first term, the factor $A_2(e_n(\alpha))/3^{n+1}$ converges to the series $C \cdot \sum_{i=0}^{+\infty} (a_i(\alpha)/3^i)$ when $n$ tends to infinity (the reasoning is similar to what was previously done for (4.5) and (4.6) in the proof of Proposition 4.20, which uses Lemma 4.22; if $\alpha < 1/2$, then $C = 1/3$, else $C = 1$). The second factor $(1/3^{\{\log_2(e_n(\alpha))\}} - 1/3^{\{\log_2(e_n(\gamma_n))\}})$ tends to 0 when $n$ tends to infinity. Indeed, let us write

$$\frac{1}{3^{\{\log_2(e_n(\alpha))\}}} - \frac{1}{3^{\{\log_2(e_n(\gamma_n))\}}} = \frac{1}{3^{\{\log_2(e_n(\alpha))\}}} \left( 1 - 3^{\{\log_2(e_n(\alpha))\} - \{\log_2(e_n(\gamma_n))\}} \right).$$

We get the conclusion since the sequence $(\{\log_2(e_n(\alpha))\})_{n \geq 1}$ of functions uniformly converges to $\log_2(\alpha+1)$ and

$$\begin{aligned} \{\log_2(e_n(\alpha))\} - \{\log_2(e_n(\gamma_n))\} &= \log_2(e_n(\alpha)) - \log_2(e_n(\gamma_n)) \\ &= \log_2 \left( \frac{e_n(\gamma_n) + 2}{e_n(\gamma_n)} \right) \leq \log_2 \left( 1 + \frac{1}{2^n} \right). \end{aligned}$$

For the second term, Corollary 3.17 gives

$$\begin{aligned} A_2(e_n(\alpha)) - A_2(e_n(\gamma_n)) &= A_2(e_n(\alpha)) - A_2(e_n(\alpha) - 2) \\ &= S_2(e_n(\alpha) - 1) + S_2(e_n(\alpha) - 2) \\ &\leq 4e_n(\alpha) \leq 2^{n+4}, \end{aligned}$$

and we have $3^{\log_2(e_n(\gamma_n))} \geq 3^{n+1}$ since $e_n(\alpha), e_n(\gamma_n) \in [2^{n+1}+1, 2^{n+2})$. Now, for $\epsilon > 0$, we have just proved that $|\phi_n(\alpha) - \phi_n(\gamma_n)| < \epsilon$ if $n$ goes to infinity. This shows that $\Phi_2$ is continuous. □

**Remark 4.25.** As stated in [BR10, Remark 9.2.2], observe that since the function $\Phi_2$ is continuous, then it is completely defined in the interval $[0, 1]$ by the values taken on the dense set of points of the form $r/2^k$. Having no error term for these values thanks to Lemma 4.24, there is no error term in Theorem 4.6.

Since a linear representation of $(A_2(n))_{n\geq 0}$ can be obtained from a linear representation of $(S_2(n))_{n\geq 0}$ (which is done in Remark 4.3), Lemma 4.24 gives a way to quickly compute $\Phi_2$ using matrices.

As already observed, Theorem 4.4 follows from Theorem 4.6.

*Proof of Theorem 4.4.* Let $\mathcal{H}_2$ be defined by $\mathcal{H}_2(x) = \Phi_2(\mathrm{relpos}_2(2^x))$ for all $x \in \mathbb{R}$. Then, for all $n \geq 1$, we have

$$\mathcal{H}_2(\log_2 n) = \Phi_2(\mathrm{relpos}_2(2^{\log_2 n})) = \Phi_2(\mathrm{relpos}_2(n)).$$

The desired properties of $\mathcal{H}_2$ are derived from those of $\Phi_2$ highlighted in Theorem 4.6. The continuity follows without difficulty. For the periodicity, we have for all $x \in \mathbb{R}$

$$\mathcal{H}_2(x+1) = \Phi_2(\mathrm{relpos}_2(2^{x+1})) = \Phi_2(\mathrm{relpos}_2(2^x)) = \mathcal{H}_2(x)$$

since, by definition,

$$\mathrm{relpos}_2(2^{x+1}) = \frac{2^{x+1} - 2^{\lfloor \log_2 2^{x+1} \rfloor}}{2^{\lfloor \log_2 2^{x+1} \rfloor}} = \frac{2^{x+1} - 2^{\lfloor x+1 \rfloor}}{2^{\lfloor x+1 \rfloor}} = \mathrm{relpos}_2(2^x).$$ □

**Remark 4.26.** It is known that the function involved in the result (4.1) is nowhere differentiable [Del75]. In our context, what can be said about $\mathcal{H}_2$ and $\Phi_2$?

A first approach is to consider the results in [Ten97]. However, one cannot directly apply the latter technique. Indeed, the summatory functions considered in this paper have a behavior of the form $n^\theta \mathcal{F}(\log n)$ with $\theta < 1$, which is not the case here.

Nevertheless, regarding the differentiability, a direction of investigation could be to use the relation between the Hölder exponent and the speed of convergence as explained in [DL92, Rio92]. To compute this Hölder exponent or at least estimate it, one could probably use the methods in [Dum13, DL92, Rio92]. Those problems remain open to this day.

## 4.2 Mashup of Different Integer Bases

In the previous section, the behavior of the 2-regular sequence $(A_2(n))_{n\geq 0}$, which exhibits a continuous periodic fluctuation, was obtained from particular decompositions in base 3. In the present section where $b \geq 2$ stands for a fixed integer, we generalize this approach by showing that the base $b$ is associated with the base $2b-1$. First, the following extends Definition 4.1.

**Definition 4.27.** Recall the sequence $(S_b(n))_{n\geq 0}$ from Definition 1.47. We consider the *summatory function* $A_b = (A_b(n))_{n\geq 0}$ of the sequence $(S_b(n))_{n\geq 0}$ defined by

$$A_b(n) = \sum_{j=0}^{n-1} S_b(j)$$

for all $n \geq 0$. As before, the quantity $A_b(n)$ is the total number of base-$b$ expansions occurring as scattered subwords in the base-$b$ expansion of integers less than $n$.

**Example 4.28.** In the view of Example 3.28, the first few terms of $(A_3(n))_{n\geq 0}$ (`A284442` in [Slo]) are

$$0, 1, 3, 5, 8, 11, 15, 18, 22, 25, 29, 34, 40, 45, 49, 55, \ldots.$$

As for the base-2 case with Proposition 4.2, the $b$-regularity of $(A_b(n))_{n\geq 0}$ follows from the one of $(S_b(n))_{n\geq 0}$ stated in Theorem 3.46. For a proof of Proposition 4.29, adapt the one of Proposition 4.2. Similarly to what was done for $(A_2(n))_{n\geq 0}$ in Remark 4.3, note that Remark 3.50 leads to obtain a linear representation with square matrices of size $2b$ for the summatory function $(A_b(n))_{n\geq 0}$.

**Proposition 4.29.** *The sequence* $(A_b(n))_{n\geq 0}$ *is* $b$*-regular.*

In order to prove the extended version of Theorem 4.4 (that is, Theorem 4.39), the goal is to decompose $(A_b(n))_{n\geq 0}$ into linear combinations of powers of $2b-1$. We will need two lemmas, namely Lemma 4.30 that immediately follows and Lemma 4.32.

**Lemma 4.30.** *For all* $\ell \geq 0$ *and all* $x \in \{1, \ldots, b-1\}$*, we have*

$$A_b(xb^\ell) = (2x-1) \cdot (2b-1)^\ell.$$

*Proof*. We proceed by induction on $\ell \geq 0$. If $\ell = 0$ and $x \in \{1, \ldots, b-1\}$, then using Table 3.7, we have

$$A_b(x) = S_b(0) + \sum_{j=1}^{x-1} S_b(j) = 1 + 2(x-1) = 2x - 1.$$

If $\ell = 1$ and $x \in \{1, \ldots, b-1\}$, then we have

$$A_b(xb) = A_b(b) + \sum_{y=1}^{x-1}\sum_{j=0}^{b-1} S_b(yb+j).$$

Using Table 3.7, we analogously get $A_b(b) = (2b-1)$ and thus

$$\begin{aligned} A_b(xb) &= (2b-1) + \sum_{y=1}^{x-1}(3+3+4(b-2)) = (2b-1) + (x-1)(4b-2) \\ &= (2x-1)(2b-1). \end{aligned}$$

Now suppose that $\ell \geq 1$, and assume that the result holds for all $\ell' \leq \ell$. We again proceed by induction on $x \in \{1, \ldots, b-1\}$. When $x = 1$, we must show that $A_b(b^{\ell+1}) = (2b-1)^{\ell+1}$. We have

$$A_b(b^{\ell+1}) = A_b(b^\ell) + \sum_{y=1}^{b-1}\sum_{j=0}^{b^\ell-1} S_b(yb^\ell + j).$$

Observe that $yb^\ell + j$ takes all the values in $[b^\ell, b^{\ell+1})$ when $y$ and $j$ vary. By decomposing the double sum into three parts with respect to the different cases in Proposition 3.29, we get

$$\begin{aligned} A_b(b^{\ell+1}) &= A_b(b^\ell) + \sum_{y=1}^{b-1}\sum_{j=0}^{b^{\ell-1}-1} S_b(yb^\ell + j) \\ &\quad + \sum_{y=1}^{b-1}\sum_{j=0}^{b^{\ell-1}-1} S_b(yb^\ell + yb^{\ell-1} + j) \\ &\quad + \sum_{y=1}^{b-1}\sum_{\substack{1\le z\le b-1 \\ z\neq y}}\sum_{j=0}^{b^{\ell-1}-1} S_b(yb^\ell + zb^{\ell-1} + j). \end{aligned}$$

Using Proposition 3.29, $A_b(b^{\ell+1})$ is thus equal to

$$A_b(b^\ell)$$

$$+\sum_{y=1}^{b-1}\sum_{j=0}^{b^{\ell-1}-1} (S_b(yb^{\ell-1} + j) + S_b(j)) \tag{4.9}$$

$$+\sum_{y=1}^{b-1}\sum_{j=0}^{b^{\ell-1}-1} (2S_b(yb^{\ell-1} + j) - S_b(j)) \tag{4.10}$$

$$+\sum_{y=1}^{b-1}\sum_{\substack{1\le z\le b-1 \\ z\neq y}}\sum_{j=0}^{b^{\ell-1}-1} (S_b(yb^{\ell-1} + j) + 2S_b(zb^{\ell-1} + j) - 2S_b(j)). \tag{4.11}$$

The definition of the summatory function $A_b$ gives

$$\sum_{j=0}^{b^{\ell-1}-1} S_b(yb^{\ell-1} + j) = A_b((y+1)b^{\ell-1}) - A_b(yb^{\ell-1}), \tag{4.12}$$

$$\sum_{j=0}^{b^{\ell-1}-1} S_b(j) = A_b(b^{\ell-1}) \tag{4.13}$$

for all $y \in \{1, \ldots, b-1\}$, and we also have the following telescoping sum

$$\sum_{y=1}^{b-1} \Big( A_b((y+1)b^{\ell-1}) - A_b(yb^{\ell-1}) \Big) = A_b(b^\ell) - A_b(b^{\ell-1}). \tag{4.14}$$

Thus, we obtain

$$\begin{aligned}
(4.9) &= A_b(b^\ell) - A_b(b^{\ell-1}) + (b-1)A_b(b^{\ell-1}) \\
&= A_b(b^\ell) + (b-2)A_b(b^{\ell-1}), \\
(4.10) &= 2(A_b(b^\ell) - A_b(b^{\ell-1})) - (b-1)A_b(b^{\ell-1}) \\
&= 2A_b(b^\ell) - (b+1)A_b(b^{\ell-1}), \\
(4.11) &= 3(b-2)(A_b(b^\ell) - A_b(b^{\ell-1})) - 2(b-1)(b-2)A_b(b^{\ell-1}) \\
&= 3(b-2)A_b(b^\ell) - (b-2)(2b+1)A_b(b^{\ell-1}).
\end{aligned}$$

Finally, by summing $A_b(b^\ell)$ and the terms (4.9), (4.10) and (4.11), we find

$$A_b(b^{\ell+1}) = (3b-2)A_b(b^\ell) - (2b^2-3b+1)A_b(b^{\ell-1}).$$

Using the induction hypothesis twice, we obtain

$$A_b(b^{\ell+1}) = (3b-2)(2b-1)^\ell - (2b^2-3b+1)(2b-1)^{\ell-1} = (2b-1)^{\ell+1},$$

which ends the case where $x = 1$.

Now suppose that $x \in \{2, \ldots, b-1\}$, and assume that the result holds for all $x' < x$. The proof follows the same lines as in the case $x = 1$ with the difference that we decompose the sum into

$$\begin{aligned}
A_b(xb^{\ell+1}) &= A_b((x-1)b^{\ell+1}) + \sum_{j=0}^{b^{\ell+1}-1} S_b((x-1)b^{\ell+1}+j) \\
&= A_b((x-1)b^{\ell+1}) \\
&\quad + \sum_{j=0}^{b^\ell-1} S_b((x-1)b^{\ell+1}+j) \\
&\quad + \sum_{j=0}^{b^\ell-1} S_b((x-1)b^{\ell+1}+(x-1)b^\ell+j) \\
&\quad + \sum_{\substack{1\le y\le b-1 \\ y\ne x-1}} \sum_{j=0}^{b^\ell-1} S_b((x-1)b^{\ell+1}+yb^\ell+j).
\end{aligned}$$

We may now apply Proposition 3.29 and use results similar to (4.12), (4.13) and (4.14) (we only need to increase the value of $\ell$ and change $y$ by $x-1$

where needed) to get the equalities

$$\begin{aligned} A_b(xb^{\ell+1}) &= A_b((x-1)b^{\ell+1}) + (A_b(xb^\ell) - A_b((x-1)b^\ell) + A_b(b^\ell)) \\ &\quad + (2A_b(xb^\ell) - 2A_b((x-1)b^\ell) - A_b(b^\ell)) \\ &\quad + (b-2)(A_b(xb^\ell) - A_b((x-1)b^\ell)) \\ &\quad + 2(A_b(b^{\ell+1}) - A_b(b^\ell) - A_b(xb^\ell) + A_b((x-1)b^\ell)) \\ &\quad - 2(b-2)A_b(b^\ell) \\ &= A_b((x-1)b^{\ell+1}) + (b-1)A_b(xb^\ell) - (b-1)A_b((x-1)b^\ell) \\ &\quad + 2A_b(b^{\ell+1}) - 2(b-1)A_b(b^\ell). \end{aligned}$$

By the induction hypotheses, we get

$$\begin{aligned} A_b(xb^{\ell+1}) &= (2x-3)(2b-1)^{\ell+1} + (b-1)(2x-1)(2b-1)^\ell \\ &\quad - (b-1)(2x-3)(2b-1)^\ell + 2(2b-1)^{\ell+1} - 2(b-1)(2b-1)^\ell. \end{aligned}$$

After few computations, we finally have $A_b(xb^{\ell+1}) = (2x-1)(2b-1)^{\ell+1}$, as expected. □

**Example 4.31.** When $b = 2$, the previous lemma coincides with Lemma 4.5 since, in this case, the only possible value of $x$ is 1. For $b = 3$, it states that $A_3(3^\ell) = 5^\ell$ and $A_3(2 \cdot 3^\ell) = 3 \cdot 5^\ell$ for all $\ell \geq 0$.

**Lemma 4.32.** *For all $\ell \geq 1$ and all $x, y \in \{1, \ldots, b-1\}$, we have*

$$A_b(xb^\ell + yb^{\ell-1}) = \begin{cases} (4xb - 2x + 4y - 2b) \cdot (2b-1)^{\ell-1}, & \text{if } y \leq x; \\ (4xb - 2x + 4y - 2b - 1) \cdot (2b-1)^{\ell-1}, & \text{if } y > x. \end{cases}$$

*Proof.* The proof of this lemma is similar to the proof of Lemma 4.30, so we only prove the formula for $A_b(xb^\ell + xb^{\ell-1})$, the others being similarly handled. We proceed by induction on $\ell \geq 1$. If $\ell = 1$, the result follows from Table 3.7. Indeed, we first have

$$A_b(xb + x) = S_b(0) + \sum_{j=1}^{b-1} S_b(j) + \sum_{y=1}^{x-1} \sum_{j=0}^{b-1} S_b(yb + j) + \sum_{j=0}^{x-1} S_b(xb + j).$$

By Table 3.7, we deduce that

$$\begin{aligned} A_b(xb + x) &= 1 + 2(b-1) + (x-1)(3 + 3 + 4(b-2)) + (3 + 4(x-1)) \\ &= 4xb + 2x - 2b, \end{aligned}$$

as desired. Assume that $\ell \geq 2$, and that the formulas hold for all $\ell' < \ell$. We have

$$\begin{aligned}A_b(xb^\ell + xb^{\ell-1}) =& A_b(xb^\ell) + \sum_{j=0}^{b^{\ell-1}-1} S_b(xb^\ell + j)\\ &+ \sum_{y=1}^{x-1} \sum_{j=0}^{b^{\ell-1}-1} S_b(xb^\ell + yb^{\ell-1} + j).\end{aligned}$$

We may apply Proposition 3.29 and use (4.12), (4.13) and (4.14) in order to obtain the equality

$$\begin{aligned}A_b(xb^\ell + xb^{\ell-1}) =& A_b(xb^\ell) + (A_b((x+1)b^{\ell-1}) - A_b(xb^{\ell-1}) + A_b(b^{\ell-1}))\\ &+ (x-1)(A_b((x+1)b^{\ell-1}) - A_b(xb^{\ell-1}))\\ &+ 2(A_b(xb^{\ell-1}) - A_b(b^{\ell-1})) - 2(x-1)A_b(b^{\ell-1}),\end{aligned}$$

which is turn yields

$$\begin{aligned}A_b(xb^\ell + xb^{\ell-1}) = & A_b(xb^\ell) + xA_b((x+1)b^{\ell-1}) + (2-x)A_b(xb^{\ell-1})\\ &+ (1-2x)A_b(b^{\ell-1}).\end{aligned}$$

Using Lemma 4.30 completes the computation:

$$\begin{aligned}A_b(xb^\ell + xb^{\ell-1}) =& (2x-1)(2b-1)^\ell + x(2x+1)(2b-1)^{\ell-1}\\ &+ (2-x)(2x-1)(2b-1)^{\ell-1} + (1-2x)(2b-1)^{\ell-1}\\ =& (4xb + 2x - 2b)(2b-1)^{\ell-1},\end{aligned}$$

as claimed. □

**Example 4.33.** For $b = 2$, the only possible case is $x = 1 = y$, and we have

$$A_2(2^\ell + 2^{\ell-1}) = 6 \cdot 3^{\ell-1} = 2 \cdot 3^\ell,$$

which corroborates Remark 4.8. When $b = 3$, Lemma 4.32 shows that for all $\ell \geq 1$,

$$\begin{aligned}A_3(3^\ell + 3^{\ell-1}) &= 8 \cdot 5^{\ell-1},\\ A_3(3^\ell + 2 \cdot 3^{\ell-1}) &= 11 \cdot 5^{\ell-1},\\ A_3(2 \cdot 3^\ell + 3^{\ell-1}) &= 18 \cdot 5^{\ell-1},\\ A_3(2 \cdot 3^\ell + 2 \cdot 3^{\ell-1}) &= 22 \cdot 5^{\ell-1}.\end{aligned}$$

Lemmas 4.30 and 4.32 give birth to recurrence relations satisfied by the summatory function $(A_b(n))_{n\geq 0}$ as stated below. In fact, this result is of the same flavor as Lemma 4.7 in the base-2 case. As for $b = 2$, this is a key result that permits us to introduce $(2b-1)$-decompositions of the summatory function $(A_b(n))_{n\geq 0}$. In that sense, Definition 4.37 below is a generalization of Definition 4.11. As a consequence, the $(2b-1)$-decompositions allow us to easily deduce the main theorem of this section (see Theorem 4.39).

**Proposition 4.34.** *For all $x, y \in \{1, \dots, b-1\}$ with $x \neq y$, all $\ell \geq 1$ and all $r \in \{0, \dots, b^{\ell-1}\}$, we have the following three equalities*

$$\begin{aligned} A_b(xb^\ell + r) &= (2b-2)\cdot(2x-1)\cdot(2b-1)^{\ell-1} \\ &\quad + A_b(xb^{\ell-1}+r) + A_b(r), \qquad (4.15) \\ A_b(xb^\ell + xb^{\ell-1} + r) &= (4xb-2x-2b+2)\cdot(2b-1)^{\ell-1} \\ &\quad + 2A_b(xb^{\ell-1}+r) - A_b(r), \qquad (4.16) \end{aligned}$$

*and*

$$A_b(xb^\ell + yb^{\ell-1} + r) = \begin{cases} (4xb-4x-2b+3)\cdot(2b-1)^{\ell-1} \\ +A_b(xb^{\ell-1}+r) \\ +2A_b(yb^{\ell-1}+r) \\ -2A_b(r), & \text{if } y < x; \\ (4xb-4x-2b+2)\cdot(2b-1)^{\ell-1} \\ +A_b(xb^{\ell-1}+r) \\ +2A_b(yb^{\ell-1}+r) \\ -2A_b(r), & \text{if } y > x. \end{cases}$$

*Proof.* We start with the proof of the first equality. Let $x \in \{1, \dots, b-1\}$, $\ell \geq 1$ and $r \in \{0, \dots, b^{\ell-1}\}$. If $r = 0$, then (4.15) holds using Lemma 4.30. Now suppose that $r \in \{1, \dots, b^{\ell-1}\}$. Applying Proposition 3.29, we have

$$\begin{aligned} A_b(xb^\ell + r) &= A_b(xb^\ell) + \sum_{j=0}^{r-1} S_b(xb^\ell + j) \\ &= A_b(xb^\ell) + \sum_{j=0}^{r-1} (S_b(xb^{\ell-1}+j) + S_b(j)) \\ &= A_b(xb^\ell) + (A_b(xb^{\ell-1}+r) - A_b(xb^{\ell-1})) + A_b(r), \end{aligned}$$

and then Lemma 4.30 gives

$$A_b(xb^\ell + r) = (2b-2)(2x-1)(2b-1)^{\ell-1} + A_b(xb^{\ell-1}+r) + A_b(r),$$

which proves (4.15).

The proof of the last two equalities are similar, thus we only prove (4.16). Let $x \in \{1,\ldots,b-1\}$, $\ell \geq 1$ and $r \in \{0,\ldots,b^{\ell-1}\}$. If $r=0$, then (4.16) holds using Lemmas 4.30 and 4.32. Now suppose that $r \in \{1,\ldots,b^{\ell-1}\}$. Applying Proposition 3.29, we have

$$\begin{aligned} A_b(xb^\ell + xb^{\ell-1} + r) &= A_b(xb^\ell + xb^{\ell-1}) + \sum_{j=0}^{r-1} S_b(xb^\ell + xb^{\ell-1} + j) \\ &= A_b(xb^\ell + xb^{\ell-1}) + \sum_{j=0}^{r-1} (2S_b(xb^{\ell-1}+j) - S_b(j)) \\ &= A_b(xb^\ell + xb^{\ell-1}) + 2(A_b(xb^{\ell-1}+r) - A_b(xb^{\ell-1})) \\ &\quad -A_b(r). \end{aligned}$$

Using Lemmas 4.30 and 4.32, we get

$$\begin{aligned} A_b(xb^\ell + xb^{\ell-1} + r) &= (4xb+2x-2b)(2b-1)^{\ell-1} - 2(2x-1)(2b-1)^{\ell-1} \\ &\quad +2A_b(xb^{\ell-1}+r) - A_b(r) \\ &= (4xb-2x-2b+2)(2b-1)^{\ell-1} \\ &\quad +2A_b(xb^{\ell-1}+r) - A_b(r), \end{aligned}$$

as expected. □

**Example 4.35.** As already noticed in Example 3.42, the relations obtained in Proposition 4.34 for $b=2$ are slightly different from those of Lemma 4.7. By Proposition 4.34 for $b=3$, the values taken by $A_3(x3^\ell + y3^{\ell-1} + r)$ are given in Table 4.8 for $1 \leq x \leq 2$, $0 \leq y \leq 2$, $\ell \geq 1$, and $r \in \{0,\ldots,3^{\ell-1}\}$.

The following corollary was conjectured in [LRS17a] and generalizes Corollary 4.9.

**Corollary 4.36.** *For all $n \geq 0$, we have $A_b(nb) = (2b-1)A_b(n)$.*

*Proof*. Let us proceed by induction on $n \geq 0$. It is easy to check that the result holds for $n \in \{0,\ldots,b-1\}$. If $n=0$, then $A_b(nb) = 0 = A_b(n)$. If

| $x$ | $y$ | $A_3(x3^\ell + y3^{\ell-1} + r)$ |
|---|---|---|
| 1 | 0 | $4 \cdot 5^{\ell-1} + A_3(3^{\ell-1} + r) + A_3(r)$ |
| 1 | 1 | $6 \cdot 5^{\ell-1} + 2A_3(3^{\ell-1} + r) - A_3(r)$ |
| 1 | 2 | $4 \cdot 5^{\ell-1} + A_3(3^{\ell-1} + r) + 2A_3(2 \cdot 3^{\ell-1} + r) - 2A_3(r)$ |
| 2 | 0 | $12 \cdot 5^{\ell-1} + A_3(2 \cdot 3^{\ell-1} + r) + A_3(r)$ |
| 2 | 1 | $13 \cdot 5^{\ell-1} + A_3(2 \cdot 3^{\ell-1} + r) + 2A_3(3^{\ell-1} + r) - 2A_3(r)$ |
| 2 | 2 | $16 \cdot 5^{\ell-1} + 2A_3(2 \cdot 3^{\ell-1} + r) - A_3(r)$ |

Table 4.8: Values of $A_3(x3^\ell + y3^{\ell-1} + r)$ for $1 \le x \le 2$, $0 \le y \le 2$, $\ell \ge 1$, and $r \in \{0, \dots, 3^{\ell-1}\}$.

$n \in \{1, \dots, b-1\}$, then Lemma 4.30 tells us that $A_b(nb) = (2n-1)(2b-1)$ and $A_b(n) = (2n-1)$. Thus consider $n \ge b$, and suppose that the result holds for all $n' < n$. The reasoning is divided into three cases according to the form of the base-$b$ expansion of $n$. As a first case, we write $n = xb^\ell + r$ with $x \in \{1, \dots, b-1\}$, $\ell \ge 1$, and $0 \le r < b^{\ell-1}$. By Proposition 4.34, we have

$$\begin{aligned} A_b(nb) - (2b-1)A_b(n) &= (2b-2) \cdot (2x-1) \cdot (2b-1)^\ell + A_b(xb^\ell + br) \\ &\quad + A_b(br) - (2b-2) \cdot (2x-1) \cdot (2b-1)^\ell \\ &\quad - (2b-1)A_b(xb^{\ell-1} + r) - (2b-1)A_b(r). \end{aligned}$$

We conclude this case by using the induction hypothesis. The other cases can be handled using the same technique, so we intentionally skip them. □

Using Proposition 4.34, we can define $(2b-1)$-decompositions as follows. Compare with Definition 4.11 for the case $b = 2$.

**Definition 4.37** ($(2b-1)$-decomposition)**.** Recall that, by Lemma 4.30, we have $A_b(n) = (2n-1)(2b-1)^0$ for all $n \in \{1, \dots, b-1\}$. We say that the single-letter word

$$(\mathsf{2b-1})\mathsf{dec}(A_b(n)) = (2n-1)$$

is the $(2b-1)$-*decomposition* of $A_b(n)$. Since $A_b(0) = 0 \cdot (2b-1)^0$, the $(2b-1)$-*decomposition* of $A_b(0)$ is $(\mathsf{2b-1})\mathsf{dec}(A_b(0)) = 0$.

Let $n \geq b$. Iteratively applying Proposition 4.34 provides a decomposition of the form

$$A_b(n) = \sum_{i=0}^{\ell_b(n)} d_i(n)\,(2b-1)^{\ell_b(n)-i},$$

where $d_i(n)$'s are integers, $d_0(n) \neq 0$, and $\ell_b(n)$ stands for $\lfloor \log_b n \rfloor - 1$. We say that the word

$$(2\mathsf{b}-1)\mathsf{dec}(A_b(n)) = d_0(n) \cdots d_{\ell_b(n)}(n)$$

is the $(2b-1)$-*decomposition* of $A_b(n)$.

When the integer $n$ is clear from the context, we simply write $d_i$ instead of $d_i(n)$. For the sake of clarity, we also write $(d_0(n), \ldots, d_{\ell_b(n)}(n))$ when necessary. Notice that the notion of $(2b-1)$-decomposition is only valid for integers in the sequence $(A_b(n))_{n\geq 0}$.

**Example 4.38.** Let us set $b = 3$. We already know that the corresponding base is $2b-1 = 5$. The goal of this example is to compute the 5-decomposition of $A_3(150) = 1665$. We have $\text{rep}_3(150) = 12120$ and $\ell_3(150) = 3$. The third formula of Proposition 4.34 (or Table 4.8) leads to

$$\begin{aligned} A_3(150) &= A_3(3^4 + 2\cdot 3^3 + 15) \\ &= 4\cdot 5^3 + A_3(3^3+15) + 2A_3(2\cdot 3^3 + 15) - 2A_3(15). \end{aligned} \tag{4.17}$$

Applying Proposition 4.34 on terms of the form $A_3(m)$ that have just appeared in the right-hand side of (4.17) (also look at Table 4.8), we get

$$\begin{aligned} A_3(3^3+15) &= A_3(3^3+3^2+6) \\ &= 6\cdot 5^2 + 2A_3(3^2+6) - A_3(6), \\ A_3(2\cdot 3^3+15) &= A_3(2\cdot 3^3+3^2+6) \\ &= 13\cdot 5^2 + A_3(2\cdot 3^2+6) + 2A_3(3^2+6) - 2A_3(6), \\ A_3(15) &= A_3(3^2+2\cdot 3^1) \\ &= 4\cdot 5^1 + A_3(3^1) + 2A_3(2\cdot 3^1) - 2A_3(0). \end{aligned}$$

Using again Proposition 4.34 on the new terms of the form $A_3(m)$ or by

Table 4.8, we find

$$\begin{aligned} A_3(3^2+6) &= A_3(3^2+2\cdot 3^1)\\ &= 4\cdot 5^1 + A_3(3^1) + 2A_3(2\cdot 3^1) - 2A_3(0),\\ A_3(2\cdot 3^2+6) &= A_3(2\cdot 3^2+2\cdot 3^1)\\ &= 16\cdot 5^1 + 2A_3(2\cdot 3^1) - A_3(0),\\ A_3(6) &= A_3(2\cdot 3^1)\\ &= 12\cdot 5^0 + A_3(2\cdot 3^0) + A_3(0). \end{aligned}$$

From Proposition 4.34 or by Table 4.8, we also have

$$\begin{aligned} A_3(3^1) &= 4\cdot 5^0 + A_3(3^0) + A_3(0),\\ A_3(2\cdot 3^1) &= 12\cdot 5^0 + A_3(2\cdot 3^0) + A_3(0). \end{aligned}$$

From Lemma 4.30, $A_3(3^0) = 5^0$ and $A_3(2\cdot 3^0) = 3\cdot 5^0$, and the procedure halts. Plugging all those values together in (4.17), we finally have

$$A_3(150) = 4\cdot 5^3 + 32\cdot 5^2 + 48\cdot 5^1 + 125\cdot 5^0.$$

The 5-decomposition of $A_3(150)$ is thus $(4, 32, 48, 125)$.

We now establish the asymptotic behavior of $A_b$. The proof of the next result follows the same lines as the proof of Theorem 4.4. Therefore, we only sketch it.

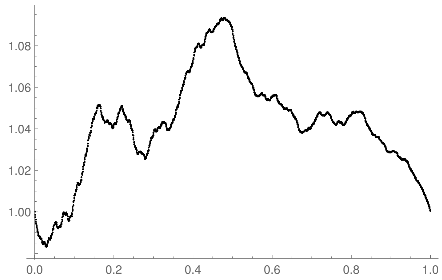
**Theorem 4.39.** *There exists a continuous and periodic function $\mathcal{H}_b$ of period 1 such that, for all large enough $n$,*

$$A_b(n) = (2b-1)^{\log_b n}\ \mathcal{H}_b(\log_b n).$$

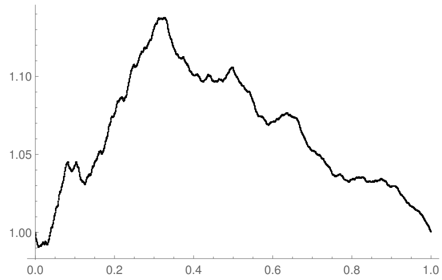
As an example, when $b \in \{3, 4, 5, 7\}$, the function $\mathcal{H}_b$ is depicted in Figure 4.9 over one period. Compare them with Figure 4.2.

*Sketch of the proof of Theorem 4.39.* Let us start by defining the function $\Phi_b$. Given any integer $n \geq 1$, we let $\phi_n$ denote the function

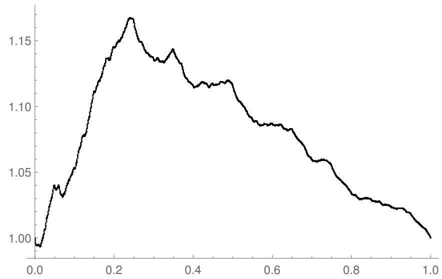
$$\alpha \in [0,1) \mapsto \phi_n(\alpha) = \frac{A_b(e_n(\alpha))}{(2b-1)^{\log_b(e_n(\alpha))}},$$



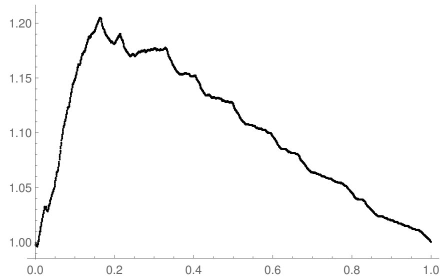

(a) $\mathcal{H}_3$




(b) $\mathcal{H}_4$




(c) $\mathcal{H}_5$




(d) $\mathcal{H}_7$

Figure 4.9: The function $\mathcal{H}_b$ over one period for $b \in \{3, 4, 5, 7\}$.

where $e_n(\alpha) = b^{n+1} + b\lfloor \alpha b^n \rfloor + 1$ (see Definition 4.17 and the beginning of Section 4.1.2 for the base-2 case). With a proof analogous to the one of Proposition 4.20, the sequence $(\phi_n)_{n\geq 1}$ of functions uniformly converges to a function $\Phi_b$. As in Theorem 4.6, this function is continuous on $[0, 1)$, and such that $\Phi_b(0) = 1 = \lim_{\alpha\to 1^-} \Phi_b(\alpha)$. Furthermore, similarly to Lemma 4.24, it satisfies

$$A_b(b^k + r) = (2b-1)^{\log_b(b^k+r)} \Phi_b\left(\frac{r}{b^k}\right) \tag{4.18}$$

for $k \geq 1$ and $0 \leq r < b^k$. Using Corollary 4.36, for all integer $n = b^j(b^k + r)$ with $j, k \geq 0$ and $r \in \{0, \ldots, b^k - 1\}$, we get

$$A_b(n) = (2b-1)^j A_b(b^k + r) = (2b-1)^{\log_b(n)} \Phi_b\left(\frac{r}{b^k}\right).$$

As in the base-2 case, we define the *relative position* $\operatorname{relpos}_b(x)$ of a positive real number $x$ inside the interval $[b^{\lfloor \log_b x \rfloor}, b^{\lfloor \log_b x \rfloor + 1})$ by

$$\operatorname{relpos}_b(x) = \frac{x - b^{\lfloor \log_b x \rfloor}}{b^{\lfloor \log_b x \rfloor}} = b^{\{\log_b x\}} - 1 \in [0, 1).$$

Note that $\operatorname{relpos}_b(n) = r/b^k$. As in the proof of Theorem 4.4 regarding the base-2 case, the function $\mathcal{H}_b$ is defined by $\mathcal{H}_b(x) = \Phi_b(\operatorname{relpos}_b(b^x))$ for all

$x \in \mathbb{R}$. It has the desired properties, and since $\mathcal{H}_b(\log_b(n)) = \Phi_b(\text{relpos}_b(n))$, we find

$$A_b(n) = (2b-1)^{\log_b(n)} \mathcal{H}_b(\log_b(n)). \qquad \square$$

**Remark 4.40.** Let us insist on the fact that the formula in Theorem 4.39 is exact. Indeed, (4.18) proves that no error term shows up on the dense set of points of the form $r/b^k$. See also Remark 4.25.

As a final comment in the general integer base case, we leave open the problem of determining whether the function $\mathcal{H}_b$ is differentiable or not, as in Remark 4.26.

## 4.3 Mixing Fibonacci and an Exotic Numeration System

As already mentioned in the introduction of this chapter, one can obtain general asymptotic estimates for summatory functions of $b$-regular sequences (see, for instance, [AS03a, BR10, Dum13, Dum14]). In this section, we show how our method can be extended to sequences that do not exhibit a regular structure in the classical sense, *i.e.*, up to our knowledge, they are not $b$-regular for any $b \geq 2$. Instead of integer base numeration systems, we use the Zeckendorf numeration system associated with the golden ratio (see Examples 1.18 and 1.30). Compared to the sequences $(S_b(n))_{n\geq 0}$, the sequence $(S_\varphi(n))_{n\geq 0}$ (see Example 1.50) only takes into account words not containing two consecutive 1's. A major difference with the integer base case is that the sequence $(S_\varphi(n))_{n\geq 0}$ is not known to be $b$-regular[3] for any integer $b \geq 2$, but is $F$-regular (see Theorem 3.63). Thus, we are no longer in the classical setting of $b$-regular sequences, and therefore known results. Nevertheless, it is striking that we are still able to mimic the same strategy, and obtain an expression for its summatory function.

**Definition 4.41.** We let $A_\varphi = (A_\varphi(n))_{n\geq 0}$ denote the *summatory function* of the sequence $(S_\varphi(n))_{n\geq 0}$ defined by

$$A_\varphi(n) = \sum_{j=0}^{n} S_\varphi(j)$$

[3] An interesting direction of investigation is to establish this fact. See the end of Chapter 3.

for all $n \geq 0$. The first few terms of $(A_\varphi(n))_{n\geq 0}$ are

$$1, 3, 6, 10, 14, 19, 25, 31, 37, 45, 54, 62, 70, 77, 87, 99, 111, 123, 133, 145, \ldots$$

(see also A282731 in [Slo]). As before, the quantity $A_\varphi(n)$ counts the total number of $F$-expansions occurring as scattered subwords in the $F$-expansion of integers less than or equal to $n$.

**Remark 4.42.** In the integer base case, the corresponding summatory function is $b$-regular (see Propositions 4.2 and 4.29). In the present context of $F$-regular sequences, it is not clear whether $(A_\varphi(n))_{n\geq 0}$ is $F$-regular or not. It is not clear either if one can adapt [Dum13, Lemma 1] and deduce a linear representation of $(A_\varphi(n))_{n\geq 0}$ from a linear representation of $(S_\varphi(n))_{n\geq 0}$. See Remark 4.3 for the integer base case.

As in the previous sections with Theorems 4.4, 4.6 and 4.39, the goal is to provide a formula for the asymptotic behavior of $A_\varphi$. To that aim, we analogously consider a convenient $B$-decomposition of $A_\varphi(n)$, with $n \geq 0$, based on the terms of a sequence $(B(n))_{n\geq 0}$ defined in Section 4.3.1 below. Using a similar technique, we prove the following result. Already observe that it contains an error term whereas the formulas are exact for integer base cases. Also, $\log_F(\cdot)$ will be defined in due time, on page 220.

**Theorem 4.43.** *Let $\lambda$ be the dominant root of $X^3 - 2X^2 - X + 1$. There exists a continuous and periodic function $\mathcal{G}$ of period 1 such that, for all large enough $n$,*

$$A_\varphi(n) = \sum_{j=0}^{n} S_\varphi(j) = c\,\lambda^{\log_F n}\mathcal{G}(\log_F n) + o(\lambda^{\lfloor \log_F n \rfloor}).$$

### 4.3.1 Introduction to $(B(n))_{n\geq 0}$

For integer base numeration systems, we are able to write the values of $A_b$ at powers of $b$ as multiples of powers of $2b-1$ (see Lemmas 4.5 and 4.30). We have a similar result in the Fibonacci case using a particular sequence defined below.

**Definition 4.44.** Let $(B(n))_{n\geq 0}$ be the sequence of integers defined by $B(0) = 1$, $B(1) = 3$, $B(2) = 6$, and $B(n+3) = 2B(n+2) + B(n+1) - B(n)$

for all $n \geq 0$ (`A006356` in [Slo]). The sequence $(B(n))_{n\geq 0}$ begins with

$$1, 3, 6, 14, 31, 70, 157, 353, 793, 1782, 4004, 8997, 20216, 45425, 102069, \ldots.$$

The characteristic polynomial $P_B(X) = X^3 - 2X^2 - X + 1$ of the linear recurrence of $(B(n))_{n\geq 0}$ has three real roots as depicted in Figure 4.10. We let $\lambda \approx 2.24698$ denote its root of maximal modulus. The other two roots are $\lambda_2 \approx -0.80194$ and $\lambda_3 \approx 0.55496$. From the classical theory of linear recurrences, there exist three constants $c \approx 1.22041$, $c_2 \approx -0.28011$, and $c_3 \approx 0.0597$ such that, for all $n \in \mathbb{N}$,

$$B(n) = c\,\lambda^n + c_2\,\lambda_2^n + c_3\,\lambda_3^n. \tag{4.19}$$

In particular, we have

$$\lim_{n\to+\infty} \frac{B(n)}{c\,\lambda^n} = 1. \tag{4.20}$$

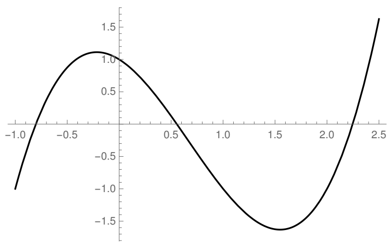



Figure 4.10: The graph of $P_B(X) = X^3 - 2X^2 - X + 1$.

**Proposition 4.45.** *For all $n \geq 0$, we have $A_\varphi(F(n) - 1) = B(n)$.*

*Proof*. The equalities $B(0) = 1 = A_\varphi(F(0) - 1)$, $B(1) = 3 = A_\varphi(F(1) - 1)$, and $B(2) = 6 = A_\varphi(F(2) - 1)$ can be checked by hand. Let us show that

$$A_\varphi(F(n+3) - 1) = 2A_\varphi(F(n+2) - 1) + A_\varphi(F(n+1) - 1) - A_\varphi(F(n) - 1)$$

holds for all $n \geq 0$. Using Definition 4.41, the equality that we want to prove is equivalent to

$$\sum_{j=F(n+2)}^{F(n+3)-1} S_\varphi(j) = \sum_{j=0}^{F(n)-1} S_\varphi(j) + 2\sum_{j=F(n)}^{F(n+1)-1} S_\varphi(j) + \sum_{j=F(n+1)}^{F(n+2)-1} S_\varphi(j). \tag{4.21}$$

Observe that

$$\{j \mid F(n+2) \leq j < F(n+3)\} = \{F(n+2) + r \mid 0 \leq r < F(n+1)\}.$$

This gives

$$\begin{aligned}\sum_{j=F(n+2)}^{F(n+3)-1} S_\varphi(j) &= \sum_{r=0}^{F(n+1)-1} S_\varphi(F(n+2)+r)\\ &= \sum_{r=0}^{F(n)-1} S_\varphi(F(n+2)+r) + \sum_{r=F(n)}^{F(n+1)-1} S_\varphi(F(n+2)+r).\end{aligned}$$

Using Proposition 3.66, we thus find

$$\begin{aligned}\sum_{j=F(n+2)}^{F(n+3)-1} S_\varphi(j) &= \sum_{r=0}^{F(n)-1} (S_\varphi(F(n+1)+r) + S_\varphi(r)) + \sum_{r=F(n)}^{F(n+1)-1} 2S_\varphi(r)\\ &= \sum_{r=0}^{F(n)-1} S_\varphi(r) + 2\sum_{r=F(n)}^{F(n+1)-1} S_\varphi(r)\\ &\quad + \sum_{r=0}^{F(n)-1} S_\varphi(F(n+1)+r).\end{aligned}$$

Then the equality (4.21) holds because

$$\{F(n+1) + r \mid 0 \leq r < F(n)\} = \{j \mid F(n+1) \leq j < F(n+2)\}.$$

We have just showed that $(A_\varphi(F(n)-1))_{n\geq 0}$ satisfies the same recurrence relation as $(B(n))_{n\geq 0}$, so we may conclude the proof by Definition 4.44. □

Thanks to Proposition 4.45, we have analogues of Lemma 4.7 and Proposition 4.34. This is the key point to obtain particular $B$-decompositions.

**Lemma 4.46.** *Let $\ell \geq 2$. If $0 \leq r < F(\ell-2)$, then*

$$A_\varphi(F(\ell)+r) = B(\ell) - B(\ell-1) + A_\varphi(F(\ell-1)+r) + A_\varphi(r).$$

*If $F(\ell-2) \leq r < F(\ell-1)$, then*

$$A_\varphi(F(\ell)+r) = 2B(\ell) - B(\ell-1) - B(\ell-2) + 2A_\varphi(r).$$

*Proof.* Assume first that $0 \le r < F(\ell-2)$. We have

$$\begin{aligned} A_\varphi(F(\ell)+r) &= \sum_{j=0}^{F(\ell)+r} S_\varphi(j) \\ &= \sum_{j=0}^{F(\ell)-1} S_\varphi(j) + \sum_{j=F(\ell)}^{F(\ell)+r} S_\varphi(j) \\ &= A_\varphi(F(\ell)-1) + \sum_{j=0}^{r} S_\varphi(F(\ell)+j). \end{aligned}$$

Applying Proposition 3.66 and Proposition 4.45, we get

$$A_\varphi(F(\ell)+r) = B(\ell) + \sum_{j=0}^{r} S_\varphi(F(\ell-1)+j) + \sum_{j=0}^{r} S_\varphi(j).$$

Using Proposition 4.45 once more, we obtain

$$\begin{aligned} A_\varphi(F(\ell)+r) &= B(\ell) + \left( \sum_{j=0}^{F(\ell-1)+r} S_\varphi(j) - \sum_{j=0}^{F(\ell-1)-1} S_\varphi(j) \right) + A_\varphi(r) \\ &= B(\ell) + A_\varphi(F(\ell-1)+r) - A_\varphi(F(\ell-1)-1) + A_\varphi(r) \\ &= B(\ell) + A_\varphi(F(\ell-1)+r) - B(\ell-1) + A_\varphi(r). \end{aligned}$$

Let us suppose that $F(\ell-2) \le r < F(\ell-1)$ to prove the second part of the result. We first have

$$\begin{aligned} A_\varphi(F(\ell)+r) &= \sum_{j=0}^{F(\ell)+r} S_\varphi(j) \\ &= \sum_{j=0}^{F(\ell)+F(\ell-2)-1} S_\varphi(j) + \sum_{j=F(\ell)+F(\ell-2)}^{F(\ell)+r} S_\varphi(j) \\ &= A_\varphi(F(\ell)+F(\ell-2)-1) + \sum_{j=F(\ell-2)}^{r} S_\varphi(F(\ell)+j). \end{aligned}$$

According to the second case of Proposition 3.66, we have

$$\begin{aligned} A_\varphi(F(\ell)+r) &= A_\varphi(F(\ell)+F(\ell-2)-1)+2\sum_{j=F(\ell-2)}^{r} S_\varphi(j) \\ &= A_\varphi(F(\ell)+F(\ell-2)-1) \\ &\quad +2\left(A_\varphi(r)-\sum_{j=0}^{F(\ell-2)-1} S_\varphi(j)\right). \end{aligned}$$

We may apply the first part of the result to the term $A_\varphi(F(\ell)+F(\ell-2)-1)$ and we find

$$\begin{aligned} A_\varphi(F(\ell)+r) &= B(\ell)-B(\ell-1)+A_\varphi(F(\ell-1)+F(\ell-2)-1) \\ &\quad +A_\varphi(F(\ell-2)-1)+2A_\varphi(r)-2A_\varphi(F(\ell-2)-1), \end{aligned}$$

and next, with Proposition 4.45, we get

$$\begin{aligned} A_\varphi(F(\ell)+r) &= B(\ell)-B(\ell-1)+B(\ell)+B(\ell-2)+2A_\varphi(r)-2B(\ell-2) \\ &= 2B(\ell)-B(\ell-1)-B(\ell-2)+2A_\varphi(r), \end{aligned}$$

as expected. □

### 4.3.2 Particular $B$-Decompositions

Similarly to the $(2b-1)$-decomposition of the function $A_b$ examined in Section 4.1.1 and Section 4.2 for integer bases, we will consider what we call the $B$-decomposition of $A_\varphi$. The idea is again to iteratively apply Lemma 4.46 to derive a decomposition of $A_\varphi$ as a particular linear combination of terms of the sequence $(B(n))_{n\geq 0}$. Indeed, each application of Lemma 4.46 provides a "leading" term of the form $B(\ell)$ or $2B(\ell)$, plus terms of smaller indices.

**Definition 4.47** ($B$-decomposition)**.** We have $A_\varphi(0) = 1\cdot B(0)$ (resp., $A_\varphi(1) = 3\cdot B(0)$; resp., $A_\varphi(2) = 6\cdot B(0)$), so we say that the single-letter word

$$\mathsf{Bdec}(A_\varphi(0)) = 1 \quad \text{(resp., } \mathsf{Bdec}(A_\varphi(1)) = 3\text{; resp., } \mathsf{Bdec}(A_\varphi(2)) = 6\text{)}$$

is the $B$-*decomposition* of $A_\varphi(0)$ (resp., $A_\varphi(1)$; resp., $A_\varphi(2)$).

Let $n \geq 3$. Iteratively applying Lemma 4.46 provides a decomposition of the form

$$A_\varphi(n) = \sum_{i=0}^{\ell_F(n)} b_i(n)\, B(\ell_F(n) - i),$$

where $b_i(n)$'s are integers, $b_0(n) \neq 0$, and $\ell_F(n) = |\,\mathrm{rep}_F(n)| - 1$. We say that the word

$$\mathsf{Bdec}(A_\varphi(n)) = b_0(n) \cdots b_{\ell_F(n)}(n)$$

is the *$B$-decomposition* of $A_\varphi(n)$.

When the integer $n$ is clear from the context, we simply write $b_i$ instead of $b_i(n)$. To avoid any confusion, we will also write $(b_0(n), \ldots, b_{\ell_F(n)}(n))$. Finally, notice that the notion of $B$-decomposition is only valid for integers in the sequence $(A_\varphi(n))_{n\geq 0}$.

**Example 4.48.** We have $\mathrm{rep}_F(42) = 10010000$, so $\ell_F(42) = 7$. Lemma 4.46 yields

$$A_\varphi(42) = B(7) + B(6) - B(5) + 2B(4) - 3B(1) + 27B(0).$$

Indeed, we have

$$\begin{aligned}
A_\varphi(42) &= A_\varphi(F(7) + 8) = B(7) - B(6) + A_\varphi(F(6) + 8) + A_\varphi(8),\\
A_\varphi(F(6) + 8) &= 2B(6) - B(5) - B(4) + 2A_\varphi(8),\\
A_\varphi(8) &= A_\varphi(F(4)) = B(4) - B(3) + A_\varphi(F(3)) + A_\varphi(0),\\
A_\varphi(F(3)) &= B(3) - B(2) + A_\varphi(F(2)) + A_\varphi(0),\\
A_\varphi(F(2)) &= B(2) - B(1) + A_\varphi(F(1)) + A_\varphi(0).
\end{aligned}$$

To get the desired equality, it suffices to write $A_\varphi(F(1)) = 6B(0)$ and $A_\varphi(0) = 1B(0)$. The $B$-decomposition of $A_\varphi(42)$ is $(1, 1, -1, 2, 0, 0, -3, 27)$. Table 4.11 displays the $B$-decomposition of $A_\varphi(0), \ldots, A_\varphi(20)$.

**Remark 4.49.** Suppose that we want to develop $A_\varphi(n)$ with the sole use of Lemma 4.46, *i.e.*, to get the $B$-decomposition of $A_\varphi(n)$. Compared to Remark 4.12, only two cases may occur.

- If $\mathrm{rep}_F(n) = 100u$, with $u \in 0^*L_F$, then we apply the first part of Lemma 4.46, and we are left with evaluations of $A_\varphi$ at integers whose $F$-expansions are shorter and given by $10u$ and $\mathrm{rep}_F(\mathrm{val}_F(u))$.

- If $\text{rep}_F(n) = 101u$, with $u \in \{\varepsilon\} \cup 0^+ L_F$, then we apply the second part of Lemma 4.46, and we are left with evaluations of $A_\varphi$ at an integer whose $F$-expansion is shorter and given by $1u$.

| $n$ | $b_0(n)$ | $b_1(n)$ | $b_2(n)$ | $b_3(n)$ | $b_4(n)$ | $b_5(n)$ | $A_\varphi(n)$ |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | $1 \times 1 = 1$ |
| 1 | 3 | | | | | | $3 \times 1 = 3$ |
| 2 | 6 | | | | | | $6 \times 1 = 6$ |
| 3 | 1 | −1 | 7 | | | | $1 \times 6 - 1 \times 3 + 7 \times 1 = 10$ |
| 4 | 2 | −1 | 5 | | | | $2 \times 6 - 1 \times 3 + 5 \times 1 = 14$ |
| 5 | 1 | 0 | −1 | 8 | | | $1 \times 14 - 1 \times 3 + 8 \times 1 = 19$ |
| 6 | 1 | 1 | −1 | 8 | | | $\vdots$ |
| 7 | 2 | −1 | −1 | 12 | | | |
| 8 | 1 | 0 | 0 | −1 | 9 | | |
| 9 | 1 | 0 | 1 | −1 | 11 | | |
| 10 | 1 | 1 | −1 | −1 | 18 | | |
| 11 | 2 | −1 | 1 | −2 | 14 | | |
| 12 | 2 | −1 | 3 | −2 | 10 | | |
| 13 | 1 | 0 | 0 | 0 | −1 | 10 | |
| 14 | 1 | 0 | 0 | 1 | −1 | 14 | |
| 15 | 1 | 0 | 1 | −1 | −1 | 24 | |
| 16 | 1 | 1 | −1 | 2 | −3 | 21 | |
| 17 | 1 | 1 | −1 | 5 | −3 | 15 | |
| 18 | 2 | −1 | 1 | 0 | −2 | 16 | |
| 19 | 2 | −1 | 1 | 2 | −2 | 16 | |
| 20 | 2 | −1 | 3 | −2 | −2 | 24 | |

Table 4.11: The $B$-decomposition of $(A_\varphi(n))_{0 \le n \le 20}$.

As in Lemma 4.13 for 3-decompositions, we can again notice similarities between certain pairs of $B$-decompositions and we can compare them.

**Lemma 4.50.** *For all finite words $u, v, v' \in \{0,1\}^*$ such that the words $1uv$ and $1uv'$ both belong to $1\{0,01\}^*$ and $|u| \ge 2$, the $B$-decompositions of $A_\varphi(\text{val}_F(1uv))$ and $A_\varphi(\text{val}_F(1uv'))$ share the same coefficients $b_0, \ldots, b_{|u|-2}$,* i.e., *their first* $|u| - 1$ *coefficients are equal.*

*Proof.* The proof is similar to the proof of Lemma 4.13 and follows from Lemma 4.46. □

**Example 4.51.** Let us consider

$$\begin{aligned} \text{rep}_F(163) &= 1(000010)1001 = 1uv, \text{ and} \\ \text{rep}_F(673) &= 1(000010)0010000 = 1uv' \end{aligned}$$

with $u = 000010$, $|u| = 6$, $v = 1001$ and $v' = 0010000$. If we compare the $B$-decompositions of $A_\varphi(163)$ and $A_\varphi(673)$ with the help of Table 4.12, they share the same first five coefficients.

| $n$ | $b_0$ | $b_1$ | $b_2$ | $b_3$ | $b_4$ | $b_5$ | $b_6$ | $b_7$ | $b_8$ | $b_9$ | $b_{10}$ | $b_{11}$ | $b_{12}$ | $b_{13}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 163 | 1 | 0 | 0 | 1 | −1 | 9 | −5 | 5 | 10 | −10 | 80 | | | |
| 673 | 1 | 0 | 0 | 1 | −1 | 4 | 0 | 5 | −5 | 15 | 0 | 0 | −20 | 180 |

Table 4.12: The $B$-decomposition of $A_\varphi(163)$ and $A_\varphi(673)$.

In the case of the integer base $b \geq 2$, the evaluation of $A_b$ at powers of $b$ is of particular importance. Here, we fully describe the $B$-decompositions of $A_\varphi$ evaluated at two sequences related to Fibonacci numbers. Recall that the real number $\lambda$ is given in Definition 4.44.

**Lemma 4.52.** *The sequence* $(\mathsf{Bdec}(A_\varphi(F(n))))_{n\geq 0}$ *converges to the infinite word* $10^\omega$, *and the sequence* $(\mathsf{Bdec}(A_\varphi(F(n)-1)))_{n\geq 0}$ *converges to the infinite word* $(g_n)_{n\geq 0}$ *where* $g_0 = 2$, $g_1 = -1$, $g_2 = 3$, *and* $g_n = 2g_{n-2}$ *for all* $n \geq 3$. *In particular,* $|g_n| \leq 2 \cdot (\sqrt{2})^n$ *for all* $n \geq 0$, *and*

$$\sum_{i=0}^{+\infty} \frac{g_i}{\lambda^i} = \lambda.$$

*Proof.* Let us prove the first part of the statement. We show that, for all $n \geq 2$, $A_\varphi(F(n)) = B(n) - B(1) + (n+5)B(0)$, or equivalently

$$\mathsf{Bdec}(A_\varphi(F(n))) = (1, \underbrace{0, \cdots, 0}_{n-2 \text{ times}}, -1, n+5). \tag{4.22}$$

We proceed by induction on $n \geq 2$. For $n \in \{2, 3\}$, the $B$-decompositions of $A_\varphi(F(2)) = A_\varphi(3)$ and $A_\varphi(F(3)) = A_\varphi(5)$ can be found in Table 4.11 and satisfy (4.22). Thus, consider $n \geq 3$, and suppose the result holds for all $m < n+1$. From Lemma 4.46, we have

$$A_\varphi(F(n+1)) = B(n+1) - B(n) + A_\varphi(F(n)) + A_\varphi(0).$$

Since $A_\varphi(0) = B(0)$, the induction hypothesis yields

$$A_\varphi(F(n+1)) = B(n+1) - B(n) + (B(n) - B(1) + (n+5) \cdot B(0)) + B(0),$$

which proves (4.22). The convergence of the sequence $(\mathsf{Bdec}(A_\varphi(F(n))))_{n\geq 0}$ of finite words to the infinite word $10^\omega$ easily follows (recall Definition 1.11).

Let us prove the second part of the statement. We show that, for all $n \geq 3$,

$$\mathsf{Bdec}(A_\varphi(F(n) - 1)) = \begin{cases} (g_0, g_1, \ldots, g_{n-2}, x), & \text{if } n \text{ is odd;} \\ (g_0, g_1, \ldots, g_{n-3}, y, z), & \text{if } n \text{ is even;} \end{cases} \tag{4.23}$$

where $x, y, z$ are integers. We proceed again by induction on $n \geq 3$. Table 4.11 provides the result for $n \in \{3, 4\}$: we have to look at the $B$-decompositions of $A_\varphi(F(3) - 1) = A_\varphi(4)$ and $A_\varphi(F(4) - 1) = A_\varphi(7)$. Thus, consider $n \geq 4$, and suppose the result holds for all $m < n+1$. Suppose first that $n$ is even. By Lemma 4.46, we have

$$\begin{aligned} A_\varphi(F(n+1) - 1) &= A_\varphi(F(n) + F(n-1) - 1) \\ &= 2B(n) - B(n-1) - B(n-2) + 2A_\varphi(F(n-1) - 1). \end{aligned}$$

Using the induction hypothesis with

$$\mathsf{Bdec}(A_\varphi(F(n-1) - 1)) = (g_0, g_1, \ldots, g_{n-3}, x)$$

and with the first value $g_0 = 2$ of $(g_n)_{n\geq 0}$, we get

$$\begin{aligned} A_\varphi(F(n+1) - 1) = &2B(n) - B(n-1) - B(n-2) + 4B(n-2) \\ &+ \sum_{j=1}^{n-3} 2g_j B(n-2-j) + 2xB(0). \end{aligned}$$

By definition of the sequence $(g_n)_{n\geq 0}$, we have $2g_j = g_{j+2}$ for all $j \geq 1$, so we finally obtain

$$A_\varphi(F(n+1) - 1) = 2B(n) - B(n-1) + 3B(n-2) + \sum_{j=3}^{n-1} g_j B(n-j) + 2xB(0),$$

which concludes the case where $n$ is even since $g_0 = 2$, $g_1 = -1$ and $g_2 = 3$. The case where $n$ is odd can be proved using the same argument.

Let us prove the last part of the statement. Using the definition of the sequence $(g_n)_{n\geq 0}$, we get

$$\sum_{i=1}^{+\infty} \frac{g_i}{\lambda^i} = \frac{g_1}{\lambda} + \frac{g_2}{\lambda^2} + \sum_{i=1}^{+\infty} \frac{g_{i+2}}{\lambda^{i+2}} = \frac{-1}{\lambda} + \frac{3}{\lambda^2} + \frac{2}{\lambda^2} \cdot \sum_{i=1}^{+\infty} \frac{g_i}{\lambda^i}$$

that is

$$\sum_{i=1}^{+\infty} \frac{g_i}{\lambda^i} = \frac{-\lambda + 3}{\lambda^2 - 2}.$$

Hence, since $P_B(\lambda) = \lambda^3 - 2\lambda^2 - \lambda + 1 = 0$, we have

$$\sum_{i=0}^{+\infty} \frac{g_i}{\lambda^i} = g_0 + \sum_{i=1}^{+\infty} \frac{g_i}{\lambda^i} = 2 + \frac{3-\lambda}{\lambda^2 - 2} = \frac{2\lambda^2 - \lambda - 1}{\lambda^2 - 2} = \lambda.$$

The fact that $|g_n| \leq 2 \cdot (\sqrt{2})^n$ follows by a smooth induction and from the definition of the sequence $(g_n)_{n\geq 0}$. □

### 4.3.3 Drifter Falling into Infinity

The idea behind the next definitions is that the real number $\alpha$ gives the relative position of an integer between two consecutive Fibonacci numbers. See Definitions 4.15 and 4.17 for the base-2 case, and the proof of Theorem 4.39 for the general integer base case. We also define an infinite word $b(\alpha)$ based on $B$-decompositions of specific integers, which is the analogue of the infinite word $a(\alpha)$ given in Definition 4.18.

**Definition 4.53.** Let $\alpha$ be a real number in $[0, 1)$, and let $d_\varphi(\alpha)$ denote its $\varphi$-expansion as in Definition 1.19. Define the sequence $(w_n(\alpha))_{n\geq 1}$ of finite words where $w_n(\alpha)$ is the length-$n$ prefix of the infinite word $10d_\varphi(\alpha)$. For each $n \geq 1$, let us define the integer

$$e_n(\alpha) = \mathrm{val}_F(w_n(\alpha)) \in [F(n-1), F(n)).$$

Note that, since the $\varphi$-expansion of $\alpha$ does not contain any factor of the form 11, we have $w_n(\alpha) = \mathrm{rep}_F(e_n(\alpha))$. Compared to the base-2 case, observe that $e_n(\alpha)$ might be an even integer.

Mimicking Remark 4.16 and thanks to Lemma 4.50, we know that the sequence $(\mathsf{Bdec}(A_\varphi(e_n(\alpha))))_{n\geq 1}$ converges to an infinite word as explained below.

**Definition 4.54.** For each $n \geq 1$, let

$$\mathsf{Bdec}(A_\varphi(e_n(\alpha))) = b_0(e_n(\alpha)) \cdots b_{n-1}(e_n(\alpha))$$

be the $B$-decomposition of $A_\varphi(e_n(\alpha))$ (note that $\ell_F(e_n(\alpha)) = n-1$). We let $b(\alpha) = b_0(\alpha)\, b_1(\alpha)\, \cdots$ denote the infinite sequence of integers that is the limit of the sequence $(\mathsf{Bdec}(A_\varphi(e_n(\alpha))))_{n\geq 1}$.

**Example 4.55.** The $\varphi$-expansion of $\alpha = \pi - 3$ is

$$d_\varphi(\alpha) = 00001010100100010101 \cdots .$$

Thus, the first ten finite words of $(w_n(\alpha))_{1\geq n}$ are

$$1, 10, 100, 1000, 10000, 100000, 1000001, 10000010, 100000101, 1000001010.$$

The first ten integers of $(e_n(\alpha))_{n\geq 1}$ are $1, 2, 3, 5, 8, 13, 22, 36, 59, 96$ and are stored in the second column of Table 4.13. In this table, we also compute the $B$-decomposition of $(A_\varphi(e_n(\alpha)))_{n\geq 1}$ for $1 \leq n \leq 10$. By examining the different rows, we conclude that the first terms of the sequence $b(\alpha)$ are $1, 0, 0, 0, 1, -1, 11, -6$.

As in the base-2 case with Lemma 4.22, a rough estimate on the coefficients in $B$-decompositions is enough to ensure a convergence.

**Lemma 4.56.** *For all $n \geq 3$ and all $0 \leq i \leq \ell_F(n)$, we have $|b_i(n)| \leq 6 \cdot 2^i$. In particular, for all $\alpha \in [0, 1)$ and all $i \geq 0$, we have $|b_i(\alpha)| \leq 6 \cdot 2^i$.*

*Proof.* The proof follows the same lines as the proof of Lemma 4.22. Let us take $n = F(\ell) + r$ with $\ell \geq 2$ and $0 \leq r < F(\ell - 1)$. Using Definition 4.47, let us write

$$A_\varphi(n) = \sum_{j=0}^{\ell} b_j(n)\, B(\ell - j),$$

| $n$ | $e_n(\alpha)$ | $b_0$ | $b_1$ | $b_2$ | $b_3$ | $\cdots$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 3 | | | | | | | | | |
| 2 | 2 | 6 | | | | | | | | | |
| 3 | 3 | 1 | −1 | 7 | | | | | | | |
| 4 | 5 | 1 | 0 | −1 | 8 | | | | | | |
| 5 | 8 | 1 | 0 | 0 | −1 | 9 | | | | | |
| 6 | 13 | 1 | 0 | 0 | 0 | −1 | 10 | | | | |
| 7 | 22 | 1 | 0 | 0 | 0 | 1 | −1 | 17 | | | |
| 8 | 36 | 1 | 0 | 0 | 0 | 1 | −1 | −1 | 36 | | |
| 9 | 59 | 1 | 0 | 0 | 0 | 1 | −1 | 11 | −6 | 30 | |
| 10 | 96 | 1 | 0 | 0 | 0 | 1 | −1 | 11 | −6 | −6 | 72 |

Table 4.13: The $B$-decomposition of $A_\varphi(e_n(\alpha))$ for $\alpha = \pi - 3$.

where $b_j(n)$'s are integers, $b_0(n) \neq 0$. Observe that $\ell_F(n) = \ell$ in this case. Let us fix $i \in \{0, 1, \ldots, \ell\}$. By Lemma 4.46, terms of the form

$$\begin{aligned} &A_\varphi(F(\ell - i) + r_1), && \text{with } 0 \leq r_1 < F(\ell - i - 1), \text{ or} \\ &A_\varphi(F(\ell - i + 1) + r_2), && \text{with } 0 \leq r_2 < F(\ell - i), \text{ or} \\ &A_\varphi(F(\ell - i + 2) + r_3), && \text{with } F(\ell - i) \leq r_3 < F(\ell - i + 1), \end{aligned}$$

are the only ones possibly contributing to $b_i(n)$, which is the coefficient of $B(\ell - i)$.

Terms of the first form give either $B(\ell - i)$, or $2B(\ell - i)$, depending on whether $0 \leq r_1 < F(\ell - i - 2)$, or $F(\ell - i - 2) \leq r_1 < F(\ell - i - 1)$ respectively.

Let us focus on terms of the second form. If $F(\ell - i - 1) \leq r_2 < F(\ell - i)$, they give $-B(\ell - i)$ with one application of the lemma. Since we also get $A_\varphi(r_2)$, there is no other contribution to $B(\ell - i)$ in further applications of Lemma 4.46. If $F(\ell - i - 2) \leq r_2 < F(\ell - i - 1)$, a first application of the lemma yields $-B(\ell - i)$ and the term $A_\varphi(F(\ell - i) + r_2)$, which is of the first form. A second application of the lemma then gives $2B(\ell - i)$, and so the final contribution is $B(\ell - i)$. Similarly, if $0 \leq r_2 < F(\ell - i - 2)$, the contributions given by two applications of the lemma cancel each other out.

Like for terms of the second form, terms of the third form need two applications of the lemma because the first application gives $-B(\ell - i)$ and the term

$$2A_\varphi(r_3) = 2A_\varphi(F(\ell - i) + r_3')$$

for some $0 \le r_3' < F(\ell-i-1)$. The final contribution is then either $B(\ell-i)$ if $0 \le r_3' < F(\ell-i-2)$, or $3B(\ell-i)$ if $F(\ell-i-2) \le r_3' < F(\ell-i-1)$.

Aping the proof of Lemma 4.22, iterating Lemma 4.46 on $A_\varphi(F(\ell)+r)$ gives a linear combination of the form

$$\sum_{j=\ell-i+1}^{\ell} y_j B(j) + \sum_{j=0}^{\ell-i+2} x_j A_\varphi(F(j)+r_j') + z_0 A_\varphi(0),$$

where

$$\sum_{j=0}^{\ell-i+2} |x_j| + |z_0| \le 2^i.$$

We conclude by observing that

$$|b_i(n)| \le 2|x_{\ell-i}| + |x_{\ell-i+1}| + 3|x_{\ell-i+2}| \le 6 \cdot 2^i.$$

The particular case follows from the definition of $b(\alpha)$. □

As in the integer base case, we introduce the *relative position* $\operatorname{relpos}_F$. Let $n$ be an integer such that $\operatorname{rep}_F(n) = 10r_1 \cdots r_k$ with $k \ge 1$ and $r_i \in \{0,1\}$ for all $i$. In particular, $n$ belongs to the interval $[F(k+1), F(k+2))$. We define

$$\operatorname{relpos}_F(n) = \sum_{i=1}^{k} \frac{r_i}{\varphi^i} \in [0,1) \quad \text{and} \quad \log_F(n) = |\operatorname{rep}_F(n)| - 1 + \operatorname{relpos}_F(n).$$

Observe that $\lfloor \log_F n \rfloor = |\operatorname{rep}_F(n)| - 1$.

By Definition 4.53, $\operatorname{rep}_F(e_n(\alpha)) = w_n(\alpha)$ is the length-$n$ prefix of the infinite word $10d_\varphi(\alpha)$. If we write $d_\varphi(\alpha) = (d_i)_{i\ge 0}$, then $w_n(\alpha) = 10d_1 \cdots d_{n-2}$ for all $n \ge 2$, and so

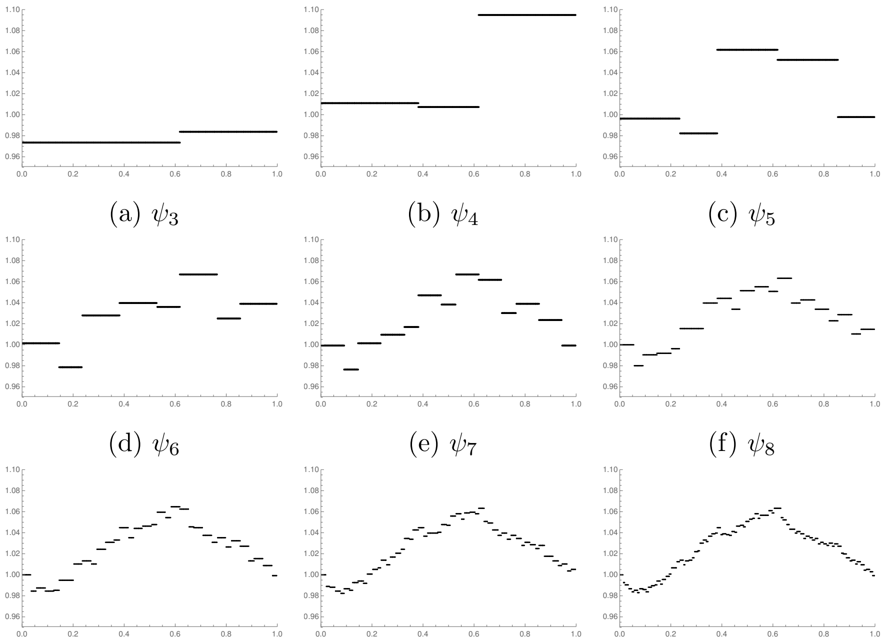
$$\lim_{n\to+\infty} \operatorname{relpos}_F(e_n(\alpha)) = \lim_{n\to+\infty} \sum_{i=1}^{n-2} \frac{d_i}{\varphi^i} = \alpha. \tag{4.24}$$

In particular, $\alpha$ gives the relative position of the integer $e_n(\alpha)$ in the interval $[F(n-1), F(n))$, as claimed at the beginning of this section.

Inspired by the integer base strategy, the technique to prove Theorem 4.43 is to make use of an auxiliary function $\Psi(\alpha)$, for $\alpha \in [0,1)$, defined as the limit of a converging sequence built on the $B$-decomposition of $A_\varphi(e_n(\alpha))$. For all $n \ge 1$, let $\psi_n$ be the step function defined by

$$\psi_n(\alpha) = \frac{A_\varphi(e_n(\alpha))}{c\,\lambda^{\log_F(e_n(\alpha))}} \quad \text{for } \alpha \in [0,1),$$

where $\lambda$ and $c$ come from (4.19). We have depicted the first functions $\psi_3, \dots, \psi_{11}$ in Figure 4.14. Similarly to the base-2 case, the number of steps in $\psi_n(\alpha)$ is given by the number of length-$n$ words over $\{0,1\}$ starting with 10 and avoiding the factor 11. For instance, $\psi_3$ is a step function built on two subintervals because $\mathrm{rep}_F(e_3(\alpha)) = w_3(\alpha)$ can only have two forms: 100 and 101. In general, $w_n(\alpha)$ can only have $F(n-2)$ distinct forms.



(a) $\psi_3$ (b) $\psi_4$ (c) $\psi_5$

(d) $\psi_6$ (e) $\psi_7$ (f) $\psi_8$

(g) $\psi_9$ (h) $\psi_{10}$ (i) $\psi_{11}$

Figure 4.14: Representation of $\psi_3, \dots, \psi_{11}$ in $[0,1]$.

Here is the analogue of Proposition 4.20.

**Proposition 4.57.** *The sequence $(\psi_n)_{n\geq 1}$ uniformly converges to the function $\Psi$ defined for $\alpha \in [0,1)$ by*

$$\Psi(\alpha) = \frac{1}{\lambda^\alpha} \sum_{i=0}^{+\infty} \frac{b_i(\alpha)}{\lambda^i}.$$

*Proof.* By Definition 4.54, the $B$-decomposition of $A_\varphi(e_n(\alpha))$ is equal to

$$A_\varphi(e_n(\alpha)) = \sum_{i=0}^{n-1} b_i(e_n(\alpha))\, B(n-1-i).$$

We know that $\log_F(e_n(\alpha)) = n - 1 + \text{relpos}_F(e_n(\alpha))$, so we have

$$\psi_n(\alpha) = \frac{A_\varphi(e_n(\alpha))}{c\lambda^{\log_F(e_n(\alpha))}} = \frac{1}{\lambda^{\text{relpos}_F(e_n(\alpha))}} \sum_{i=0}^{n-1} b_i(e_n(\alpha)) \frac{B(n-1-i)}{c\lambda^{n-1}}. \quad (4.25)$$

Firstly, the sum is converging when $n$ tends to infinity to the convergent series

$$\sum_{i=0}^{+\infty} \frac{b_i(\alpha)}{\lambda^i}.$$

Indeed, the sequence $(\mathsf{Bdec}(A_\varphi(e_n(\alpha))))_{n\geq 1}$ of finite words converges to the infinite word $b(\alpha)$ thanks to Lemma 4.50 (see also Definition 4.54). Moreover, due to Lemma 4.56 and the equality (4.19) or (4.20), the sequence of partial sums uniformly converges to the series.

Secondly, the sequence $(\text{relpos}_F(e_n(\alpha)))_{n\geq 1}$ of functions is uniformly convergent. Indeed, if $d_\varphi(\alpha) = d_1d_2d_3\cdots$ is the $\varphi$-expansion of $\alpha$, then we particularly know that, for all $j \geq 1$,

$$\sum_{i\geq j} d_i\varphi^{-i} < \varphi^{-j+1}$$

from Definition 1.19. The definition of the relative position gives

$$|\text{relpos}_F(e_n(\alpha)) - \alpha| = \left|\sum_{i=1}^{n-2} \frac{d_i}{\varphi^i} - \sum_{i=1}^{+\infty} \frac{d_i}{\varphi^i}\right| < \frac{1}{\varphi^{n-2}} \quad (4.26)$$

for all $\alpha \in [0, 1)$.

To conclude the proof, we use the same reasoning as in the proof of Proposition 4.20. Using (4.25), we have

$$\begin{aligned}|\psi_n(\alpha) - \Psi(\alpha)| \leq & \left|\frac{1}{\lambda^{\text{relpos}_F(e_n(\alpha))}}\right| \cdot \left|\sum_{i=0}^{n-1} b_i(e_n(\alpha))\frac{B(n-1-i)}{c\lambda^{n-1}} - \sum_{i=0}^{+\infty} \frac{b_i(\alpha)}{\lambda^i}\right| \\ & + \left|\sum_{i=0}^{+\infty} \frac{b_i(\alpha)}{\lambda^i}\right| \cdot \left|\frac{1}{\lambda^{\text{relpos}_F(e_n(\alpha))}} - \frac{1}{\lambda^\alpha}\right|.\end{aligned}$$

Now let $\epsilon > 0$. Then $|\psi_n(\alpha) - \Psi(\alpha)| < \epsilon$ holds for all $\alpha \in [0, 1)$ and $n$ large enough. □

Instead of considering rational numbers of the form $r/b^k$ as in the integer base case, we use the set

$$D = \left\{\sum_{i=1}^{k} \frac{r_i}{\varphi^i} \mid k \geq 1,\ r_1 \cdots r_k \in \{1, \varepsilon\}\{0, 01\}^*, r_i \in \{0, 1\}\right\},$$

which is dense in $[0, 1]$. As an example, in Figure 4.15, we have only considered the suitable words $r_1 \cdots r_k$ of length 8. The next result makes explicit the values taken by $\Psi$ on the set $D$ (see Lemma 4.24 and the equality (4.18) for counterparts in integer bases).

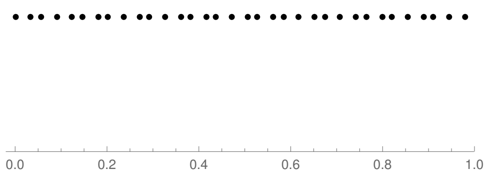



Figure 4.15: An estimation of the set $D$ for length-8 words.

**Lemma 4.58.** *Let $r_1 \cdots r_k \in \{1, \varepsilon\}\{0, 01\}^*$ with $k \geq 1$, $r_i \in \{0, 1\}$ for all $1 \leq i \leq k$, and let $\alpha = \sum_{i=1}^{k} r_i/\varphi^i$. We have*

$$\Psi(\alpha) = \sum_{i=0}^{k-1} \frac{b_i(m)}{\lambda^{i+\alpha}} + \frac{b_k(\alpha)}{\lambda^{k+\alpha}} + \frac{b_{k+1}(\alpha)}{\lambda^{k+1+\alpha}},$$

*where $m = \mathrm{val}_F(10r_1 \cdots r_k)$ and $b_0(m) \cdots b_{k+1}(m)$ is the $B$-decomposition of $A_\varphi(m)$.*

*Proof.* By hypothesis, $\mathrm{rep}_F(m) = 10r_1 \cdots r_k$ and $10d_\varphi(\alpha) = 10r_1 \cdots r_k 0^\omega$. By Definition 4.53, $w_n(\alpha)$ is the length-$n$ prefix of the latter infinite word. For large enough $n$, $\mathrm{rep}_F(m)$ and $w_n(\alpha)$ have a common prefix of length $k+2$, namely $10r_1 \cdots r_k$. Due to Lemma 4.50, $\mathsf{Bdec}(A_\varphi(e_n(\alpha)))$ has thus a prefix equal to $b_0(m) \cdots b_{k-1}(m)$. More precisely, in the view of Definition 4.54, it is of the form

$$b_0(m) \cdots b_{k-1}(m)\, b_k(e_n(\alpha))\, b_{k+1}(e_n(\alpha))\, 0^{n-k-4}\, b_{n-2}(e_n(\alpha))\, b_{n-1}(e_n(\alpha)).$$

This is again a consequence of Lemma 4.46. Applying recursively this lemma to $A_\varphi(e_n(\alpha))$, we will be left with the evaluation of $A_\varphi(F(n-k-2))$. Indeed, thanks to Remark 4.49, one has to progressively delete letters in the word $w_n(\alpha) = 10r_1 \cdots r_k 0^{n-k-2}$ to finally reach $10^{n-k-2}$. Now, as in the proof of

Lemma 4.52 with (4.22), we have

$$A_\varphi(F(n-k-2)) = B(n-k-2) - B(1) + (n-k+3)B(0),$$

which explains the block of zeroes. Also observe that $b_{n-2}(e_n(\alpha))$ (resp., $b_{n-1}(e_n(\alpha))$) might be different from $-1$ (resp., $n-k+3$) because other contributions to $B(1)$ (resp., $B(0)$) might come into play.

Due to Proposition 4.57, we know that

$$\Psi(\alpha) = \lim_{n\to+\infty} \psi_n(\alpha) = \lim_{n\to+\infty} \frac{A_\varphi(e_n(\alpha))}{c\,\lambda^{\log_F(e_n(\alpha))}},$$

so splitting the $B$-decomposition of $A_\varphi(e_n(\alpha))$ yields

$$\begin{aligned}
\Psi(\alpha) = &\lim_{n\to+\infty} \frac{1}{c\,\lambda^{\log_F(e_n(\alpha))}} \left(\sum_{i=0}^{k-1} b_i(m) B(n-1-i)\right)\\
&+ \lim_{n\to+\infty} \frac{1}{c\,\lambda^{\log_F(e_n(\alpha))}}\, b_k(e_n(\alpha))B(n-1-k)\\
&+ \lim_{n\to+\infty} \frac{1}{c\,\lambda^{\log_F(e_n(\alpha))}}\, b_{k+1}(e_n(\alpha))B(n-2-k)\\
&+ \lim_{n\to+\infty} \frac{1}{c\,\lambda^{\log_F(e_n(\alpha))}}\, b_{n-2}(e_n(\alpha))B(1)\\
&+ \lim_{n\to+\infty} \frac{1}{c\,\lambda^{\log_F(e_n(\alpha))}}\, b_{n-1}(e_n(\alpha))B(0).
\end{aligned}$$

Now we analyze each term of the right-hand side of the previous equality. Recall that $\log_F(e_n(\alpha)) = n - 1 + \mathrm{relpos}_F(e_n(\alpha))$. Using (4.19) or (4.20), and (4.24), we get

$$\lim_{n\to+\infty} \frac{B(n-1-i)}{c\,\lambda^{\log_F(e_n(\alpha))}} = \lim_{n\to+\infty} \frac{B(n-1-i)}{c\,\lambda^{n-1}} \frac{1}{\lambda^{\mathrm{relpos}_F(e_n(\alpha))}} = \frac{1}{\lambda^{i+\alpha}},$$

in turn giving

$$\lim_{n\to+\infty} \frac{1}{c\,\lambda^{\log_F(e_n(\alpha))}} \left(\sum_{i=0}^{k-1} b_i(m) B(n-1-i)\right) = \sum_{i=0}^{k-1} \frac{b_i(m)}{\lambda^{i+\alpha}}.$$

Similarly, for $j \in \{1,2\}$, we find

$$\begin{aligned}
\lim_{n\to+\infty} \frac{b_{k+j-1}(e_n(\alpha))B(n-j-k)}{c\,\lambda^{\log_F(e_n(\alpha))}} &= \lim_{n\to+\infty} \frac{B(n-j-k)}{c\,\lambda^{n-1}} \frac{b_{k+j-1}(e_n(\alpha))}{\lambda^{\mathrm{relpos}_F(e_n(\alpha))}}\\
&= \frac{b_{k+j-1}(\alpha)}{\lambda^{k+j-1+\alpha}}
\end{aligned}$$

with the additional help of Definition 4.54. Now, it remains to show that

$$\lim_{n\to+\infty} \frac{b_{n-2}(e_n(\alpha))B(1)}{c\,\lambda^{\log_F(e_n(\alpha))}} = 0 \quad \text{and} \quad \lim_{n\to+\infty} \frac{b_{n-1}(e_n(\alpha))B(0)}{c\,\lambda^{\log_F(e_n(\alpha))}} = 0,$$

which both hold since $|b_{n-2}(e_n(\alpha))| \leq 6 \cdot 2^{n-2}$ and $|b_{n-1}(e_n(\alpha))| \leq 6 \cdot 2^{n-1}$ by Lemma 4.56. □

In the remaining of the section, we prove the following result, which is an equivalent version of Theorem 4.43.

**Theorem 4.59.** *The function* $\Psi$ *defined in Proposition 4.57 is continuous on* $[0, 1)$ *such that* $\Psi(0) = 1$ *and* $\lim_{\alpha\to 1^-} \Psi(\alpha) = 1$. *The sequence* $(A_\varphi(n))_{n\geq 0}$ *satisfies, for* $n \geq 3$,

$$A_\varphi(n) = c\,\lambda^{\log_F n}\Psi(\text{relpos}_F(n)) + o(\lambda^{\lfloor \log_F n \rfloor}),$$

*where* $\lambda$ *is the dominant root of* $P_B(X) = X^3 - 2X^2 - X + 1$.

A representation of $\Psi$ is given in Figure 4.16. It has been obtained by estimating $A_\varphi(n)/(c\,\lambda^{\log_F n})$ for $F(16) = 2584 \leq n \leq 4180 = F(17)$.

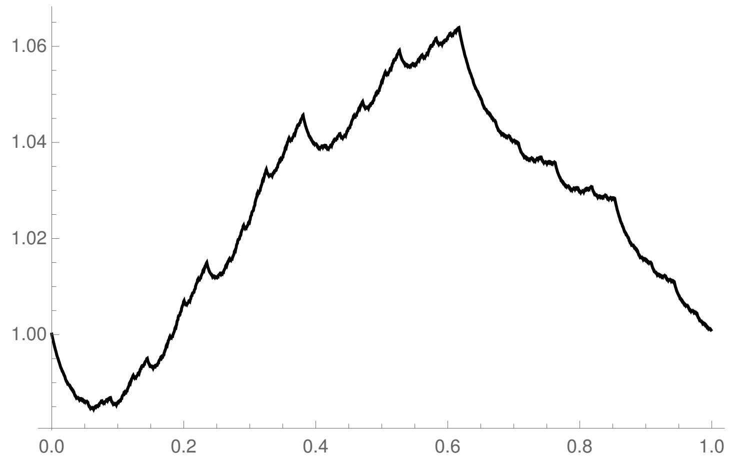



Figure 4.16: The graph of $\Psi$.

*Proof of Theorem 4.59.* This proof is divided into four parts: the error term for the sequence $(A_\varphi(n))_{n\geq 0}$, the fact that $\Psi(0) = 1$, the computation of the limit $\lim_{\alpha\to 1^-} \Psi(\alpha) = 1$, and the continuity of the function $\Psi$.

**Error term.** We first focus on the error term. Let $\mathrm{rep}_F(n) = 10r_1 \cdots r_k$ with $k \geq 1$ and $r_1 \cdots r_k \in \{1, \varepsilon\}\{0, 01\}^*$. In this case, observe that $k$ depends on $n$ since $k + 2 = |\,\mathrm{rep}_F(n)| = \lfloor \log_F n \rfloor + 1$. By definition of the relative position, we have

$$\mathrm{relpos}_F(n) = \sum_{i=1}^{k} \frac{r_i}{\varphi^i}.$$

On the one hand, Lemma 4.58 (with $\alpha = \mathrm{relpos}_F(n)$ and $m = n$) gives

$$\begin{aligned} c\,\lambda^{\log_F n}\Psi(\mathrm{relpos}_F(n)) =& c\,\lambda^{(k+2)-1+\mathrm{relpos}_F(n)}\Psi(\mathrm{relpos}_F(n)) \\ =& c\sum_{i=0}^{k-1} b_i(n)\,\lambda^{k+1-i} + c\,\lambda\, b_k(\mathrm{relpos}_F(n)) \\ &+ c\, b_{k+1}(\mathrm{relpos}_F(n)). \end{aligned}$$

On the other hand, since $\ell_F(n) = k + 1$, we know from Definition 4.47 that

$$A_\varphi(n) = \sum_{i=0}^{k+1} b_i(n) B(k + 1 - i).$$

Thus, the error term $R(n) = A_\varphi(n) - c\,\lambda^{\log_F n}\Psi(\mathrm{relpos}_F(n))$ is equal to

$$\begin{aligned} R(n) \quad = \quad & \sum_{i=0}^{k-1} b_i(n) \left(B(k + 1 - i) - c\,\lambda^{k+1-i}\right) \\ & + (b_k(n)B(1) - c\,\lambda\, b_k(\mathrm{relpos}_F(n))) \\ & + (b_{k+1}(n)B(0) - c\, b_{k+1}(\mathrm{relpos}_F(n)))\,. \end{aligned}$$

Using (4.19), $B(k + 1 - i) - c\,\lambda^{k+1-i} = c_2\lambda_2^{k+1-i} + c_3\lambda_3^{k+1-i}$. Dividing $R(n)$ by $\lambda^{\lfloor \log_F n \rfloor} = \lambda^{k+1}$ and recalling that $B(0) = 1$, we get

$$\begin{aligned} \frac{R(n)}{\lambda^{k+1}} \quad = \quad & \sum_{i=0}^{k-1} \frac{b_i(n)}{\lambda^i}\ \frac{c_2\lambda_2^{k+1-i} + c_3\lambda_3^{k+1-i}}{\lambda^{k+1-i}} \\ & + \frac{b_k(n)B(1) - c\,\lambda\, b_k(\mathrm{relpos}_F(n))}{\lambda^{k+1}} \\ & + \frac{b_{k+1}(n) - c\, b_{k+1}(\mathrm{relpos}_F(n))}{\lambda^{k+1}}. \end{aligned}$$

Firstly, by Definition 4.44, we have

$$\frac{|c_2\lambda_2^{k+1-i} + c_3\lambda_3^{k+1-i}|}{\lambda^{k+1-i}} \leq 2|c_2| \left(\frac{|\lambda_2|}{\lambda}\right)^{k+1-i},$$

and from Lemma 4.56,

$$\frac{|b_i(n)|}{\lambda^i} \leq 6\left(\frac{2}{\lambda}\right)^i.$$

Secondly, since $B(1) = 3$ and by Lemma 4.56, we also have

$$\begin{aligned}\frac{|b_k(n)B(1) - c\,\lambda\, b_k(\mathrm{relpos}_F(n))|}{\lambda^{k+1}} &\leq \frac{9 \cdot 2^{k+1} + c\,\lambda\, 3 \cdot 2^{k+1}}{\lambda^{k+1}} \\ &\leq 3\,(3 + c\lambda)\left(\frac{2}{\lambda}\right)^{k+1},\end{aligned}$$

and

$$\begin{aligned}\frac{|b_{k+1}(n) - c\, b_{k+1}(\mathrm{relpos}_F(n))|}{\lambda^{k+1}} &\leq \frac{3 \cdot 2^{k+2} + c\, 3 \cdot 2^{k+2}}{\lambda^{k+1}} \\ &\leq 3\,(1 + c)\,\lambda\left(\frac{2}{\lambda}\right)^{k+2}.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\frac{|R(n)|}{\lambda^{k+1}} \leq \frac{12|c_2|\,|\lambda_2|^2}{\lambda^2} \sum_{i=0}^{k-1} \left(\frac{2}{\lambda}\right)^i \left(\frac{|\lambda_2|}{\lambda}\right)^{k-1-i} + 3\,(3 + c\lambda)\left(\frac{2}{\lambda}\right)^{k+1} \\ + 3\,(1 + c)\,\lambda\left(\frac{2}{\lambda}\right)^{k+2}.\end{aligned}$$

Since the Cauchy product $\sum_{i=0}^{k-1} a^i b^{k-1-i}$ is equal to $(a^k - b^k)/(a - b)$, we deduce that

$$\begin{aligned}\frac{|R(n)|}{\lambda^{k+1}} \leq \frac{12|c_2|\,|\lambda_2|^2}{\lambda(2 - |\lambda_2|)} \left(\left(\frac{2}{\lambda}\right)^k - \left(\frac{|\lambda_2|}{\lambda}\right)^k\right) + 3\,(3 + c\lambda)\left(\frac{2}{\lambda}\right)^{k+1} \\ + 3\,(1 + c)\,\lambda\left(\frac{2}{\lambda}\right)^{k+2}.\end{aligned}$$

Consequently, $|R(n)|/\lambda^{k+1}$ tends to zero when $k$ goes to infinity since we know that $\lambda > 2 > |\lambda_2|$. This implies that $R(n) = o(\lambda^{k+1})$.

**Value of** $\Psi(0)$**.** We show that $\Psi(0) = 1$. By definition, $w_n(0)$ is the length-$n$ prefix of the infinite word $10d_\varphi(0) = 10^\omega$, and $e_n(0)$ is thus equal to $F(n-1)$. In this case, the relative position $\mathrm{relpos}_F(e_n(0))$ is 0. By Proposition 4.57 that defines $\Psi$ and using (4.22) and (4.20), we have

$$\begin{aligned}\Psi(0) = \lim_{n\to+\infty} \psi_n(0) &= \lim_{n\to+\infty} \frac{A_\varphi(F(n-1))}{c\,\lambda^{n-1}} \\ &= \lim_{n\to+\infty} \frac{B(n-1) - B(1) + (n+4)\,B(0)}{c\,\lambda^{n-1}} = 1.\end{aligned}$$

**Limit for $1^-$.** To show that $\lim_{\alpha\to 1^-}\Psi(\alpha)=1$, we make use of the uniform convergence in Proposition 4.57. By definition, recall that $\log_F(e_n(\alpha))$ is equal to $n-1+\mathrm{relpos}_F(e_n(\alpha))$ and $\mathrm{relpos}_F(e_n(\alpha))\to\alpha$ with $n\to+\infty$ (see (4.24)), so we have

$$\begin{aligned}
\lim_{\alpha\to 1^-}\Psi(\alpha) &= \lim_{\alpha\to 1^-}\lim_{n\to+\infty}\psi_n(\alpha)\\
&= \lim_{\alpha\to 1^-}\lim_{n\to+\infty}\frac{1}{\lambda^{\mathrm{relpos}_F(e_n(\alpha))}}\,\frac{A_\varphi(e_n(\alpha))}{c\,\lambda^{n-1}}\\
&= \lim_{\alpha\to 1^-}\frac{1}{\lambda^\alpha}\lim_{n\to+\infty}\frac{A_\varphi(e_n(\alpha))}{c\,\lambda^{n-1}}\\
&= \frac{1}{\lambda}\lim_{\alpha\to 1^-}\lim_{n\to+\infty}\frac{A_\varphi(e_n(\alpha))}{c\,\lambda^{n-1}}\\
&= \frac{1}{\lambda}\lim_{n\to+\infty}\lim_{\alpha\to 1^-}\frac{A_\varphi(e_n(\alpha))}{c\,\lambda^{n-1}}.
\end{aligned}$$

Recall that $d_\varphi(1)=110^\omega$. For any fixed integer $n\geq 3$, we can chose $\alpha\in[0,1)$ close enough to 1 such that

$$d_\varphi(\alpha)\in(10)^n\{0,1\}^\omega.$$

Since $w_n(\alpha)$ is the length-$n$ prefix of $10d_\varphi(\alpha)$, which is also the length-$n$ prefix of $d^*_\varphi(1)=(10)^\omega$ (see Example 1.21), we find $e_n(\alpha)=F(n)-1$. Using (4.23), we have

$$\begin{aligned}
A_\varphi(e_n(\alpha)) &= A_\varphi(F(n)-1)\\
&= \sum_{i=0}^{n-3} g_i B(n-1-i)+b_{n-2}(e_n(\alpha))B(1)+b_{n-1}(e_n(\alpha))B(0).
\end{aligned}$$

Due to Lemma 4.56, both $b_{n-1}(e_n(\alpha))$ and $b_{n-2}(e_n(\alpha))$ are smaller than $3\cdot 2^n$, which yields

$$\lim_{n\to+\infty}\frac{b_{n-2}(e_n(\alpha))B(1)+b_{n-1}(e_n(\alpha))B(0)}{c\lambda^{n-1}}=0.$$

To complete the proof, our aim is thus to show that

$$\lim_{n\to+\infty}\frac{1}{c\,\lambda^{n-1}}\sum_{i=0}^{n-3} g_i B(n-1-i)=\lim_{n\to+\infty}\sum_{i=0}^{n-3}\frac{g_i}{\lambda^i}\,\frac{B(n-1-i)}{c\lambda^{n-1-i}}=\lambda.$$

By Lemma 4.52, we have

$$\left|\sum_{i=0}^{n-3}\frac{g_i}{\lambda^i}\,\frac{B(n-1-i)}{c\lambda^{n-1-i}}-\lambda\right| \leq \left|\sum_{i=0}^{n-3}\frac{g_i}{\lambda^i}\left(\frac{B(n-1-i)}{c\lambda^{n-1-i}}-1\right)\right|+\left|\sum_{i=n-2}^{+\infty}\frac{g_i}{\lambda^i}\right|.$$

By Lemma 4.52 again, we know that $|g_k| \leq 2(\sqrt{2})^k$ for all $k \geq 0$, which shows that the second term of the right-hand side of the previous inequality goes to 0 as $n$ goes to infinity. Using (4.19), we have

$$\left| \frac{B(n-1-i)}{c\lambda^{n-1-i}} - 1 \right| \leq \frac{2|c_2||\lambda_2|^{n-1-i}}{c\lambda^{n-1-i}},$$

and thus

$$\begin{aligned} \left| \sum_{i=0}^{n-3} \frac{g_i}{\lambda^i} \left( \frac{B(n-1-i)}{c\lambda^{n-1-i}} - 1 \right) \right| &\leq \frac{4|c_2||\lambda_2|^{n-1}}{c\lambda^{n-1}} \sum_{i=0}^{n-3} \left( \frac{\sqrt{2}}{|\lambda_2|} \right)^i \\ &\leq \frac{4|c_2||\lambda_2|^2}{c(\sqrt{2}-|\lambda_2|)} \frac{(\sqrt{2})^{n-2} - |\lambda_2|^{n-2}}{\lambda^{n-1}}, \end{aligned}$$

which also tends to 0 as $n$ tends to infinity. All in all, we have just shown that $\lim_{\alpha \to 1^-} \Psi(\alpha) = 1$, as expected.

**Continuity.** To finish the proof, let us show that $\Psi$ is continuous. Let $\alpha \in [0, 1)$, and let us consider its $\varphi$-expansion $d_\varphi(\alpha) = (d_i)_{i \geq 1}$. We make use of the uniform convergence of the sequence $(\psi_n)_{n \geq 0}$ in Proposition 4.57, and we write

$$\begin{aligned} \lim_{\gamma \to \alpha} |\Psi(\gamma) - \Psi(\alpha)| &= \lim_{\gamma \to \alpha} \lim_{n \to +\infty} |\psi_n(\gamma) - \psi_n(\alpha)| \\ &= \lim_{n \to +\infty} \lim_{\gamma \to \alpha} |\psi_n(\gamma) - \psi_n(\alpha)|. \end{aligned}$$

First, assume that $\alpha$ is not of the form $\sum_{i=1}^{k} r_i/\varphi^i$ where the letters $r_i$ are not all 0, *i.e.*, $(d_i)_{i \geq 1}$ does not belong to $\{0,1\}^* 10^\omega$ (note that this case includes $\alpha = 0$). For any fixed integer $n$, we can chose $\gamma_n$ close enough to $\alpha$ such that $d_\varphi(\gamma_n) \in d_1 d_2 \cdots d_n \{0,1\}^\omega$. Therefore, we have $w_n(\gamma_n) = w_n(\alpha)$, hence $e_n(\gamma_n) = e_n(\alpha)$. Thus, $\psi_n(\gamma_n) = \psi_n(\alpha)$, and $\lim_{\gamma \to \alpha} |\Psi(\gamma) - \Psi(\alpha)| = 0$.

Now suppose that $d_\varphi(\alpha) = d_1 d_2 \cdots d_k 0^\omega$ with $d_k = 1$. For any fixed integer $n > k + 1$, we can take $\gamma_n$ close enough to $\alpha$ such that

$$d_\varphi(\gamma_n) \in \begin{cases} d_1 d_2 \cdots d_k 0^n \{0,1\}^\omega, & \text{if } \gamma_n \geq \alpha; \\ d_1 d_2 \cdots d_{k-1} (01)^n \{0,1\}^\omega, & \text{if } \gamma_n < \alpha. \end{cases}$$

If $\gamma_n \geq \alpha$, we get $\psi_n(\gamma_n) = \psi_n(\alpha)$ as in the first case, and the conclusion is similar. If $\gamma_n < \alpha$, we get

$$\begin{aligned} e_n(\alpha) &= \mathrm{val}_F(10 d_1 d_2 \cdots d_k 0^{n-k-2}), \\ e_n(\gamma_n) &= \mathrm{val}_F(10 d_1 d_2 \cdots d_{k-1} (01)^{\frac{n-k-1}{2}}) \end{aligned}$$

(recall that fractional powers of words are defined in Definition 1.3 in Chapter 1). In this case, we get $e_n(\alpha) = e_n(\gamma_n) + 1$, and

$$\begin{aligned} |\psi_n(\alpha) - \psi_n(\gamma_n)| &= \left| \frac{A_\varphi(e_n(\alpha))}{c\,\lambda^{\log_F(e_n(\alpha))}} - \frac{A_\varphi(e_n(\gamma_n))}{c\,\lambda^{\log_F(e_n(\gamma_n))}} \right| \\ &\leq \left| \frac{A_\varphi(e_n(\alpha))}{c\lambda^{n-1}} \right| \left| \frac{1}{\lambda^{\operatorname{relpos}_F(e_n(\alpha))}} - \frac{1}{\lambda^{\operatorname{relpos}_F(e_n(\gamma_n))}} \right| \\ &\quad + \left| \frac{1}{c\lambda^{\log_F(e_n(\gamma_n))}} \right| |A_\varphi(e_n(\alpha)) - A_\varphi(e_n(\gamma_n))| \,. \end{aligned}$$

Let us now bound each term. For the first term, the special form of $\gamma_n$ leads to $|\alpha - \gamma_n| < 1/\varphi^{n-2}$. Using (4.26), we find

$$|\operatorname{relpos}_F(e_n(\alpha)) - \operatorname{relpos}_F(e_n(\gamma_n))| < 3/\varphi^{n-2}.$$

By continuity,

$$\lim_{n\to+\infty} \left| \frac{1}{\lambda^{\operatorname{relpos}_F(e_n(\alpha))}} - \frac{1}{\lambda^{\operatorname{relpos}_F(e_n(\gamma_n))}} \right| = 0.$$

We claim that $A_\varphi(e_n(\alpha))/(c\lambda^{n-1})$ converges to some real number when $n$ goes to infinity. In fact, it follows from Proposition 4.57 and equality (4.24) because

$$\lim_{n\to+\infty} \frac{A_\varphi(e_n(\alpha))}{c\lambda^{n-1}} = \lim_{n\to+\infty} \lambda^{\operatorname{relpos}_F(e_n(\alpha))} \frac{A_\varphi(e_n(\alpha))}{c\lambda^{n-1+\operatorname{relpos}_F(e_n(\alpha))}} = \lambda^\alpha \Psi(\alpha).$$

Consequently, the first term tends to zero when $n$ increases. The second term also tends to zero as, by Corollary 3.67,

$$\begin{aligned} A_\varphi(e_n(\alpha)) - A_\varphi(e_n(\gamma_n)) &= A_\varphi(e_n(\alpha)) - A_\varphi(e_n(\alpha) - 1) \\ &= S_\varphi(e_n(\alpha)) \leq 2^n, \end{aligned}$$

and $c\lambda^{\log_F(e_n(\gamma_n))} \geq c\lambda^{n-1}$. This shows that $\Psi$ is continuous. □

**Remark 4.60.** Similarly to the open questions left in Remarks 4.26 and 4.40, are $\mathcal{G}$ and $\Psi$ nowhere differentiable?

## 4.4 Perspectives

The reader can wonder whether the method presented in this chapter can be applied to classical digital sequences. Consider the example of the sum-of-digits function $s_2$ for base-2 expansions of integers mentioned in the introduction. Its summatory function $(A(n))_{n\geq 0}$ (see `A000788` in [Slo]) defined, for all $n \geq 0$, by

$$A(n) = \sum_{j=0}^{n-1} s_2(j)$$

verifies for all $\ell \geq 0$ and for all $0 \leq r < 2^\ell$,

$$A(2^\ell + r) = U(\ell) + A(r) + r \cdot U(1), \tag{4.27}$$

where $U(0) = 0$, $U(1) = 1$, and the sequence $U = (U(n))_{n \geq 0}$ satisfies the linear recurrence relation

$$U(n+2) = 4\,U(n+1) - 4\,U(n) \text{ for all } n \geq 0.$$

Indeed, let us show that (4.27) holds. First, it is not difficult to show that

$$U(n) = 2^{n-1} n$$

for all $n \geq 0$. The characteristic polynomial of the recurrence relation satisfied by $(U(n))_{n \geq 0}$ has 2 as double root. Thus, there exist constants $c_1, c_2 \in \mathbb{C}$ such that $U(n) = (c_1 n + c_2)\, 2^n$ for all $n \geq 0$. The values of $c_1$ and $c_2$ can be determined by solving the system of equations

$$\begin{cases} U(0) & = & (c_1 0 + c_2)\, 2^0 \\ U(1) & = & (c_1 1 + c_2)\, 2^1, \end{cases}$$

which leads to the expected result. Then a result of [AS03a, Section 3.2] states that

$$A(2^n) = \frac{2-1}{2}\, 2^n n = U(n)$$

for all $n \geq 0$. Now, let $\ell \geq 0$ and $0 \leq r < 2^\ell$. If $r = 0$, the desired result (4.27) holds. Otherwise, we find

$$\begin{aligned} A(2^\ell + r) &= \sum_{j=0}^{2^\ell - 1} s_2(j) + \sum_{j=0}^{r-1} s_2(2^\ell + j) \\ &= A(2^\ell) + \sum_{j=0}^{r-1} s_2(2^\ell + j) \\ &= U(\ell) + \sum_{j=0}^{r-1} (s_2(j) + 1) \\ &= U(\ell) + A(r) + r \cdot U(1) \end{aligned}$$

since $U(1) = 1$. The next step of our method is to use (4.27) in order to obtain $U$-decompositions of the sequence $(A(n))_{n\geq 0}$. For instance, we obtain

$$\begin{aligned}A(7) &= A(2^2+3) = U(2) + A(3) + 3U(1)\\ &= U(2) + A(2^1+1) + 3U(1) = U(2) + (U(1) + A(1) + 1\cdot U(1)) + 3U(1)\\ &= U(2) + A(2^0+0) + 5U(1) = U(2) + (U(0) + A(0) + 0\cdot U(1)) + 5U(1)\\ &= U(2) + 5U(1) + U(0),\end{aligned}$$

so the $U$-decomposition of $A(7)$ would be $(1, 5, 1)$ (also notice that we have $A(7) = 9 = 1\cdot 4 + 5\cdot 1 + 1\cdot 0$). Roughly, our method implies that there exists a continuous and periodic function $\mathcal{F}$ of period 1 such that $A(n)$ behaves like

$$\frac{1}{2}\, n\ \log_2 n\ \mathcal{F}(\log_2 n).$$

One has to view the dominant component $(x2^x)/2$ of the second base $U$ as a function of $x$, and evaluate it at $\log_2(n)$, which is the logarithm in the first base 2. To draw a parallel with the Fibonacci case, $x \mapsto c\,\lambda^x$ would be the dominant component function for the sequence $B$ (acting as the second base), which is evaluated at $\log_F(n)$ (where $F$ is the first base). The function $\mathcal{F}$ is depicted in Figure 4.17. Numerical experiments then suggest that our method gives a result similar to (4.1).

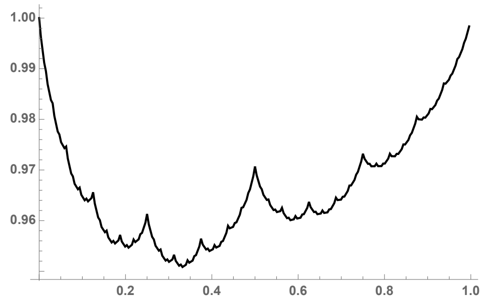
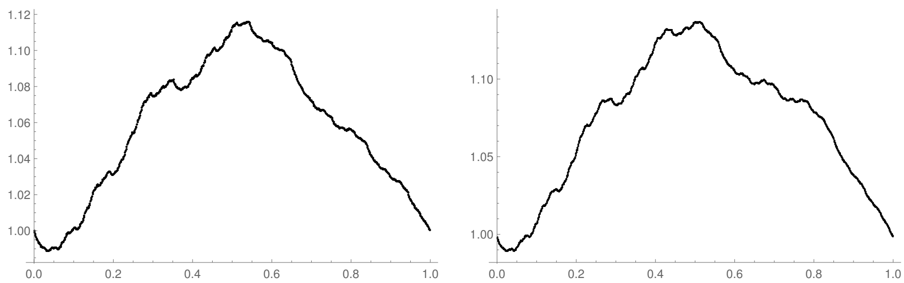



Figure 4.17: The graph of $\mathcal{F}$ over one period.

Other examples can be considered with sequences defined analogously to $(S_b(n))_{n\geq 0}$ and $(S_\varphi(n))_{n\geq 0}$, *i.e.*, sequences associated with binomial coefficients of representations of integers in some numeration system. The main problem is that we do not have a statement similar to Propositions 3.15, 3.29 and 3.66. If we leave the $b$-regular setting and try to replace the Fibonacci sequence with another linear recurrent sequence, the situation seems to be more intricate. For the Tribonacci numeration system (see Remark 3.57), we conjecture that a result similar to Theorem 4.43 should hold for the corresponding summatory function $A_{\beta_T}$. Computing the first values of $A_{\beta_T}(T(n))$, the sequence $(B(n))_{n\geq 0}$ should be replaced by the sequence $(V(n))_{n\geq 0}$ satisfying, for all $n \geq 0$,

$$V(n+5) = 3V(n+4) - V(n+3) + V(n+2) - 2V(n+1) + 2V(n),$$

with initial conditions $1, 3, 9, 23, 63$ (see A282732 in [Slo]). The dominant root $\lambda_T$ of the characteristic polynomial of the recurrence is close to 2.703. There should exist a continuous and periodic function $\mathcal{G}_T$ of period 1 whose




(a) Tribonacci case. (b) Quadribonacci case.

Figure 4.18: The conjectured functions $\mathcal{G}_T$ and $\mathcal{G}_Q$ over one period.

graph is depicted in Figure 4.18a such that the corresponding summatory function has a main term in $c_T\, \lambda_T^{\log_T(n)} \mathcal{G}_T(\log_T(n))$, where the definition of $\log_T$ is straightforward. We are also able to handle the same computations with the Quadribonacci numeration system where the factor $1^4$ is avoided. In that case, the analogue of the sequence $(B(n))_{n\geq 0}$ should be a linear recurrent sequence of order 6 whose characteristic polynomial is

$$X^7 - 4X^6 + 4X^5 - 2X^4 - X^3 + 3X^2 - 6X + 2.$$

Again, we conjecture a similar behavior involving a function $\mathcal{G}_Q$ depicted in Figure 4.18b. This reasoning leads to the following open question.

**Question 1.** Let us consider the following framework: for a Parry number $\beta > 1$, consider the Parry–Bertrand numeration system $U_\beta$ associated with $\beta$ from Definition 1.29, and let us also take the sequence $(S_\beta(n))_{n\geq 0}$ from Definition 1.47 and its summatory function $(A_\beta(n))_{n\geq 0}$. Is the asymptotic behavior of $(A_\beta(n))_{n\geq 0}$ similar to the ones of $(A_b(n))_{n\geq 0}$ with $b \geq 2$ and $(A_\varphi(n))_{n\geq 0}$? Also, prove or disprove that the sequence $(A_\beta(n))_{n\geq 0}$ is $U_\beta$-regular as it was the case for $\beta \in \mathbb{N}_{>1}$.

Let us have a closer look at Figures 4.2 and 4.9. It seems that when the base $b$ increases, the function $\mathcal{H}_b$ tends to a limit function. Similarly, Figures 4.16 and 4.18 also appear to be related. This gives the following question.

**Question 2.** Does the sequence $(\mathcal{H}_b)_{b\geq 2}$ of functions converge to a limit function? What about the sequence of functions obtained in the $m$-bonacci case if they are well defined? More generally, would it be possible to classify the functions we get?

As a final point, let us exit the scope of our generalized Pascal triangles, and recall the non-exhaustive list of other extensions of the Pascal triangle in Section 1.5.

**Question 3.** Would it be possible to apply our method to such existing extensions? Could we handle the case of other sequences, not especially related to the context of Pascal triangles?

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