

Looking for the Tangent Portfolio: Risk Optimization Techniques on Equity Style Buckets

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AGENDA

THE PAPER IN A NUTSHELL

OUR CONTRIBUTION

DATA

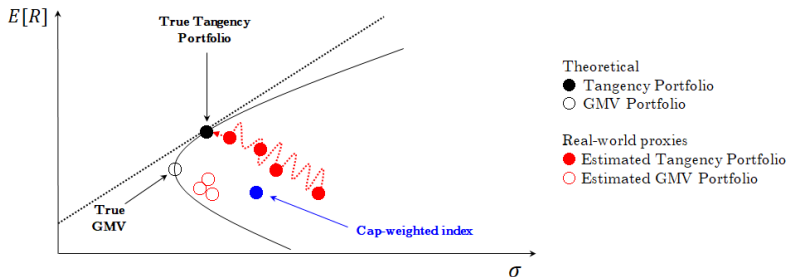
EMPIRICAL RESULTS

CONCLUSION

RESEARCH OBJECTIVE

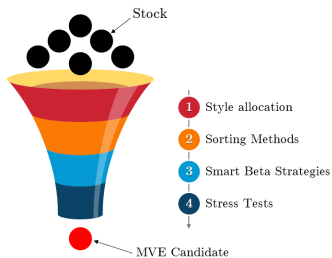
Looking for the tangency portfolio using risk-optimization techniques.

Our objective is to propose a simple intermediary method to proxy for ●



METHODOLOGY AND RESULTS

- ▶ Stratification of the US equity universe (NYSE, AMEX, Nasdaq) into size and book-to-markets equity style buckets
 - ▶ Extension to momentum
- ▶ Risk-based investment strategies (MV, MD, RP) are shown to provide
 - ▶ Better pricing of characteristic-sorted portfolios than existing multifactor models
 - ▶ Higher Sharpe ratio than a portfolio made of:
 - ▶ market portfolio (Mkt)
 - ▶ 30-year US treasury bond (B30)
 - ▶ size (SMB) and value (HML) factors



MOTIVATION

We ground our research into the following papers:

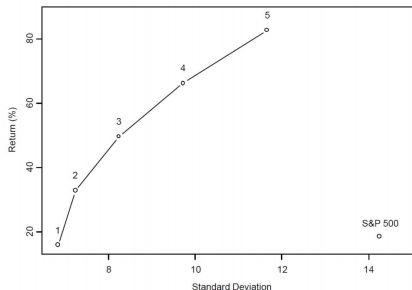
- ▶ Daniel et al. (2017, JF)
- ▶ Grinblatt and Saxena (2018, JFQA Forthcoming)
- ▶ Ao, Li, and Zheng (2018, RFS)

We rely on the following facts and evidence:

1. Caveats over the cap-weighted market benchmarks
2. Sharp rise in multi-factor models and in the number of index-funds and ETFs
3. Inefficiencies of long-short factors
4. Finding MSR is a noisy exercise

MOTIVATION - CAVEATS OVER THE CAP-WEIGHTED MARKET BENCHMARKS

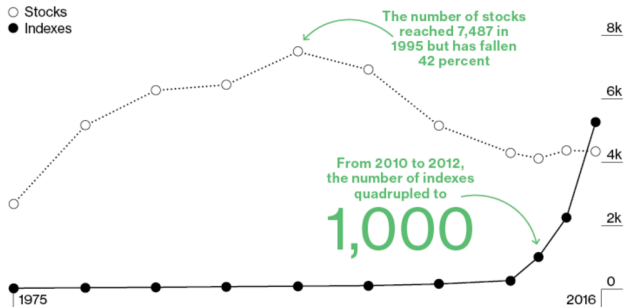
- ▶ “Market indices [...] are if anything inside that [mean-variance] frontier” (Cochrane 2001, Asset Pricing)
- ▶ “Cap-weighted stock portfolios are inefficient investments. [...] Even the most comprehensive cap-weighted portfolios occupy positions inside the efficient set” (Haugen and Baker 1991, JPM, p.35)



Based on data for the period 1979-1998. The efficient frontier assumes a perfect forecast of the future covariance matrix and of the future mean return. Figure taken from Schwartz(2000, Figure 3, p. 19).

MOTIVATION - SHARP RISE IN THE NUMBER OF ETFs VERSUS LISTED STOCKS

The Rise of the Benchmark



*BLOOMBERG LP (WHICH OWNS BLOOMBERG BUSINESSWEEK) AND ITS AFFILIATES PROVIDE INDEXES TRACKING VARIOUS ASSET CLASSES. DATA: BLOOMBERG INTELLIGENCE, SANFORD C. BERNSTEIN, WORLD BANK. CASH FLOWS AS OF MARCH 31; GRAPHIC BY BLOOMBERG BUSINESSWEEK

Source: Bloomberg.com

MOTIVATION – INEFFICIENCIES OF LONG / SHORT FACTORS

Daniel et al. (2017, WP, p. 3):

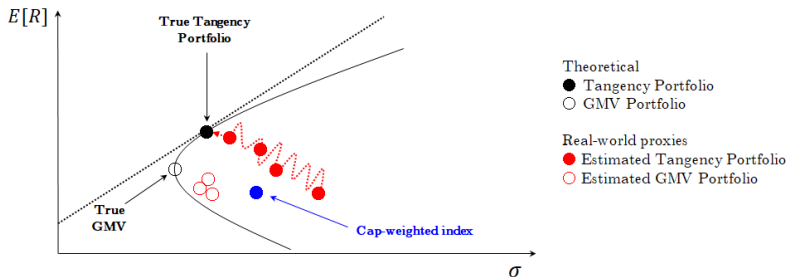
“This set of portfolios will explain the returns of portfolios sorted on the same characteristics, but are unlikely to span the MVE portfolio of all assets, because they do not take into account the asset covariance structure.”

Grinblatt and Saxena (2018, JFQA Forthcoming, p. 5):

“The optimal combination of the factor mimicking portfolios has a significantly lower Sharpe ratio than the optimal combination of the basis portfolios they are created from.”

MOTIVATION – SAMPLE ERRORS WITH MSR ESTIMATE

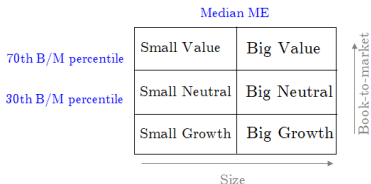
- ▶ Sample and specification errors
- ▶ Low-risk portfolios : giving up on estimating expected returns
- ▶ Robust variance-covariance matrix



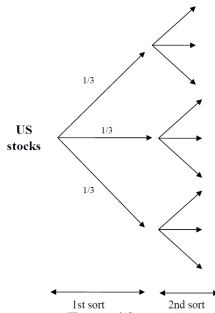
FINDING A CANDIDATE FOR THE MVE PORTFOLIO

- The opportunity sets: the DNS versus the original Fama-French sorting procedure
 - Independent versus dependent (**D**) sorting
 - NYSE breakpoints vs all names (**N**) breakpoints
 - Double and triple sort (size, value and momentum): 2x3, 3x3 and 3x3x3 (Asymmetric versus Symmetric sort)

(a) Independent Sort



(b) Dependent Sort



FINDING A CANDIDATE FOR THE MVE PORTFOLIO

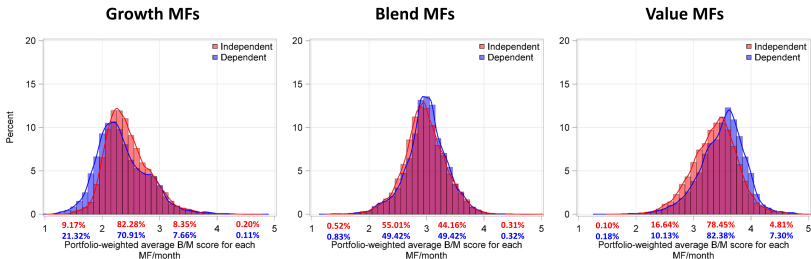
1. The opportunity sets
2. MSR weights replaced by smart beta (risk-based) optimization
 - ▶ Minimum Variance (MV) (Clarke, Silva, and Thorley 2013, JPM)
 - ▶ Maximum Diversification (MD) (Choueifaty and Coignard 2008, JPM)
 - ▶ Risk parity (RP) (Maillard, Roncalli, and Teiletche 2010, JPM)

Strategy	Objective Function	Constraints
Minimum Variance (MV)	$\min f(w) = \sum_i^N \sum_j^N w_i \sigma_{ij} w_j$	
Maximum Diversification (MD)	$\max f(w) = \frac{\sum_i^N w_i \sigma_i}{\sqrt{\sum_i^N \sum_j^N w_i \sigma_{ij} w_j}}$	$w_i \in [0, 1]$ and $\sum_{i=1}^N w_i = 1$
Risk parity (RP)	$\min f(w) = \sum_i^N \sum_j^N (w_i \times (\Sigma w)_i - w_j \times (\Sigma w)_j)^2$	

FINDING A CANDIDATE FOR THE MVE PORTFOLIO - THE OPPORTUNITY SETS

The DNS sorting procedure allows for:

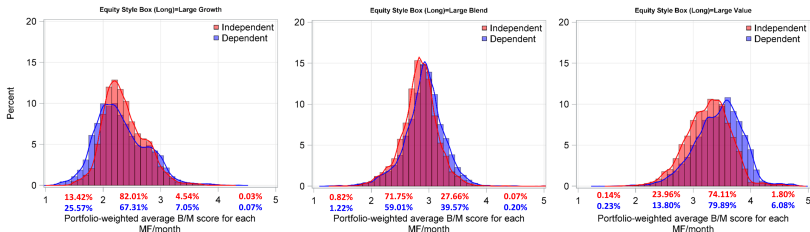
- A better stratification of the US equity universe



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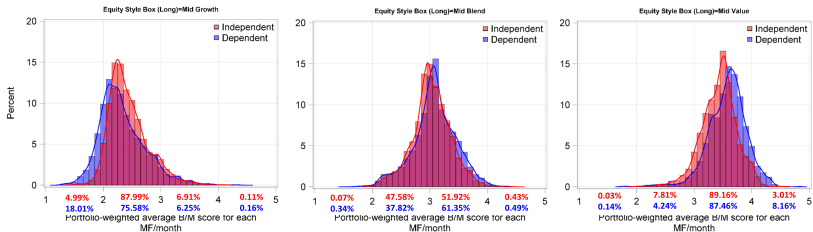
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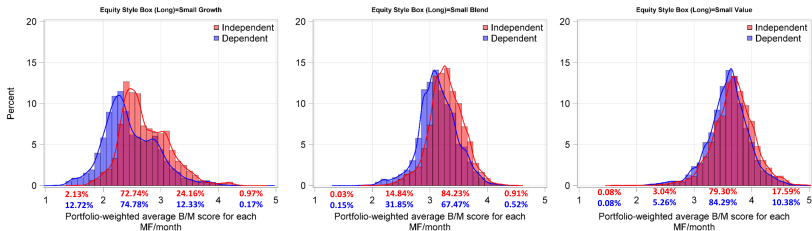
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FINDING A CANDIDATE FOR THE MVE PORTFOLIO - THE OPPORTUNITY SETS

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FINDING A CANDIDATE FOR THE MVE PORTFOLIO - THE OPPORTUNITY SETS

The DNS sorting procedure allows for:

- ▶ A better stratification of the US equity universe
- ▶ Better diversification

# Portfolios	Independent Sort (1)	Dependent Sort (2)	Difference (1)-(2)
Panel A: Cap-weighted Portfolios			
2x3	84.99	78.00	6.99
3x3	84.99	75.81	9.18
3x3x3	78.38	66.8	11.58

FINDING A CANDIDATE FOR THE MVE PORTFOLIO - THE OPPORTUNITY SETS

The DNS sorting procedure allows for:

- ▶ A better stratification of the US equity universe
- ▶ Better diversification
- ▶ Similar to other portfolio sorts, a reduction of the complexity of the universe (consistent with the categorization process of Barberis and Shleifer (2003))

Level of Risk		Investment Style			Average Market Capitalization
		Value	Blend	Growth	
Low	○	Large-Cap Value	Large-Cap Blend	Large-Cap Growth	Large
Moderate	○	Mid-Cap Value	Mid-Cap Blend	Mid-Cap Growth	Medium
High	●	Small-Cap Value	Small-Cap Blend	Small-Cap Growth	Small

FINDING A CANDIDATE FOR THE MVE PORTFOLIO - MSR WEIGHTS REPLACED BY SMART BETA (RISK-BASED) OPTIMIZATION

- ▶ Long-only investment scheme
- ▶ Avoid the empirical challenge of estimating expected returns

US EQUITIES

We employ:

- ▶ Dataset from the merge of CRSP and Compustat.
- ▶ All stocks listed on NYSE, NASDAQ, and AMEX stocks and share code of 10 or 11.
- ▶ Sample period ranges from July 1963 to December 2015.

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Filtering criteria following Fama and French (1993, JF):

- ▶ Shares (SHROUT) and price (PRC)
- ▶ Stock return (RET) data for month t
- ▶ 2 years of listing on COMPUSTAT (survival bias)



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Characteristics:

- ▶ Market equity (firm size) as $SHARE \times PRICE$
- ▶ Book-to-market equity as BE/ME
- ▶ Momentum is the $t-2$ to $t-12$ cumulative return of stock

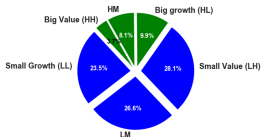
CHARACTERISTIC-SORTED PORTFOLIOS

Each year in June, we sort US stocks on the following traditional characteristics.

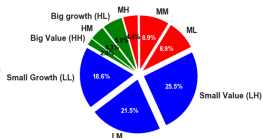
- ▶ size and value (2x3)
- ▶ size and value (3x3)
- ▶ size and value and momentum (3x3x3)

Average distribution of stock in portfolios

Independent (2x3)



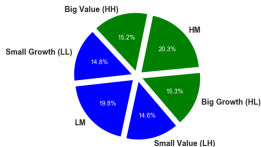
Independent (3x3)



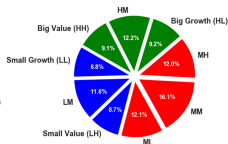
Independent (3x3x3)



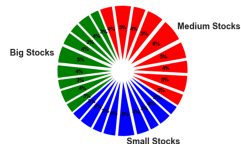
Dependent (2x3)



Dependent (3x3)



Dependent (3x3x3)



EMPIRICAL TESTS AND RESULTS IN A NUTSHELL

1. Smart investment strategies on DSN portfolios achieve better diversification return than other smart investment strategies and equally weighted scheme
 - ▶ Diversification return framework of Booth and Fama (1992, FAJ) and Willenbrock (2011, FAJ)
2. Strategic beta portfolios constructed on dependent equity style buckets outperform a single-index model (using CW factor), a multi-factor model (FF-3 Factors) and other strategic beta portfolios
 - ▶ Mean-variance spanning of Kan and Zhou (2012, AEF)
 - ▶ Bootstrap procedure similar to Fama and French (2010, JF) and Harvey and Liu (2016, WP)
 - ▶ Factor selection technique from Harvey and Liu (2016, WP)

DIVERSIFICATION RETURN

Following Booth and Fama (1992, FAJ) and Willenbrock (2011, FAJ), the diversification return is given by,

$$DR^{FW} = \underbrace{\mu_p - \sum_i^N w_i \mu_i}_{DR_1^{FW} = 0 \text{ if weights are constant}} + \underbrace{\frac{1}{2} \left(\sum_i^N w_i \sigma_i^2 - \sigma_p^2 \right)}_{DR_2^{FW} = \text{variance reduction benefit}} \quad (1)$$

The relationship assumes that ,

- ▶ weights w_i are held constant over the estimation period,
- ▶ i stands for the i^{th} security in the portfolio p ,
- ▶ FW denotes Fixed-Weight.

DIVERSIFICATION RETURN

Issues:

- ▶ Weights of the low risk strategies are not constant over time. For rebalancing strategies (non fixed weight), Erb and Harvey (2006, FAJ) use of the average of the weights over the sample period ($\bar{w}_i = \frac{1}{T} \sum_1^T w_i^t$).

DIVERSIFICATION RETURN

We test the difference in the diversification components (DR_1 , DR_2 , and DR) using the bootstrap method of Ledoit and Wolf (2008, JEF).

Fixed-Weight (FW) Benchmark

	DR_1^{FW}				DR_2^{FW}				DR^{FW}			
	Ind	Dep	Δ Dep-Ind	p-val ^b	Ind	Dep	Δ Dep-Ind	p-val ^b	Ind	Dep	Δ Dep-Ind	p-val ^b
MD _{2x3}	-0.005	-0.003	0.002	0.936	0.025	0.038	0.012	0.000	0.020	0.034	0.014	0.524
MD _{3x3}	0.004	-0.032	-0.035	0.257	0.031	0.050	0.019	0.000	0.034	0.018	-0.016	0.594
MD _{3x3x3}	-0.041	-0.075	-0.034	0.428	0.046	0.080	0.034	0.000	0.005	0.005	0.000	0.993
MV _{2x3}	0.013	-0.025	-0.038	0.554	0.024	0.031	0.007	0.165	0.037	0.006	-0.031	0.635
MV _{3x3}	-0.001	-0.036	-0.035	0.581	0.024	0.045	0.020	0.003	0.023	0.009	-0.014	0.807
MV _{3x3x3}	-0.018	-0.115	-0.097	0.093	0.047	0.070	0.023	0.001	0.029	-0.045	-0.074	0.191
RP _{2x3}	0.005	0.005	0.000	0.992	0.024	0.036	0.012	0.000	0.030	0.041	0.012	0.308
RP _{3x3}	0.005	0.000	-0.005	0.679	0.027	0.042	0.014	0.000	0.032	0.041	0.009	0.460
RP _{3x3x3}	-0.001	-0.010	-0.009	0.522	0.043	0.065	0.023	0.000	0.041	0.056	0.014	0.307

*number of bootstraps=4999

** figures are from gross return on a monthly basis (in %)

*** Block size for bootstrap = 10

Bootstrap

DIVERSIFICATION RETURN

In this framework, results suggests that dependent-sorted portfolios provide significantly

- ▶ greater variance reduction benefits
- ▶ greater diversification return

Equal-Weight (EW) Benchmark

	DR_1^{EW}				DR_2^{EW}				DR^{EW}			
	Ind	Dep	Δ Dep-Ind	p-val	Ind	Dep	Δ Dep-Ind	p-val	Ind	Dep	Δ Dep-Ind	p-val
MD _{2x3}	0.005	0.088	0.083	0.012	0.027	0.038	0.011	0.003	0.032	0.126	0.094	0.007
MD _{3x3}	0.010	0.070	0.060	0.125	0.033	0.048	0.015	0.000	0.043	0.118	0.075	0.055
MD _{3x3x3}	-0.034	0.063	0.097	0.107	0.046	0.061	0.014	0.010	0.012	0.124	0.112	0.067
MV _{2x3}	0.082	0.218	0.136	0.071	0.033	0.028	-0.004	0.407	0.115	0.246	0.131	0.075
MV _{3x3}	0.050	0.143	0.093	0.163	0.034	0.046	0.012	0.066	0.084	0.189	0.105	0.112
MV _{3x3x3}	0.022	0.019	-0.003	0.955	0.061	0.080	0.018	0.002	0.084	0.099	0.015	0.785
RP _{2x3}	0.020	0.056	0.036	0.013	0.026	0.036	0.010	0.001	0.046	0.092	0.047	0.005
RP _{3x3}	0.019	0.048	0.029	0.063	0.029	0.042	0.013	0.000	0.048	0.090	0.041	0.015
RP _{3x3x3}	0.011	0.035	0.024	0.102	0.046	0.068	0.022	0.000	0.057	0.102	0.046	0.007

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** figures are from gross return on a monthly basis (in %)

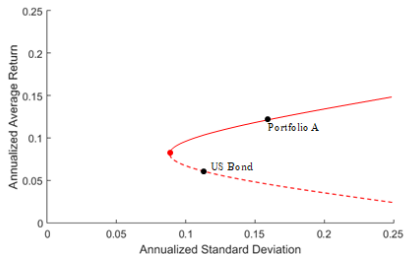
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Bootstrap

TEST OF MEAN-VARIANCE SPANNING

Illustration of Kan and Zhou (2012, AEF) mean-variance spanning test :

H_0^1 : Tangency Portfolio



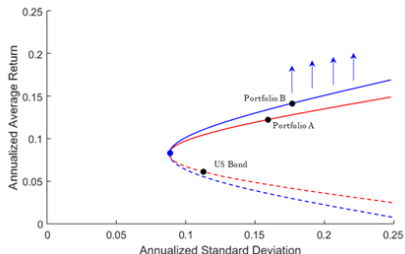
Benchmark Assets (R_1):

- ▶ US Bond
- ▶ Portfolio A

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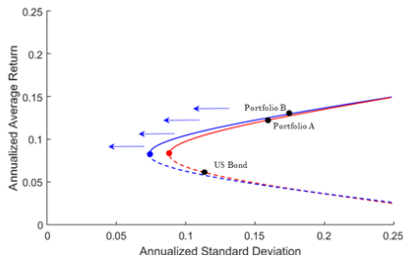
Test Asset (R_2):

- ▶ Portfolio B

TEST OF MEAN-VARIANCE SPANNING

Illustration of Kan and Zhou (2012, AEF) test of mean-variance spanning :

H_0^2 : GMV Portfolio



Benchmark Assets (R_1):

- ▶ US Bond
- ▶ Portfolio A

Test Asset (R_2):

- ▶ Portfolio B

TEST OF MEAN-VARIANCE SPANNING

Huberman and Kandel (1987, JF) define the following regression test:

$$R_2^t = \alpha + \beta R_1^t + e^t \quad (3)$$

The null hypothesis H_0 sets $\alpha = 0$ and $\delta = 1 - \beta = 0$.

Considering an efficient frontier comprising $K + N$ assets, the weights of the N assets into the tangent (Qw_1) and GMV (Qw_2) portfolios are defined as:

$$Qw_1 = \frac{QV^{-1}\mu}{1'_{N+K}V^{-1}\mu} = \frac{\Sigma^{-1}\alpha}{1'_{N+K}V^{-1}\mu} \quad (4)$$

$$Qw_2 = \frac{QV^{-1}1_{N+K}}{1'_{N+K}V^{-1}1_{N+K}} = \frac{\Sigma^{-1}\delta}{1'_{N+K}V^{-1}1_{N+K}}$$

where $Q = [0_{N \times K}, I_N]$, I_N is an $N \times N$ identity matrix, $\Sigma = V_{22} - V_{21}V_{11}^{-1}V_{12}$, and V is the variance-covariance matrix of the K benchmark assets (R_1) plus the N test assets (R_2) such that,

$$V = \text{Var}[R_1, R_2] = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$$

TEST OF MEAN-VARIANCE SPANNING

Step-down procedure to test the spanning hypothesis (Kan and Zhou 2012, AEF):

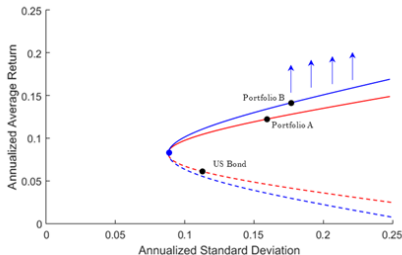
$$H_0^1 = \alpha = 0_N, \text{ such that } Qw_1 = 0.$$

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$$H_0^1 = \alpha = 0_N, \text{ such that } Qw_1 = 0.$$

H_0^1 : Tangency Portfolio



The F -test (H_0^1):

$$F_1 = \frac{T - K - N}{N} \frac{\hat{a} - \hat{a}_1}{1 + \hat{a}_1}$$

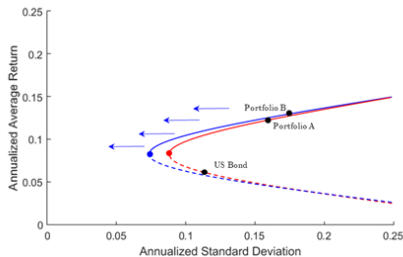
- ▶ T is the number of observations
- ▶ K is the number of benchmark assets
- ▶ N is the number of test assets
- ▶ $\hat{a}_1 = \hat{\mu}_1' \hat{V}_{11}^{-1} \hat{\mu}_1$
- ▶ \hat{V}_{11} : the variance of the benchmark assets
- ▶ $\hat{\mu}_1$: the vector of mean return of the benchmark assets
- ▶ \hat{a} but refers to the benchmark assets (R_1) plus the new test asset (R_2)

TEST OF MEAN-VARIANCE SPANNING

Step-down procedure to test the spanning hypothesis (Kan and Zhou 2012, AEF):

$$H_0^2 : \delta = 1_N - \beta 1_K = 0_N | \alpha = 0_N, \text{ such that } Qw_2 = 0 \text{ conditional on } Qw_1 = 0.$$

H_0^2 : GMV Portfolio



The F -test (H_0^2):

$$F_2 = \frac{T - K - N + 1}{N} \left[\frac{\hat{c} + \hat{d}}{\hat{c}_1 + \hat{d}_1} \frac{1 + \hat{a}_1}{1 + \hat{a}} - 1 \right]$$

- ▶ $\hat{a}_1 = \hat{\mu}'_1 \hat{V}_{11}^{-1} \hat{\mu}_1$
- ▶ $\hat{b}_1 = \hat{\mu}'_1 \hat{V}_{11}^{-1} 1_K$
- ▶ $\hat{c}_1 = 1'_K \hat{V}_{11}^{-1} 1_K$
- ▶ $\hat{d}_1 = \hat{a}_1 \hat{c}_1 - \hat{b}_1^2$
- ▶ \hat{V}_{11} : the variance of the benchmark assets
- ▶ $\hat{\mu}_1$: the vector of mean return of the benchmark assets

\hat{a} , \hat{b} , \hat{c} and \hat{d} refers to the benchmark assets (R_1) plus the new test asset (R_2)

BOOTSTRAP APPROACH ON TEST OF MEAN-VARIANCE SPANNING

The null hypothesis should be true in-sample (Harvey and Liu (2016, WP) and White (2000, ECO)):

$$\rightarrow Qw_1 = 0 \text{ and } Qw_2 = 0.$$

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Step 1: Orthogonalization under the null

$$R_2^t = \alpha + \beta R_1^t + R_2^{t,e} \tag{5}$$

- ▶ $R_2^{orth} = R_2^{t,e} + R_{1,MVE}^t$ by construction $\alpha = 0$ and $\beta_{MVE} = 1 \rightarrow Qw_1 = 0$ and $Qw_2 = 0$
- ▶ $R_{1,MVE}$ is the proxy for the market portfolio present in R_1

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- ▶ $R_{1,MVE}$ is the proxy for the market portfolio present in R_1

Step 2: Bootstrap (Harvey and Liu 2016, WP)

- ▶ preserves the cross-sectional correlations among the benchmark (R_1) and test (R_2^{orth}) assets
- ▶ preserves the uncertainty of the time-series: bootstrap sampling length=original time-series length

BOOTSTRAP APPROACH ON TEST OF MEAN-VARIANCE SPANNING

Step 3: Test of Mean-Variance Spanning

- ▶ Apply the test of mean-variance spanning from Kan and Zhou (2012, AEF)
- ▶ Outputs:
 - ▶ Tangency portfolio: range of $F_{1,ind}^b$ and $F_{1,dep}^b$ with $\{b = 1, 2, \dots, B\}$
 - ▶ GMV portfolio: range of $F_{2,ind}^b$ and $F_{2,dep}^b$ with $\{b = 1, 2, \dots, B\}$
 - ▶ Where,

$$F_1 = \frac{T - K - N}{N} \frac{\hat{a} - \hat{a}_1}{1 + \hat{a}_1}$$

$$F_2 = \frac{T - K - N + 1}{N} \left[\frac{\hat{c} + \hat{d}}{\hat{c}_1 + \hat{d}_1} \frac{1 + \hat{a}_1}{1 + \hat{a}} - 1 \right]$$

BOOTSTRAP APPROACH ON TEST OF MEAN-VARIANCE SPANNING

Step 3: Mean-Variance Spanning Test (cont'd)

- ▶ Harvey and Liu (2016, WP)'s bootstrap approach robust for multiple testing
- ▶ Conservative **reference point**:
 - ▶ $F_1^b = \max(F_{1,ind}^b, F_{1,dep}^b)$ with $\{b = 1, 2, \dots, B\}$
 - ▶ $F_2^b = \max(F_{2,ind}^b, F_{2,dep}^b)$ with $\{b = 1, 2, \dots, B\}$
- ▶ Bootstrap p-value for the F-tests

F_1^o p-value:

$$\text{p-val}_{ind}^b = \frac{\#\{F_1^b > F_{1,ind}^o\}}{B}$$

$$\text{p-val}_{dep}^b = \frac{\#\{F_1^b > F_{1,dep}^o\}}{B}$$

F_2^o p-value:

$$\text{p-val}_{ind}^b = \frac{\#\{F_2^b > F_{1,ind}^o\}}{B}$$

$$\text{p-val}_{dep}^b = \frac{\#\{F_2^b > F_{1,dep}^o\}}{B}$$

SPANNING OF SINGLE INDEX AND MULTIFACTOR MODELS

Sample period: July 1963 - December 2015

- ▶ Benchmark assets (R_1)= CW-Market Portfolio (Mkt) + 30-Year US Treasury Bond (B30)
- ▶ Test asset (R_2)= Smart Beta (SB)

$$R_2^t = \alpha + \beta_1 B30^t + \beta_2 Mkt^t + R_2^{t,e}$$

- ▶ $R_2^{orth} = R_2^{t,e} + Mkt^t$

MVE candidates	α_{ind}	$F_{1,ind}$	p-val ^b	$F_{2,ind}$	p-val ^b	α_{dep}	$F_{1,dep}$	p-val ^b	$F_{2,dep}$	p-val ^b
Panel A: $R_1 = MKT + B30$, $R_2 = SB$, $R_2^{orth} = SB^e + Mkt$										
MD _{2x3}	0.0020	11.69	0.00	2.37	0.31	0.0031	13.00	0.00	13.52	0.01
MD _{3x3}	0.0022	11.89	0.00	2.96	0.25	0.0034	10.66	0.00	21.30	0.00
MD _{3x3x3}	0.0019	7.21	0.01	1.42	0.58	0.0038	9.45	0.00	19.39	0.00
MV _{2x3}	0.0032	13.63	0.00	11.83	0.01	0.0049	15.55	0.00	23.12	0.00
MV _{3x3}	0.0029	10.60	0.00	6.68	0.07	0.0044	13.39	0.00	28.21	0.00
MV _{3x3x3}	0.0029	13.60	0.00	10.24	0.02	0.0034	10.02	0.00	19.87	0.00
RP _{2x3}	0.0021	13.03	0.00	1.62	0.42	0.0027	10.60	0.00	7.48	0.04
RP _{3x3}	0.0022	11.57	0.00	0.49	0.72	0.0029	8.70	0.00	8.49	0.02
RP _{3x3x3}	0.0023	12.09	0.00	0.25	0.84	0.0031	10.28	0.00	7.34	0.04

* Mkt, B30 and SB are taken in excess of the risk-free rate (one-month T-bill from Ibbotson)

SPANNING OF SINGLE INDEX AND MULTIFACTOR MODELS

Sample period: July 1963 - December 2015

- ▶ Benchmark assets (R_1)= CW-Market Portfolio (Mkt) + 30-Year US Treasury Bond (B30) + SMB + HML
- ▶ Test asset (R_2)= Smart Beta (SB)

$$R_2^t = \alpha + \beta_1 B30^t + \beta_2 Mkt^t + \beta_3 SMB^t + \beta_4 HML^t + R_2^{t,e}$$

- ▶ $R_2^{orth} = R_2^{t,e} + Mkt^t$

MVE candidates	α_{ind}	$F_{1,ind}$	p-val ^b	$F_{2,ind}$	p-val ^b	α_{dep}	$F_{1,dep}$	p-val ^b	$F_{2,dep}$	p-val ^b
Panel B: $R_1 = MKT + B30 + SMB + HML$, $R_2 = SB$, $R_2^{orth} = SB^e + Mkt$										
MD _{2x3}	0.0001	0.21	0.85	1419.96	0.00	0.0011	4.89	0.05	578.95	0.00
MD _{3x3}	0.0002	0.66	0.63	1392.77	0.00	0.0011	3.13	0.14	365.03	0.00
MD _{3x3x3}	-0.0003	0.65	0.63	824.06	0.00	0.0012	2.22	0.22	285.69	0.00
MV _{2x3}	0.0009	2.22	0.22	317.78	0.00	0.0022	6.21	0.02	249.22	0.00
MV _{3x3}	0.0006	1.02	0.47	438.93	0.00	0.0018	4.59	0.06	254.99	0.00
MV _{3x3x3}	0.0007	2.22	0.23	564.35	0.00	0.0010	1.92	0.28	310.91	0.00
RP _{2x3}	0.0002	0.92	0.47	1786.24	0.00	0.0007	3.21	0.11	882.21	0.00
RP _{3x3}	0.0002	0.88	0.52	2015.93	0.00	0.0007	2.36	0.20	801.67	0.00
RP _{3x3x3}	0.0003	1.16	0.44	1895.22	0.00	0.0010	4.27	0.06	811.01	0.00

* Mkt, B30 and SB are taken in excess of the risk-free rate (one-month T-bill from Ibbotson)

SPANNING OF SINGLE INDEX AND MULTIFACTOR MODELS

Sample period: July 1963 - December 2015

- ▶ Benchmark assets (R_1)= Smart Beta (SB) + 30-Year US Treasury Bond (B30)
- ▶ Test asset (R_2)= CW-Market Portfolio (Mkt)

$$R_2^t = \alpha + \beta_1 B30^t + \beta_2 SB^t + R_2^{t,e}$$

▶ $R_2^{orth} = R_2^{t,e} + SB^t$

MVE candidates	α_{ind}	$F_{1,ind}$	p-val ^b	$F_{2,ind}$	p-val ^b	α_{dep}	$F_{1,dep}$	p-val ^b	$F_{2,dep}$	p-val ^b
Panel C: $R_1 = SB + B30$, $R_2 = Mkt$, $R_2^{orth} = Mkt^e + SB$										
MD _{2x3}	-0.0013	5.92	0.01	7.93	0.03	-0.0016	4.20	0.05	8.11	0.03
MD _{3x3}	-0.0014	5.51	0.02	9.40	0.02	-0.0013	1.99	0.22	10.40	0.01
MD _{3x3x3}	-0.0010	2.22	0.18	17.25	0.01	-0.0010	0.92	0.49	19.97	0.01
MV _{2x3}	-0.0017	3.97	0.05	12.51	0.01	-0.0016	2.31	0.17	21.01	0.00
MV _{3x3}	-0.0014	2.63	0.13	18.31	0.00	-0.0014	1.87	0.22	15.30	0.00
MV _{3x3x3}	-0.0017	4.60	0.05	10.31	0.01	-0.0012	1.45	0.30	14.54	0.00
RP _{2x3}	-0.0014	6.82	0.01	9.52	0.02	-0.0014	3.44	0.07	10.80	0.01
RP _{3x3}	-0.0014	5.35	0.03	16.33	0.00	-0.0011	1.81	0.21	16.07	0.00
RP _{3x3x3}	-0.0014	5.50	0.02	19.14	0.00	-0.0013	2.46	0.14	17.64	0.00

* Mkt, B30 and SB are taken in excess of the risk-free rate (one-month T-bill from Ibbotson)

SPANNING OF SINGLE INDEX AND MULTIFACTOR MODELS

Sample period: July 1993 - December 2015

- ▶ Benchmark assets (R_1)= CW-Market Portfolio (Mkt) + 30-Year US Treasury Bond (B30)
- ▶ Test asset (R_2)= Smart Beta (SB)

$$R_2^t = \alpha + \beta_1 B30^t + \beta_2 Mkt^t + R_2^{t,e}$$

- ▶ $R_2^{orth} = R_2^{t,e} + Mkt^t$

MVE candidates	α_{ind}	$F_{1,ind}$	p-val ^b	$F_{2,ind}$	p-val ^b	α_{dep}	$F_{1,dep}$	p-val ^b	$F_{2,dep}$	p-val ^b
Panel A: $R_1 = MKT + B30$, $R_2 = SB$, $R_2^{orth} = SB^e + Mkt$										
MD _{2x3}	0.0019	3.77	0.11	6.48	0.09	0.0048	12.70	0.00	16.12	0.00
MD _{3x3}	0.0024	5.08	0.06	7.26	0.05	0.0049	9.14	0.01	16.55	0.00
MD _{3x3x3}	0.0016	1.86	0.33	5.87	0.17	0.0053	9.38	0.01	16.66	0.02
MV _{2x3}	0.0039	9.03	0.00	19.93	0.00	0.0071	14.13	0.00	17.80	0.00
MV _{3x3}	0.0038	7.55	0.02	11.39	0.01	0.0066	13.10	0.00	21.37	0.00
MV _{3x3x3}	0.0042	11.36	0.00	18.15	0.00	0.0050	9.92	0.00	19.77	0.00
RP _{2x3}	0.0021	4.73	0.07	4.94	0.12	0.0039	9.39	0.01	9.07	0.03
RP _{3x3}	0.0024	5.02	0.05	3.44	0.21	0.0039	6.64	0.02	7.98	0.05
RP _{3x3x3}	0.0027	6.19	0.03	4.37	0.13	0.0043	8.69	0.01	9.42	0.02

* Mkt, B30 and SB are taken in excess of the risk-free rate (one-month T-bill from Ibbotson)

SPANNING OF SINGLE INDEX AND MULTIFACTOR MODELS

Sample period: July 1993 - December 2015

- ▶ Benchmark assets (R_1)= CW-Market Portfolio (Mkt) + 30-Year US Treasury Bond (B30) + SMB + HML
- ▶ MVE market portfolio proxy in is Mkt ($R_{1,MVE}$)
- ▶ Test asset (R_2)= Smart Beta (SB)

$$R_2^t = \alpha + \beta_1 B30^t + \beta_2 Mkt^t + \beta_3 SMB^t + \beta_4 HML^t + R_2^{t,e}$$

- ▶ $R_2^{orth} = R_2^{t,e} + Mkt^t$

MVE candidates	α_{ind}	$F_{1,ind}$	p-val ^b	$F_{2,ind}$	p-val ^b	α_{dep}	$F_{1,dep}$	p-val ^b	$F_{2,dep}$	p-val ^b
	Panel B: $R_1 = MKT + B30 + SMB + HML$, $R_2 = SB, R_2^{orth} = SB^e + Mkt$									
MD _{2x3}	0.0006	1.52	0.40	384.37	0.00	0.0036	16.78	0.00	118.87	0.00
MD _{3x3}	0.0010	3.71	0.11	382.82	0.00	0.0036	9.31	0.00	63.08	0.00
MD _{3x3x3}	0.0001	0.02	0.99	249.23	0.00	0.0041	8.88	0.01	39.76	0.00
MV _{2x3}	0.0027	6.52	0.03	42.73	0.00	0.0056	14.39	0.00	44.38	0.00
MV _{3x3}	0.0025	5.89	0.02	92.31	0.00	0.0052	13.19	0.00	43.67	0.00
MV _{3x3x3}	0.0028	11.43	0.00	124.93	0.00	0.0038	9.86	0.00	51.57	0.00
RP _{2x3}	0.0008	3.50	0.11	563.63	0.00	0.0026	14.80	0.00	217.80	0.00
RP _{3x3}	0.0010	5.07	0.05	652.92	0.00	0.0026	9.23	0.00	174.22	0.00
RP _{3x3x3}	0.0013	7.72	0.02	618.11	0.00	0.0031	13.91	0.00	158.94	0.00

* Mkt, B30 and SB are taken in excess of the risk-free rate (one-month T-bill from Ibbotson)

HORSE RACE BETWEEN SMART INVESTMENT PORTFOLIOS

- ▶ Panel A: MVE market portfolio proxy is SB_{dep}^{net} ($R_{1,MVE}$)
- ▶ Panel B: MVE market portfolio proxy is SB_{ind}^{net} ($R_{1,MVE}$)

	Panel A:				Panel B:				MVE Candidate	GMV Candidate
	$R_1 = B30 + SB_{dep}^{net}$				$R_1 = B30 + SB_{ind}^{net}$					
	$F_{1,ind}^o$	p-val ^b	$F_{2,ind}^o$	p-val ^b	$F_{1,dep}^o$	p-val ^b	$F_{2,dep}^o$	p-val ^b		
MD _{2x3}	0.801	0.448	0.064	0.944	5.823	0.012	11.175	0.013	Dependent	Dependent
MD _{3x3}	0.016	0.973	0.121	0.914	4.160	0.049	17.576	0.000	Dependent	Dependent
MD _{3x3x3}	0.001	1.000	2.470	0.361	5.959	0.026	18.900	0.005	Dependent	Dependent
MV _{2x3}	0.118	0.871	6.515	0.054	5.254	0.015	10.170	0.015	Dependent	Dependent
MV _{3x3}	0.035	0.956	0.090	0.954	6.095	0.011	24.206	0.000	Dependent	Dependent
MV _{3x3x3}	1.490	0.285	3.210	0.178	0.788	0.471	9.723	0.010	Dep ≈ Ind	Dependent
RP _{2x3}	0.006	0.984	0.299	0.773	1.743	0.173	5.814	0.061	Dep ≈ Ind	Dependent
RP _{3x3}	0.035	0.931	0.116	0.885	1.497	0.204	9.125	0.019	Dep ≈ Ind	Dependent
RP _{3x3x3}	0.009	0.984	0.087	0.922	2.315	0.109	9.240	0.021	Dep ≈ Ind	Dependent

* SB strategies are **net of transactions costs** estimated according to Hasbrouck (2009, JF)'s model as in Novy-Marx and Velikov (2016, RFS) Transaction Costs

CROSS-SECTIONAL ASSET PRICING TEST

Harvey and Liu (2016, WP) define a scaled intercept (SI) to

- ▶ measure the incremental contribution of an augmented model *w.r.t.* a baseline model to explain the cross-sectional variations of the J test assets returns
- ▶ overcome the over-rejection issues of the GRS test

$$SI_{ew}^{med} = \frac{\text{median}(\{|a_i^g|/s_i^b\}_{i=1}^J) - \text{median}(\{|a_i^b|/s_i^b\}_{i=1}^J)}{\text{median}(\{|a_i^b|/s_i^b\}_{i=1}^J)} \quad (6)$$

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where,

- ▶ $\text{median}(\cdot)$ is the median value of the ratio $|a_i^g|/s_i^b$ or $|a_i^b|/s_i^b$
- ▶ s denotes the standard errors for the regression intercept a
- ▶ superscript b is for the baseline model
- ▶ superscript g is for the augmented model
- ▶ subscript i refers to the i -th portfolio among the J test assets

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Outputs:

- ▶ if $SI < 0$ then augmented > baseline model

CROSS-SECTIONAL ASSET PRICING TEST

To test the significance of the model improvement, Harvey and Liu (2016, WP) define the following bootstrap procedure,

Step 1: Orthogonalization of the list of K candidates

- ▶ Baseline assets (R_1)= 30-Year US Treasury Bond (B30) + SMB + HML
- ▶ Test asset (R_2^i)= i -th candidate among the list of K candidates

CROSS-SECTIONAL ASSET PRICING TEST

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- ▶ Baseline assets (R_1)= 30-Year US Treasury Bond (B30) + SMB + HML
- ▶ Test asset (R_2^i)= i -th candidate among the list of K candidates

$$\begin{aligned} R_2^i &= \alpha^i + \beta^i R_1 + e^i \\ R_2^{\alpha,i} &= R_2^i - \alpha^i = \beta^i R_1 + e^i \end{aligned} \tag{7}$$

Such that, $R_2^{\alpha,i}$ does not bring any additional information to the baseline model.

CROSS-SECTIONAL ASSET PRICING TEST

Step 2: Bootstrap (Similar to the method presented earlier)

In each sample of the B bootstrap:

- ▶ a score for the scaled intercept SI_{ew}^{med} can be obtained for the K number of orthogonalized candidates (i.e, $R_2^{\alpha,i}$ with the $i = \{1, 2, \dots, K\}$ candidate)
- ▶ take the minimum value among the b -th bootstrap to control for multiple testing

$$SI^{b,*} = \underbrace{\min}_{i \in \{1, 2, \dots, K\}} \{SI^{b,i}\} \quad (8)$$

CROSS-SECTIONAL ASSET PRICING TEST

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$$SI^{b,*} = \underbrace{\min}_{i \in \{1, 2, \dots, K\}} \{SI^{b,i}\} \quad (8)$$

Step 3: Single test p-value

Select the candidate with the lowest SI^0 value and significant p-val

$$\text{p-val} = \frac{\#\{SI^0 > SI^b\}}{B} \quad (9)$$

CROSS-SECTIONAL ASSET PRICING TEST

Step 2: Bootstrap (Similar to the method presented earlier)

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Step 3: Single test p-value

Select the candidate with the lowest SI^o value and significant p-val

$$\text{p-val} = \frac{\#\{SI^o > SI^b\}}{B} \quad (9)$$

Step 4: Multiple test p-value

$$\text{p-val} = \frac{\#\{SI^o > SI^{b,*}\}}{B} \quad (10)$$

CROSS-SECTIONAL ASSET PRICING TEST

MVE Candidates →	Mkt	MV_{dep}	MD_{dep}	RP_{dep}	MV_{ind}	MD_{ind}	RP_{ind}
Baseline = US30 + SMB + HML							
Panel A: 2x3 cap-weighted independent portfolios as test assets							
GRS	4.836	4.189	4.391	4.538	4.155	4.445	4.341
p-value	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Scaled intercept (SI)	0.042	0.070	0.041	0.036	0.066	-0.012	-0.842
Single test p-value	0.701	0.984	0.929	0.894	0.893	0.325	0.000
SI sequence	2	2	2	2	2	2	1
Selected candidate(s)	RP_{ind}						
Multiple test p-value	[0.000]						
Panel B: 2x3 cap-weighted dependent portfolios as test assets							
GRS	12.058	10.947	11.344	11.589	11.552	11.888	11.962
p-value	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Scaled intercept (SI)	0.049	0.061	-0.838	0.023	0.009	-0.001	-0.008
Single test p-value	0.880	0.865	0.000	0.782	0.569	0.547	0.468
SI sequence	2	2	1	2	2	2	2
Selected candidate(s)	MD_{dep}						
Multiple test p-value	[0.000]						

CROSS-SECTIONAL ASSET PRICING TEST

MVE Candidates →	Mkt	MV_{dep}	MD_{dep}	RP_{dep}	MV_{ind}	MD_{ind}	RP_{ind}
Baseline = US30 + SMB + HML							
Panel C: 3x3 cap-weighted independent portfolios as test assets							
GRS	3.403	2.792	2.892	2.975	2.909	2.918	2.929
p-value	0.000	0.003	0.002	0.002	0.002	0.002	0.002
Scaled intercept (SI)	-0.006	0.041	0.255	0.373	-0.876	0.100	0.169
Single test p-value	0.542	0.755	0.883	0.895	0.000	0.697	0.755
SI sequence	2	2	2	2	1	2	2
Selected candidate(s)				MV_{ind}			
Multiple test p-value				[0.000]			
Panel D: 3x3 cap-weighted dependent portfolios as test assets							
GRS	8.228	7.757	7.959	7.950	8.116	8.108	8.108
p-value	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Scaled intercept (SI)	0.210	0.249	0.221	-0.806	0.001	0.150	0.107
Single test p-value	0.937	0.979	0.916	0.000	0.528	0.861	0.849
SI sequence	2	2	2	1	2	2	2
Selected candidate(s)				RP_{dep}			
Multiple test p-value				[0.000]			

CROSS-SECTIONAL ASSET PRICING TEST

MVE Candidates →	Mkt	MV_{dep}	MD_{dep}	RP_{dep}	MV_{ind}	MD_{ind}	RP_{ind}
Baseline = US30 + SMB + HML							
Panel E: 3x3x3 cap-weighted independent portfolios as test assets							
GRS	2.262	2.135	2.164	2.213	2.085	2.332	2.110
p-value	0.000	0.001	0.001	0.000	0.001	0.000	0.001
Scaled intercept (SI)	0.245	0.030	-0.790	0.301	0.283	0.111	0.225
Single test p-value	0.943	0.661	0.000	0.970	0.962	0.816	0.952
SI sequence	2	2	1	2	2	2	2
Selected candidate(s)				MD_{dep}			
Multiple test p-value				[0.000]			
Panel F: 3x3x3 cap-weighted dependent portfolios as test assets							
GRS	4.591	4.366	4.335	4.283	4.335	4.608	4.379
p-value	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Scaled intercept (SI)	-0.015	0.007	-0.668	-0.039	0.006	0.013	0.020
Single test p-value	0.566	0.571	0.000	0.431	0.610	0.686	0.739
SI sequence	2	2	1	2	2	2	2
Selected candidate(s)				MD_{dep}			
Multiple test p-value				[0.000]			

CONCLUSION AND TAKEAWAYS

Testing the MVE of Smart Beta strategies on characteristic-sorted portfolios

► Context

- Multidimensional market risks, especially after 1993
- Inefficiencies of long-short factors
- Sample errors for estimating MVE
- Need for long-only solutions

► Contribution

- Risk-based optimization on DNS opportunity sets span a single-index model, other MVE candidates (market-cap and other risk-based strategies) and improves a 3-factor model
- Risk-based strategies on DNS opportunity sets have incremental significance for pricing characteristics-sorted portfolios
- **Dependent**-sorted portfolios provide a better investment opportunity set to investors compared to **independent**-sorted portfolio

► Robustness

- Out-of-sample, multiple testing

END...

Thank you for your attention!
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DIVERSIFICATION RETURN: BOOTSTRAP

Method:

- ▶ block-bootstrap method from Politis and Romano (1992)
- ▶ studentized test statistic following Ledoit and Wolf (2008)

Bootstrap:

1. block length = 10 observations (robust to other length {2, 4, 6, 8, 10})
2. match the length of the original time-series (630 observations)
3. randomly resample with replacement the original time-series
4. keep the same sequence for all assets in each sample (cross-dependence)
5. 4999 simulations similar to Ledoit and Wolf (2008)

TESTING THE INCREMENTAL DIVERSIFICATION RETURN

Hypothesis testing the spread in Sharpe ratio between the strategy i and j ,

$$\hat{\Delta} = \frac{\hat{\mu}_i}{\hat{\sigma}_i} - \frac{\hat{\mu}_j}{\hat{\sigma}_j} \quad (11)$$

Assumption:

Difference between the first and second moments of the distributions between the two series converge towards zero

$$\sqrt{T}(\hat{u} - u) \xrightarrow{d} N(0, \Omega) \quad (12)$$

- ▶ $\hat{u} = (\hat{\mu}_i, \hat{\mu}_j, \hat{\sigma}_i^2, \hat{\sigma}_j^2)$ are the sample estimates of $u = (\mu_i, \mu_j, \sigma_i^2, \sigma_j^2)$
- ▶ \xrightarrow{d} refers to the convergence in distribution of the parameters
- ▶ Ω not valid when returns exhibit non-normal distribution or serial autocorrelation

Solution:

$$\sqrt{T}(\hat{v} - v) \xrightarrow{d} N(0, \Psi) \quad (13)$$

where $\hat{v} = (\hat{\mu}_i, \hat{\mu}_j, \hat{\gamma}_i, \hat{\gamma}_j)$ is the sample estimates of $v = (\mu_i, \mu_j, \gamma_i, \gamma_j)$, $\hat{\gamma}_i = E(r_i^2)$ and $\hat{\gamma}_j = E(r_j^2)$ and a HAC kernel estimate of Ψ .

DIVERSIFICATION RETURN: HYPOTHESIS TESTING

Spread in Sharpe ratio:

$$f(a, b, c, d) = \frac{a}{\sqrt{c - a^2}} - \frac{b}{\sqrt{d - b^2}} \quad (14)$$

With $a = \hat{\mu}_i$, $b = \hat{\mu}_j$, $c = \hat{\gamma}_i$, and $d = \hat{\gamma}_j$.

Gradient of this function (delta-method) is

$$\nabla' f(\hat{v}) = \left(\frac{c}{(c - a^2)^{1.5}}, -\frac{d}{(d - b^2)^{1.5}}, -\frac{1}{2} \frac{a}{(c - a^2)^{1.5}}, \frac{1}{2} \frac{b}{(d - b^2)^{1.5}} \right)$$

The standard error of delta estimate is,

$$s(\hat{\Delta}) = \sqrt{\frac{\nabla' f(\hat{v}) \hat{\Psi} \nabla f(\hat{v})}{T}} \quad (15)$$

The kernel estimator $\hat{\Psi}$ ensures that the estimation of the standard error is robust to heteroskedasticity and autocorrelation (HAC).

DIVERSIFICATION RETURN: HYPOTHESIS TESTING

Studentized test statistic:

On the original time-series,

$$d = \frac{|\hat{\Delta}|}{s(\hat{\Delta})} \quad (16)$$

On the b -th bootstrap sample,

$$d^b = \frac{|\hat{\Delta}^b - \hat{\Delta}|}{s(\hat{\Delta}^b)} \quad (17)$$

The bootstrap $1-\alpha$ confidence interval is defined as:

$$\left[\hat{\Delta} - z_{|\cdot|, 1-\alpha/2}^b s(\hat{\Delta}), \hat{\Delta} + z_{|\cdot|, 1-\alpha/2}^b s(\hat{\Delta}) \right] \quad (18)$$

with $z_{|\cdot|, 1-\alpha}^b$ the quantile of the distribution of d^b denoted $\mathcal{L}(d^b)$.

p-value:

$$\text{p-val} = \frac{\#\{d^b \geq d\} + 1}{B + 1} \quad (19)$$

TRANSACTION COSTS: GIBBS ESTIMATES

Hasbrouck (2009) extend Roll (1984)'s price dynamics model with a market factor

$$\Delta p_t = c\Delta q_t + \beta_{rm}rm_t + u_t \quad (20)$$

with

$$\begin{aligned} \Delta p_t &= p_t - p_{t-1} \\ &= m_t + cq_t - m_{t-1} - cq_{t-1} \\ &= c\Delta q_t + u_t \end{aligned} \quad (21)$$

- ▶ m_t is the log midpoint of the prior bid-ask price
- ▶ p_t is the log trade price
- ▶ q_t is the sign of the last trade of the day (+1 for a buy and -1 for a sale)
- ▶ u_t is assumed to be unrelated to the sign of the trade (q_t)
- ▶ rm_t is the market return on day t
- ▶ β_{rm} is the slope on the market return
- ▶ c is the effective cost

TRANSACTION COSTS: GIBBS ESTIMATES

$$\Delta p_t = c\Delta q_t + \beta_{rm}rm_t + u_t \quad (22)$$

Iterative Bayesian methodology to estimate the effective costs (c):

1. Initialize q_1 to +1 and σ_u^2 to 0.001.
 - ▶ if no trade $q_t=0$ (in CRSP, $PRC < 0$) else $q_t = \text{sign}(\Delta p_t)$
 - ▶ minimum of 60 to a max 250 daily observations
2. Initialize the distribution from where the values c , β_{rm} , and σ_u^2 will be drawn:
 - ▶ $c \sim N^+ (\mu = 0.01, \sigma^2 = 0.01^2)$
 - ▶ $\beta_{rm} \sim N (\mu = 1, \sigma^2 = 1)$
 - ▶ $\sigma_u^2 \sim IG (\alpha = 10^{-12}, \beta = 10^{-12})$

TRANSACTION COSTS: GIBBS ESTIMATES

for 1 to 1000 sweeps

1. Perform a Bayesian OLS regression on a 250-day of lagged observations to estimate the new values of c and β_{rm} , update the posterior distribution of the parameters and make a new draw of the coefficients.
2. Back out u_t according to $c, \beta_{rm}, \Delta p_t, rm_t, \Delta q_t$

$$u_t = \Delta p_t - \beta_{rm} rm_t - c \Delta q_t \quad (23)$$

► update σ_u^2

3. Draw new series of q_t according to the posterior σ_u^2

$$u_t = \Delta p_t - \beta_{rm} rm_t - c q_t + c q_{t-1} \quad (24)$$

► estimate $u_t(q_t = +1)$ and $u_t(q_t = -1)$ given $u_t \sim N(0, \sigma_u^2)$

$$\text{Odds} = \frac{f(u_t(q_t = +1))}{f(u_t(q_t = -1))} \begin{cases} q_t = +1 & \text{if Odds} > 1 \\ q_t = -1 & \text{if Odds} < 1 \end{cases} \quad (25)$$

end

→ c is the average of the last 800 estimations ("burn in" the 200 first obs.)

PORTFOLIO SORTING AND FACTOR PERFORMANCE

A small exercise:

Sorting on two characteristics with a perfect negative correlation (Lambert, Fays, and Hübner 2018, WP)

- ▶ First sort on market capitalization, ME: SMB factor
- ▶ Second sort on $-ME$: HML factor
- ▶ The factors should be similar

Factor construction	# Obs	Mean (in %)	SD (in %)	t-stat	Correlation Matrix	
Panel A: Independent - 2x3 - NYSE						
SMB	618	0.136	1.651	2.045	1	
HML	618	0.164	1.962	2.078	0.98041***	1
Panel B: Independent - 2x3 - Name						
SMB	618	0.151	2.102	1.783	1	
HML	618	0.204	2.411	2.105	0.97861***	1
Panel C: Independent - 3x3 - NYSE						
SMB	618	0.4	4.105	2.421	1	
HML	618	-0.244	3.248	-1.869	-0.0184	1
Panel D: Independent - 3x3 - Name						
SMB	618	0.705	7.601	2.304	1	
HML	618	-0.097	4.112	-0.585	0.0579	1

PORTFOLIO SORTING AND FACTOR PERFORMANCE

A small exercise:

Sorting on two characteristics with a perfect negative correlation (Lambert, Fays, and Hübner 2018, WP)

Panel A: Independent - 2x3 - NYSE									
	SL	SM	SH	BL	BM	BH			
Average (# stocks)	0	416	2226	394	309	0			
Min (# stocks)	0	133	439	182	124	0			
Max (# stocks)	0	696	4085	638	525	0			
Panel B: Independent - 2x3 - Name									
	SL	SM	SH	BL	BM	BH			
Average (# stocks)	0	671	977	1021	674	0			
Min (# stocks)	0	177	255	269	177	0			
Max (# stocks)	0	1152	1726	1727	1151	0			
Panel C: Independent - 3x3 - NYSE									
	SL	SM	SH	BL	BM	BH	ML	MM	MH
Average (# stocks)	0	1	2226	393	0	0	1	724	0
Min (# stocks)	0	1	439	181	0	0	1	256	0
Max (# stocks)	0	2	4085	637	0	0	1	1220	0
Panel D: Independent - 3x3 - Name									
	SL	SM	SH	BL	BM	BH	ML	MM	MH
Average (# stocks)	0	1	977	1021	0	0	1	1345	0
Min (# stocks)	0	1	255	268	0	0	1	353	0
Max (# stocks)	0	1	1726	1726	0	0	1	2302	0

PORTFOLIO SORTING AND FACTOR PERFORMANCE

A small exercise:

Sorting on two characteristics with a perfect negative correlation (Lambert, Fays, and Hübner 2018, WP)

- ▶ First sort on market capitalization, ME: SMB factor
- ▶ Second sort on $-ME$: HML factor
- ▶ The factors should be similar

Factor construction	# Obs	Mean (in %)	SD (in %)	t-stat	Correlation Matrix	
Panel E: Dependent - 2x3 - NYSE						
SMB	618	0.138	1.264	2.706	1	
HML	618	0.193	2.351	2.038	0.88938***	1
Panel F: Dependent - 2x3 - Name						
SMB	618	0.117	1.367	2.128	1	
HML	618	0.231	2.36	2.434	0.8237***	1
Panel G: Dependent - 3x3 - NYSE						
SMB	618	0.104	1.476	1.743	1	
HML	618	0.104	1.478	1.748	0.9993***	1
Panel H: Dependent - 3x3 - Name						
SMB	618	0.190	1.765	2.670	1	
HML	618	0.193	1.763	2.724	0.99953***	1