Computing the k-binomial complexity of the Thue–Morse word

January 17, 2019

Marie Lejeune (FNRS grantee)

Plan

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Definition

A *morphism* on the alphabet A is an application

$$\sigma : A^* \to A^*$$

such that, for every word $u = u_1 \cdots u_n \in A^*$,

$$\sigma(u) = \sigma(u_1) \cdots \sigma(u_n).$$

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If there exists a letter $a \in A$ such that $\sigma(a)$ begins by a, and if $\lim_{n\to+\infty} |\sigma^n(a)| = +\infty$, then one can define

$$\sigma^\omega(a) = \lim_{n\to+\infty} \sigma^n(a).$$

This word is called a *fixed point* of the morphism σ.

Example (Thue–Morse)

Let us define the *Thue–Morse morphism*

$$\varphi : \{0,1\}^* \to \{0,1\}^* : \begin{cases} 0 \mapsto 01 = 0\overline{0}; \\ 1 \mapsto 10 = 1\overline{1}. \end{cases}$$

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We have

$$\begin{aligned} \varphi(0) &= 01, \\ \varphi^2(0) &= 0110, \\ \varphi^3(0) &= 01101001, \\ &\dots \end{aligned}$$

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We have

$$\begin{aligned} \varphi(0) &= 01, \\ \varphi^2(0) &= 0110, \\ \varphi^3(0) &= 01101001, \\ &\dots \end{aligned}$$

We can thus define the *Thue–Morse word* as one of the fixed points of the morphism φ :

$$\mathbf{t} := \varphi^\omega(0) = 0110100110010110\cdots$$

Remark

Since $\mathbf{t}$ is a fixed point of φ, we have

$$\mathbf{t} = \varphi(\mathbf{t}) = \varphi^2(\mathbf{t}) = \varphi^3(\mathbf{t}) = \cdots .$$

Hence, every factor of $\mathbf{t}$ can be written as

$$p\varphi^k(z)s,$$

where $k \geq 1$, p (resp., s) is a proper suffix (resp., prefix) of one of the words in $\{\varphi^k(0), \varphi^k(1)\}$, and z is also a factor of $\mathbf{t}$.

Plan

1. Preliminary definitions
   - Morphisms and the Thue–Morse word
   - Complexity functions
   - k-binomial complexity
2. Why to compute $\mathbf{b}_t^{(k)}$?
3. Computing the function $\mathbf{b}_\mathbf{t}^{(k)}$
   - Binomial coefficients of (iterated) images
   - Factorizations of order k
   - Types of order k

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Definition

Let $u = u_1 \cdots u_m \in A^m$ be a word ($m \in \mathbb{N}^+ \cup \{\infty\}$).
A *(scattered) subword of* u is a finite subsequence of the sequence $(u_j)_{j=1}^m$.
A *factor of* u is a subword made with consecutive letters.
Otherwise stated, every (non empty) factor of u is of the form $u_i u_{i+1} \cdots u_{i+\ell}$, with $1 \leq i \leq m$, $0 \leq \ell \leq m - i$.

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Example

Let $u = 0102010$.
The word 021 is a subword of u, but it is not a factor of u.

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The word 0201 is a factor of u, thus also a subword of u.

Let $\binom{u}{x}$ denote the number of times x appears as a subword in u and $|u|_x$ the number of times it appears as a factor in u.

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The *factor complexity* of the word w is the function

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where $u \sim_= v \Leftrightarrow u = v$.

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Let us compute the first values of the Thue–Morse's factor complexity. We have

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Then, for every $n \geq 3$, it is known that

$$p_{\mathbf{t}}(n) = \begin{cases} 4n - 2 \cdot 2^m - 4, & \text{if } 2 \cdot 2^m < n \leq 3 \cdot 2^m; \\ 2n + 4 \cdot 2^m - 2, & \text{if } 3 \cdot 2^m < n \leq 4 \cdot 2^m. \end{cases}$$

Different equivalence relations from $\sim_=$ can be considered :

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If $k \in \mathbb{N}^+$,

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We will most of the time deal with the last one.

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Let u and x be two words. The *binomial coefficient* $\binom{u}{x}$ is the number of times that x appears as a subword in u.

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Example

The words $u = bbaabb$ and $v = babbab$ are 2-binomially equivalent. Indeed,

$$\binom{u}{a} = 1 = \binom{v}{a}.$$

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The words $u = bb\color{red}{a}a\color{red}{b}b$ and $v = b\color{red}{a}b\color{red}{b}ab$ are 2-binomially equivalent. Indeed,

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Proposition

For all words u, v and for every nonnegative integer k,

$$u \sim_{k+1} v \Rightarrow u \sim_k v.$$

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The words u and v are 1-abelian equivalent if

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If w is an infinite word, we can define the function

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Example

For the Thue–Morse word $\mathbf{t}$, we have $\mathbf{b}_{\mathbf{t}}^{(1)}(0) = 1$ and, for every $n \geq 1$,

$$\mathbf{b}_{\mathbf{t}}^{(1)}(n) = \begin{cases} 3, & \text{if } n \equiv 0 \pmod{2}; \\ 2, & \text{otherwise.} \end{cases}$$

Example (proof)

- If $n = 2\ell$, every factor of $\mathbf{t}$ is of the form $\varphi(z)$ (with $z \in \mathsf{Fac}_{\mathbf{t}}(\ell)$) or of one of the following forms, where $z' \in \mathsf{Fac}_{\mathbf{t}}(\ell - 1)$:

$$0\varphi(z')0,\ 0\varphi(z')1,\ 1\varphi(z')0,\ 1\varphi(z')1.$$

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We have

$$\binom{\varphi(z)}{0} = \binom{0\varphi(z')1}{0} = \binom{1\varphi(z')0}{0} = \ell,$$

$$\binom{0\varphi(z')0}{0} = \ell + 1 \quad \text{and} \quad \binom{1\varphi(z')1}{0} = \ell - 1,$$

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hence $\mathbf{b}_{\mathbf{t}}^{(1)}(n) = 3$.

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- If $n = 2\ell - 1$, every factor of $\mathbf{t}$ is of one of the following forms, where $z' \in \mathrm{Fac}_t(\ell - 1)$:

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hence $\mathbf{b}_{\mathbf{t}}^{(1)}(n) = 2$.

Plan

We have an order relation between the different complexity functions.

Proposition

$$\rho_w^{ab}(n) \leq \mathbf{b}_w^{(k)}(n) \leq \mathbf{b}_w^{(k+1)}(n) \leq p_w(n) \quad \forall n \in \mathbb{N}, k \in \mathbb{N}^+$$

where ρ_w^{ab} is the abelian complexity function of the word w.

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Moreover, a lot of properties about the factor complexity are known.

Theorem (Morse–Hedlund)

Let w be an infinite word on an ℓ-letter alphabet. The three following assertions are equivalent.

1. The word w is ultimately periodic : there exist finite words u and v such that $w = u \cdot v^\omega$.
2. There exists $n \in \mathbb{N}$ such that $p_w(n) < n + \ell - 1$.
3. The function p_w is bounded by a constant.

One natural application of the previous theorem is to define aperiodic words with the minimal factor complexity.

Definition

A *Sturmian word* is an infinite word having, as factor complexity, $p(n) = n + 1$ for all $n \in \mathbb{N}$.

One natural application of the previous theorem is to define aperiodic words with the minimal factor complexity.

Definition

A *Sturmian word* is an infinite word having, as factor complexity, $p(n) = n + 1$ for all $n \in \mathbb{N}$.

Let w be a Sturmian word. We have, for every $n \geq 2$,

$$n < p_w(n) < p_{\mathbf{t}}(n).$$

However, results are quite different when regarding the k-binomial complexity function.

Theorem (M. Rigo, P. Salimov)

Let w be a Sturmian word. We have $\mathbf{b}_w^{(2)}(n) = p_w(n) = n + 1$.

Thus, since $\mathbf{b}_w^{(k)}(n) \leq \mathbf{b}_w^{(k+1)}(n) \leq p_w(n)$, we obtain

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This is not the case for the Thue–Morse word.

Theorem (M. Rigo, P. Salimov)

For every $k \geq 1$, there exists a constant $C_k > 0$ such that, for every $n \in \mathbb{N}$,

$$\mathbf{b}_{\mathbf{t}}^{(k)}(n) \leq C_k.$$

This result holds for every infinite word which is a fixed point of a Parikh-constant morphism.

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A morphism $\sigma : A^* \to A^*$ is *Parikh-constant* if, for all $a, b, c \in A$, $|\sigma(a)|_c = |\sigma(b)|_c$. Otherwise stated, images of the different letters have to be equal up to a permutation.

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Example

The morphism

$$\sigma : \{0,1,2\}^* \to \{0,1,2\}^* : \begin{cases} 0 & \mapsto & 0112; \\ 1 & \mapsto & 1201; \\ 2 & \mapsto & 1120; \end{cases}$$

is Parikh-constant.

Theorem (M. L., J. Leroy, M. Rigo)

Let k be a positive integer. For every $n \leq 2^k - 1$, we have

$$\mathbf{b}_{\mathbf{t}}^{(k)}(n) = p_{\mathbf{t}}(n),$$

while for every $n \geq 2^k$,

$$\mathbf{b}_{\mathbf{t}}^{(k)}(n) = \begin{cases} 3 \cdot 2^k - 3, & \text{if } n \equiv 0 \pmod{2^k}; \\ 3 \cdot 2^k - 4, & \text{otherwise.} \end{cases}$$

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Cases where $k = 1$ or $k = 2$ can be computed by hand. We will thus assume that $k \geq 3$.

Plan

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1. Preliminary definitions
   - Morphisms and the Thue–Morse word
   - Complexity functions
   - k-binomial complexity
2. Why to compute $\mathbf{b}_t^{(k)}$?
3. Computing the function $\mathbf{b}_{\mathbf{t}}^{(k)}$
   - Binomial coefficients of (iterated) images
   - Factorizations of order k
   - Types of order k

All our reasonings need to compute certain binomial coefficients explicitely. We thus need some tools.

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Proposition

Let u, v be some finite words over A and let a, b be letters of A. We have

$$\binom{ua}{vb} = \binom{u}{vb} + \delta_{a,b}\binom{u}{v},$$

where $\delta_{a,b}$ equals 1 if $a = b$, 0 otherwise.

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Proposition

Let u, u' be some finite words over A, and let $v = v_1 \cdots v_m$ be a word in A^*. We have

$$\binom{uu'}{v} = \sum_{j=0}^{m} \binom{u}{v_1 \cdots v_j}\binom{u'}{v_{j+1} \cdots v_m}.$$

Example

Let us first illustrate the computation of a coefficient $\binom{p\varphi^k(z)s}{v}$ on an example.

$$\binom{0\varphi^3(011)1}{01} =$$

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How could we compute coefficients of the form $\binom{\varphi(u)}{v}$ and, more generally, $\binom{\varphi^\ell(u)}{v}$?

Each time a factor 01 or 10 occurs in v, either we can see it appearing in $\varphi(u)$ as the image of a unique letter of u, or we can see it appearing as a subword of the image of two different letters of u.
We will thus study the different factorizations of v.

Definition : φ-factorization

Let v be a finite word over $A = \{0, 1\}$. If v contains at least one factor in $\{01, 10\}$, it can be factorized as follows :

$$\begin{aligned} v &= w_0 a_1 \overline{a_1} w_1 \cdots w_{\ell-1} a_\ell \overline{a_\ell} w_\ell \\ &= w_0 \varphi(a_1) w_1 \cdots w_{\ell-1} \varphi(a_\ell) w_\ell \end{aligned}$$

where $\ell \geq 1$, $a_1, \ldots, a_\ell \in A$ and $w_0, \ldots w_\ell \in A^*$.

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where $\ell \geq 1$, $a_1, \ldots, a_\ell \in A$ and $w_0, \ldots w_\ell \in A^*$.
This factorization is called a φ-*factorization* of v and is coded by the tuple

$$\kappa = (|w_0|, |w_0 \varphi(a_1) w_1|, \ldots, |w_0 \varphi(a_1) w_1 \ldots \varphi(a_{\ell-1}) w_{\ell-1}|).$$

The set of all tuples coding φ-factorizations of v is denoted by φ-Fac(v).

Example

Let $v = 01101$. The tree of all φ-factorizations of v is the following.

$$\boxed{01101}$$

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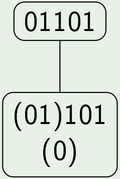

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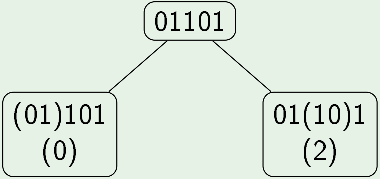

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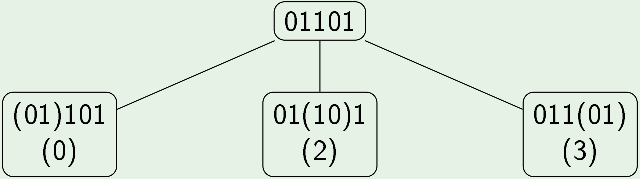

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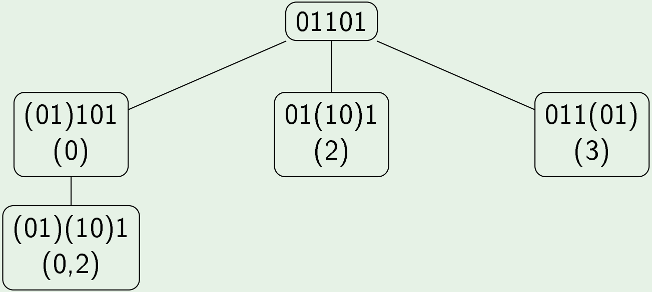

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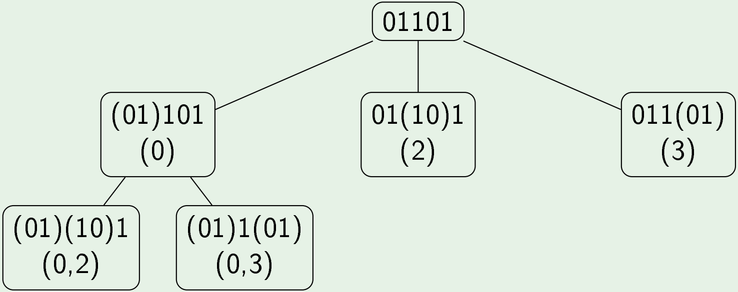

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$$\binom{\varphi(01101001)}{01101} = \binom{|u|}{5}$$

- The 5 letters of v come from 5 different letters of u.
  This case could correspond to the trivial factorization $\kappa = ()$.

Let us illustrate the computation of $\binom{\varphi(u)}{v}$ on an example. Let us compute $\binom{\varphi(01101001)}{01101}$.

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- The 5 letters of v come from 5 different letters of u.
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- The two first letters of v come from the image (by φ) of a letter 0 in u, while the three last ones come from three different letters of u. This case corresponds to $\kappa = (0)$.

Let us illustrate the computation of $\binom{\varphi(u)}{v}$ on an example. Let us compute $\binom{\varphi(01101001)}{01101}$.

$$\binom{\varphi(01101001)}{01(10)1} = \binom{|u|}{5} + \sum_{z \in A^3} \binom{u}{0z} + \sum_{z \in A^2, z' \in A} \binom{u}{z1z'}$$

- The 5 letters of v come from 5 different letters of u.
  This case could correspond to the trivial factorization $\kappa = ()$.
- The two first letters of v come from the image (by φ) of a letter 0 in u, while the three last ones come from three different letters of u. This case corresponds to $\kappa = (0)$.
- Letters v_3 and v_4 come from a block $\varphi(1)$ while the three other ones come from different letters of u. The associated factorization is $\kappa = (2)$.

$$\binom{\varphi(01101001)}{011(01)} = \binom{|u|}{5} + \sum_{z\in A^3}\binom{u}{0z} + \sum_{z\in A^2, z'\in A}\binom{u}{z1z'} + \sum_{z\in A^3}\binom{u}{z0}$$

- Letters v_4 and v_5 come from a block $\varphi(0)$ in u, which corresponds to the factorization $\kappa = (3)$.

$$\binom{\varphi(01101001)}{(01)(10)1} = \binom{|u|}{5} + \sum_{z\in A^3} \binom{u}{0z} + \sum_{z\in A^2, z'\in A} \binom{u}{z1z'} + \sum_{z\in A^3} \binom{u}{z0} + \sum_{z\in A} \binom{u}{01z}$$

- Letters v_4 and v_5 come from a block $\varphi(0)$ in u, which corresponds to the factorization $\kappa = (3)$.
- Letters v_1 and v_2 come from a block $\varphi(0)$ while v_3 and v_4 come from $\varphi(1)$. The associated factorization is $\kappa = (0, 2)$.

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- Letters v_4 and v_5 come from a block $\varphi(0)$ in u, which corresponds to the factorization $\kappa = (3)$.
- Letters v_1 and v_2 come from a block $\varphi(0)$ while v_3 and v_4 come from $\varphi(1)$. The associated factorization is $\kappa = (0, 2)$.
- Letters v_1 and v_2 come from a block $\varphi(0)$, exactly like v_4 and v_5. The associated factorization is $\kappa = (0, 3)$.

We will associate to every φ-factorization $\kappa \in \varphi\text{-Fac}(v)$ of the form

$$w_0\varphi(a_1)w_1 \cdots w_{\ell-1}\varphi(a_\ell)w_\ell,$$

the language

$$\mathcal{L}(v, \kappa) := A^{|w_0|}a_1A^{|w_1|} \cdots A^{|w_{\ell-1}|}a_\ell A^{|w_\ell|},$$

in such a way that $v = w_0\varphi(a_1)w_1 \cdots w_{\ell-1}\varphi(a_\ell)w_\ell$ (factorized in this way) can be seen in any $\varphi(z)$, where $z \in \mathcal{L}(v, \kappa)$.

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We then define

$$f(v) = \biguplus_{\kappa\in\varphi\text{-Fac}(v)} \mathcal{L}(v,\kappa)$$

if φ-Fac(v) contains at least one (non trivial) factorization. Otherwise, $f(v) = \emptyset$.
The union $\biguplus$ has to be considered as a multiset union, where the multiplicities of an element are summed up.

Example (continuing)

Let $v = 01101$; we had

$$\varphi\text{-Fac}(v) = \{(0),(2),(3),(0,2),(0,3)\}$$

and we thus obtain

$$f(01101) = \mathcal{L}(v,(0)) \uplus \mathcal{L}(v,(2)) \uplus \mathcal{L}(v,(3)) \uplus \mathcal{L}(v,(0,2)) \uplus \mathcal{L}(v,(0,3)).$$

Reminder

To every φ-factorization of the form $w_0\varphi(a_1)w_1 \cdots w_{\ell-1}\varphi(a_\ell)w_\ell$ coded by $\kappa = (|w_0|, |w_0\varphi(a_1)w_1|, \ldots)$, we associate the language

$$\mathcal{L}(v,\kappa) := A^{|w_0|}a_1A^{|w_1|}\ldots A^{|w_{\ell-1}|}a_\ell A^{|w_\ell|}.$$

Example (continuing)

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and we thus obtain

$$\begin{aligned} f((01)101) &= \mathcal{L}(v,(0)) \uplus \mathcal{L}(v,(2)) \uplus \mathcal{L}(v,(3)) \uplus \mathcal{L}(v,(0,2)) \uplus \mathcal{L}(v,(0,3)) \\ &= 0A^3 \end{aligned}$$

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$$\begin{aligned} f(011(01)) &= \mathcal{L}(v,(0)) \uplus \mathcal{L}(v,(2)) \uplus \mathcal{L}(v,(3)) \uplus \mathcal{L}(v,(0,2)) \uplus \mathcal{L}(v,(0,3)) \\ &= 0A^3 \uplus A^2 1A \uplus A^3 0 \end{aligned}$$

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To every φ-factorization of the form $w_0\varphi(a_1)w_1 \cdots w_{\ell-1}\varphi(a_\ell)w_\ell$ coded by $\kappa = (|w_0|, |w_0\varphi(a_1)w_1|, \ldots)$, we associate the language

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Example (continuing)

Let $v = 01101$; we had

$$\varphi\text{-Fac}(v) = \{(0), (2), (3), (0,2), (0,3)\}$$

and we thus obtain

$$\begin{aligned} f(01101) =& \mathcal{L}(v,(0)) \uplus \mathcal{L}(v,(2)) \uplus \mathcal{L}(v,(3)) \uplus \mathcal{L}(v,(0,2)) \uplus \mathcal{L}(v,(0,3)) \\ =& 0A^3 \uplus A^2 1A \uplus A^3 0 \uplus 01A \uplus 0A0 \\ =& \{0000_2, 0001_1, 0010_3, 0011_2, 0100_2, 0101_1, 0110_3, 0111_2, \\ & 1010_2, 1011_1, 1110_2, 1111_1, 1000_2, 1100_2, 010_2, 011_1, 000_1\}. \end{aligned}$$

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We can now state the formal proposition.

Proposition

For all finite words u and v, we have

$$\binom{\varphi(u)}{v} = \binom{|u|}{|v|} + \sum_{\kappa \in \varphi\text{-Fac}(v)} \sum_{y \in \mathcal{L}(v,\kappa)} \binom{u}{y}.$$

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Example (continuing)

We computed

$$\binom{\varphi(01101001)}{01101} = \binom{|u|}{5} + \sum_{z \in A^3} \binom{u}{0z} + \sum_{z \in A^2, z' \in A} \binom{u}{z1z'} + \sum_{z \in A^3} \binom{u}{z0} + \sum_{z \in A} \binom{u}{01z} + \sum_{z \in A} \binom{u}{0z0}$$

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For all finite words u and v, we have

$$\binom{\varphi(u)}{v} = \binom{|u|}{|v|} + \sum_{\kappa \in \varphi\text{-Fac}(v)} \sum_{y \in \mathcal{L}(v,\kappa)} \binom{u}{y} = \binom{|u|}{|v|} + \sum_{y \in f(v)} m_{f(v)}(y) \binom{u}{y}.$$

Example (continuing)

$$\binom{\varphi(01101001)}{01101} = \binom{|u|}{5} + \sum_{y \in \mathcal{L}(v,(0))} \binom{u}{y} + \sum_{y \in A^2 1 A} \binom{u}{y} + \sum_{z \in A^3} \binom{u}{z0} + \sum_{z \in A} \binom{u}{01z} + \sum_{z \in A} \binom{u}{0z0}.$$

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Applying several times the previous proposition, we can obtain a formula allowing us to compute coefficients of the form $\binom{\varphi^\ell(u)}{v}$.

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For all finite words u, v and for all $\ell \geq 1$, we have

$$\binom{\varphi^\ell(u)}{v} = \sum_{i=0}^{\ell-1} \sum_{y \in f^i(v)} m_{f^i(v)}(y) \binom{|\varphi^{\ell-i-1}(u)|}{|v|} + \sum_{y \in f^\ell(v)} m_{f^\ell(v)}(y) \binom{u}{y}.$$

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Corollary

If u and u' are two finite words of the same length, then, for every finite word v, we have

$$\binom{\varphi^\ell(u)}{v} - \binom{\varphi^\ell(u')}{v} = \sum_{y \in f^\ell(v)} m_{f^\ell(v)}(y) \left[\binom{u}{y} - \binom{u'}{y} \right].$$

Plan

How could we compute $\mathbf{b}_{\mathbf{t}}^{(k)}(n)$? We have to look, for each pair of words $u, v \in \mathrm{Fac}_n(\mathbf{t})$, if $u \sim_k v$ or not.
Recall that every factor u of $\mathbf{t}$ can be written as

$$p\varphi^k(z)s.$$

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Definition : factorization of order k

Let $u \in \mathrm{Fac}(\mathbf{t})$. If there exist $(p, s) \in A^{<2^k} \times A^{<2^k}$, $a, b \in A$ and $z \in \mathrm{Fac}(\mathbf{t})$ such that

- $u = p\varphi^k(z)s$;
- p is a proper suffix of $\varphi^k(a)$;
- s is a proper prefix of $\varphi^k(b)$;

then (p, s) is called a *factorization of order* k of u while the triple (a, z, b) is called a *desubstitution of order* k of u.

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No : the word 010 appears as a factor of $\mathbf{t}$ several times; it can be factorized as $0\varphi(1)$ or as $\varphi(0)0$.

$$\mathbf{t} = 01 \cdot 10 \cdot 10 \cdot 01 \cdot 10 \cdot 01 \cdot 01 \cdot 10 \cdots$$

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Let u be a factor of **t** of length at least $2^k - 1$. The word u has exactly two different factorizations of order k if and only if it is a factor of $\varphi^{k-1}(010)$ or $\varphi^{k-1}(101)$.

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Because we will use this result, we will only consider words of length at least $2^k - 1$.

Exemple

Let us consider the factor $u = 01001011$.

$$\mathbf{t} = \varphi^3(\mathbf{t}) = 01101001 \cdot 10010110 \cdot 10010110 \cdot 01101001 \cdot 10010110 \cdot 01101001 \cdot 01101001 \cdots$$

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Observe that

$$(0, 1001011) = (0, \varphi^2(1)011)$$

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How can we deal with factors having two factorizations ? We will define an equivalence relation on factorizations, in such a way that if a word has two factorizations, these two are equivalent.

Plan

1. Preliminary definitions
   - Morphisms and the Thue–Morse word
   - Complexity functions
   - k-binomial complexity
2. Why to compute $\mathbf{b}_t^{(k)}$?
3. Computing the function $\mathbf{b}_\mathbf{t}^{(k)}$
   - Binomial coefficients of (iterated) images
   - Factorizations of order k
   - Types of order k

Definition : equivalence $\equiv_k$

Let (p_1, s_1) and (p_2, s_2) be couples of $A^{<2^k} \times A^{<2^k}$. These two are equivalent for $\equiv_k$ if there exist $a \in A$, $x, y \in A^*$ such that one of these cases occurs :

1. $|p_1| + |s_1| = |p_2| + |s_2|$ and
   1. $(p_1, s_1) = (p_2, s_2)$;
   2. $(p_1, s_1) = (x\varphi^{k-1}(a), y)$ and $(p_2, s_2) = (x, \varphi^{k-1}(a)y)$;
   3. $(p_1, s_1) = (x, \varphi^{k-1}(a)y)$ and $(p_2, s_2) = (x\varphi^{k-1}(a), y)$;
   4. $(p_1, s_1) = (\varphi^{k-1}(a), \varphi^{k-1}(\bar{a}))$ and $(p_2, s_2) = (\varphi^{k-1}(\bar{a}), \varphi^{k-1}(a))$;
2. $\big|(|p_1| + |s_1|) - (|p_2| + |s_2|)\big| = 2^k$ and
   1. $(p_1, s_1) = (x, y)$ and $(p_2, s_2) = (x\varphi^{k-1}(a), \varphi^{k-1}(\bar{a})y)$;
   2. $(p_1, s_1) = (x\varphi^{k-1}(a), \varphi^{k-1}(\bar{a})y)$ and $(p_2, s_2) = (x, y)$.

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Example (continuing)

The word $u = 01001011$ has the two factorizations $(0, \varphi^2(1)011)$ and $(0\varphi^2(1), 011)$. This corresponds to case (1.3), where $x = 0$, $y = 011$.

Proposition

If a word $u \in A^{\geq 2^k - 1}$ has two factorizations (p_1, s_1) and (p_2, s_2), then these two are equivalent for $\equiv_k$.

Let $u \in A^{\geq 2^k - 1}$. We can thus define the *type of u of order k* as the equivalence class of its factorizations. We denote by (p_u, s_u) the type of order k of u, with $|p_u|$ minimal.

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We can also have two different words having equivalent factorizations. In this case, the two words they come from are k-binomially equivalent. This result is even stronger.

Theorem

Let u and v be two factors of $\mathbf{t}$ of length $n \geq 2^k - 1$. We have

$$u \sim_k v \Leftrightarrow (p_u, s_u) \equiv_k (p_v, s_v).$$

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The reasoning used in the proof can be adapted to show that for all factors $u, v \in \mathrm{Fac}(\mathbf{t})$ of length at most $2^k - 1$, we have $u \not\sim_k v$.
Hence, for all $n \leq 2^k - 1$, for all $k \geq 3$, we have $\mathbf{b}_{\mathbf{t}}^{(k)}(n) = p_{\mathbf{t}}(n)$.

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Corollary

Let $k \geq 3$ and $n \geq 2^k$. We have

$$\mathbf{b}_{\mathbf{t}}^{(k)}(n) = \#(\mathrm{Fac}_n(\mathbf{t})/\sim_k) = \#(\{(p_u, s_u) \ : \ u \in \mathrm{Fac}_n(\mathbf{t})\}/\equiv_k)$$

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The last part of the reasoning consists in computing this quantity. Fix $n \in \mathbb{N}_0$.

For all $\ell \in \{0, \dots, 2^{k-1} - 1\}$, define

$$P_\ell = \{(p_u, s_u) : u \in \mathrm{Fac}_n(\mathbf{t}), |p_u| = \ell \text{ or } |p_u| = 2^{k-1} + \ell\}.$$

Hence,

$$\{(p_u, s_u) \ : \ u \in \mathrm{Fac}_n(\mathbf{t})\} = \bigcup_{\ell=0}^{2^{k-1}-1} P_\ell \quad \text{and} \quad \mathbf{b}_{\mathbf{t}}^{(k)}(n) = \sum_{\ell=0}^{2^{k-1}-1} \#(P_\ell / \equiv_k).$$

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$$P_{\ell_0} = \{(p_u, s_u) : u \in \mathrm{Fac}_n(\mathbf{t}), |s_u| = 0 \text{ or } |s_u| = 2^{k-1}\}.$$

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Denote by λ the quantity $n \bmod 2^k$. We have

$$\#\{0, \ldots, 2^{k-1} - 1\} \setminus \{0, \ell_0\} = \begin{cases} 2^{k-1} - 1, & \text{if } \lambda = 0 \text{ or } \lambda = 2^{k-1}; \\ 2^{k-1} - 2, & \text{otherwise.} \end{cases}$$

Moreover, we can show that

$$\#\left((P_0 \cup P_{\ell_0})/\equiv_k\right) = \begin{cases} 3, & \text{if } \lambda = 0; \\ 2, & \text{if } \lambda = 2^{k-1}; \\ 8, & \text{otherwise}; \end{cases}$$

Moreover, we can show that

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and that, for all $\ell \notin \{0, \ell_0\}$,

$$\#(P_\ell/\equiv_k) = 6.$$

Hence, putting all the information together,

$$\#\left(\{(p_u, s_u) : u \in \mathrm{Fac}_n(\mathbf{t})\}/\equiv_k\right) = \# \bigcup_{\ell=0}^{2^{k-1}-1} P_\ell$$
$$= \begin{cases} 6\,(2^{k-1}-1)+3, & \text{if } \lambda = 0; \\ 6\,(2^{k-1}-1)+2, & \text{if } \lambda = 2^{k-1}; \\ 6\,(2^{k-1}-2)+8, & \text{otherwise,} \end{cases}$$
$$= \begin{cases} 3\cdot 2^k - 3, & \text{if } \lambda = 0; \\ 3\cdot 2^k - 4, & \text{otherwise,} \end{cases}$$

which leads to the result that was announced in the beginning of the talk.

Is there a possible generalisation of our results ?

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The formula used to compute $\binom{\varphi(u)}{v}$ was generalized to an arbitrary non-erasing morphism.

Proposition

Let $\Psi : A^* \to B^*$ be a non-erasing morphism and $u \in A^+, v \in B^+$ be two words.

$$\binom{\Psi(u)}{v} = \sum_{k=1}^{|v|} \sum_{\substack{v_1,\ldots,v_k \in B^+ \\ v=v_1\cdots v_k}} \sum_{a_1,\ldots,a_k \in A} \binom{\Psi(a_1)}{v_1} \cdots \binom{\Psi(a_k)}{v_k} \binom{u}{a_1 \cdots a_k}.$$

Definition

Let $\mathbf{t}_\ell$ be the fixed point $\varphi_\ell^\infty(0)$ on the alphabet $B := \{0, 1, \ldots, \ell - 1\}$, where

$$\varphi_\ell : B^* \to B^* : \begin{cases} 0 \mapsto 01 \cdots (\ell - 1); \\ \cdots \\ i \mapsto i(i+1) \cdots (\ell - 1)01 \cdots (i - 1); \\ \cdots \\ \ell - 1 \mapsto (\ell - 1)01 \cdots (\ell - 2). \end{cases}$$

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Conjecture

Let $k \in \mathbb{N}_0$. We have, for all $n < 3^k$, $\mathbf{b}_{\mathbf{t}_3}^{(k)}(n) = p_{\mathbf{t}_3}(n)$ and, for all $n \geq 3^k$,

$$\mathbf{b}_{\mathbf{t}_3}^{(k)}(n) = \begin{cases} 7 \cdot 3^k - 14, & \text{if } n \equiv 0 \pmod{3^k}; \\ 7 \cdot 3^k - 15, & \text{if } n \equiv 3^{k-1} \text{ or } 2 \cdot 3^{k-1} \pmod{3^k}; \\ 7 \cdot 3^k - 19 & \text{otherwise.} \end{cases}$$