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METRIC TENSORS INVARIANT UNDER SOME SUBGROUPS OF THE CONFORMAL GROUP OF SPACE-TIME

J. BECKERS and M. JAMINON

*Physique Théorique et Mathématique, Université de Liège, Liège, Belgium**

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General metric tensors invariant under some maximal subgroups of the conformal group of space-time are determined through an infinitesimal method based on conformal *point* transformations (discussed by Fulton, Rohrlich and Witten). The cases of the following maximal subgroups are successively studied: SO(3) ⊗ SO(2, 1), O(3, 2) and O(2) ⊗ O(4).

1. Introduction

The conformal invariance requirement is one of the current main properties under study in different fields of particle physics. As it has been known for a long time that Maxwell's equations are invariant[1,2]) not only under the Poincaré group but also under the conformal group of space–time, conformal invariance has also been extended[3–6]) to the *other zero-mass* relativistic wave equations. General references, historical survey and specific considerations can be found in the following works[7–12]).

The very recent contributions taking advantage of or dealing with "conformal symmetry" are numerous and we refer the reader to the specialized literature, Let us only mention the new approach of Fubini and collaborators[13–16]) where solutions invariant under O(3, 2) or O(2) ⊗ O(4) were particularly studied. Furthermore, let us emphasize the current interests of scale and conformal transformations in very high energy strong interaction processes[11]). These transformations also enter in recent attempts of unification of different types of interactions through gauge theories[17–18]): in fact, *nonzero-masses* are then generated through spontaneous breaking of gauge symmetry and so lead to the breaking of conformal symmetry. Thus the conformal group plays a really prominent part in the understanding of our physical world.

* Postal address: Institut de Physique au Sart Tilman, Bâtiment B.5, Université de Liège, B-4000 Liège 1, Belgique.

In this work, we want to analyse what metric tensors are invariant under some *maximal* subgroups of the conformal group of space–time. This study works out an *infinitesimal* method essentially based on the elements of the Fulton–Rohrlich–Witten (F.R.W.) work[8]) associated with *conformal point transformations* of tensor fields and, more particularly, of the metric tensor. Such developments appear as particular cases of a more general study[19]) on invariant tensor fields, realized from a *global* point of view and intimately connected with relatively elaborated elements of differential geometry (differential forms, . . .).

The contents of this paper is distributed as follows: in section 2, we recall some elements of the conformal group and of some of its maximal subgroups of special interest in physics and in our study. Section 3 is devoted to conformal point transformations as discussed by F.R.W.[8]) and to their implications on metric tensors through invariance arguments. So, section 3.1 will consider the general condition of invariance under conformal point transformations; section 3.2 will give its explicit form and, in section 3.3, we shall find an equivalent set of differential equations. Section 4 is essentially an illustration of the preceding considerations when only *Poincaré invariance* is taken into account. In section 5, we apply the method to the following three *maximal* subgroups of the conformal group: $SO(3)\otimes SO(2,1)$ (section 5.1), $O(3,2)$ (section 5.2) and $O(2)\otimes O(4)$ (section 5.3). So, we establish the forms of the more general metric tensors invariant under each of these subgroups. Section 6 contains our conclusions, particularly in connection with recent results[13–16,19]) and with differential geometrical arguments.

2. The conformal group and its maximal subgroups

It is well known[7]) that the conformal group of space–time ($x\equiv\{x^\mu, \mu=0,1,2,3\}=\{t, \boldsymbol{r}\}$) has fifteen essential parameters associated with *four* translations (a^μ), *six* homogeneous Lorentz transformations ($\omega^{\mu\nu}=-\omega^{\nu\mu}$), *one* dilatation ($\rho>0$) and *four* special conformal transformations (c^μ). Explicitly, this 15-parameter conformal group is a transformation group corresponding to the following coordinate transformations: the *in*homogeneous Lorentz transformations*

$$x^{\mu'}=\Lambda^\mu{}_\nu x^\nu+a^\mu \quad (\mu=0,1,2,3), \tag{2.1}$$

* We use the metric tensor $G_M=\{\eta^{\mu\nu}: \eta^{00}=-\eta^{ii}=1\ (i=1,2,3),\ \eta^{\mu\nu}=0\ (\mu\neq\nu)\}$. The summation convention on repeated indices is used throughout this work except specific mention. Let us also notice that we only consider *continuous* transformations, i.e. the restricted part of the corresponding groups.

the dilatations

$$x^{\mu'} = \rho x^{\mu} \quad (\rho > 0), \tag{2.2}$$

and the special conformal transformations

$$x^{\mu'} = \lambda^{-1}(x)(x^{\mu} - c^{\mu}x^2), \tag{2.3}$$

where

$$\lambda(x) = 1 - 2(c \cdot x) + c^2x^2, \tag{2.4}$$

$$x^2 = x^{\mu}x_{\mu} = t^2 - r^2, \quad r^2 = x^2 + y^2 + z^2.$$

Now if, as usual, the set $\{P^{\mu}, M^{\mu\nu} \equiv (\boldsymbol{J}, \boldsymbol{K})\}$, where $\boldsymbol{J}$ and $\boldsymbol{K}$ refer respectively to true rotations and Lorentz boosts represents the (10) generators of the Poincaré group and if we denote by D and C^{μ} the generators associated with dilatations and special conformal transformations respectively, the Lie algebra of the conformal group is[9]:

$$\begin{aligned}
&[M^{\mu\nu}, M^{\rho\sigma}] = \mathrm{i}(\eta^{\rho\mu}M^{\sigma\nu} + \eta^{\sigma\nu}M^{\rho\mu} - \eta^{\sigma\mu}M^{\rho\nu} - \eta^{\rho\nu}M^{\sigma\mu}),\\
&[M^{\mu\nu}, P^{\rho}] = \mathrm{i}(\eta^{\rho\nu}P^{\mu} - \eta^{\rho\mu}P^{\nu}), \quad [P^{\mu}, P^{\nu}] = 0,\\
&[D, M^{\mu\nu}] = 0, \quad [D, P^{\mu}] = -\mathrm{i}P^{\mu}, \quad [D, C^{\mu}] = \mathrm{i}C^{\mu},\\
&[C^{\mu}, C^{\nu}] = 0, \quad [C^{\mu}, P^{\nu}] = -2\,\mathrm{i}(\eta^{\mu\nu}D + M^{\mu\nu}),\\
&[C^{\rho}, M^{\mu\nu}] = \mathrm{i}(\eta^{\rho\mu}C^{\nu} - \eta^{\rho\nu}C^{\mu}).
\end{aligned} \tag{2.5}$$

The O(4, 2) structure of this conformal group is very well known[7,9]. Furthermore, as a part of a general program for *subalgebra structure analyses*[20–23], the subalgebras of the conformal Lie algebra have been recently studied. More particularly, all the *maximal* subalgebras were identified[22] and their substructures completely determined[21,23]. There are *nine* O(4, 2) *conjugacy classes* of maximal subalgebras of the conformal algebra. They correspond to the following types[22] of Lie groups: Sim(3, 1), Opt(3, 1), O(3, 2), O(4, 1), S[U(1) ⊗ U(2, 1)], O(2) ⊗ O(4), O(2) ⊗ O(2, 2), O(3) ⊗ O(2, 1) and O(2, 1) ⊗ O(2, 1), orders of which are respectively n = 11, 10, 10, 10, 9, 7, 7, 6 *and* 6.

The present study (cf. section 5) shall concern more particularly *three maximal subgroups* among the nine preceding ones: these are

a) SO(3) ⊗ SO(2, 1): $\{\boldsymbol{J}, P^0, D, C^0\}$, (2.6)

b) O(3, 2): $\{\boldsymbol{J}, \boldsymbol{K}, P^{\mu} + C^{\mu}\}$, (2.7)

c) O(2) ⊗ O(4): $\{\boldsymbol{J}, \boldsymbol{P} - \boldsymbol{C}, P^0 + C^0\}$, (2.8)

where {...} refers to the corresponding sets of generators, their commutation relations being easily deduced from the general ones (2.5) of the conformal group. Evidently, the six other maximal subgroups can be studied in a complete parallel way but, for those cases, we refer to the global method and results[19]). The specific choice of the subgroups (2.6), (2.7) and (2.8) is made in correspondence with the physical interests proper to them[24]) and to recent developments[13-16]).

3. Conformal point transformations and metric tensors[8])

3.1. General condition of invariance under conformal point transformations

According to F.R.W.'s notations*, we will denote by $x, \bar{x}, \dots$ different points of a four dimensional Riemannian space in the *same* coordinate system S while usually $x \equiv \{x^\mu\}$, $x' \equiv \{x^{\mu'}\}, \dots$ will refer to components of the same point x measured in *different* coordinate systems $S, S', \dots$. Then, it can be shown that an "active" conformal point transformation implies, on the metric tensor $G(x) \equiv \{g_{\mu\nu}(x)\}$, the property:

$$g_{\mu\nu}(\bar{x})\partial_\alpha \bar{x}^\mu \partial_\beta \bar{x}^\nu = \sigma(x) g_{\alpha\beta}(x), \quad \partial_\alpha \equiv \frac{\partial}{\partial x^\alpha}, \tag{3.1}$$

with the restriction that the scalar function $\sigma(x)$ is positive definite. Such a relation can be used[8]) as a *defining equation* for conformal point transformations when x and $\bar{x}$ refer to two different points in the same coordinate system. Now, let us impose that $g_{\mu\nu}(x)$ are the components of a *covariant* tensor which has to be invariant under conformal point transformations. If we define[8]) $\bar{g}_{\alpha\beta}(x)$ as:

$$\bar{g}_{\alpha\beta}(x) = \partial_\alpha \bar{x}^\mu \partial_\beta \bar{x}^\nu g_{\mu\nu}(\bar{x}), \tag{3.2}$$

this invariance implies that:

$$\bar{g}_{\alpha\beta}(x) = g_{\alpha\beta}(x), \tag{3.3}$$

that is

$$g_{\mu\nu}(\bar{x}) = \bar{\partial}_\mu x^\alpha \bar{\partial}_\nu x^\beta g_{\alpha\beta}(x). \tag{3.4}$$

This relation is the condition of invariance of the metric tensor at the level of its covariant components. After F.R.W., we also note that, in correspondence with (3.4), the contravariant components of the metric tensor $G(x)$ then

* For brevity, we refer to section 3 of ref. 8. Note only that the Minkowski metric used here has signature -2.

transform as:

$$g^{\mu\nu}(\bar{x}) = \sigma^{-2}(x)\partial_\alpha\bar{x}^\mu\partial_\beta\bar{x}^\nu g^{\alpha\beta}(x), \tag{3.5}$$

where the scalar function $\sigma(x)$ is simply given by:

$$\sigma(x) = \tfrac{1}{4}\partial^\alpha\bar{x}_\nu\partial_\alpha\bar{x}^\nu. \tag{3.6}$$

The relations (3.5) and (3.6) will be fundamental in our study: eq. (3.5) corresponds to the general condition of invariance of contravariant components of the metric tensor under conformal point transformations.

3.2. *Explicit form of the general condition* (3.5)

Let us consider an *infinitesimal* conformal[7]) point transformation

$$\begin{aligned} &x \to \bar{x} = x + \xi: \\ &\bar{x}^\mu = x^\mu + \xi^\mu = x^\mu + a^\mu + \omega^\mu{}_\sigma x^\sigma + \rho x^\mu + 2x^\mu(c\cdot x) - c^\mu x^2, \end{aligned} \tag{3.7}$$

where a^μ, $\omega^{\mu\nu}$, ρ and c^μ are small ($\ll 1$) parameters defined in the transformations (2.1)–(2.3). An explicit calculation of $\sigma(x) \equiv (3.6)$ gives:

$$\sigma(x) = 1 + 2\rho + 4(c\cdot x), \tag{3.8}$$

if we only keep first order terms. We also get:

$$\sigma^{-2}(x) = 1 - 4\rho - 8(c\cdot x). \tag{3.9}$$

Then, with eqs. (3.7) and (3.9), the right-hand side of (3.5) becomes:

$$\begin{aligned} g^{\mu\nu}(\bar{x}) &= [1 - 4\rho - 8(c\cdot x)]\{\delta^\mu_\alpha\delta^\nu_\beta[1 + 2\rho + 4(c\cdot x)] + \omega^\nu{}_\sigma\delta^\sigma_\beta\delta^\mu_\alpha \\ &\quad + \omega^\mu{}_\sigma\delta^\sigma_\alpha\delta^\nu_\beta + 2(x\times c)^\nu{}_\beta\delta^\mu_\alpha + 2(x\times c)^\mu{}_\alpha\delta^\nu_\beta\}g^{\alpha\beta}(x) \\ &= [1 - 2\rho - 4(c\cdot x)]g^{\mu\nu}(x) + (\omega \dot{\times} G)^{\mu\nu} + 2[(x\times c)\dot{\times} G]^{\mu\nu}, \end{aligned} \tag{3.10}$$

where we used the well-known definition of the exterior product of two fourvectors:

$$[a\times b]^\rho{}_\sigma = a^\rho b_\sigma - a_\sigma b^\rho, \tag{3.11}$$

and where

$$(\omega\times G)^{\mu\nu} = \omega^\mu{}_\sigma g^{\mu\nu} + \omega^\nu{}_\sigma g^{\sigma\mu}, \tag{3.12}$$

$$[(x\times c)\dot{\times} G]^{\mu\nu} = (x\times c)^\mu{}_\sigma g^{\sigma\nu} + (x\times c)^\nu{}_\sigma g^{\sigma\mu}. \tag{3.13}$$

Furthermore, if we expand the left-hand side of (3.5) by a first order Taylor development (in the neighbourhood of x), we get:

$$g^{\mu\nu}(\bar{x}) = g^{\mu\nu}(x) + [a + \omega\cdot x + \rho x + 2x(c\cdot x) - cx^2]^\sigma\nabla_\sigma g^{\mu\nu}(x), \tag{3.14}$$

so that, by (3.10) and (3.14), the condition (3.5) takes the final form:

$$\mathscr{D}G(x)+\omega \underset{\cdot}{\times} G(x)+2(x\times c)\underset{\cdot}{\times} G(x)-[\rho+2(c\cdot x)][2+(x\cdot\nabla)]G(x)$$
$$+x^2(c\cdot\nabla)G(x)=0, \tag{3.15}$$

where

$$\mathscr{D}\equiv x\cdot\omega\cdot\nabla-a\cdot\nabla, \tag{3.16}$$

with

$$\{\nabla_\mu\}\equiv\{\partial_\mu\}=\left\{\frac{\partial}{\partial t},\frac{\partial}{\partial \boldsymbol{r}}\right\},$$
$$\{\nabla^\mu\}\equiv\{\partial^\mu\}=\left\{\frac{\partial}{\partial t},-\frac{\partial}{\partial \boldsymbol{r}}\right\}, \tag{3.17}$$

and

$$x\cdot\omega\cdot\nabla=-\omega\cdot x\cdot\nabla,$$

as a consequence of the skew-symmetric character of the parameters of the Lorentz transformations. Finally, if as usual we see $\omega=\{\omega^{\mu\nu}\}$ as a bivector $\omega\equiv(\boldsymbol{\phi},\boldsymbol{\theta})$ with

$$\phi^i=\omega^{0i},\quad \theta^i=\tfrac{1}{2}\epsilon^{ijk}\omega_{jk}\quad(\epsilon^{123}=1), \tag{3.18}$$

the differential operator $\mathscr{D}\equiv(3.16)$ can also be written:

$$\mathscr{D}\equiv(\boldsymbol{r}\cdot\boldsymbol{\phi})\frac{\partial}{\partial t}+(t\boldsymbol{\phi}+\boldsymbol{r}\wedge\boldsymbol{\theta})\cdot\frac{\partial}{\partial \boldsymbol{r}}-a\cdot\nabla. \tag{3.19}$$

The equations (3.15) and (3.16) or (3.19) are the necessary and sufficient conditions that the metric tensor $G(x)$ has to satisfy in order to be invariant under the arbitrary (infinitesimal) conformal transformation (3.7).

3.3. *Equivalent set of necessary and sufficient differential equations*

Let us define a specific characterization of the (ten) significative components of the *symmetric* metric tensor $G(x)$. For example, we can choose:

$$G(x)\equiv\{\boldsymbol{\alpha}(x),\boldsymbol{\beta}(x),\boldsymbol{\gamma}(x),\gamma^0\}, \tag{3.20}$$

with

$$\alpha^i=\tfrac{1}{2}(g^{0i}+g^{i0}),$$
$$\beta^i=\tfrac{1}{2}(g^{jk}+g^{kj}),\quad(i=1,2,3) \tag{3.21}$$
$$\gamma^i=g^{ii},\quad\gamma^0=g^{00}.$$

After lengthy but elementary calculations, it is easy to show that these ten "components" of G have to satisfy the following set of equations in correspondence with eq. (3.15):

$$\begin{aligned}\mathscr{D}\alpha^i - \phi^i(\gamma^i+\gamma^0) - (\phi^j\beta^k + \phi^k\beta^j) - (\boldsymbol{\alpha}\wedge\boldsymbol{\theta})^i - 2[\boldsymbol{\alpha}\wedge(\boldsymbol{r}\wedge\boldsymbol{c})]^i \\ - 2t(c^i\gamma^i + c^i\gamma^0 + c^j\beta^k + c^k\beta^j) + 2c^0(x^i\gamma^i + x^i\gamma^0 + x^j\beta^k + x^k\beta^j) \\ - [\rho + 2(c\cdot x)][(x\cdot\nabla)+2]\alpha^i + x^2(c\cdot\nabla)\alpha^i = 0 \quad (i=1,2,3),\end{aligned} \tag{3.22}$$

$$\begin{aligned}\mathscr{D}\beta^i + \theta^k(\gamma^i-\gamma^j) - (\phi^i\alpha^j + \phi^j\alpha^i) + (\boldsymbol{\beta}\wedge\boldsymbol{\theta})^k + 2[\boldsymbol{\beta}\wedge(\boldsymbol{r}\wedge\boldsymbol{c})]^k \\ - 2t(c^i\alpha^j + c^j\alpha^i) + 2c^0(x^i\alpha^j + x^j\alpha^i) + 2(\boldsymbol{r}\wedge\boldsymbol{c})^k(\gamma^i-\gamma^j) \\ - [\rho + 2(c\cdot x)][(x\cdot\nabla)+2]\beta^i + x^2(c\cdot\nabla)\beta^i = 0 \quad (i=1,2,3),\end{aligned} \tag{3.23}$$

$$\begin{aligned}\mathscr{D}\gamma^i - 2\phi^i\alpha^i + 2(\theta^j\beta^j - \theta^k\beta^k) + 4[(\boldsymbol{r}\wedge\boldsymbol{c})^j\beta^j - (\boldsymbol{r}\wedge\boldsymbol{c})^k\beta^k] \\ - 4tc^i\alpha^i + 4c^0x^i\alpha^i - [\rho + 2(c\cdot x)][(x\cdot\nabla)+2]\gamma^i + x^2(c\cdot\nabla)\gamma^i = 0 \\ (i=1,2,3),\end{aligned} \tag{3.24}$$

$$\begin{aligned}\mathscr{D}\gamma^0 - 2\boldsymbol{\phi}\cdot\boldsymbol{\alpha} - 4t(\boldsymbol{c}\cdot\boldsymbol{\alpha}) + 4c^0(\boldsymbol{r}\cdot\boldsymbol{\alpha}) - [\rho + 2(c\cdot x)][(x\cdot\nabla)+2]\gamma^0 \\ + x^2(c\cdot\nabla)\gamma^0 = 0,\end{aligned} \tag{3.25}$$

where there is, here, no summation on repeated indices and where (i, j, k) is an even permutation of $(1, 2, 3)$.

These relations (3.22)–(3.25) and (3.19) form an manageable set of differential equations, equivalent to (3.15) and (3.19), in order to determine an invariant metric tensor. Let us first illustrate the method in a particularly simple case: the determination of the invariant metric tensor under the restricted Poincaré group (section 4), and, secondly, in some cases concerning maximal subgroups of the conformal group (section 5).

4. Invariant metric tensor under the Poincaré group

The Poincaré group can evidently be seen as a nonmaximal subgroup of the conformal group. All the preceding developments are then meaningful when dilatations and special conformal transformations are excluded. So, with $\rho = 0$ and $c^\mu = 0$ $(\mu = 0, 1, 2, 3)$, the condition (3.15) becomes:

$$\mathscr{D}G(x) + \omega \underset{\cdot}{\times} G(x) = 0, \tag{4.1}$$

where $\mathscr{D}$ is given by (3.19). With regard to the set of eqs. (3.22)–(3.25), we get:

$$\mathscr{D}\alpha^i - \phi^i(\gamma^i+\gamma^0) - (\phi^j\beta^k + \phi^k\beta^j) - (\boldsymbol{\alpha}\wedge\boldsymbol{\theta})^i = 0, \tag{4.2}$$

$$\mathscr{D}\beta^i + \theta^k(\gamma^i-\gamma^j) - (\phi^i\alpha^j + \phi^j\alpha^i) + (\boldsymbol{\beta}\wedge\boldsymbol{\theta})^k = 0, \tag{4.3}$$

$$\mathscr{D}\gamma^i - 2\phi^i\alpha^i + 2(\theta^j\beta^j - \theta^k\beta^k) = 0, \tag{4.4}$$

$$\mathscr{D}\gamma^0 - 2\boldsymbol{\phi}\cdot\boldsymbol{\alpha} = 0. \tag{4.5}$$

Now, if we search for a *constant and uniform* metric tensor G (invariance under the translation generators P^{μ}), we remark that:

$$\mathscr{D} \equiv 0, \tag{4.6}$$

and, consequently, we only have to solve the condition:

$$\omega \times G = 0, \tag{4.7}$$

or, equivalently:

$$\phi^i(\gamma^i + \gamma^0) + \phi^j\beta^k + \phi^k\beta^j) + (\boldsymbol{\alpha} \wedge \boldsymbol{\theta})^i = 0, \tag{4.8}$$

$$\theta^k(\gamma^i - \gamma^j) - (\phi^i\alpha^j + \phi^j\alpha^i) + (\boldsymbol{\beta} \wedge \boldsymbol{\theta})^k = 0, \tag{4.9}$$

$$\phi^i\alpha^i - (\theta^j\beta^j - \theta^k\beta^k) = 0, \tag{4.10}$$

$$\boldsymbol{\phi} \cdot \boldsymbol{\alpha} = 0. \tag{4.11}$$

These equations correspond to the invariance conditions of G under the *homogeneous* Lorentz transformations, i.e. under the rotations $\boldsymbol{J} \equiv (J^x, J^y, J^z)$ and the "boosts" $\boldsymbol{K} \equiv (K^x, K^y, K^z)$. They can be solved without difficulty. Invariance under J^x and J^y already gives:

$$\boldsymbol{\alpha} = \boldsymbol{\beta} = 0, \quad \gamma^1 = \gamma^2 = \gamma^3, \tag{4.12}$$

and invariance under K^x leads to:

$$\gamma^0 = -\gamma^1. \tag{4.13}$$

Invariance under J^z, K^y, K^z does not give supplementary constraints. We then conclude that the corresponding invariant metric tensor has to be of the (expected) form:

$$G = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & -a \end{pmatrix} = aG_M \quad (a = \text{constant}). \tag{4.14}$$

Let us add the following comments: eqs. (4.1) and (4.7) are respectively the analogous Combe–Sorba[25]) and Bacry–Combe–Richard[26]) conditions *when symmetric tensor fields* are under study. So, very simple connections between their methods and the present one can be discussed. In fact, as the preceding works[25,26]) refer essentially to the determination of invariant "electromagnetic" *skew-symmetric tensor fields*, such a discussion will take place elsewhere[27]) but it is interesting to point out these correspondences already here. For completeness, let us also mention that other such determinations of electromagnetic fields invariant under certain subgroups of the Poincaré group have already been realized[28,29,30]).

5. Invariant metric tensor under maximal subgroups of O(4, 2)

Let us determine the invariant metric tensors corresponding to the announced three maximal subgroups of O(4, 2). Thus, let us illustrate the infinitesimal method and solve the set of eqs. (3.22)–(3.25) in these specific cases.

Given the corresponding generators (2.6)–(2.8), it has to be noted that all the three subgroups have the true rotations in common. Thus, let us treat this invariance prior to more specific considerations. Invariance under the $\boldsymbol{J} \equiv (J^x, J^y, J^z) \equiv \{J^i, i = 1, 2, 3\}$ corresponds to the following set of parameters:

$$\boldsymbol{\theta}^{(i)} = (\delta^i_1, \delta^i_2, \delta^i_3), \quad \boldsymbol{\phi} = 0, \quad \rho = 0, \quad a^\mu = c^\mu = 0, \tag{5.1}$$

and to the differential operator (3.19) which takes the form:

$$\mathscr{D} \equiv (\boldsymbol{r} \wedge \boldsymbol{\theta}^{(i)} \cdot \frac{\partial}{\partial \boldsymbol{r}} = -\left(\boldsymbol{r} \wedge \frac{\partial}{\partial \boldsymbol{r}}\right)^i \equiv \mathscr{D}^{(i)}. \tag{5.2}$$

The explicitations of eq. (3.22) with the conditions (5.1) and (5.2) give three systems of equations in terms of the α^i $(i = 1, 2, 3)$, leading rather easily to the conclusion that

$$\boldsymbol{\alpha} \wedge \boldsymbol{r} = 0, \tag{5.3}$$

so that

$$\boldsymbol{\alpha}(x) = k(x)\boldsymbol{r}. \tag{5.4}$$

By eq. (3.25) and the conditions (5.1) and (5.2), we immediately get:

$$\mathscr{D}^1\gamma^0 = \mathscr{D}^2\gamma^0 = \mathscr{D}^3\gamma^0 = 0, \tag{5.5}$$

or

$$\boldsymbol{r} \wedge \frac{\partial \gamma^0}{\partial \boldsymbol{r}} = \boldsymbol{r} \wedge \nabla\gamma^0 = 0. \tag{5.6}$$

So, if we recall that

$$\nabla \wedge \gamma^0\boldsymbol{r} = \gamma^0\nabla \wedge \boldsymbol{r} + \boldsymbol{r} \wedge \nabla g^0 = \boldsymbol{r} \wedge \nabla\gamma^0,$$

the condition (5.6) can also be written:

$$\nabla \wedge \gamma^0\boldsymbol{r} = 0,$$

or

$$\gamma^0\boldsymbol{r} = \nabla T(\boldsymbol{r}, t) = \frac{\boldsymbol{r}}{r}\frac{\partial T(r, t)}{\partial r}, \tag{5.7}$$

where T is an arbitrary function of r and t so that we obtain:

$$\gamma^0 = \frac{1}{r}\frac{\partial T(r, t)}{\partial r}. \tag{5.8}$$

Finally, by eqs. (3.23), (3.24) and the conditions (5.1) and (5.2), we can conclude, after some manipulations of the corresponding systems of equations that, if $\boldsymbol{r} \neq 0$, we have:

$$\gamma^1(x) = \gamma^2(x) = \gamma^3(x), \tag{5.9}$$

and, moreover,

$$\boldsymbol{\beta}(x) = 0. \tag{5.10}$$

In conclusion, invariance of the metric tensor under the rotation group leads to the conditions:

$$\boldsymbol{\alpha}(x) = k(x)\boldsymbol{r}, \quad \boldsymbol{\beta}(x) = 0, \quad \gamma^1(x) = \gamma^2(x) = \gamma^3(x),$$
$$\gamma^0(x) = \frac{1}{r}\frac{\partial T(r, t)}{\partial r}. \tag{5.11}$$

5.1. *Metric tensor $G^{(a)}(x)$ invariant under* SO(3) ⊗ SO(2, 1)

This maximal subgroup (of order 6) contains beside the rotation generators, the P^0, D and C^0 ones.

i) *Invariance under P^0* $[a = (a^0, 0, 0, 0) = (1, 0, 0, 0),\ \boldsymbol{\theta} = \boldsymbol{\phi} = 0,\ \rho = 0,\ c^\mu = 0]$ simply implies:

$$\mathscr{D} \equiv -(a \cdot \nabla) = -\frac{\partial}{\partial t}, \quad \frac{\partial \boldsymbol{\alpha}}{\partial t} = \frac{\partial \boldsymbol{\beta}}{\partial t} = \frac{\partial \gamma}{\partial t} = \frac{\partial \gamma^0}{\partial t} = 0, \tag{5.12}$$

so that the arbitrary functions, k [in eq. (5.4)], T [in eq. (5.8)] and the γ^i's [in eq. (5.9)] are time independent.

ii) *Invariance under D* $[\rho > 0,\ \boldsymbol{\theta} = \boldsymbol{\phi} = 0,\ a^\mu = c^\mu = 0]$ leads, through eqs. (3.22)–(3.25) with $\mathscr{D} \equiv 0$, to the simplified relations:

$$(x \cdot \nabla + 2)\alpha = 0, \quad (x \cdot \nabla + 2)\gamma = 0, \quad (x \cdot \nabla + 2)\gamma^0 = 0,$$

or, with (5.12),

$$\left(\boldsymbol{r} \cdot \frac{\partial}{\partial \boldsymbol{r}} + 2\right)\boldsymbol{\alpha} = 0, \quad \left(\boldsymbol{r} \cdot \frac{\partial}{\partial \boldsymbol{r}} + 2\right)\gamma = 0, \tag{5.13}$$

$$\left(\boldsymbol{r} \cdot \frac{\partial}{\partial \boldsymbol{r}} + 2\right)\gamma^0 = 0. \tag{5.14}$$

These Euler equations give the dimension ($d = -2$) of the metric tensor and its homogeneous character, so that, by combining (5.8), (5.12) and (5.14), we immediately deduce that:

$$T(r) = C \ln r + C', \tag{5.15}$$

where C and C' are arbitrary constants, and, that finally:

$$\gamma^0 = C \frac{1}{r^2}. \tag{5.16}$$

iii) *Invariance under* C^0 $[\boldsymbol{\theta} = \boldsymbol{\phi} = 0,\ \rho = 0,\ a^\mu = 0,\ c = (1, 0, 0, 0)]$ gives, through eqs. (3.22) (3.25) and with $\mathscr{D} \equiv 0$,

$$x^i \gamma^i + x^i \gamma^0 = 0, \quad i = 1, 2, 3, \tag{5.17a}$$

$$\boldsymbol{r} \cdot \boldsymbol{\alpha} = 0, \tag{5.17b}$$

in such a way that we obtain, from (5.17a),

$$\gamma^1 = \gamma^2 = \gamma^3 = -\gamma^0 = -C \frac{1}{r^2}, \tag{5.18}$$

and

$$\boldsymbol{\alpha} = 0, \tag{5.19}$$

this last result being obtained by combining (5.4) and (5.17b).

Collecting all these results, we get the final conclusion that the metric tensor invariant under $SO(3) \otimes SO(2, 1)$ is necessarily of the form

$$G^{(a)} = \frac{C}{r^2} G_M, \quad C = \text{constant}. \tag{5.20}$$

5.2. *Metric tensors* $G^{(b)}(x)$ *invariant under* O(3, 2)

This maximal subgroup (of order 10) contains, beside the J^i ($i = 1, 2, 3$), the following generators: $\boldsymbol{K}$ and $P^\mu + C^\mu$ ($\mu = 0, 1, 2, 3$). So, with the precisions (5.11), let us determine $G^{(b)}$ by its invariance conditions under the above seven generators.

i) *Invariance under* K^x *and* K^y is sufficient to establish that

$$\gamma^1 = -\gamma^0, \quad \alpha^2 = \alpha^3 = 0 \quad \text{and} \quad \alpha^1 = 0;$$

we then obtain

$$\gamma^1(x) = \gamma^2(x) = \gamma^3(x) = -\gamma^0(x), \tag{5.21}$$

and

$$\boldsymbol{\alpha} = 0. \tag{5.22}$$

ii) *Invariance under* $P^\mu + C^\mu$ corresponds to the following set of parameters

$$a^{(\mu)} = (\delta_0^\mu, \delta_1^\mu, \delta_2^\mu, \delta_3^\mu), \quad c^{(\mu)} = (\delta_0^\mu, \delta_1^\mu, \delta_2^\mu, \delta_3^\mu),$$
$$\boldsymbol{\theta} = \boldsymbol{\phi} = 0, \quad \rho = 0, \tag{5.23}$$

and to the differential operator (3.19)

$$\mathscr{D}^{(\mu)} \equiv -a^{(\mu)} \cdot \nabla = -\partial_\mu. \tag{5.24}$$

Eqs. (3.22)–(3.25) and the conditions (5.23) and (5.24) lead to the four relations

$$(x^2 - 1)\partial_\mu \gamma^0(x) - 2x_\mu(x \cdot \nabla + 2)\gamma^0(x) = 0, \quad (\mu = 0, 1, 2, 3) \tag{5.25}$$

A few manipulations on this set of eqs. (5.25) give the relation:

$$(x^2 + 1)\frac{\partial \gamma^0(x)}{\partial t} + 4t\gamma^0(x) = 0; \tag{5.26}$$

its general solution is[31]):

$$\gamma^0(x) = C(\boldsymbol{x}) \exp\left[-\int \mathrm{d}t \frac{4t}{t^2 - r^2 + 1}\right] \tag{5.27}$$

or

$$\gamma^0(x) = C(\boldsymbol{x}) \frac{1}{(t^2 - r^2 + 1)^2} = C(\boldsymbol{x}) \frac{1}{(x^2 + 1)^2}. \tag{5.28}$$

If we introduce (5.28) in the original equations (5.25), we get that $C(\boldsymbol{x})$ has to be constant. Collecting this result with (5.11), (5.21), (5.22), and (5.28), we obtain the following final form of the metric tensor $G^{(b)}$ invariant under O(3, 2):

$$G^{(b)} = \frac{C}{(x^2 + 1)^2} G_M, \quad C = \text{constant}. \tag{5.29}$$

5.3. *Metric tensor* $G^{(c)}(x)$ *invariant under* O(2)⊗O(4)

Beside the J^i $(i = 1, 2, 3)$, the maximal subgroup O(2)⊗O(4) of order 7 contains the supplementary generators $\boldsymbol{P} - \boldsymbol{C}$ and $P^0 + C^0$. Taking (5.11) into account, let us consider invariance under these four generators.

i) *Invariance under* $P^i - C^i$ $(i = 1, 2, 3)$ corresponds to the following set of parameters

$$a^{(i)} = (0, \delta_1^i, \delta_2^i, \delta_3^i), \quad c^{(i)} = (0, \delta_1^i, \delta_2^i, \delta_3^i), \quad \boldsymbol{\theta} = \boldsymbol{\phi} = 0, \quad \rho = 0 \tag{5.30}$$

and to the differential operator (3.19)

$$\mathscr{D}^{(i)} \equiv -a^{(i)} \cdot \nabla = -\partial_i. \tag{5.31}$$

From eq. (3.22) when $i = 1$ and from (3.23) and (3.24), we easily deduce

$$\alpha = 0, \quad \gamma^1(x) = \gamma^2(x) = \gamma^3(x) = -\gamma^0(x). \tag{5.32}$$

Moreover, from eq. (3.25), we point out the three following equations on $\gamma^0(x)$:

$$(1 + x^2)\partial_i\gamma^0(x) - 2x_i[(x \cdot \nabla) + 2]\gamma^0(x) = 0, \quad (i = 1, 2, 3). \tag{5.33}$$

ii) *Invariance under* $P^0 + C^0$ corresponds to the set of parameters

$$a = (1, 0, 0, 0), \quad c = (1, 0, 0, 0), \quad \boldsymbol{\theta} = \boldsymbol{\phi} = 0, \quad \rho = 0 \tag{5.34}$$

and to the differential operator (3.19)

$$\mathscr{D} \equiv -\frac{\partial}{\partial t} \tag{5.35}$$

so that eq. (3.25) leads to:

$$(1 - x^2)\frac{\partial}{\partial t}\gamma^0(x) + 2t[(x \cdot \nabla) + 2]\gamma^0(x) = 0. \tag{5.36}$$

By combining eqs. (5.33) and (5.36), we obtain:

$$\frac{(1 - x^2)^2 + 4t^2}{2t}\frac{\partial\gamma^0(x)}{\partial t} + 2(1 + x^2)\gamma^0(x) = 0; \tag{5.37}$$

its general solution is[31])

$$\gamma^0(x) = C(\boldsymbol{x}) \exp\left[-4\int \mathrm{d}t \,\frac{(1 + t^2 - r^2)t}{(1 - t^2 + r^2)^2 + 4t^2}\right] \tag{5.38}$$

or

$$\gamma^0(x) = C(\boldsymbol{x}) \exp\{-\ln[(1 - x^2)^2 + 4t^2]\}$$
$$= \frac{C'(\boldsymbol{x})}{t^2 + \frac{1}{4}(1 - x^2)^2}. \tag{5.39}$$

If we introduce (5.39) in the original equations (5.33) and (5.36), we get that $C'(\boldsymbol{x})$ has to be constant. Collecting this result with (5.11), (5.32) and (5.39), the final form of the metric tensor $G^{(c)}$ invariant under $O(2) \otimes O(4)$ is:

$$G^{(c)} = \frac{C'}{t^2 + \frac{1}{4}(1 - x^2)^2} G_M, \quad C' = \text{constant}. \tag{5.40}$$

6. Conclusions

We thus obtained three different types of metric tensors $G^{(a)} \equiv (5.20)$, $G^{(b)} \equiv (5.29)$ and $G^{(c)} \equiv (5.40)$ invariant under the subgroups $SO(3) \otimes SO(2,1)$, $O(3,2)$ and $O(2) \otimes O(4)$ respectively, these subgroups being maximal ones of the conformal group of space–time. These results correspond to conformally flat metrics related to the general results obtained through the global method[19]).

The group $SO(3) \otimes SO(2,1)$ is by itself an interesting group in connection with the Coulomb problem[24]) and the corresponding metric tensor is well determined by (5.20). With regard to the group $O(3,2)$, Fubini[13]) has shown that, in introducing a fundamental scale of hadron phenomena in the framework of a scale-invariant Lagrangian field theory ($\lambda\phi^4, \ldots$), the "vacuum" is still invariant under this group. One of the main points of his study is the consideration of operators

$$R^{\mu} = \frac{1}{2}\left(aP^{\mu} + \frac{1}{a}C^{\mu}\right) \quad (\mu = 0, 1, 2, 3), \tag{6.1}$$

which do introduce a fundamental length. If, in our study, we consider these R^{μ} and their invariance conditions corresponding to a new set of parameters [in analogy with (5.23)], we get the metric tensor

$$G_a^{(b)} = \frac{4a^2 C}{(a^2 + x^2)^2} G_M, \tag{6.2}$$

which coincides with $G^{(b)}$ if $a = 1$. Eq. (6.2) is also the result obtained by the global method[19]) when general coordinates are introduced.

Analogous considerations can also be made in connection with the group $O(2) \otimes O(4)$ where different combinations of P^{μ} and C^{μ} also appear. If $a \neq 1$, the corresponding metric tensor is then:

$$G_a^{(c)} = \frac{C'}{t^2 + (a^2/4)[1 - (x^2/a^2)]^2} G_M, \tag{6.3}$$

a result which may also be interesting in connection with recent work[16]).

Let us end this section by some comments on differential geometrical elements[32]). From this point of view, the determination of (metric) tensors invariant under some Lie groups of infinitesimal transformations is equivalent to the expression of the fact that the Lie derivative of these tensors with respect to the vector fields X induced by the one-parameter subgroups should vanish. So, this corresponds to

$$\mathscr{L}_X G(x) = \bar{G}(x) - G(x) = 0, \tag{6.4}$$

where X is the vector field associated with the infinitesimal conformal transformation (3.7):

$$X \equiv \xi^{\mu}\partial_{\mu}. \tag{6.5}$$

Applied to a covariant (0, 2) tensor field, the condition (6.4) takes the explicit form:

$$g_{\mu\nu,\alpha}\xi^{\alpha} + g_{\mu\alpha}\xi^{\alpha}{}_{,\nu} + g_{\alpha\nu}\xi^{\alpha}{}_{,\mu} = 0. \tag{6.6}$$

This leads to differential equations describing *local* invariance of the metric tensor $G(x) \equiv \{g_{\mu\nu}(x)\}$, which are equivalent to those established in section 3, but on contravariant components. Note that this differential geometrical point of view has been discussed in some parts of another work[19]) including, besides the metric tensor case, the study of 1-forms, 2-forms and of scalar densities.

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