

# Critical exponents of infinite balanced words

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## Abstract

Over an alphabet of size 3 we construct an infinite balanced word with critical exponent  $2 + \sqrt{2}/2$ . Over an alphabet of size 4 we construct an infinite balanced word with critical exponent  $(5 + \sqrt{5})/4$ . Over larger alphabets, we give some candidates for balanced words (found computationally) having small critical exponents. We also explore a method for proving these results using the automated theorem prover Walnut.

*Key words:* infinite word, balanced word, Sturmian word, critical exponent

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## 1. Introduction

A word  $w$  (finite or infinite) is *balanced* if, for any two factors  $u$  and  $v$  of  $w$  of the same length, the number of occurrences of each alphabet symbol in  $u$  and  $v$  differ by at most 1. Vuillon [25] gives a survey of some of the work done previously on balanced words. It is well-known that over a binary alphabet the class of infinite aperiodic balanced words is exactly the class of Sturmian

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words. Sturmian words have been studied extensively (see the survey by Berstel and Séébold [2]), and in particular, much is known about the repetitions that occur in Sturmian words. The *critical exponent* of an infinite word  $w$  is the supremum of the set of exponents of fractional powers appearing in  $w$ . Damanik and Lenz [4] and Justin and Pirillo [12] gave an exact formula for the critical exponent of a Sturmian word. The Fibonacci word has critical exponent  $(5 + \sqrt{5})/2$  [15], and furthermore, by the formula previously mentioned [4, 12], this is minimal over all Sturmian words. In other words, over a binary alphabet, the least critical exponent among all infinite balanced words is  $(5 + \sqrt{5})/2$ . However, little is known about the critical exponents of infinite balanced words over larger alphabets. In this paper we aim to construct infinite balanced words over a given alphabet having the smallest critical exponent possible. Over an alphabet of size 3 we construct a balanced word with critical exponent  $2 + \sqrt{2}/2$ . Over an alphabet of size 4 we construct a balanced word with critical exponent  $(5 + \sqrt{5})/4$ . Over larger alphabets, we give some candidates for balanced words (found computationally) having small critical exponents. We also explore a method for proving these results using the automated theorem prover Walnut [17].

## 2. Preliminaries

For basic definitions of standard terms used in combinatorics on words, the reader may consult the book by Lothaire [14]. We let  $|w|$  denote the length of a finite word  $w$ , and if  $a$  is a letter of the alphabet, we let  $|w|_a$  denote the number of occurrences of  $a$  in  $w$ .

*Definition 1.* A word  $w$  (finite or infinite) over an alphabet  $A$  is *balanced* if for every  $a \in A$  and every pair  $u, v$  of factors of  $w$  with  $|u| = |v|$  we have

$$||u|_a - |v|_a| \leq 1.$$

Let  $u$  be a finite word and write  $u = u_0u_1 \cdots u_{n-1}$ , where the  $u_i$  are letters. A positive integer  $p$  is a *period* of  $u$  if  $u_i = u_{i+p}$  for all  $i$ . Let  $e = |u|/p$  and let

$z$  be the prefix of  $u$  of length  $p$ . We say that  $u$  has *exponent*  $e$  and write  $u = z^e$ . The word  $z$  is called a *fractional root* of  $u$ . Note that a word may have multiple periods, and consequently, multiple exponents and fractional roots. The word  $u$  is *primitive* if the only integer exponent of  $u$  is 1. Let  $w$  be a finite or infinite word. The largest  $r \in \mathbb{N}$  (if it exists) such that  $u^r$  is a factor of  $w$  is the (integral) *index* of  $u$  in  $w$ .

*Definition 2.* The *critical exponent* of an infinite word  $w$  is

$$\begin{aligned} E(w) &= \sup\{r \in \mathbb{Q} : \text{there is a finite, non-empty factor of } w \text{ with exponent } r\} \\ &= \inf\{r \in \mathbb{Q} : \text{there is no finite, non-empty factor of } w \text{ with exponent } r\}. \end{aligned}$$

The infinite words studied in this paper are constructed by modifying Sturmian words. The structure of such words are determined by a parameter  $\alpha$ , which is an irrational real number between 0 and 1, called the *slope*, and more specifically, by the continued fraction expansion  $\alpha = [0, d_1, d_2, d_3, \dots]$ , where  $d_i \in \mathbb{Z}$  for  $i \geq 0$ .

*Definition 3.* The *characteristic Sturmian word with slope*  $\alpha$  (see [1, Chapter 9]) is the infinite word  $c_\alpha$  obtained as the limit of the sequence of *standard words*  $s_n$  defined by

$$s_0 = 0, \quad s_1 = 0^{d_1-1}1, \quad s_n = s_{n-1}^{d_n} s_{n-2}, \quad n \geq 2.$$

We write  $c_\alpha[i]$  to denote the  $i$ -th letter of  $c_\alpha$ , where we index starting from 1. For  $n \geq 2$ , we also define the *semi-standard words*

$$s_{n,t} = s_{n-1}^t s_{n-2},$$

where  $t \in \mathbb{Z}$  and  $1 \leq t < d_n$ .

We also make use of the *convergents* of  $\alpha$ , namely

$$\frac{p_n}{q_n} = [0, d_1, d_2, d_3, \dots, d_n],$$

where

$$\begin{aligned} p_{-2} &= 0, & p_{-1} &= 1, & p_n &= d_n p_{n-1} + p_{n-2} \text{ for } n \geq 0; \\ q_{-2} &= 1, & q_{-1} &= 0, & q_n &= d_n q_{n-1} + q_{n-2} \text{ for } n \geq 0. \end{aligned}$$

The convergents have the following approximation property:

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}. \quad (1)$$

The following fact is classical:

$$\frac{q_{n+1}}{q_n} = [d_{n+1}, d_n, \dots, d_1]. \quad (2)$$

Another classical result is that the continued fraction expansion for  $\alpha$  is ultimately periodic if and only if  $\alpha$  is a quadratic irrational. We will need the reverse direction of the following result as well.

**Theorem 1** ([13]). *The sequences  $(q_n)_{n \geq 0}$  and  $(p_n)_{n \geq 0}$  satisfy a linear recurrence with constant coefficients if and only if  $\alpha$  is a quadratic irrational. For both sequences the linear recurrence is the same and has the form*

$$q_{n+2s} - tq_{n+s} + (-1)^s q_n = 0, \quad n \geq r, \quad (3)$$

for some  $r, s, t$ .

It is easy to see that  $|s_n| = q_n$  for  $n \geq 1$ . We need a more precise result:

**Lemma 2** ([1, Lemma 9.1.9]). *For  $n \geq 0$  we have  $|s_n|_0 = q_n - p_n$  and  $|s_n|_1 = p_n$ .*

Next we introduce the *Ostrowski  $\alpha$ -numeration system* [19] (see also [1, Section 3.9]). Each non-negative integer  $N$  can be represented uniquely as  $b_j b_{j-1} \cdots b_0$ , where

$$N = \sum_{0 \leq i \leq j} b_i q_i,$$

where the  $b_i$  are integers satisfying:

1.  $0 \leq b_0 < d_1$ ,
2.  $0 \leq b_i \leq d_{i+1}$ , for  $i \geq 1$ , and
3. for  $i \geq 1$ , if  $b_i = d_{i+1}$ , then  $b_{i-1} = 0$ .

We call the string  $b_j b_{j-1} \cdots b_0$  the *Ostrowski  $\alpha$ -representation* of  $N$ . The next two results give the connection between characteristic Sturmian words and the Ostrowski numeration system.

**Theorem 3** ([1, Theorem 9.1.13]). *Let  $N \geq 1$  be an integer with Ostrowski  $\alpha$ -representation  $b_j b_{j-1} \cdots b_0$ . Then the length- $N$  prefix of  $c_\alpha$  is equal to  $s_j^{b_j} s_{j-1}^{b_{j-1}} \cdots s_0^{b_0}$ .*

**Theorem 4** ([1, Theorem 9.1.15]). *Let  $N \geq 1$  be an integer with Ostrowski  $\alpha$ -representation  $b_j b_{j-1} \cdots b_0$ . Then  $c_\alpha[N] = 1$  if and only if  $b_j b_{j-1} \cdots b_0$  ends with an odd number of 0's.*

One characteristic Sturmian word is of particular significance. Let  $\phi = (1 + \sqrt{5})/2$ . The *Fibonacci word* is the characteristic Sturmian word

$$c_\theta = 010010100100101001010010010100 \cdots$$

with slope  $\theta := 1/\phi^2 = [0, 2, \bar{1}]$ . We call the corresponding standard words the *finite Fibonacci words*:

$$f_0 = 0, \quad f_1 = 01, \quad f_2 = 010, \quad \dots$$

It is easy to verify that for each  $i \geq 2$  the finite Fibonacci word  $f_i$  has length  $F_{i+2}$  (the  $(i+2)$ -th Fibonacci number) and has  $F_{i+1}$  0's and  $F_i$  1's. It is also well-known that the infinite Fibonacci word is fixed by the morphism that maps  $0 \rightarrow 01$  and  $1 \rightarrow 0$ . Mignosi and Pirillo [15] showed that  $E(c_\theta) = 2 + \phi$ . The more general results of Damanik and Lenz [4] and Justin and Pirillo [12] show that this is minimal over all Sturmian words.

### 3. Constructing balanced words

We need a characterization of recurrent aperiodic balanced words due to Hubert [11] (Graham [9] gave an equivalent characterization), which is based on the following notion: a word  $y$  has the *constant gap* property if, for each letter  $a$ , there is some number  $d$  such that the distance between successive occurrences of  $a$  in  $y$  is always  $d$ . Constant gap words are periodic, but not all periodic words are constant gap words. For example, the periodic word  $(0120)^\omega$  is *not* a constant gap word, since occurrences of 0's are sometimes at distance 3 from the previous occurrence and sometimes at distance 1. On the other hand, the

word  $(0102)^\omega$  is a constant gap word, since the distance between successive 0's is always 2, and the distance between successive 1's (and similarly 2's) is always 4.

**Theorem 5** ([11]). *A recurrent aperiodic word  $x$  is balanced if and only if  $x$  is obtained from a Sturmian word  $u$  over  $\{0, 1\}$  by the following procedure: Let  $y = y_0y_1 \cdots$  and  $y' = y'_0y'_1 \cdots$  be constant gap sequences over disjoint alphabets  $A$  and  $B$  respectively. For each  $i \geq 0$ , replace the  $i$ -th occurrence of 0 (resp. 1) in  $u$  by the letter  $y_i$  (resp.  $y'_i$ ).*

For example, if  $u$  is a Sturmian word that begins with 01010010100101010010100,  $y = (0102)^\omega$ , and  $y' = (34)^\omega$ , then we have  $x = 03140230410324031042301 \cdots$ .

Note that we shall only consider recurrent infinite words in this paper, since a well-known result of Furstenberg [7] states that for every infinite word  $x$  there exists a uniformly recurrent infinite word  $x'$  such that every factor of  $x'$  is a factor of  $x$ .

For  $3 \leq k \leq 10$  we define an infinite word  $x_k$  constructed “à la Hubert” from a Sturmian word  $c_\alpha$ , where we set  $\alpha$ ,  $y$  and  $y'$  according to Table 1.

$k$	$\alpha$	c.f.	$y$	$y'$
3	$\sqrt{2} - 1$	$[0, \bar{2}]$	$(01)^\omega$	$2^\omega$
4	$1/\phi^2$	$[0, 2, \bar{1}]$	$(01)^\omega$	$(23)^\omega$
5	$\sqrt{2} - 1$	$[0, \bar{2}]$	$(0102)^\omega$	$(34)^\omega$
6	$(78 - 2\sqrt{6})/101$	$[0, 1, 2, 1, 1, \bar{1}, \bar{1}, \bar{2}]$	$0^\omega$	$(123415321435)^\omega$
7	$(63 - \sqrt{10})/107$	$[0, 1, 1, 3, \bar{1}, \bar{2}, \bar{1}]$	$(01)^\omega$	$(234526432546)^\omega$
8	$(23 + \sqrt{2})/31$	$[0, 1, 3, 1, \bar{2}]$	$(01)^\omega$	$(234526732546237526432576)^\omega$
9	$(23 - \sqrt{2})/31$	$[0, 1, 2, 3, \bar{2}]$	$(01)^\omega$	$(234567284365274863254768)^\omega$
10	$(109 + \sqrt{13})/138$	$[0, 1, 4, 2, \bar{3}]$	$(01)^\omega$	$(234567284963254768294365274869)^\omega$

Table 1:  $c_\alpha$  and constant gap words  $y$  and  $y'$  for the construction of  $x_k$

We will show that

$$E(x_3) = 2 + \frac{\sqrt{2}}{2} \text{ and } E(x_4) = 1 + \frac{\phi}{2}.$$

For  $k \geq 5$ , computer calculations suggest that

$$E(x_k) = \frac{k-2}{k-3}.$$

For  $5 \leq k \leq 9$ , we have established via backtracking searches that there are no infinite balanced words with critical exponent less than this value; consequently, if we could prove that  $E(x_k)$  has the claimed value for these alphabet sizes, this would show that this is indeed the least possible critical exponent for balanced words over a  $k$ -letter alphabet. However, the backtracking algorithm used to establish this is not the usual one, but instead involves backtracking over the tree of *standard pairs*. This an (infinite) binary tree with root  $(0, 1)$ ; each vertex  $(u, v)$  has children  $(u, uv)$  and  $(vu, v)$  (see [5, p. 254] for more details). Every finite balanced binary word appears as a factor of either  $u$  or  $v$  for some node  $(u, v)$  appearing in this tree.

Pseudocode for the backtracking search is given in Algorithm 1. We assume the existence of two subroutines: `CriticalExponent( $x$ )`, which computes the largest exponent over all factors of  $x$ , and `GenerateConstantGapWords( $A$ )`, which returns the set<sup>2</sup> of all primitive roots of words with the constant gap property over the alphabet  $A$ . In Step 6 we do not consider  $i > k/2$  because the standard pairs as enumerated contain both the word  $x$  and its complement. If the algorithm with input  $\eta$  and  $k$  terminates, we have proven that no balanced word over a  $k$ -letter alphabet has critical exponent less than  $\eta$ .

#### 4. Establishing the critical exponent

We first observe that if  $x$  is an infinite ternary word obtained from a Sturmian word  $u$  as described by Theorem 5, then  $E(u) \geq E(x) \geq E(u)/2$ . The first

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<sup>2</sup>For more information on how to enumerate this set, see the paper by Goulden et al. [8]. Constant gap words over a  $k$ -letter alphabet are equivalent to *exact covering systems* of size  $k$ . In the paper by Goulden et al., they count a subset of these consisting of the *natural exact covering systems*; however, the numbers of exact covering systems and natural exact covering systems are equal up to size 12.

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**Algorithm 1** Find balanced words with critical exponent  $< \eta$

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1: procedure STANDARDPAIRSBACKTRACK( $\eta, k$ )     $\triangleright k$  is the alphabet size
2:    $Q \leftarrow \text{Queue}((0, 1))$ 
3:   while  $Q$  is non-empty do
4:      $(u, v) \leftarrow Q.\text{dequeue}()$ 
5:      $x \leftarrow$  the longer of  $u$  and  $v$ 
6:     for  $i \leftarrow 1, \dots, \lfloor k/2 \rfloor$  do
7:        $CG1 \leftarrow \text{GenerateConstantGapWords}(\{0, 1, \dots, i-1\})$ 
8:        $CG2 \leftarrow \text{GenerateConstantGapWords}(\{i, i+1, \dots, k-1\})$ 
9:        $x' \leftarrow x$ 
10:      for all  $w \in CG1$  do
11:        for all  $w' \in CG2$  do
12:          for  $j \leftarrow 0, \dots, |x| - 1$  do
13:             $count0 \leftarrow 0; count1 \leftarrow 0$ 
14:            if  $x[j] = 0$  then
15:               $count0 \leftarrow count0 + 1$ 
16:               $x'[j] \leftarrow w[(count0 - 1) \bmod |w|]$ 
17:            else
18:               $count1 \leftarrow count1 + 1$ 
19:               $x'[j] \leftarrow w'[(count1 - 1) \bmod |w'|]$ 
20:            end if
21:          end for
22:          if  $\text{CriticalExponent}(x') < \eta$  then
23:             $Q.\text{enqueue}((u, uv)); Q.\text{enqueue}((vu, v))$ 
24:            goto 3
25:          end if
26:        end for
27:      end for
28:    end for
29:  end while
30: end procedure
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inequality is clear, since the Hubert construction cannot create repetitions in  $x$  of higher exponent than the ones in  $u$ . For the second, note that up to permutation of the alphabet, the only choices for  $y$  and  $y'$  are  $(01)^\omega$  and  $2^\omega$  (or vice-versa), and thus, any repetition in  $u$  of exponent  $e \geq 2$  will produce a repetition in  $x$  of exponent  $e/2$ .

**Proposition 6.** *The word  $x_3$  has critical exponent*

$$E(x_3) = 2 + \frac{\sqrt{2}}{2} \approx 2.7071.$$

*Proof.* As stated in Table 1, let  $\alpha = \sqrt{2} - 1 = [0, \bar{2}]$  and let  $c_\alpha$  be the characteristic Sturmian word with slope  $\alpha$ . That is, the word  $c_\alpha$  is the infinite word obtained as the limit of the sequence of standard words  $s_k$  defined by

$$s_0 = 0, \quad s_1 = s_01, \quad s_k = s_{k-1}^2 s_{k-2}, \quad k \geq 2$$

Let  $x_3$  be the word over  $\{0, 1, 2\}$  obtained from  $c_\alpha$  by replacing the 0's in  $c_\alpha$  by the periodic sequence  $(01)^\omega$  and by replacing the 1's with 2's. We have  $s_1 = 01$ ,  $s_2 = 01010$ ,  $s_3 = (01010)^2 01$ , etc., and

$$c_\alpha = 01010010100101010010100101010010100101001010010 \dots$$

and

$$x_3 = 0212012021021202102120120212012021201202102120 \dots$$

Let  $(z')^e$  be a repetition of exponent  $e \geq 2$  in  $x_3$  ( $e \in \mathbb{Q}$ ). Then, by applying the morphism that sends  $\{0, 1\} \rightarrow 0$  and  $2 \rightarrow 1$  to  $x_3$ , we see that there is a corresponding repetition  $z^e$  of the same length in  $c_\alpha$ . Suppose that  $z$  is primitive. By [20, Corollary 4.6] (originally due to Damanik and Lenz [4])  $z$  is either a conjugate of one of the standard words  $s_k$  defined above or a conjugate of one of the semi-standard words

$$s_{k,1} = s_{k-1} s_{k-2}, \quad k \geq 2.$$

Suppose that  $z$  is a conjugate of a standard word. Note that  $|s_k|_0$  is odd for every  $k \geq 1$ . Hence  $|z|_0$  is odd. It follows that  $z'z'$  cannot occur in  $x_3$  (the

alternations of 0 and 1 will not “match up” in the two copies of  $z'$ ), and so there is no repetition  $(z')^e$  in  $x_3$ .

Now suppose that  $z$  is a conjugate of a semi-standard word. Then  $|z| = q_{k-2} + q_{k-1}$  for some  $k \geq 2$ . From [12, Theorem 4(i)], one finds that the longest factor of  $c_\alpha$  with this period has length  $2(q_{k-2} + q_{k-1}) + q_{k-1} - 2$ . It follows that

$$\begin{aligned} e &\leq \frac{2(q_{k-2} + q_{k-1}) + q_{k-1} - 2}{q_{k-2} + q_{k-1}} \\ &= 2 + \frac{q_{k-1} - 2}{q_{k-2} + q_{k-1}} \\ &= 2 + \frac{1 - 2/q_{k-1}}{1 + q_{k-2}/q_{k-1}}. \end{aligned}$$

It is easy to show that  $q_{k-2} = p_{k-1}$ ; using this fact together with (1) we find that  $q_{k-2}/q_{k-1} > \sqrt{2} - 1 - 1/q_{k-1}^2$ . Thus, we have

$$e < 2 + \frac{1 - 2/q_{k-1}}{\sqrt{2} - 1/q_{k-1}^2}.$$

The fraction on the right clearly tends to  $\sqrt{2}/2$  as  $k \rightarrow \infty$ , and is increasing for  $k \geq 1$ , so the convergence is from below. Thus  $e < 2 + \sqrt{2}/2$ .

Indeed, for every  $k \geq 2$ , there are such factors with exponent

$$2 + \frac{1 - 2/q_{k-1}}{1 + q_{k-2}/q_{k-1}} \xrightarrow{k \rightarrow \infty} 2 + \frac{\sqrt{2}}{2},$$

where the convergence is from below. Note that  $|s_{k,1}|_0$  is even for every  $k \geq 2$  and so every such repetition  $z^e$  in  $c_\alpha$  gives rise to a repetition  $(z')^e$  in  $x_3$ , since  $|z|_0$  in this case is even.

Finally, suppose that  $z$  is not primitive. By [21, Proposition 4.6.12], the critical exponent of  $c_\alpha$  is  $3 + \sqrt{2}$ , so clearly  $z$  cannot have exponent  $\geq 3$ . If  $z$  is a square we have

$$e < \frac{3 + \sqrt{2}}{2} < 2 + \frac{\sqrt{2}}{2}.$$

Thus  $E(x_3) = 2 + \frac{\sqrt{2}}{2}$ . □

Next we show that the exponent in the previous result is the least possible over 3 letters.

**Proposition 7.** *Every balanced word  $x$  over a 3-letter alphabet has critical exponent*

$$E(x) \geq 2 + \frac{\sqrt{2}}{2} \approx 2.7071.$$

*Proof.* Let  $\alpha = [0, d_1, d_2, d_3, \dots]$  and let  $c_\alpha$  be the Sturmian word with slope  $\alpha$ . We substitute the 0's in  $c_\alpha$  with periodic word  $y$  and the 1's with periodic word  $y'$ . We don't have much choice on 3 letters, so let  $y = (01)^\omega$  and  $y' = 2^\omega$ . Let  $x$  be the resulting word and suppose that  $x$  is cubefree.

Consider the standard words  $s_0 = 0$ ,  $s_1 = 0^{d_1-1}1$  and  $s_k = s_{k-1}^{d_k} s_{k-2}$  for  $k \geq 2$ . Let  $k \geq 2$ . We apply [20, Theorem 4.5(iii)] (which is a restatement of a result of [4]), which states that the index of the reversal of  $s_k$  (denoted  $s_k^R$ ) in  $c_\alpha$  is  $d_{k+1} + 2$ .

The first observation is that the number of 0's in  $s_k$  must be odd. If  $|s_k|_0$  is even then  $(s_k^R)^3$  occurs in  $c_\alpha$  and produces a cube in  $x$ , even after replacement of the 0's by  $(01)^\omega$ .

Next we observe that  $d_{k+2}$  must be even. The number of 0's in  $s_{k+2}$  is  $d_{k+2}|s_{k+1}|_0 + |s_k|_0$ , and since  $|s_k|_0$  and  $|s_{k+1}|_0$  are odd, we must have  $d_{k+2}$  even. Now if  $d_{k+2} \geq 4$ , then the index of  $s_{k+1}^R$  is at least 6, and  $(s_{k+1}^R)^6$  will produce a cube in  $x$  after replacement of the 0's with  $(01)^\omega$ .

So  $d_k = 2$  for  $k \geq 4$ ; i.e.,  $\alpha = [0, d_1, d_2, d_3, \bar{2}]$ . As in Proposition 6, we apply [12, Theorem 4(i)], which states that  $c_\alpha$  contains repetitions whose fractional roots are conjugates of the semi-standard words  $s_{k-1}s_{k-2}$  (which have an even number of 0's) and whose exponents are of the form  $2 + (q_k - 2)/(q_{k-1} + q_k)$ . Because the fractional roots have an even number of 0's, after replacement of the 0's by  $(01)^\omega$ , the corresponding factor of the word  $x$  still has exponent  $2 + (q_k - 2)/(q_{k-1} + q_k)$ . A calculation similar to the one in the proof of Proposition 6 shows that as  $k \rightarrow \infty$ , this quantity again converges to  $2 + \sqrt{2}/2$  (regardless of the first few  $d_i$ 's). The same argument (counting parities of 1's this time) applies if  $y = 2^\omega$  and  $y' = (01)^\omega$ .  $\square$

In order to establish the critical exponent for  $x_4$ , we first need a technical

lemma concerning repetitions in the Fibonacci word<sup>3</sup>.

**Lemma 8.** *Let  $w$  be a factor of the Fibonacci word. Write  $w = x^f$ , where  $f \in \mathbb{Q}$  and  $|x|$  is the least period of  $w$ . Suppose that  $w$  has another representation  $w = z^e$ , where  $e \in \mathbb{Q}$ ,  $z$  is primitive, and  $|x| < |z|$ . Then  $e < 1 + \phi/2$ .*

*Proof.* First, recall that the critical exponent of the Fibonacci word is  $2 + \phi$  [15]. Let  $g = \lceil f - 1 \rceil$  and write  $x^f = x^g x'$  (so  $x'$  may possibly equal  $x$ ). We must have  $e < 2$ , since otherwise we would have  $|w| \geq |z| + |x|$ , and so by the Fine–Wilf Theorem (see [1, Theorem 1.5.6]),  $w$  would have period  $\gcd(|x|, |z|)$ , contradicting the minimality of  $x$  ( $|z|$  cannot be a multiple of  $|x|$ , since  $z$  is primitive). We thus write  $w = z z'$ , where  $z'$  is a non-empty prefix of  $z$ . For the same reason we have  $e < 2$  we must also have  $|z'| < |x|$ . We now consider several cases.

Case 1:  $|z'| < |x'|$ . Then  $x'$  has  $z'$  as both a prefix and a suffix. It follows that  $x'$  has period  $|x'| - |z'|$  and hence exponent

$$\frac{|x'|}{|x'| - |z'|} < 2 + \phi.$$

This implies that

$$|z'| < \left(\frac{1 + \phi}{2 + \phi}\right) |x'| < \left(\frac{1 + \phi}{2 + \phi}\right) \cdot \frac{1}{2} |w|.$$

Thus

$$\frac{|z'|}{|w|} < \left(\frac{1 + \phi}{2 + \phi}\right) \cdot \frac{1}{2},$$

and so

$$e = \frac{|w|}{|z|} < \frac{1}{1 - \left(\frac{1 + \phi}{2 + \phi}\right) \cdot \frac{1}{2}} \approx 1.5669 < 1 + \frac{\phi}{2}.$$

Case 2:  $|z'| > |x'|$ . Then we have  $g > 1$ .

---

<sup>3</sup>Arseny Shur has pointed out to us that a result of Pirillo [22] can be used in place of Lemma 8 in the argument that follows to obtain our desired result. Nevertheless, we give the proof of this lemma here, since it motivated us to prove the last results of this section.

Subcase 2a:  $g = 2$  and  $|x'| \geq |x|/2$ . Then since  $|z'| < |x|$ , we have

$$\frac{|z'|}{|w|} < \frac{|x|}{|xx'|} \leq \frac{2}{5},$$

and so

$$e = \frac{|w|}{|z|} \leq \frac{1}{1 - \frac{2}{5}} = \frac{5}{3} < 1 + \frac{\phi}{2}.$$

Subcase 2b:  $g = 2$  and  $|x'| < |x|/2$ . Then  $xx'$  has  $z'$  as both a prefix and a suffix. It follows that  $xx'$  has period  $|xx'| - |z'|$  and hence exponent

$$\frac{|xx'|}{|xx'| - |z'|} < 2 + \phi.$$

This implies that

$$|z'| < \left( \frac{1 + \phi}{2 + \phi} \right) |xx'|.$$

Thus

$$\frac{|z'|}{|w|} < \frac{\left( \frac{1 + \phi}{2 + \phi} \right) |xx'|}{|xx'|} < \left( \frac{1 + \phi}{2 + \phi} \right) \cdot \frac{3}{5},$$

where we have used the fact that  $|x'| < |x|/2$  to obtain the last inequality.

Therefore,

$$e = \frac{|w|}{|z|} < \frac{1}{1 - \left( \frac{1 + \phi}{2 + \phi} \right) \cdot \frac{3}{5}} \approx 1.7673 < 1 + \frac{\phi}{2}.$$

Subcase 2c:  $g = 3$ . Then  $|xx'| < |w|/2$  and  $|z'| < |xx'|$ , so we can apply the argument of Case 1 with  $xx'$  in place of  $x'$ .  $\square$

**Proposition 9.** *The word  $x_4$  has critical exponent  $E(x_4) = 1 + \frac{\phi}{2} \approx 1.8090$ .*

*Proof.* Let  $c_\theta$  be the Fibonacci word and let

$$x_4 = 021031201301203102130120130210310213012 \dots$$

be constructed from  $c_\theta$  as described above with  $y = (01)^\omega$  and  $y' = (23)^\omega$ .

Consider a repetition of the form  $(z')^e$  in  $x_4$ , where  $e \in \mathbb{Q}$  and  $e > 1$ . If this repetition occurs at some position in  $x_4$ , then there is a repetition  $z^e$  of the same length at the same position in  $c_\theta$ . We will assume that  $|(z')^{e-1}| \geq 3$ :

for the case  $|(z')^{e-1}| < 3$ , one can verify with a little thought (or by computer) that  $e < 1 + \phi/2$ . We therefore have that  $z^{e-1}$  contains at least one 0 and one 1, and so  $(z')^{e-1}$  contains at least one letter from  $\{0, 1\}$  and at least one letter from  $\{2, 3\}$ . We claim that

$$|z'|_0 + |z'|_1 \equiv 0 \pmod{2} \text{ and } |z'|_2 + |z'|_3 \equiv 0 \pmod{2}. \quad (4)$$

To see this, note that the first occurrence of a letter from  $\{0, 1\}$  in  $z'$  must match the first occurrence of such a letter in  $z'^{e-1}$ . Since occurrences of 0 and 1 alternate in  $z'$ , the total number of such letters in  $z'$  must therefore be even. The same argument applies to the letters from  $\{2, 3\}$ .

Currie and Saari [3, Theorem 3] characterized the minimal fractional roots of factors of Sturmian words. In the case of the Fibonacci word, this characterization states that the minimal fractional root of any factor of the Fibonacci word is a conjugate of a finite Fibonacci word.

Thus, if  $z$  is the minimal fractional root of the repetition  $z^e$ , then  $|z|$  equals  $F_i$  (the  $i$ -th Fibonacci number) for some  $i$ , and furthermore, we have  $|z|_0 = F_{i-1}$  and  $|z|_1 = F_{i-2}$ . Equation (4) thus implies that

$$F_{i-1} = |z|_0 = |z'|_0 + |z'|_1 \equiv 0 \pmod{2} \text{ and } F_{i-2} = |z|_1 = |z'|_2 + |z'|_3 \equiv 0 \pmod{2},$$

which is impossible.

Now suppose that  $z$  is primitive but is not the minimal fractional root of the repetition  $z^e$ . Then by Lemma 8, we have  $e < 1 + \phi/2$ , as required.

Finally, if  $z$  is not primitive, then there exists a word  $v$  such that  $z = v^2$  or  $z = v^3$ . Thus either  $z^e = v^{2e}$  or  $z^e = v^{3e}$ . Since  $E(c_\theta) = 2 + \phi$ , it follows that either  $2e < 2 + \phi$  or  $3e < 2 + \phi$ . Both cases lead to the inequality  $e < 1 + \phi/2$ , as required.

Finally, note that since  $E(c_\theta) = 2 + \phi$ , there is a sequence of repetitions in  $c_\theta$  with exponents  $2 + r_j/s_j$ , with  $r_j/s_j$  converging to  $\phi$  from below. Consider such a repetition and write it as  $V^{2+r_j/s_j} = Z^{1+r_j/(2s_j)}$ . Then  $V$  is a conjugate of a finite Fibonacci word, and so for some  $i$  we have

$$2F_{i-1} = 2|V|_0 = |Z|_0 \equiv 0 \pmod{2} \text{ and } 2F_{i-2} = 2|V|_1 = |Z|_1 \equiv 0 \pmod{2},$$

so there is a corresponding repetition  $(Z')^{1+r_j/(2s_j)}$  in  $x_4$ . Thus  $E(x_4) = 1 + \phi/2$ .  $\square$

A backtracking computer search (see Algorithm 1.) shows that there is no infinite balanced word over a 4-letter alphabet with critical exponent  $< 1.8088$ . The minimal possible critical exponent among all infinite balanced words over 4 letters is thus between 1.8088 and  $1 + \frac{\phi}{2} \approx 1.8090$ .

Note that an important element of the previous proof was the result of Lemma 8 concerning non-minimal periods of the Fibonacci word. Little seems to be known on this topic, so we conclude this section with the following result.

**Proposition 10.** *Let  $w$  be a factor of a Sturmian word  $s$  and let  $\{p_1 < p_2 < \dots < p_r\}$  be the set of periods of  $w$ . For  $i \in \{1, \dots, r\}$  let  $P_i$  be the prefix of  $w$  of length  $p_i$  and for  $i \geq 2$ , write  $P_i = P_{i-1}E_i$ . Then for  $i \in \{2, \dots, r\}$  the word  $E_i$  is a conjugate of either a standard word  $s_k$  or a semi-standard word  $s_{k,t}$  for some  $k, t$ . Furthermore, the lengths of the  $E_i$  are non-increasing.*

*Proof.* We need the following fact (see [16]): If a word  $w$  has periods  $p$  and  $q$  with  $q < p$ , then the prefix of  $w$  of length  $|w| - q$  has period  $p - q$ . We also use the result of [3, Theorem 3]: If  $w$  is a non-empty factor of  $s$  then the minimal fractional root of  $w$  is a conjugate of a standard or semi-standard word.

For  $i \in \{2, \dots, r\}$ , consider the prefix of  $w$  of length  $|w| - p_{i-1}$ . This prefix has period  $p_i - p_{i-1}$  and corresponding fractional root  $E_i$ . We claim that this is its minimal fractional root. Suppose to the contrary that it has a smaller fractional root of length  $q$ . Then  $w$  has period  $p_{i-1} + q$ , which is less than  $p_i$ , which is a contradiction. By [3, Theorem 3], we have that  $E_i$  is a conjugate of a standard word  $s_k$  or a semi-standard word  $s_{k,t}$ .

To show that the lengths of the  $E_i$  are non-increasing, suppose to the contrary that  $|E_{i+1}| > |E_i|$ . Then the suffix of  $w$  of length  $|w| - p_{i-1}$  has period  $|E_i|$  and so does the suffix of  $w$  of length  $|w| - p_i$ . As  $w$  has period  $p_i$ , it therefore has period  $p_i + |E_i| < p_{i+1}$ , which is a contradiction. This completes the proof.  $\square$

**Corollary 11.** *Let  $w$  be a factor of the Fibonacci word  $c_\theta$  and let  $\{p_1 < p_2 <$*

$\dots < p_r\}$  be the set of periods of  $w$ . For  $i \in \{1, \dots, r\}$  let  $P_i$  be the prefix of  $w$  of length  $p_i$  and for  $i \geq 2$ , write  $P_i = P_{i-1}E_i$ . Then for  $i \in \{2, \dots, r\}$  the word  $E_i$  is a conjugate of a finite Fibonacci word.

For more on words with multiple periods, the reader might consult [23, 24].

## 5. A computational approach

We can also attempt to establish the critical exponents of each  $x_k$  using the Walnut theorem-proving software [17] and the methods of Du, Mousavi, Schaeffer, and Shallit [6, 18]. The method is based on two things. The first is the fact that the Fibonacci word  $c_\theta$  is a *Fibonacci-automatic sequence*: that is, the terms of  $c_\theta$  can be computed by a finite automaton that takes a number  $n$  written in the Fibonacci numeration system as input and outputs the  $n$ -th term of  $c_\theta$ . The second important element to the method of Mousavi et al. is that the addition relation  $\{(x, y, z) \in \mathbb{N}^3 : x + y = z\}$  can be recognized by a finite automaton that reads its input in the Fibonacci numeration system.

To extend this method to an arbitrary characteristic word  $c_\alpha$ , we first need to show that the terms of  $c_\alpha$  can be computed by a finite automaton that takes the Ostrowski  $\alpha$ -representation of  $n$  as input. This is immediate from Theorem 4 above. The second thing we need is the recognizability of the addition relation for the Ostrowski  $\alpha$ -numeration system. Hieronymi and Terry [10] showed that addition is indeed recognizable in the case when  $\alpha$  is a quadratic irrational. We now show that when  $\alpha$  is a quadratic irrational, the Hubert construction applied to  $c_\alpha$  results in a word that is automatic for the Ostrowski  $\alpha$ -numeration system.

**Theorem 12.** *Let  $\alpha$  be a quadratic irrational and let  $c_\alpha$  be the characteristic Sturmian word with slope  $\alpha$ . Let  $x$  be any word obtained by replacing the 0's in  $c_\alpha$  with a periodic sequence  $y$  and replacing the 1's with a periodic sequence  $y'$ . Then  $x$  is Ostrowski  $\alpha$ -automatic.*

*Proof.* Let  $p$  and  $p'$  be the periods of  $y$  and  $y'$  respectively. We need to show that there is a deterministic finite automaton with output that takes the Ostrowski



```
eval periods_of_high_powers "?msd_fib Ei (p>=1) &
(Aj (3*j <= 2*p) => X[i+j]=X[i+j+p])";
```

The language accepted by the resulting automaton is  $0^*1001000^*$ ; i.e., representations of numbers of the form  $F_n + F_{n-3} = 2F_{n-1}$ .

```
reg pows msd_fib "0*1001000*";
```

Next we compute pairs  $(n, p)$  such that  $x_4$  has a factor of length  $n + p$  with period  $p$ , and furthermore that factor cannot be extended to a longer factor of length  $n + p + 1$  with the same period.

```
def maximal_reps "?msd_fib Ei (Aj (j<n) =>
X[i+j]=X[i+j+p]) & (X[i+n]!=X[i+n+p])";
```

We now compute pairs  $(n, p)$  where  $p$  has to be of the form  $0^*1001000^*$  and  $n + p$  is the longest length of any factor having that period.

```
eval highest_powers "?msd_fib (p >= 1) & $pows(p) & $maximal_reps(n,p) &
(Am $maximal_reps(m,p) => m <= n)";
```

The output of this last command is an automaton accepting pairs  $(n, p)$  having the form

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}^* \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}^* \begin{pmatrix} 0 \\ 0 \end{pmatrix} \left\{ \epsilon, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

So when  $p = 2F_{i-1}$  we see that  $n = F_i - 2$ . Thus, the maximal repetitions of “large exponent” in  $x_4$  have exponent of the form  $1 + (F_i - 2)/(2F_{i-1})$ . Using (1) and (2) we find that

$$\frac{F_i - 2}{2F_{i-1}} = \frac{F_i}{2F_{i-1}} - \frac{1}{F_{i-1}} < \frac{1}{2} \left( \phi + \frac{1}{F_{i-1}^2} \right) - \frac{1}{F_{i-1}} < \frac{\phi}{2}.$$

Hence these exponents converge to  $1 + \phi/2$  from below, and we conclude that  $x_4$  has critical exponent  $1 + \phi/2$ .

In principle, if the addition automaton of [10] were implemented in Walnut, it would be possible to carry out a similar proof for  $x_5$ ,  $x_6$ , etc., provided the resulting computation was feasible.

## 6. Future work

The obvious open problem is to establish the claimed critical exponents for  $x_k$  for  $5 \leq k \leq 10$ , and more generally, show that for  $k \geq 5$ , the least critical exponent for an infinite balanced word over a  $k$ -letter alphabet is  $(k-2)/(k-3)$ . A first, possibly difficult, step would be to get a better understanding of the non-minimal periods of factors of Sturmian words.

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