High-Frequency Integral Equation Solver for 3-D Wave Scattering Around Convex Obstacles

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***Résumé* — We present a numerical method for the computation of wave scattering by convex three-dimensional obstacles, which exhibits a bounded computational complexity for arbitrarily high frequencies. The method is based on a combination of a high-frequency ansatz for the unknown density in a boundary integral formulation of the problem, and an extension of the ideas of the method of stationary phase to allow for the efficient integration of oscillatory functions. The method builds upon the ideas proposed in [2] and [1].**

I. INTRODUCTION

Classical convergent numerical methods for solving scattering problems require a given number of discretization points per wavelength λ, and thus necessarily require a computational expense of $O(k^n)$ operations, where $k = 2\pi/\lambda$ denotes the spatial frequency and where n is the dimensionality of the discretized manifold. In particular, an integral-equation time-harmonic surface-scattering solver runs at (at least) a computational cost of $O(k^2)$ operations per simulation—an unacceptably high cost for high-frequency applications (i.e., for large values of k). Asymptotic methods such as the geometrical theory of diffraction [7], on the other hand, use frequency-independent discretizations, and thus require a fixed computational cost, independent of frequency—at least for evaluation of surface currents. Unfortunately, these methods are inherently non-convergent for any given frequency, and are thus unsuitable for applications that require error-controllable solutions. Thus numerical methods have been sought which would remain efficient for large frequencies without compromising error-controllability.

In previous contributions [2, 5, 1] we have presented the main tools for building convergent numerical methods for scattering problems which exhibit a computational cost that is independent of the frequency. In [2, 5] we presented full solutions for scattering by one or several convex two-dimensional obstacles and in [1] we presented a three-dimensional extension of the localized integrator for the singular kernels arising in the surface integral equations.

In this paper we present for the first time full three-dimensional scattering results combining a simpler, fully numerical alternative to the integrator proposed in [1] and a GMRES iterative solver. Numerical results are given on a three-dimensional test case.

II. SURFACE SCATTERING BY CONVEX OBSTACLES

We consider the problem of evaluating the scattering of an incident plane wave $\psi^{\text{inc}}(\boldsymbol{r}) = e^{ik\boldsymbol{\alpha}\cdot\boldsymbol{r}}$, $|\boldsymbol{\alpha}| = 1$, from a convex sound-soft obstacle $D \subseteq \mathbb{R}^3$. We thus look for the solution $\psi(\boldsymbol{r}) = \psi^{\text{inc}}(\boldsymbol{r}) + \psi^{\text{scat}}(\boldsymbol{r})$ of the Helmholtz equation under Dirichlet boundary conditions

$$\Delta\psi(\boldsymbol{r}) + k^2\psi(\boldsymbol{r}) = 0 \quad \text{in} \quad \mathbb{R}^3\backslash\bar{D}, \tag{1}$$

$$\psi = 0 \quad \text{on} \quad \partial D, \tag{2}$$

where the scattered field $\psi^{\text{scat}}(\boldsymbol{r})$ satisfies the Sommerfeld radiation condition [4].

III. HIGH-FREQUENCY INTEGRAL EQUATION

The corresponding scalar combined-field boundary integral equation formulation reads [8]

$$\frac{1}{2}\mu(\boldsymbol{r}) - \int_{\partial D} H(\boldsymbol{r},\boldsymbol{r}')\mu(\boldsymbol{r}')\,ds(\boldsymbol{r}') = \frac{\partial\psi^{\text{inc}}(\boldsymbol{r})}{\partial\boldsymbol{\nu}(\boldsymbol{r})} + \mathrm{i}\gamma\psi^{\text{inc}}(\boldsymbol{r}), \quad \boldsymbol{r}\in\partial D, \tag{3}$$

where

$$H(\boldsymbol{r},\boldsymbol{r}') = \frac{\partial\Phi(\boldsymbol{r},\boldsymbol{r}')}{\partial\boldsymbol{\nu}(\boldsymbol{r})} + \mathrm{i}\gamma\Phi(\boldsymbol{r},\boldsymbol{r}'). \tag{4}$$

In this formulation, $\mu(\boldsymbol{r}) = \partial\psi(\boldsymbol{r})/\partial\boldsymbol{\nu}(\boldsymbol{r})$ denotes the normal derivative of $\psi(\boldsymbol{r})$, $\boldsymbol{\nu}(\boldsymbol{r})$ is the outward unit normal to ∂D at the observation point $\boldsymbol{r}$, γ is a coupling constant and

$$\Phi(\boldsymbol{r},\boldsymbol{r}') = \frac{e^{\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} \tag{5}$$

is the radiating free-space Green function.

When D is convex, our formulation of the problem is based on the observation that, away from shadow regions, the unknown $\mu(\boldsymbol{r})$ oscillates like the incoming wave [9, 6], that is:

$$\mu(\boldsymbol{r}) = \mu_{\text{slow}}(\boldsymbol{r})e^{\mathrm{i}k\boldsymbol{\alpha}\cdot\boldsymbol{r}}. \tag{6}$$

It follows that (3) can be rewritten in the form

$$\frac{1}{2}\mu_{\text{slow}}(\boldsymbol{r}) - \int_{\partial D}\tilde{H}(\boldsymbol{r},\boldsymbol{r}')\mu_{\text{slow}}(\boldsymbol{r}')\,ds(\boldsymbol{r}') = \mathrm{i}(k\boldsymbol{\alpha}\cdot\boldsymbol{\nu}(\boldsymbol{r}) + \gamma), \quad \boldsymbol{r}\in\partial D, \tag{7}$$

with

$$\tilde{H}(\boldsymbol{r},\boldsymbol{r}') = H(\boldsymbol{r},\boldsymbol{r}')e^{\mathrm{i}k\boldsymbol{\alpha}\cdot(\boldsymbol{r}'-\boldsymbol{r})}. \tag{8}$$

Throughout the illuminated region of ∂D, the variations in the slowly oscillatory envelope $\mu_{\text{slow}}(\boldsymbol{r})$ in (7) do not accentuate with increasing frequency and thus, even for very short wavelengths, $\mu_{\text{slow}}(\boldsymbol{r})$ can be represented on a coarse grid.

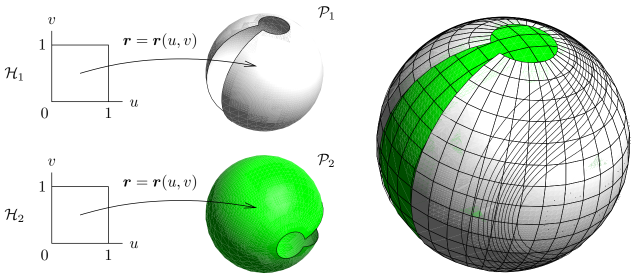

Fig. 1. Two-patch covering of a sphere. Left: mapping from the unit square to the patches $\mathcal{P}_1$ and $\mathcal{P}_2$. Right: coarse Cartesian discretization grids and iso-value lines of the partition of unity on $\mathcal{P}_1$.

IV. Numerical Solution

The slowly oscillatory envelope $\mu_{\text{slow}}(\boldsymbol{r})$ can be represented very efficiently using the patching and smooth-partitioning strategies presented in [3, 1]. The unknowns we seek to obtain when we solve the discretized scattering problem are then the values on Cartesian sets of nodes on the overlapping patches (see Fig. 1).

For $\boldsymbol{r} \neq \boldsymbol{r'}$, the kernel $\tilde{H}(\boldsymbol{r}, \boldsymbol{r}')$ in the boundary integral equation (7) behaves like the kernel of a generalized Fourier integral with phase

$$\phi(\boldsymbol{r}, \boldsymbol{r}') = |\boldsymbol{r} - \boldsymbol{r}'| + \boldsymbol{\alpha} \cdot (\boldsymbol{r}' - \boldsymbol{r}). \qquad (9)$$

It follows that, *asymptotically*, the only significant contributions to the oscillatory integral in (7) arise from values of μ_{slow} and its derivatives at the critical points; in the present context, these critical points are the observation point $\boldsymbol{r}$, where the kernel is singular, and the stationary phase points $\boldsymbol{r}_i^s$, $i = 1, \ldots, N^s(\boldsymbol{r})$, where the gradient $\nabla_{\boldsymbol{r}'}\phi(\boldsymbol{r}, \boldsymbol{r}')$ of the phase (9) vanishes.

In order to obtain a *convergent* (not merely asymptotic) method for *arbitrary frequencies*, we introduced the following localized integration procedure *around* the critical points in [2]: For each observation point $\boldsymbol{r}$ on the surface ∂D the corresponding set of critical points is covered by a number of small regions:

1. The observation point $\boldsymbol{r}$ is covered by a region of radius proportional to the wavelength $\lambda = 2\pi/k$;
2. The i-th stationary phase point $\boldsymbol{r}_i^s$, $i = 1, \ldots, N^s(\boldsymbol{r})$, is covered by a region U_i of radius proportional to $\sqrt[3]{\lambda}$ (at the shadow boundaries) or $\sqrt{\lambda}$ (away from the shadow boundaries).

Plugging a numerical integrator based on the trapezoidal rule in each of these small regions (with a polar change of coordinates to remove the singularity around the observation point $\boldsymbol{r}$) into a GMRES linear solver provides the sought-after scattering solver.

Table I shows the L^2 error on $\mu_{\text{slow}}(\boldsymbol{r})$, i.e.,

$$\left\{ \int_{\partial D} |\mu_{\text{slow}}^{\text{exact}}(\boldsymbol{r}) - \mu_{\text{slow}}(\boldsymbol{r})|^2 \, ds(\boldsymbol{r}) \right\}^{1/2} \Bigg/ \left\{ \int_{\partial D} |\mu_{\text{slow}}^{\text{exact}}(\boldsymbol{r})|^2 \, ds(\boldsymbol{r}) \right\}^{1/2}, \qquad (10)$$

TABLE I. Scattering of a plane wave e^{ikz} by a sphere of radius $R = 1$, with $16 \times 32 \times 2$ grid points. (CPU times are given in seconds, on a single 3 GHz CPU.)

k	L^2 Error	GMRES Iter.	CPU	Memory
1000	0.64%	11	53568 s	147 MB
2000	0.72%	14	44659 s	147 MB
5000	1.56%	15	41852 s	147 MB
10000	1.85%	15	41650 s	147 MB
20000	2.37%	15	45189 s	147 MB
50000	2.76%	14	48784 s	147 MB

when computing the scattering of a plane wave by a unit sphere at wavenumbers ranging from 1,000 to 50,000. We can observe that the accuracy, the computational time and the memory usage remain almost constant in the whole frequency range.

Additional results as well as details about the numerical integrator will be presented in the extended paper.

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